Multivariable Control Systems

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Lecture 5

References are appeared in the last slide.

Limitation on Performance in MIMO Systems

Topics to be covered include:

- A Brief Review of Linear Control Systems
- Scaling and Performance
- Shaping Closed-loop Transfer Functions
- Fundamental Limitation on Performance (Frequency domain)
    - Fundamental Limitation on Sensitivity
    - Limitations Imposed by RHP Zeros
    - Limitations Imposed by Unstable (RHP) Poles
    - Limitations Imposed by Time Delays
- Fundamental Limitation on Performance (Time domain)

A Brief Review of Linear Control Systems

- Time Domain Performance
- Frequency Domain Performance
- Bandwidth and Crossover Frequency

Time Domain Performance

- **Nominal stability NS**: The system is stable with no model uncertainty.
- **Nominal Performance NP**: The system satisfies the performance specifications with no model uncertainty.
- **Robust stability RS**: The system is stable for all perturbed plants about the nominal model up to the worst case model uncertainty.
- **Robust performance RP**: The system satisfies the performance specifications for all perturbed plants about the nominal model up to the worst case model uncertainty.

Time Domain Performance

Although closed loop stability is an important issue,

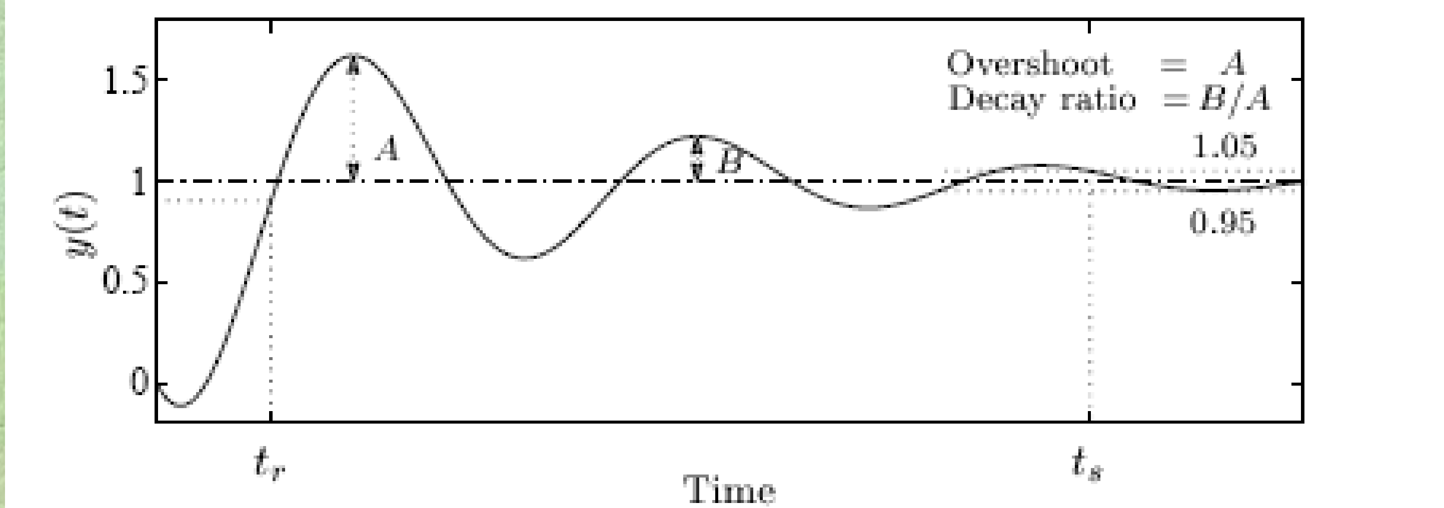
the real objective of control is to improve performance, that is, to make the output y(t) behave in a more desirable manner.

Actually, the possibility of inducing instability is one of the disadvantages of feedback control which has to be traded off against performance improvement.

The objective of this section is to discuss the ways of evaluating closed loop performance.

Time Domain Performance

Step response of a system



- **Rise time, t_r**
- **Settling time, t_s**
- **Overshoot, P.O**
- **Decay ratio**
- **Steady state offset, e_{ss}**

- **ISE** : Integral squared error $$ISE = \int_0^\infty e(\tau)^2 d\tau$$
- **IAE** : Integral absolute error $$IAE = \int_0^\infty |e(\tau)| d\tau$$
- **ITSE** : Integral time weighted squared error $$ITSE = \int_0^\infty \tau e(\tau)^2 d\tau$$
- **ITAE** : Integral time weighted absolutesquared error

Frequency Domain Performance

Let *L(s)* denote the loop transfer function of a system which is closed-loop stable under negative feedback.

$$\angle L(j\omega_{180}) = -180$$

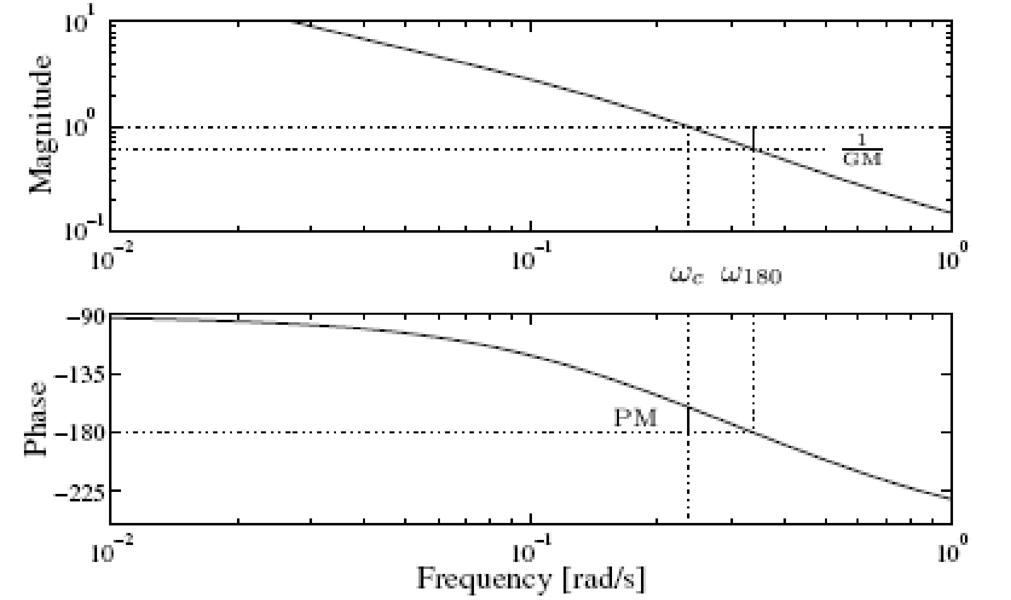
$$GM = \frac{1}{|L(j\omega_{180})|}$$

$$|L(j\omega_c)| = 1$$

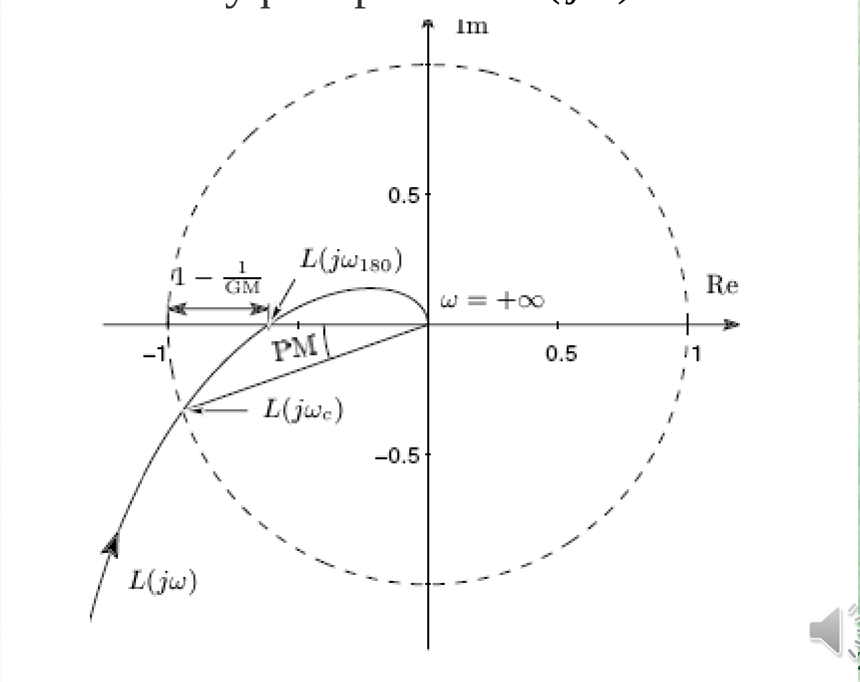
$$PM = \angle L(j\omega_c) + 180$$

$$\theta_{max} = PM / \omega_c$$

Bode plot of $L(j\omega)$



Nyquist plot of $L(j\omega)$



Frequency Domain Performance

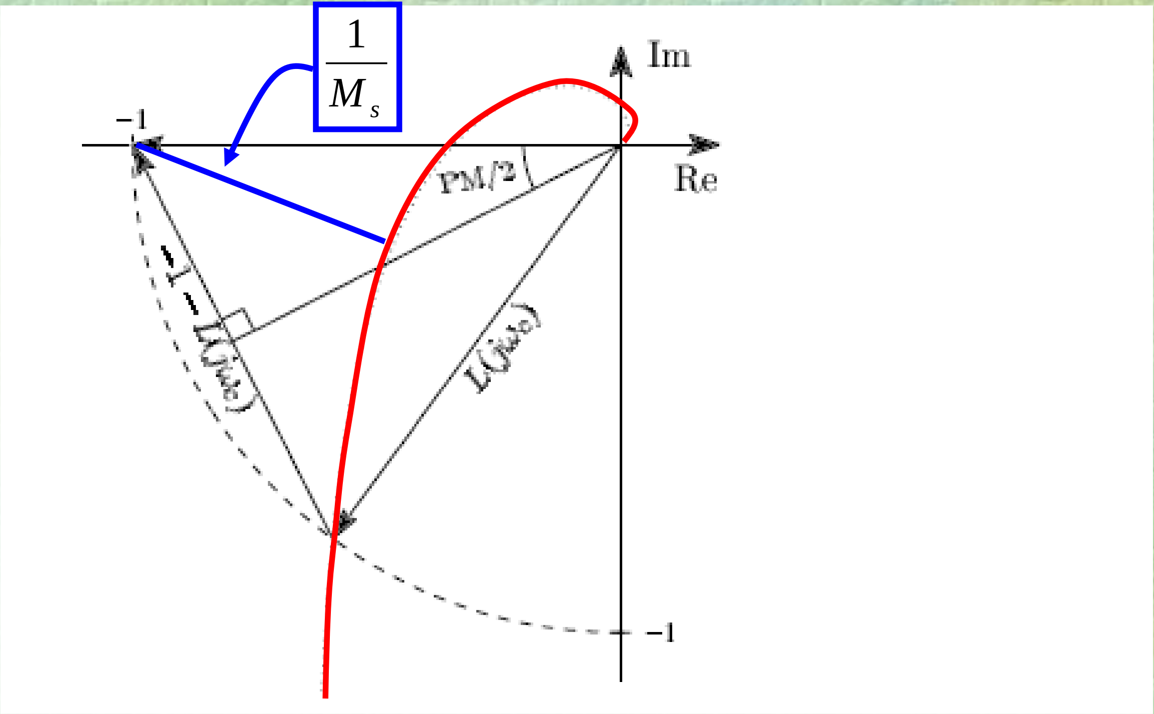
Stability margins are measures of how close a stable closed-loop system is to instability.

From the above arguments we see that the *GM* and *PM* provide stability margins for gain and delay uncertainty.

More generally, to maintain closed-loop stability, the Nyquist stability condition tells us that the number of encirclements of the critical point -1 by $L(j\omega)$ must not change.

Thus the actual closest distance to -1 is a measure of stability

Frequency Domain Performance

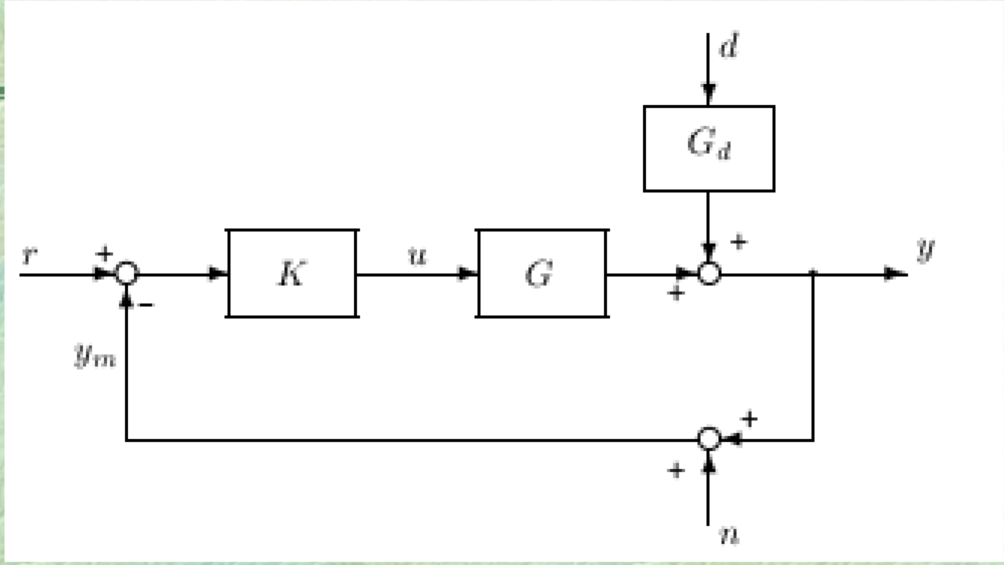


$$M_s = \max_{\omega} |S(j\omega)|$$

Thus one may also view M_s as a robustness measure.

Frequency Domain Performance

One degree-of-freedom configuration



$$y(s) = GK(I+GK)^{-1}r + (I+GK)^{-1}G_d d - GK(I+GK)^{-1}n$$

Complementary sensitivity function $T(s)$

Sensitivity function $S(s)$

The maximum peaks of the sensitivity and complementary sensitivity functions are defined as

$$M_s = \max_{\omega}|S(j\omega)| \qquad M_T = \max_{\omega}|T(j\omega)|$$

Frequency Domain Performance

There is a close relationship between M_S and M_T and the GM and PM.

$$GM \geq \frac{M_s}{M_s - 1}; \qquad PM \geq 2\sin^{-1}\left(\frac{1}{2M_s}\right) \geq \frac{1}{M_s}[rad]$$

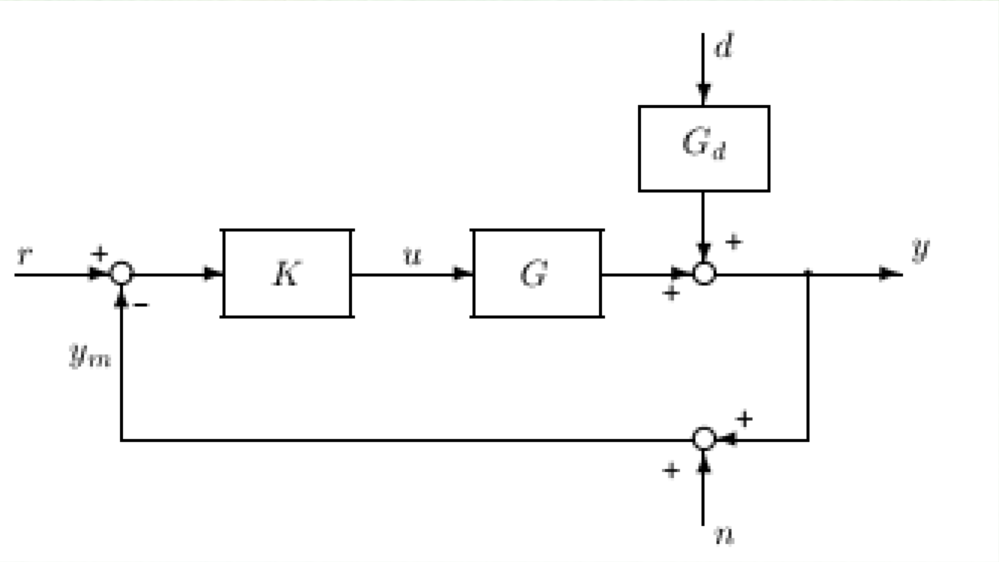
For example, with $M_S = 2$ we are guaranteed GM > 2 and PM >29˚.

$$GM \geq 1 + \frac{1}{M_T}; \qquad PM \geq 2\sin^{-1}\left(\frac{1}{2M_T}\right) \geq \frac{1}{M_T}[rad]$$

For example, with $M_T = 2$ we are guaranteed GM > 1.5 and PM >29˚.

Trade-offs in Frequency Domain

One degree-of-freedom configuration



$$y(s) = GK(I+GK)^{-1} r + (I+GK)^{-1} G_d d - GK(I+GK)^{-1} n$$

Complementary sensitivity function $T(s)$

Sensitivity function $S(s)$

$T(s)$

$$T(s) + S(s) = I$$

$$e = y - r = -Sr + SG_d d - Tn \qquad u = KSr - KSG_d d - KSn$$

Trade-offs in Frequency Domain

$$e = y - r = -Sr + SG_d d - Tn \qquad u = KSr - KSG_d d - KSn$$

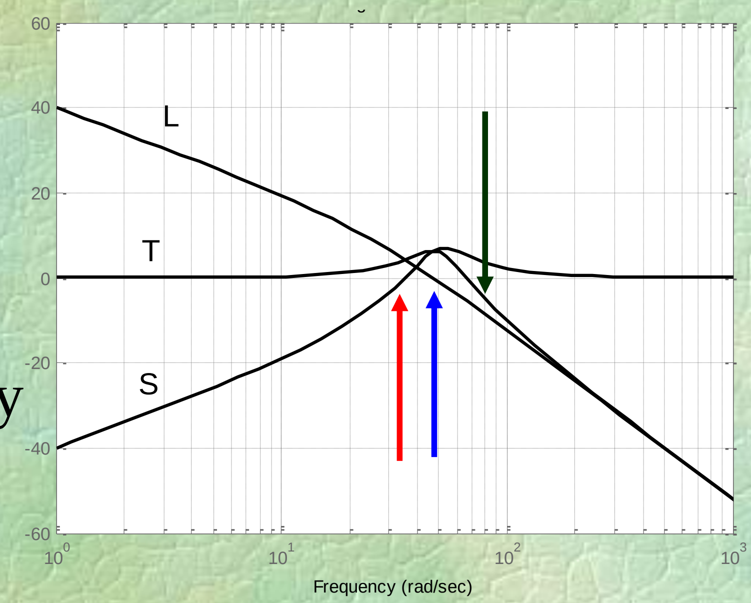
$$T(s) + S(s) = I \qquad S(s) = \left(I + L(s)\right)^{-1}$$

- Performance, good disturbance rejection $S \to 0$ or $T \to I$ or $L \to \infty$
- Performance, good command following $S \to 0$ or $T \to I$ or $L \to \infty$
- Mitigation of measurement noise on output $T \to 0$ or $S \to I$ or $L \to 0$
- Small magnitude of input signals $K \to 0$ or $T \to 0$ or $\mathrm{L} \to 0$
- Physical controller must be strictly proper $K \to 0$ or $L \to 0$ or $\mathrm{T} \to 0$
- Nominal stability (stable plant) L be small
- Stabilization of unstable plant L be large or $T \to I$

Bandwidth and Crossover Frequency

Definition 5-1

The bandwidth, ω_B, is the frequency where $|S(j\omega)|$ first crosses -3 db from below.



Definition 5-2

The bandwidth, ω_{BT}, is the highest frequency at which $|T(j\omega)|$ crosses -3 db from above.

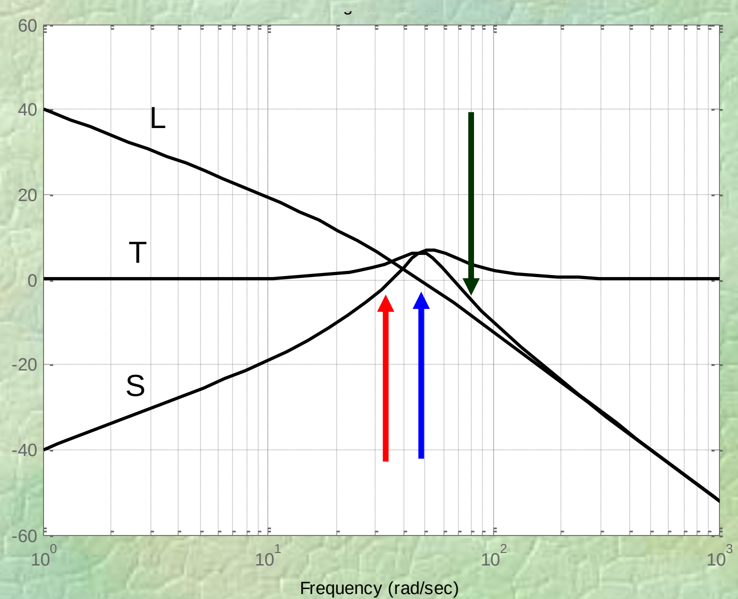
Definition 5-3

The gain crossover frequency, ω_c, is the frequency where $|L(j\omega)|$ crosses 0 db from above.

Bandwidth and Crossover Frequency

Specifically, for systems with PM < 90° (most practical systems) we have

$$\omega_B \le \omega_c \le \omega_{BT}$$



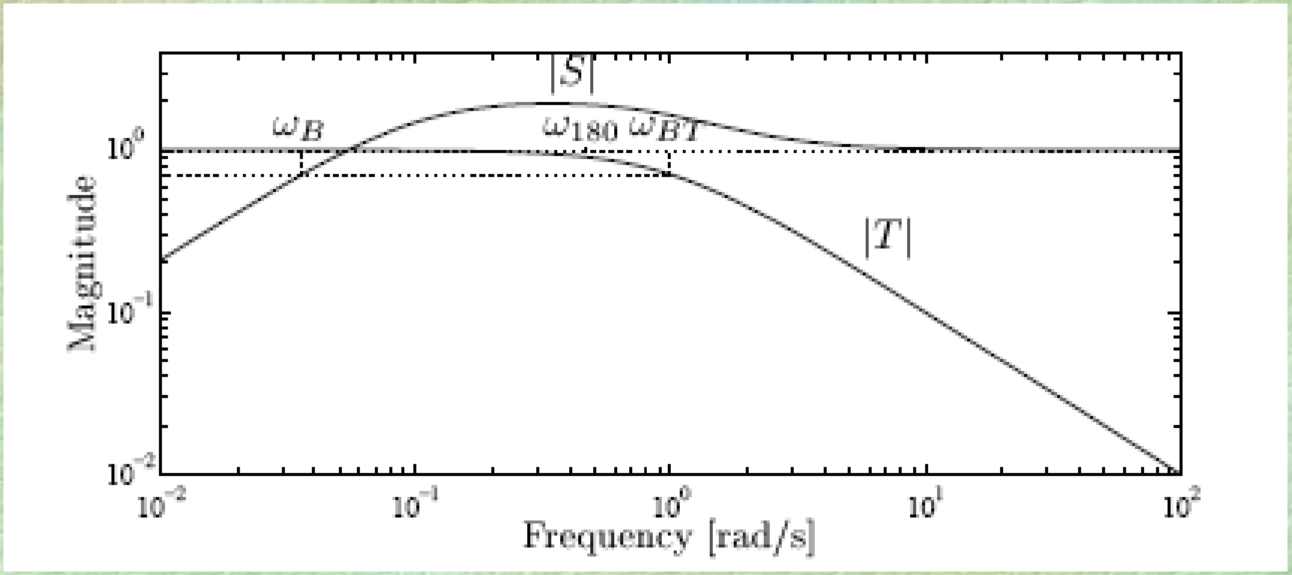
In conclusion ω_B (which is defined in terms of *S*) and also ω_c (in terms of *L*) are good indicators of closed-loop performance, while ω_{BT} (in terms of *T*) may be misleading in some cases.

Bandwidth and Crossover Frequency

Example 5-1 : Comparison of ω_B and ω_{BT} as indicators of performance.

Let $L = \dfrac{-s+z}{s(\tau s + \tau z + 2)}$; $T = \dfrac{-s+z}{s+z}\dfrac{1}{\tau s + 1}$; $z = 0.1,\ \tau = 1$

$\omega_c = 0.054$ $\omega_B = 0.036$ $\omega_{BT} = 1$ There is an RHP zero $z = 0.1$



Bandwidth and Crossover Frequency

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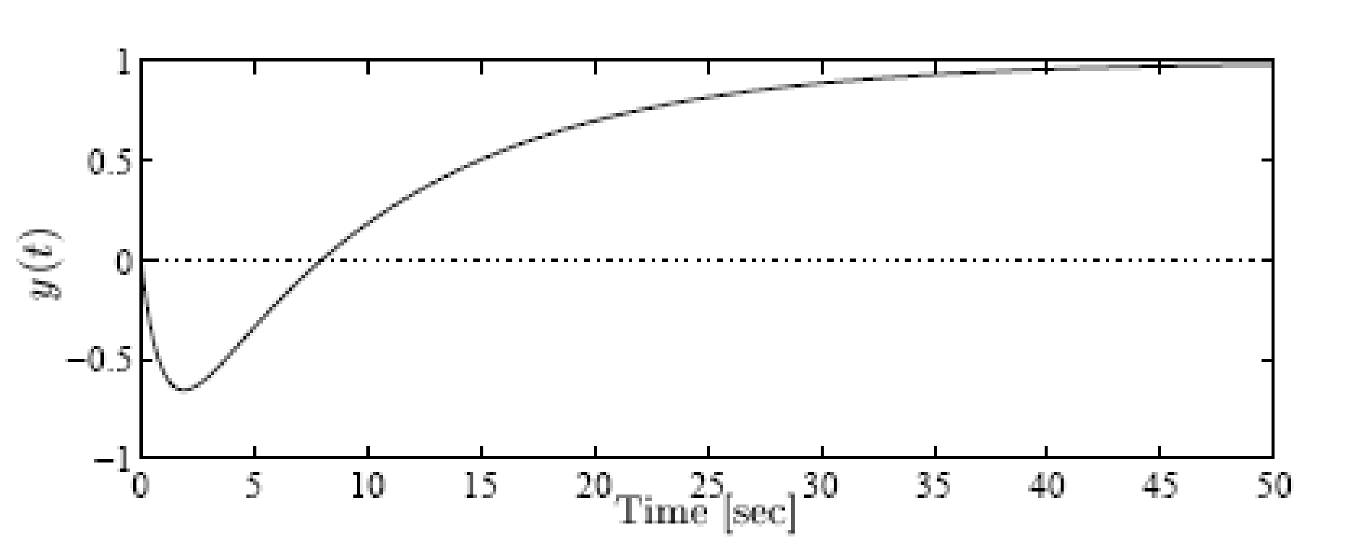
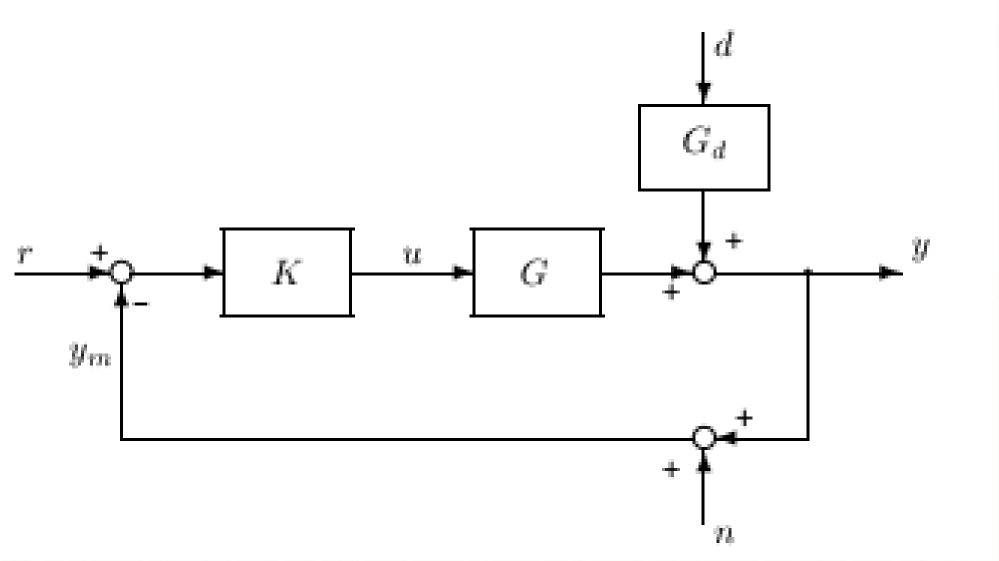


Introduction

One degree-of-freedom configuration



$$y(s) = GK(I+GK)^{-1}\ r + (I+GK)^{-1}\ G_d d - GK(I+GK)^{-1} n$$

$$y(s) = T(s)r + S(s)\ G_d d - T(s)n$$

- Performance, good disturbance rejection $S \to 0$ or $T \to I$ or $L \to \infty$
- Performance, good command following $T \to I$ or $S \to 0$ or $L \to \infty$
- Mitigation of measurement noise on output $T \to 0$ or $S \to I$ or $L \to 0$

Limitation on Performance in MIMO Systems

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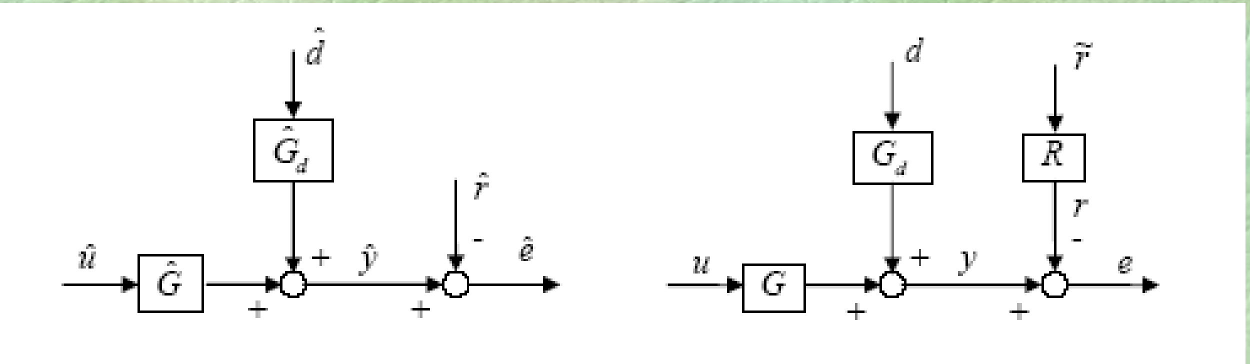
Scaling

$$\hat{y} = \hat{G}\hat{u} + \hat{G}_d \hat{d}\,; \qquad \hat{e} = \hat{y} - \hat{r}$$

$$d = D_d^{-1}\hat{d}\ , u = D_u^{-1}\hat{u}\ ,\ y = D_e^{-1}\hat{y}\ , e = D_e^{-1}\hat{e}\ , r = D_e^{-1}\hat{r}$$

$$D_e y = \hat{G} D_u u + \hat{G}_d D_d d\,; \qquad D_e e = D_e y - D_e r$$

$$G = D_e^{-1}\hat{G}D_u\ , G_d = D_e^{-1}\hat{G}_d D_d \qquad y = Gu + G_d d\,; \qquad e = y - r$$



Shaping Closed-loop Transfer Functions

Many design procedure act on the shaping of the **open-loop transfer function** *L.*

An alternative design strategy is to directly shape the magnitudes of **closed-loop transfer functions**, such as *S(s) and T(s).*

Such a design strategy can be formulated as an H_∞ optimal control problem, thus automating the actual controller design and leaving the engineer with the task of selecting reasonable bounds **"weights"** on the desired closed-loop transfer functions.

The terms H_∞ and H_2

The **H_∞ norm** of a stable transfer function matrix $F(s)$ is simply define as,

$$\|F(s)\|_\infty \cong \max_{\omega} \bar{\sigma}\left(F(j\omega)\right)$$

We are simply talking about a design method which aims to **press down the peak(s)** of one or more selected transfer functions.

Now, the term H_∞ which is purely mathematical, has now established itself in the control community.

In literature the symbol H_∞ stands for the transfer function matrices with bounded H_∞-norm which is the set of stable and proper transfer function matrices.

The terms H_∞ and H_2

The ***H_2* norm** of a stable transfer function matrix *F(s)* is simply define as,

$$\|F(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} tr\left[F(j\omega)F(j\omega)^H\right]d\omega}$$

Similarly, the symbol H_2 stands for the transfer function matrices with bounded ***H_2*-norm**, which is the set of stable and strictly proper transfer function matrices.

Note that the H_2 norm of a semi-proper transfer function is infinite, whereas its H_∞ norm is finite.

Why?

Weighted Sensitivity

As already discussed, the **sensitivity function *S*** is a very good indicator of closed-loop performance (both for SISO and MIMO systems).

Why ***S*** is a very good indicator of closed-loop performance in many literatures?

The main advantage of considering *S* is that because we ideally want ***S* to be small**, it is sufficient to consider just its **magnitude**, $\|S\|$ that is, we need not worry about its **phase**.

Weighted Sensitivity

Typical specifications in terms of *S* include:

- Minimum bandwidth frequency ω_B*.
- Maximum tracking error at selected frequencies.
- System type, or alternatively the maximum steady-state tracking error, *A*.
- Shape of *S* over selected frequency ranges.
- Maximum peak magnitude of S, $\|S(j\omega)\|_\infty \le M$

The peak specification prevents amplification of noise at high frequencies, and also introduces a margin of robustness; typically we select $M=2$

Weighted Sensitivity

Mathematically, these specifications may be captured simply by an upper bound

$$\frac{1}{|w_p(s)|}$$

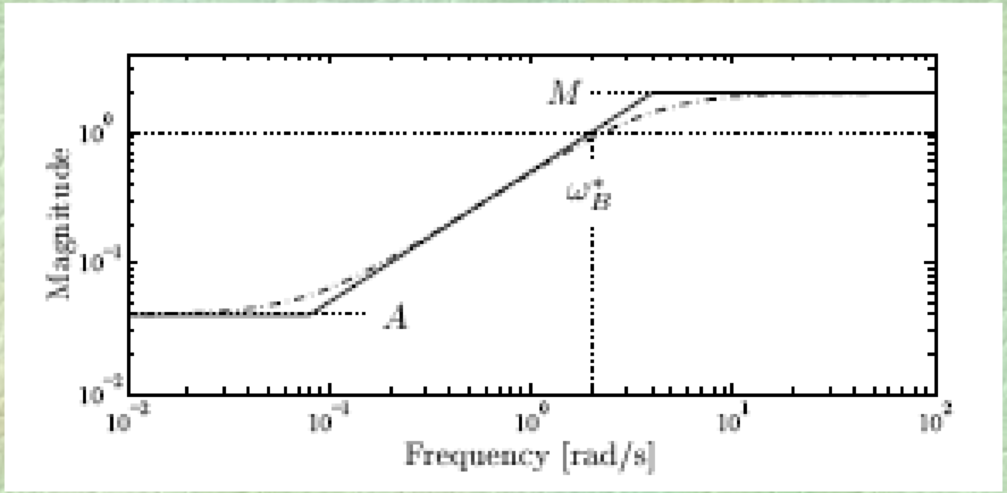
$$\bar{\sigma}\big(S(j\omega)\big) \le \frac{1}{|w_P(j\omega)|}, \qquad \forall \omega$$

$$\Leftrightarrow \qquad \bar{\sigma}\big(w_P(j\omega)S(j\omega)\big) \le 1, \quad \forall \omega \qquad \Leftrightarrow \qquad \|w_P(j\omega)S(j\omega)\|_\infty \le 1$$

The subscript P stands for performance

Weight Selection

Performance at Low Frequencies

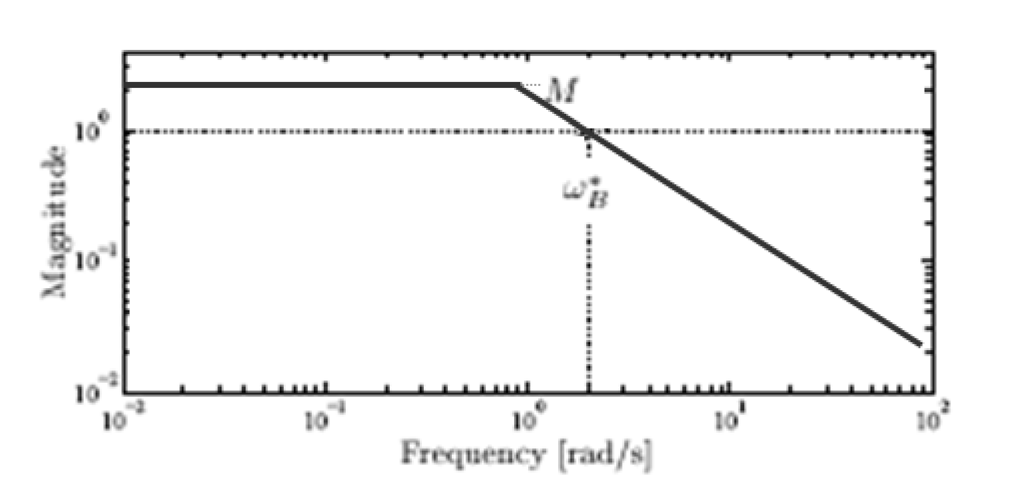


plot of $\dfrac{1}{|w_P(j\omega)|}$

$$w_P(s) = \frac{s/M + \omega_B{}^*}{s + \omega_B{}^* A}$$

Weight Selection

Performance at High Frequencies



plot of $\dfrac{1}{|w_P(j\omega)|}$

$$w_P(s) = \frac{1}{M} + \frac{s}{\omega_B^{\ *}}$$

Weight Selection

A weight which asks for a slope -2 for L at lower frequencies is

$$w_P(s) = \frac{\left(s/M^{1/2} + \omega_B^{\ *}\right)^2}{\left(s + \omega_B^{\ *} A^{1/2}\right)^2}$$

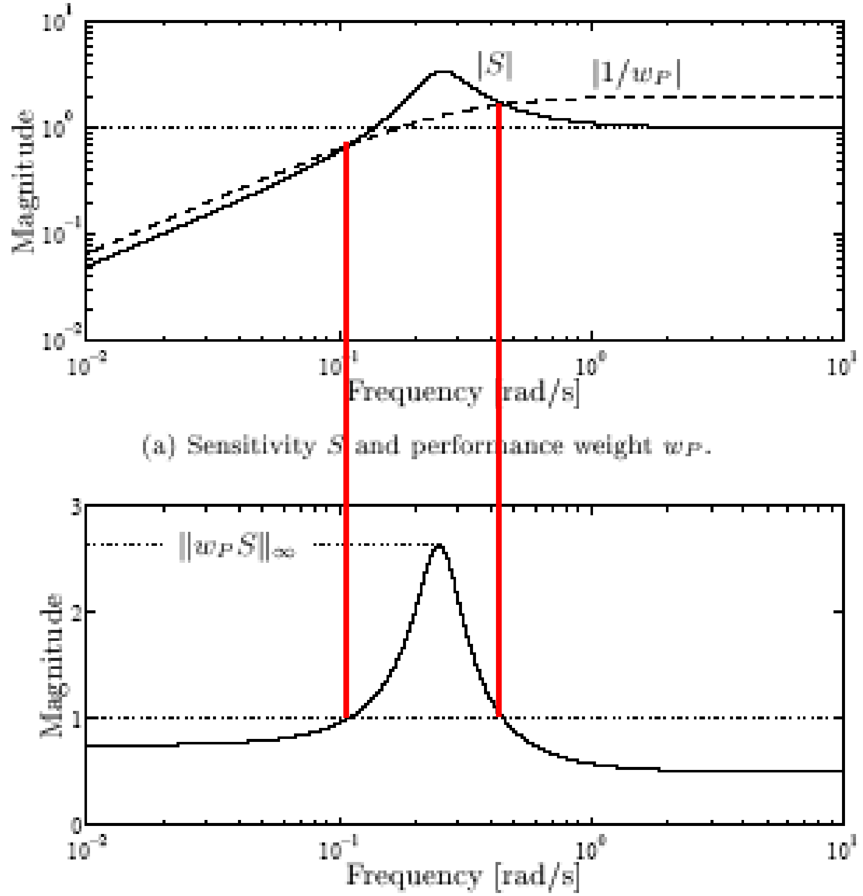
The **insight gained** from the previous section on loop-shaping design is very useful for selecting weights.

For example, for disturbance rejection

$$\bar{\sigma}\left(SG_d(j\omega)\right) < 1$$

It then follows that a good initial choice for the performance weight is to let $w_P(s)$ look like $|G_d(j\omega)|$ at frequencies where $|G_d(j\omega)| > 1$

Weighted Sensitivity



(a) Sensitivity S and performance weight w_P.

(b) Weighted sensitivity $w_P S$.

Stacked Requirements: Mixed Sensitivity

The specification $\|w_P S\|_\infty < 1$ puts a lower bound on the bandwidth, but not an upper one, and nor does it allow us to specify the roll-off of $L(s)$ above the bandwidth.

To do this one can make demands on another closed-loop transfer function

$$\|N\|_\infty = \max_\omega \bar{\sigma}\big(N(j\omega)\big) < 1, \quad N = \begin{bmatrix} w_P S \\ w_T T \\ w_u KS \end{bmatrix}$$

For SISO systems, N is a vector and

$$\bar{\sigma}(N) = \sqrt{|w_P S|^2 + |w_T T|^2 + |w_u KS|^2}$$

Solving H_∞ Optimal Control Problem

After selecting the form of N and the weights, the H_∞ optimal controller is obtained by solving the problem

$$\min_K \|N(K)\|_\infty$$

Let $\gamma_0 = \min_K \|N(K)\|_\infty$ denote the optimal H_∞ norm.

The practical implication is that, except for at most a factor $\sqrt{n}$ the transfer functions will be close to γ_0 times the bounds selected by the designer.

This gives the designer a mechanism for directly shaping the magnitudes of

$$\bar{\sigma}(S) \quad , \quad \bar{\sigma}(T) \quad \text{and} \quad \bar{\sigma}(KS)$$

Solving H_∞ Optimal Control Problem

Example 5-2

$$G(s)=\frac{200}{10s+1}\frac{1}{(0.05s+1)^2}\ , \qquad G_d(s)=\frac{100}{10s+1}$$

The control objectives are:

1. Command tracking: The rise time (to reach 90% of the final value) should be less than 0.3 second and the overshoot should be less than 5%.

2. Disturbance rejection: The output in response to a unit step disturbance should remain within the range [-1,1] at all times, and it should return to 0 as quickly as possible (|y(t)| should at least be less than 0.1 after 3 seconds).

3. Input constraints: *u*(t) should remain within the range [-1,1] at all times to avoid input saturation (this is easily satisfied for most designs).

Solving H_∞ Optimal Control Problem

Consider an H_∞ mixed sensitivity *S/KS design in which*

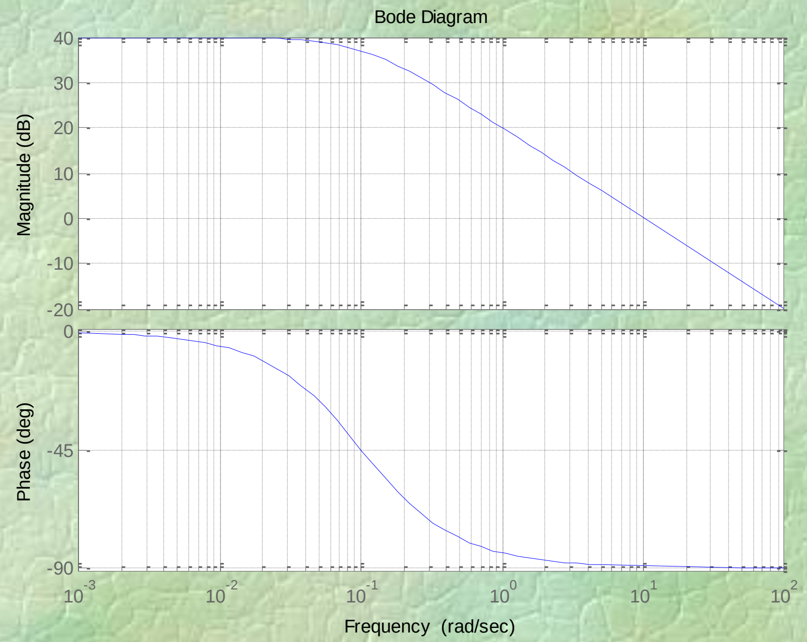
$$N = \begin{bmatrix} w_P S \\ w_u KS \end{bmatrix}$$

It was stated earlier that appropriate scaling has been performed so that the inputs should be about 1 or less in magnitude, and we therefore

$$w_u = 1 \quad \text{and} \quad w_P(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A}$$

Solving H$_\infty$ Optimal Control Problem

See the Bode diagram of $G_d(s) = \dfrac{100}{10s+1}$



We need control till 10 rad/sec to reduce disturbance and a suitable rise time.

So let $\omega_B \approx \omega_c \approx 10\,rad/\sec$

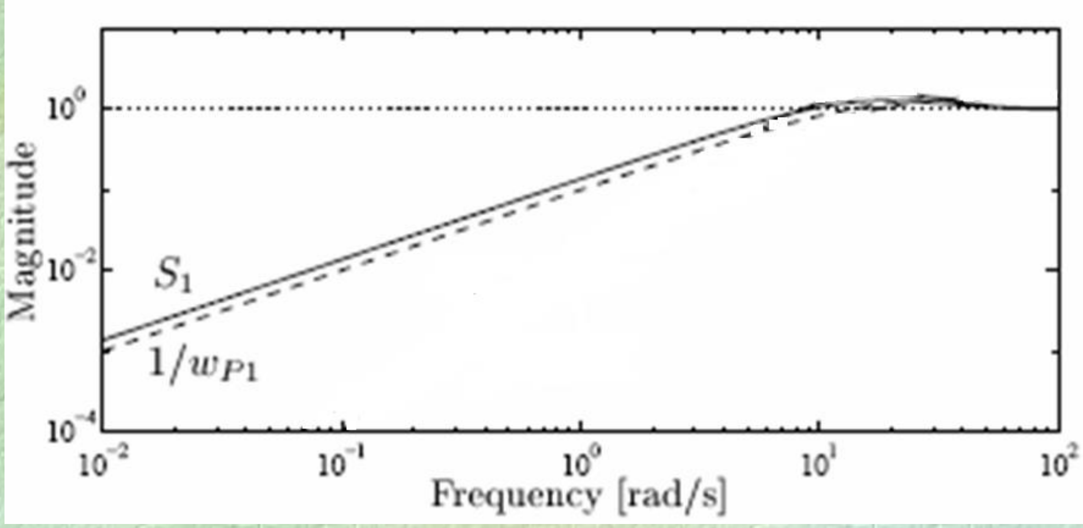
Overshoot should be less than 5% so let $M_S < 1.5$

Solving H_∞ Optimal Control Problem

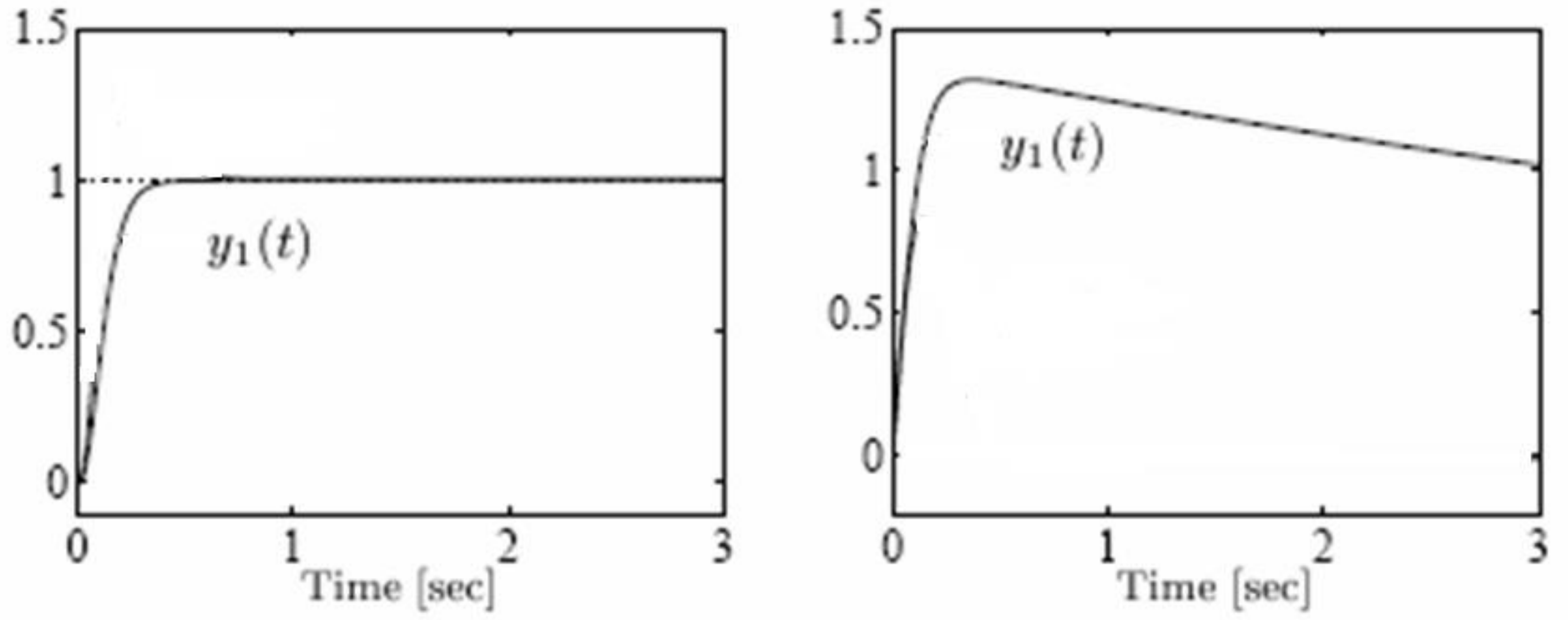
$$w_P(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A}, \quad M = 1.5, \quad \omega_B^* = 10, \quad A = 10^{-4}$$

For this problem, we achieved an optimal H_∞ norm of 1.37, so the weighted sensitivity requirements are not quite satisfied. Nevertheless, the design seems good with

$M_S = 1.30,\ M_T = 1.0,\ GM = 8.04,\ PM = 71.2^\circ$ and $\omega_c = 5.22$ rad/sec



Solving H_∞ Optimal Control Problem



(a) Tracking response. (b) Disturbance response.

The tracking response is very good as shown by curve in Figure. However, we see that the disturbance response is very sluggish.

Solving H_∞ Optimal Control Problem

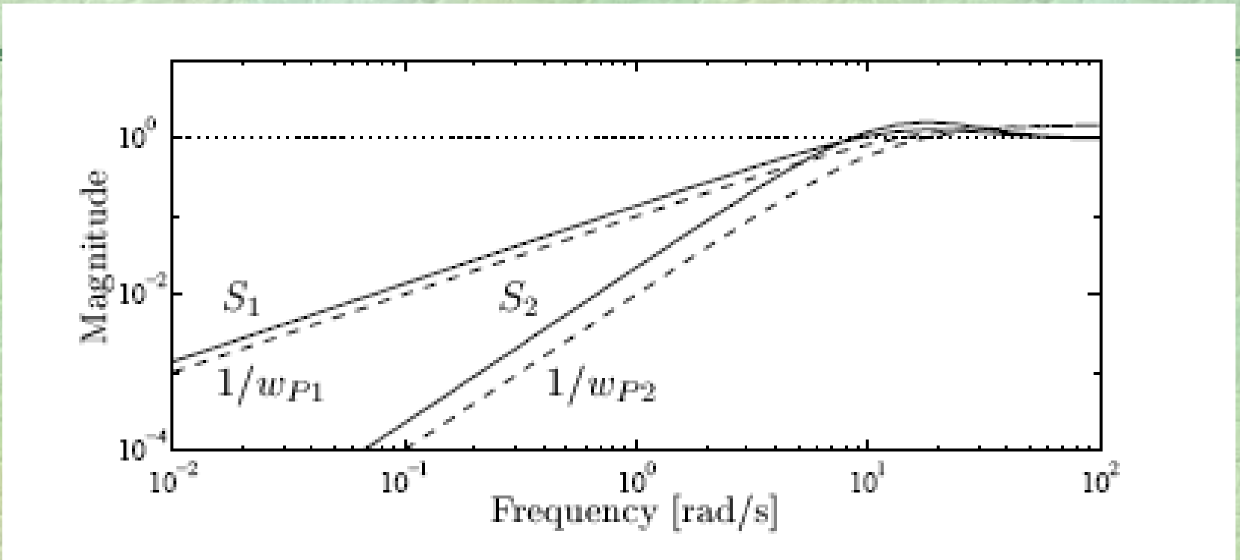
If disturbance rejection is the main concern, then from our earlier discussion we need for a performance weight that specifies higher gains at low frequencies. We therefore try

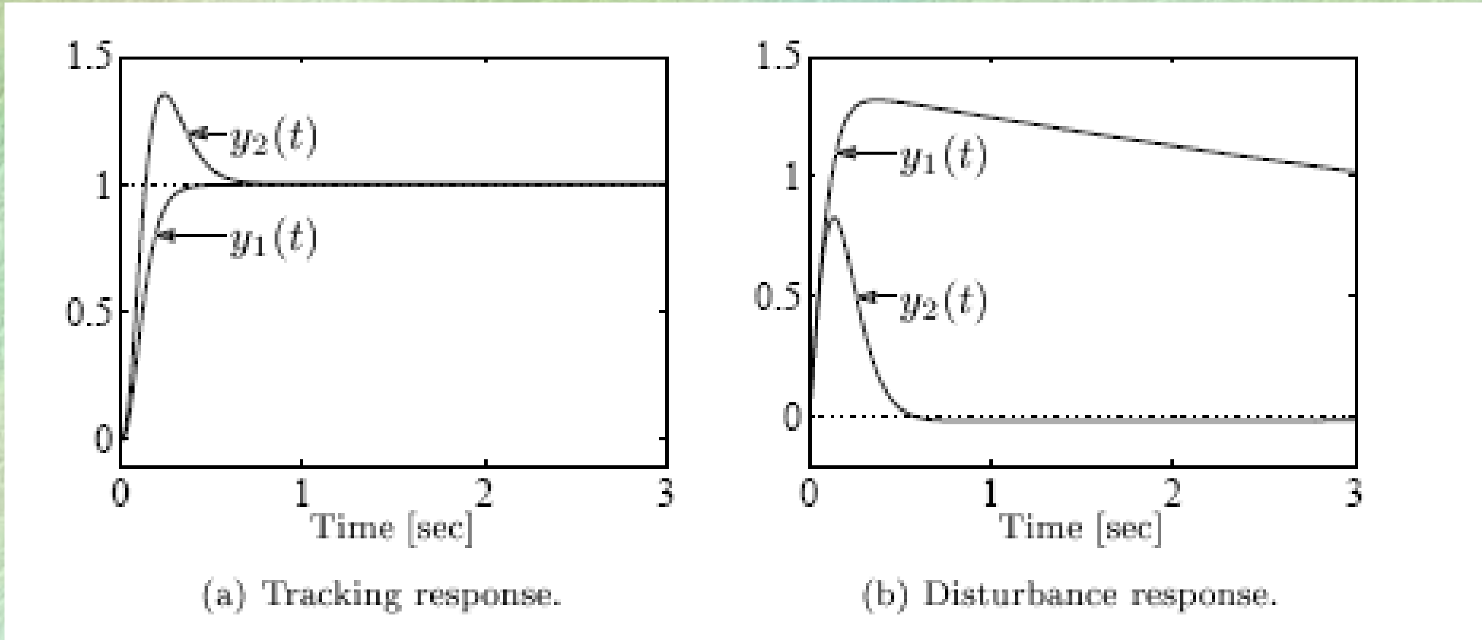
$$w_P(s) = \frac{\left(s / M^{1/2} + \omega_B{}^*\right)^2}{\left(s + \omega_B{}^* A^{1/2}\right)^2}, \quad M = 1.5,\ \omega_B{}^* = 10,\ A = 10^{-6}$$

For this problem, we achieved an optimal H_∞ norm of 2.21, so the weighted sensitivity requirements are not quite satisfied. Nevertheless, the design seems good with

$$M_S = 1.63,\ M_T = 1.43,\ GM = 4.76,\ PM = 43.3^\circ \text{ and } \omega_c = 11.2 \text{ rad/sec}$$

Solving H_∞ Optimal Control Problem





(a) Tracking response.

(b) Disturbance response.

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Fundamental Limitation on Sensitivity (Frequency domain)

_S_ plus _T_ is the identity matrix

$$S + T = I$$

$$\left|\bar{\sigma}(S) - 1\right| \le \bar{\sigma}(T) \le \bar{\sigma}(S) + 1$$

$$\left|\bar{\sigma}(T) - 1\right| \le \bar{\sigma}(S) \le \bar{\sigma}(T) + 1$$

Fundamental Limitation on Sensitivity (Frequency domain)

Interpolation Constraints

RHP-zero:

If $G(s)$ has a RHP-zero at z with output direction y_z then for internal stability of the feedback system the following interpolation constraints must apply:

In SISO Case: $T(z)=0; \quad S(z)=1$

In MIMO Case: $y_z{}^H T(z)=0; \quad y_z{}^H S(z)=y_z{}^H$

Proof:

$y_z{}^H G(z)=0 \qquad y_z{}^H L(z)=0 \qquad T=LS \qquad$ ***S*** has no RHP-pole

$$y_z{}^H T(z)=0 \qquad\qquad y_z{}^H (I-S(z))=0$$

Limitations Imposed by RHP Zeros

Moving the Effect of a RHP-zero to a Specific Output

Example 5-3

$$G(s) = \frac{1}{(0.2s+1)(s+1)}\begin{bmatrix} 1 & 1 \\ 1+2s & 2 \end{bmatrix}$$

which has a RHP-zero at $s = z = 0.5$

The output zero direction is $y_z = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.89 \\ -0.45 \end{bmatrix}$

Interpolation constraint is

$$2t_{11}(z) - t_{21}(z) = 0; \quad 2t_{12}(z) - t_{22}(z) = 0$$

Limitations Imposed by RHP Zeros

Moving the Effect of a RHP-zero to a Specific Output

$$2t_{11}(z) - t_{21}(z) = 0; \quad 2t_{12}(z) - t_{22}(z) = 0$$

$$T_0(s) = \begin{bmatrix} \frac{-s+z}{s+z} & 0 \\ 0 & \frac{-s+z}{s+z} \end{bmatrix}$$

$$T_1(s) = \begin{bmatrix} 1 & 0 \\ \frac{\beta_1 s}{s+z} & \frac{-s+z}{s+z} \end{bmatrix} \quad ??? \quad \Rightarrow \quad \beta_1 = 4$$

$$T_2(s) = \begin{bmatrix} \frac{-s+z}{s+z} & \frac{\beta_2 s}{s+z} \\ 0 & 1 \end{bmatrix} \quad ??? \quad \Rightarrow \quad \beta_2 = 1$$

$$y_z = \begin{bmatrix} 0.89 \\ -0.45 \end{bmatrix}$$

Limitations Imposed by RHP Zeros

Moving the Effect of a RHP-zero to a Specific Output

$$2t_{11}(z)-t_{21}(z)=0; \quad 2t_{12}(z)-t_{22}(z)=0$$

$$T_0(s)=\begin{bmatrix} \frac{-s+z}{s+z} & 0 \\ 0 & \frac{-s+z}{s+z} \end{bmatrix}$$

$$T_1(s)=\begin{bmatrix} 1 & 0 \\ \frac{\beta_1 s}{s+z} & \frac{-s+z}{s+z} \end{bmatrix} \quad ??? \quad \Rightarrow \quad \beta_1 = 4$$

$$T_2(s)=\begin{bmatrix} \frac{-s+z}{s+z} & \frac{\beta_2 s}{s+z} \\ 0 & 1 \end{bmatrix} \quad ??? \quad \Rightarrow \quad \beta_2 = 1$$

$$y_z = \begin{bmatrix} 0.89 \\ -0.45 \end{bmatrix}$$

Limitations Imposed by RHP Zero

Theorem 5-1 Assume that $G(s)$ is square, functionally controllable and stable and has a single RHP-zero at $s = z$ and no RHP-pole at $s = z$. Then if the k'th Element of the output zero direction is non-zero, i.e. $y_{zk} \neq 0$ it is possible to obtain "perfect" control on all outputs $j \neq k$ with the remaining output exhibiting no steady-state offset. Specifically, T can be chosen of the form

$$T(s) = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ . & . & \dots & . & . & . & \dots & . \\ . & . & \dots & . & . & . & \dots & . \\ \frac{\beta_1 s}{s+z} & \frac{\beta_2 s}{s+z} & \dots & \frac{\beta_{k-1} s}{s+z} & \frac{-s+z}{s+z} & \frac{\beta_{k+1} s}{s+z} & \dots & \frac{\beta_n s}{s+z} \\ . & . & \dots & . & . & . & \dots & . \\ . & . & \dots & . & . & . & \dots & . \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\beta_j = -2\frac{y_{zj}}{y_{zk}} \quad for \;\; j \neq k$$

Fundamental Limitation on Sensitivity (Frequency domain)

Interpolation Constraints

RHP-pole:

If $G(s)$ has a RHP pole at p with output direction y_p then for internal stability the following interpolation constraints apply

In SISO Case: $S(p)=0; \quad T(p)=1$

In MIMO Case: $S(p)y_p=0; \quad T(p)y_p=y_p$

Proof:

$L^{-1}(p)y_p=0$ $\quad$ $T=SL$ $\quad$ T has no RHP-pole $\quad$ S has a RHP-zero

$S=TL^{-1}$ $\quad$ $T(p)L^{-1}(p)y_p=S(p)y_p=0$ $\quad$ $T(p)y_p=(I-S(p))y_p=y_p$

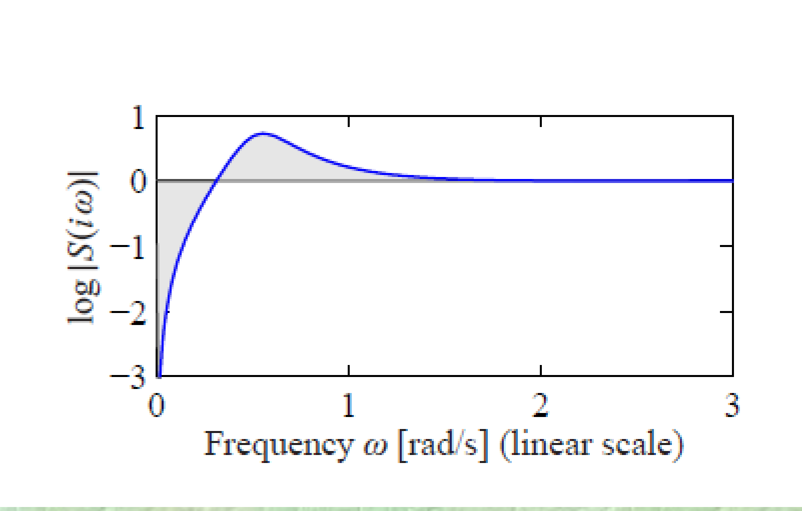
Fundamental Limitation on Sensitivity (Frequency domain)

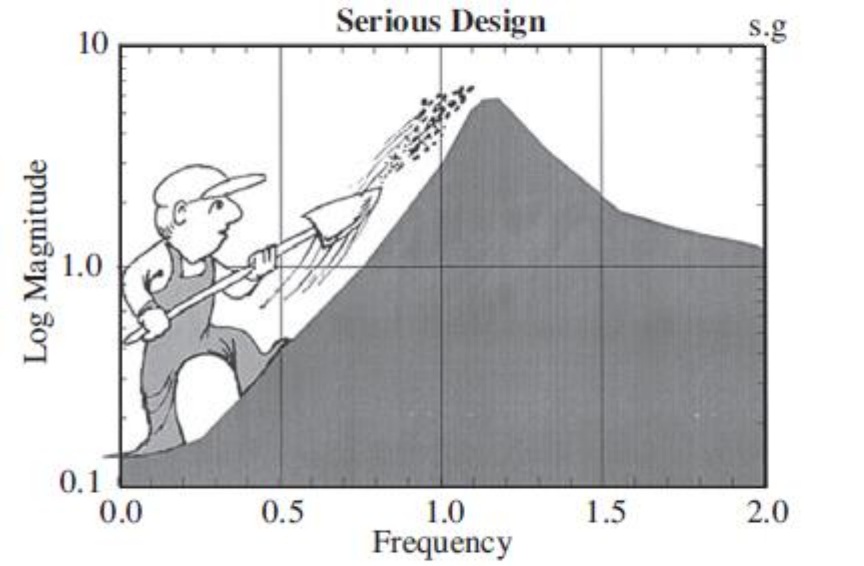
Sensitivity Integrals

If $L(s)$ has two more poles than zeros (the loop transfer function L(s) of a feedback system goes to zero faster than 1/s as s → ∞),(Bode integral)

In SISO Case:

$$\int_0^{\infty} \ln|S(j\omega)|\,d\omega = \pi.\sum_{i=1}^{N_p} \mathrm{Re}(p_i)$$





Figures are derived from:
"Feedback Systems" Karl Johan Astrom, Richard M. Murray, Princeton university press 2009

Fundamental Limitation on Sensitivity (Frequency domain)

Sensitivity Integrals

If $L(s)$ has two more poles than zeros (the loop transfer function L(s) of a feedback system goes to zero faster than 1/s as s → ∞),(Bode integral)

In SISO Case:

$$\int_0^{\infty} \ln|S(j\omega)|\,d\omega = \pi.\sum_{i=1}^{N_p} \mathrm{Re}(p_i)$$

In MIMO Case: (Generalization of SISO case)

$$\int_0^{\infty} \ln|\det S(j\omega)|\,d\omega = \sum_i \int_0^{\infty} \ln \sigma_i\big(S(j\omega)\big)\,d\omega = \pi.\sum_{i=1}^{N_p} \mathrm{Re}(p_i)$$

Fundamental Limitation: Bounds on Peaks

In the following, $M_{S,\min}$ and $M_{T,\min}$ denote the lowest achievable values for $\|S\|_\infty$ and $\|T\|_\infty$, respectively, using any stabilizing controller K.

$$M_{S,\min} \cong \min\|S\|_\infty \ , \ M_{T,\min} \cong \min\|T\|_\infty$$

Fundamental Limitation: Bounds on Peaks

Theorem 5-2 Sensitivity and Complementary Sensitivity Peaks

Consider a rational plant $G(s)$ (with no time delay). Suppose $G(s)$ has N_z RHP-zeros with output zero direction vectors $y_{z,i}$ and N_p RHP-poles with output pole direction vectors $y_{p,i}$. Suppose all z_i and p_i are distinct.

Then we have the following tight lower bound on $\|T\|_\infty$ and $\|S\|_\infty$

$$M_{S,\min} = M_{T,\min} = \sqrt{1+\bar{\sigma}^2\left(Q_z^{-1/2} Q_{zp} Q_p^{-1/2}\right)}$$

$$[Q_z]_{ij} = \frac{y_{z,i}^H y_{z,j}}{z_i + \bar{z}_j}, \quad [Q_p]_{ij} = \frac{y_{p,i}^H y_{p,j}}{\bar{p}_i + p_j}, \quad [Q_{zp}]_{ij} = \frac{y_{z,i}^H y_{p,j}}{z_i - p_j}$$

Fundamental Limitation: Bounds on Peaks

Example 5-4

$$G(s) = \frac{(s-1)(s-3)}{(s-2)(s+1)^2}$$

Derive lower bounds on $\|T\|_\infty$ and $\|S\|_\infty$

$$z_1 = 1,\ z_2 = 3,\ p_1 = 2$$

$$Q_z = \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 1/6 \end{bmatrix},\ Q_p = 1/4,\ Q_{z,p} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$M_{S,\min} = M_{T,\min} = \sqrt{1 + \bar{\sigma}^2\left(\begin{bmatrix} -7.9531 \\ 12.6786 \end{bmatrix}\right)} = 15$$

Fundamental Limitation: Bounds on Peaks

$$M_{S,\min} = M_{T,\min} = \sqrt{1+\bar{\sigma}^2\left(Q_z^{-1/2} Q_{zp} Q_p^{-1/2}\right)}$$

$$[Q_z]_{ij} = \frac{y_{z,i}^H y_{z,j}}{z_i + \bar{z}_j},\ [Q_p]_{ij} = \frac{y_{p,i}^H y_{p,j}}{\bar{p}_i + p_j},\ [Q_{zp}]_{ij} = \frac{y_{z,i}^H y_{p,j}}{z_i - p_j}$$

One RHP-pole and one RHP-zero

$$M_{S,\min} = M_{T,\min} = \sqrt{\sin^2\phi + \frac{|z+p|^2}{|z-p|^2}\cos^2\phi} \qquad \phi = \cos^{-1}\left|y_z^H y_p\right|$$

Exercise5-1 : Proof above equation.

Fundamental Limitation: Bounds on Peaks

Example 5-5

$$G_\alpha(s) = \begin{bmatrix} \frac{1}{s-p} & 0 \\ 0 & \frac{1}{s+3} \end{bmatrix} \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}}_{U_\alpha} \begin{bmatrix} \frac{s-z}{0.1s+1} & 0 \\ 0 & \frac{s+2}{0.1s+1} \end{bmatrix}; \quad z=2,\ p=3$$

$$U_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad and \quad G_0(s) = \begin{bmatrix} \frac{s-z}{(0.1s+1)(s-p)} & 0 \\ 0 & \frac{s+2}{(0.1s+1)(s+3)} \end{bmatrix}$$

$$U_{90} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad and \quad G_{90}(s) = \begin{bmatrix} 0 & -\frac{s+2}{(0.1s+1)(s-p)} \\ \frac{s-z}{(0.1s+1)(s+3)} & 0 \end{bmatrix}$$

Fundamental Limitation: Bounds on Peaks

α	0°	30°	60°	90°
y_z	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}0.33\\-0.94\end{bmatrix}$	$\begin{bmatrix}0.11\\-0.99\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$
$\phi = \cos^{-1}\left(y_z^H y_p\right)$	0°	70.9°	83.4°	90°
$M_{S,\min} = M_{T,\min}$	5.00	1.89	1.15	1.00
$\Vert S\Vert_\infty$	7.00	2.60	1.59	1.98
$\Vert T\Vert_\infty$	7.40	2.76	1.60	1.31

$$W_u = I, \; W_P = \left(\frac{s/M + \omega_B^*}{s}\right) I, \; M = 2, \; \omega_B^* = 0.5$$

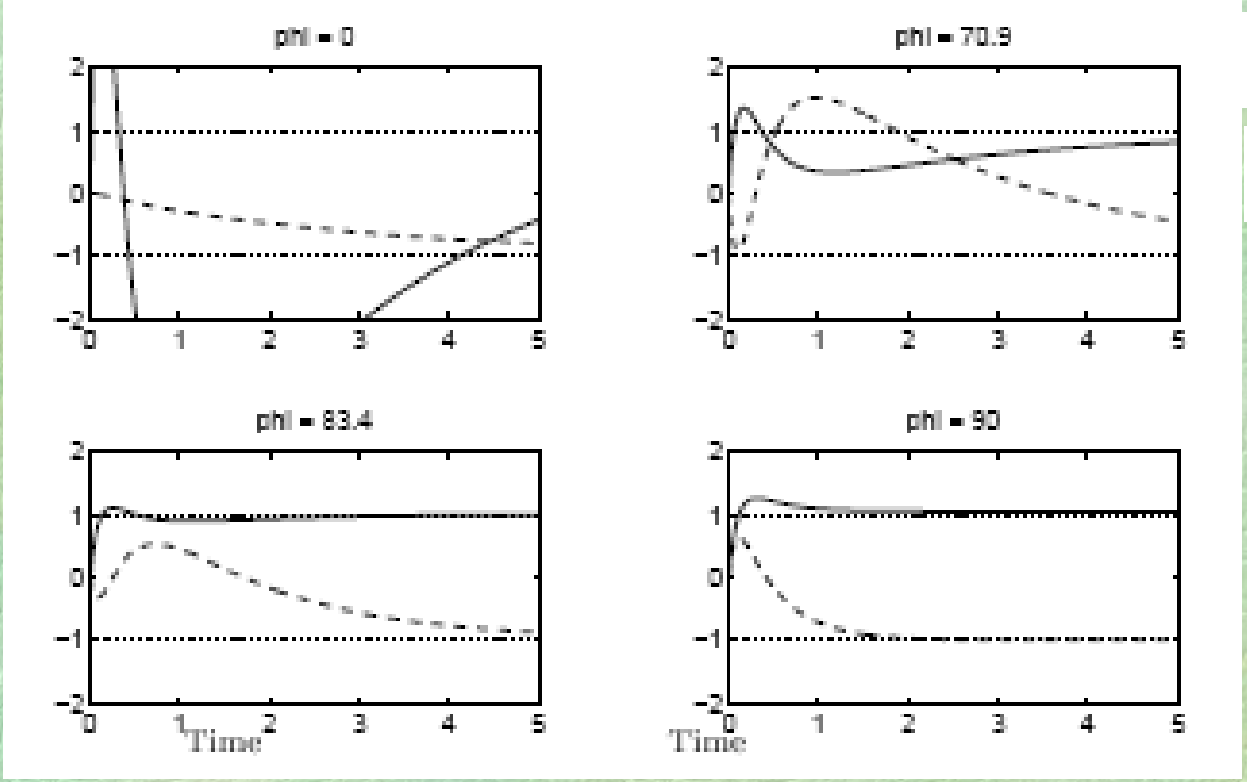
Fundamental Limitation: Bounds on Peaks

α	0°	30°	60°	90°
y_z	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}0.33\\-0.94\end{bmatrix}$	$\begin{bmatrix}0.11\\-0.99\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$
$\phi=\cos^{-1}\left(y_z^H y_p\right)$	0°	70.9°	83.4°	90°
$M_{S,\min}=M_{T,\min}$	5.00	1.89	1.15	1.00
$\lVert S\rVert_\infty$	7.00	2.60	1.59	1.98
$\lVert T\rVert_\infty$	7.40	2.76	1.60	1.31

$$W_u = I,\; W_P = \left(\frac{s/M+\omega_B^*}{s}\right)I,\; M=2,\; \omega_B^*=0.5$$

Fundamental Limitation: Bounds on Peaks

The corresponding responses to a step change in the reference $r = [\ 1\ \ -1\ ]$, are shown



Solid line: y_1

Dashed line: y2

1- For α = 0 there is one RHP-pole and zero so the responses for y_1 is very poor.

2- For α = 90 the RHP-pole and zero do not interact but y_2 has an undershoot since of …

3- For α = 0 and 30 the H_∞ controller is unstable since of …

Limitations Imposed by RHP Zeros

$$\left\|w_p(s)S(s)\right\|_\infty = \max_\omega \left|w_p(j\omega)\right|.\bar{\sigma}(\,S(j\omega))$$

Let a RHP-zero located at z so by maximum module theorem

$$\left\|w_p(s)S(s)\right\|_\infty = \max_\omega \left|w_p(j\omega)\right|.\bar{\sigma}(S(j\omega)) \geq \left|w_p(s)\right|.\bar{\sigma}(S(s)) \;\; \forall s \in RHP$$

$$\left\|w_p(s)S(s)\right\|_\infty \geq \left|w_p(z)\right|.\bar{\sigma}(S(z))$$

$$\left\|w_P(s)S(s)\right\|_\infty < 1 \quad \Rightarrow \quad \left|w_P(z)\right| < 1$$

Limitations Imposed by RHP Zeros

Performance at Low Frequencies

$$\left\| w_P(s)S(s) \right\|_\infty < 1$$

$$\left| w_P(z) \right| < 1$$

$$w_P(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A} \quad \Rightarrow \quad \left| w_P(z) \right| = \left| \frac{z/M + \omega_B^*}{z + \omega_B^* A} \right| < 1$$

Real zero: $\omega_B^*(1-A) < z(1-\frac{1}{M}) \quad \Rightarrow \quad \omega_B^* < \frac{z}{2} \quad (I)$

Imaginary zero $\omega_B^* < |z|\sqrt{1-\frac{1}{M^2}} \quad \Rightarrow \quad \omega_B^* < 0.87|z| \quad (II)$

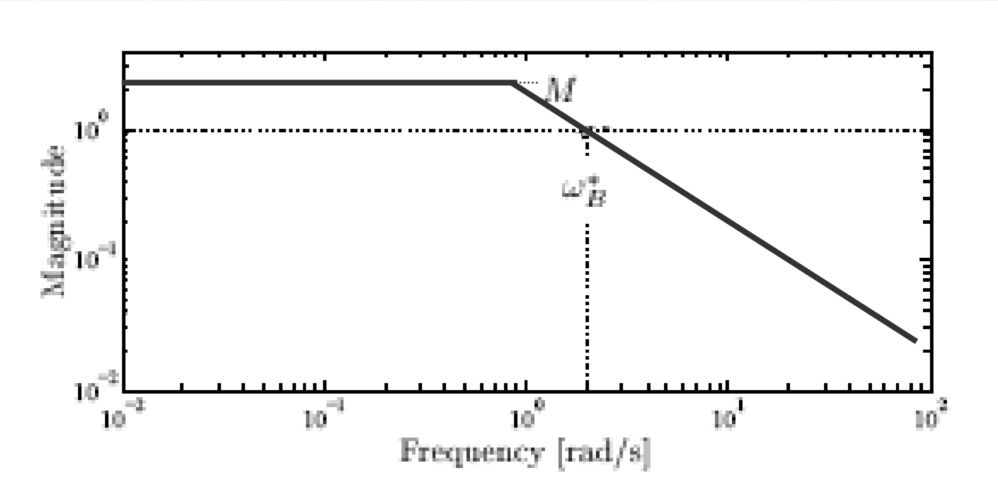
Exercise5-2 : Derive (I) and (II).

Limitations Imposed by RHP Zeros

Performance at High Frequencies

$$\|w_P(s)S(s)\|_\infty < 1$$

$$|w_P(z)| < 1$$



$$w_P(s) = \frac{1}{M} + \frac{s}{\omega_B^{\ *}}$$

$$w_P(s) = \frac{1}{M} + \frac{s}{\omega_B^{\ *}} \quad \Rightarrow \quad |w_P(z)| = \left|\frac{1}{M} + \frac{z}{\omega_B^{\ *}}\right| < 1$$

Real zero: $\omega_B^{\ *} > z\dfrac{1}{1-1/M}$ $\Rightarrow$ $\omega_B^{\ *} > 2z$

Limitations Imposed by Unstable (RHP) Poles

$$\left\| w_T(s)T(s) \right\|_\infty = \max_{\omega} \left| w_T(j\omega) \right| . \bar{\sigma}(T(j\omega)) \geq \left| w_T(p) \right|$$

$$\left\| w_T(s)T(s) \right\|_\infty < 1 \quad \Rightarrow \quad \left| w_T(p) \right| < 1$$

$$w_T(s) = \frac{s}{\omega_{BT}^*} + \frac{1}{M_T}$$

Real RHP-pole

$$\omega_{BT}^* > p \frac{M_T}{M_T - 1} \quad \Rightarrow \quad \omega_{BT}^* > 2p \quad \Rightarrow \quad \omega_c > 2p$$

Imaginary RHP-pole

$$\omega_{BT}^* > 1.15 \left| p \right|$$

Limitation on Performance in MIMO Systems

- Scaling and Performance
- Shaping Closed-loop Transfer Functions
- Fundamental Limitation on Performance (Frequency domain)
  - Fundamental Limitation on Sensitivity
  - Limitations Imposed by RHP Zeros
  - Limitations Imposed by Unstable (RHP) Poles
  - Limitations Imposed by Time Delays
- Fundamental Limitation on Performance (Time domain)

Fundamental Limitation on Performance (Time domain)

Consider the system:

$$Y(s) = G(s)U(s)$$
$$U(s) = C(s)\big(R(s) - Y(s)\big)$$

Let a step response signal $\left(\hat{r}u(t)\right)$ at i^{th} input but other inputs are zero so

Overshoot in output i is:

$$y_i^o = \sup_{t>0}\{y_i(t) - r_i(t), 0\}$$

Undershoot is defined as:

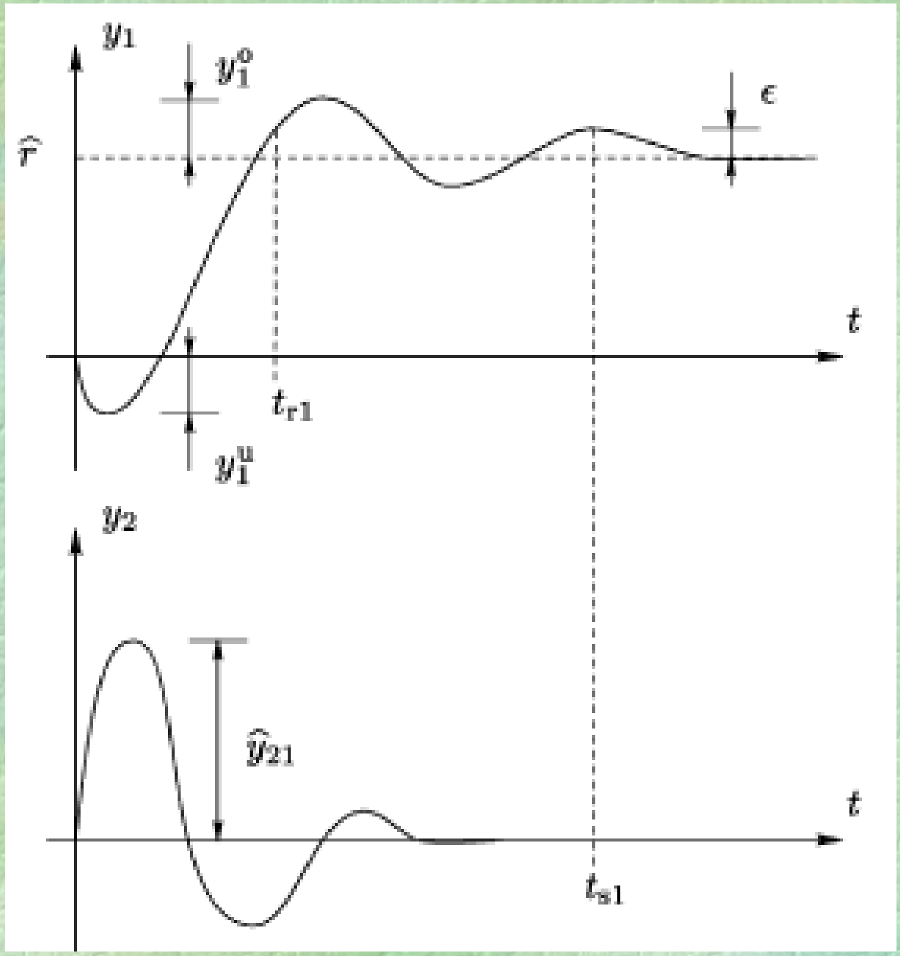
$$y_i^u = \sup_{t>0}\{-y_i(t), 0\}$$

Rise time is defined as:

$$\hat{y}_{ki} = \sup_{t>0}\{|y_k(t)|\}$$

Settling time is defined as:

$$t_{si} = \max_{k\in\{1,\dots,m\}} \inf_{\delta>0}\{\delta : |y_k(t) - r_k(t)| \le \varepsilon, t > \delta\}$$



Reference: “Interaction Bounds in Multivariable Control Systems” K H Johanson, Automatica, vol 38,pp 1045-1051, 2002

Fundamental Limitation on Performance (Time domain)

Theorem5-3: Consider the stable closed loop system with zero initial conditions at t=0 and let $r(t)=(\hat{r},0,\ldots,0)^T$ for t>0. Assume that the open loop transfer function G has a real RHP zero $z>0$ with zero direction y_z and $y_{z1}>0$. then we have:

$$y_{z1}y_1^u+\sum_{k=2}^{m}|y_{zk}|\hat{y}_{k1}\geq\frac{1}{e^{zt_{s1}}-1}\left[y_{z1}(\hat{r}-\varepsilon)-\varepsilon\sum_{k=2}^{m}|y_{zk}|\right]$$

- undershoot: y_1^u
- elements of zero direction: y_{z1}, y_{zk}
- interaction: $\hat{y}_{k1}$
- settling time: t_{s1}
- settling level: ε

Fundamental Limitation on Performance (Time domain)

Theorem5-3: Consider the stable closed loop system with zero initial conditions at t=0 and let $r(t)=\left(\hat{r},0,\ldots,0\right)^T$ for t>0. Assume that the open loop transfer function G has a real RHP zero $z>0$ with zero direction y_z and $y_{z1}>0$. then we have:

undershoot

$$y_{z1}y_1^u+\sum_{k=2}^{m}\left|y_{zk}\right|\hat{y}_{k1}\geq\frac{1}{e^{zt_{s1}}-1}\left[y_{z1}(\hat{r}-\varepsilon)-\varepsilon\sum_{k=2}^{m}\left|y_{zk}\right|\right]$$

elements of zero direction — interaction — settling time — settling level

Theorem5-4: Consider the stable closed loop system with zero initial conditions at t=0. Assume that the open loop transfer function G has a real RHP pole $p>0$ with pole direction y_p and $y_{p1}>0$. Consider m independent responses with $r_i(t)=\hat{r}$ for t>0. Then we have:

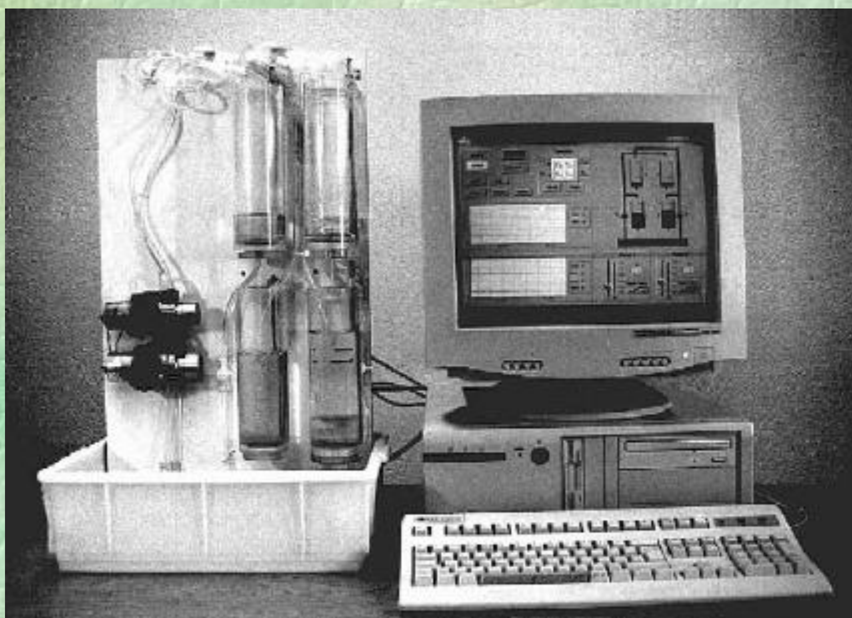
overshoot

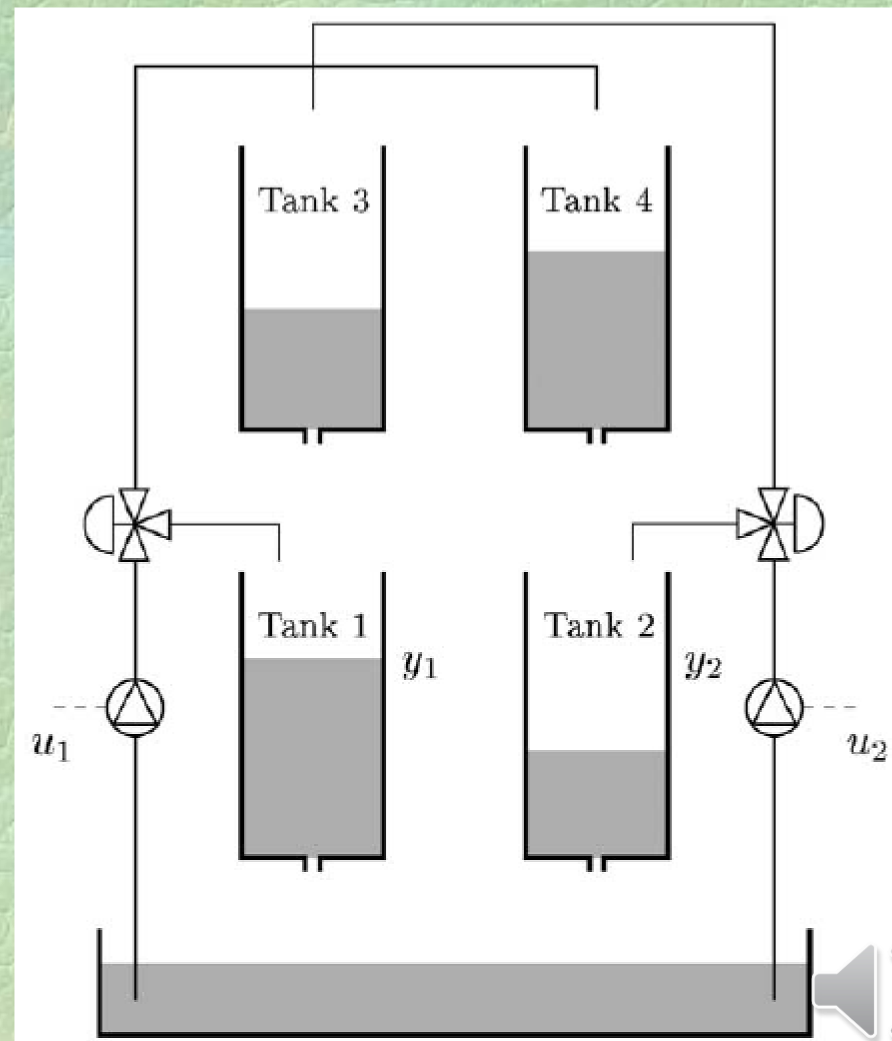
$$y_{p1}y_1^o+\sum_{k=2}^{m}\left|y_{pk}\right|\hat{y}_{1k}\geq\frac{\hat{r}pt_{r1}}{2}y_{p1}-\left(e^{pt_{r1}}-1\right)\sum_{k=2}^{m}\left|y_{pk}\right|\hat{y}_{1k}$$

elements of pole direction — interaction — rise time

Fundamental Limitation on Performance (Time domain)

Example5-6: Experimental set-up for the quadruple-tank process.





Fundamental Limitation on Performance (Time domain)

Example5-6(Continue): Experimental set-up for the quadruple-tank process.

Valve set points are used to make the process more or less difficult to control.

If $\gamma_1 + \gamma_2 \in [1,2]$ no RHP zero, here $\gamma_1 = 0.7, \gamma_2 = 0.6$

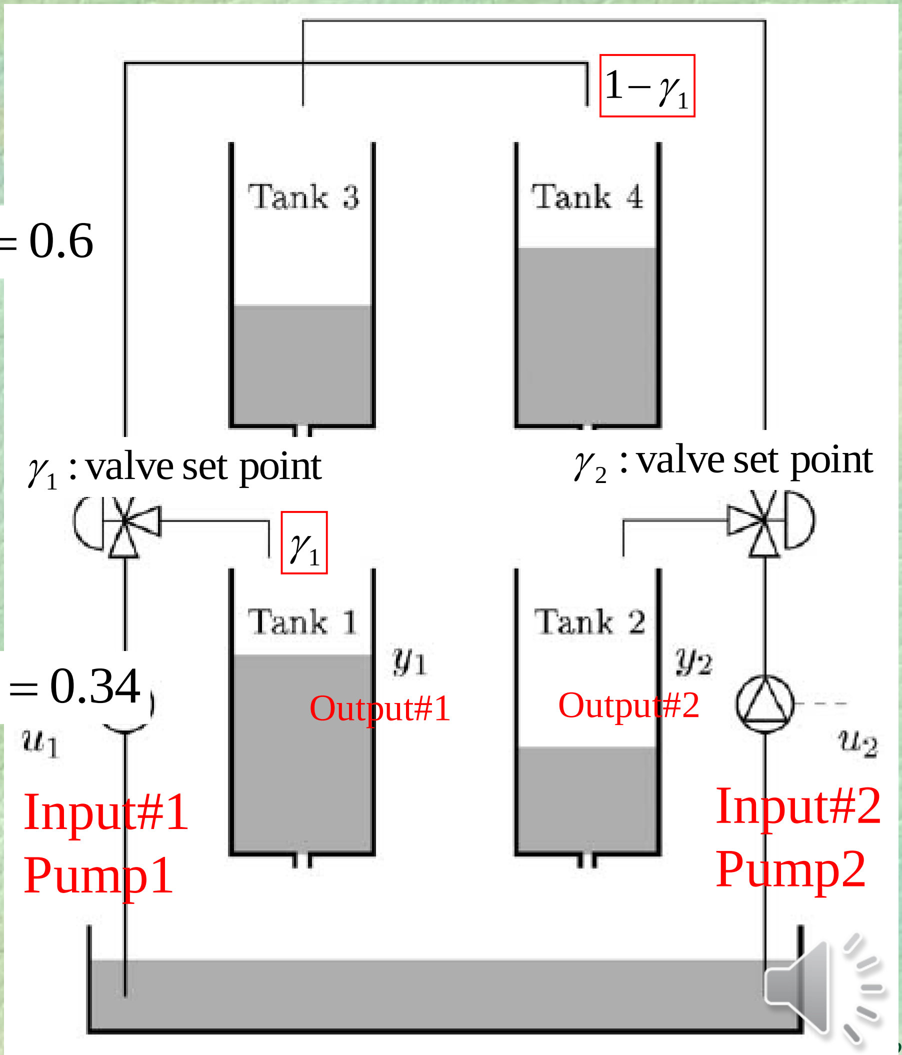
$$G_-(s) = \begin{bmatrix} \dfrac{3.11}{1+95.57s} & \dfrac{2.04}{(1+32.05s)(1+95.57s)} \\ \dfrac{1.71}{(1+38.90s)(1+98.67s)} & \dfrac{3.24}{1+98.67s} \end{bmatrix}$$

$z_1 = -0.012 \qquad z_2 = -0.045$

If $\gamma_1 + \gamma_2 \in [0,1]$ one RHP zero, here $\gamma_1 = 0.43, \gamma_2 = 0.34$

$$G_+(s) = \begin{bmatrix} \dfrac{1.69}{1+76.75s} & \dfrac{3.33}{(1+52.30s)(1+76.75s)} \\ \dfrac{3.11}{(1+56.3s)(1+111.55s)} & \dfrac{1.97}{1+111.55s} \end{bmatrix}$$

$z_1 = +0.014 \qquad z_2 = -0.051$



Fundamental Limitation on Performance (Time domain)

For a unit step in r_1 we have:

$$y_{z1} y_1^{u} + |y_{z2}| \hat{y}_{21} \geqslant \frac{y_{z1}}{e^{z t_{s1}} - 1},$$

$$y_1^{u} + 1.20 \hat{y}_{21} \geqslant \frac{1}{e^{0.014 t_{s1}} - 1}.$$

For a settling time of t_{s1}=100 we have:

$$y_1^{u} + 1.20 \hat{y}_{21} \geqslant 0.32.$$

High undershoot for small interaction.

Fundamental Limitation on Performance (Time domain)

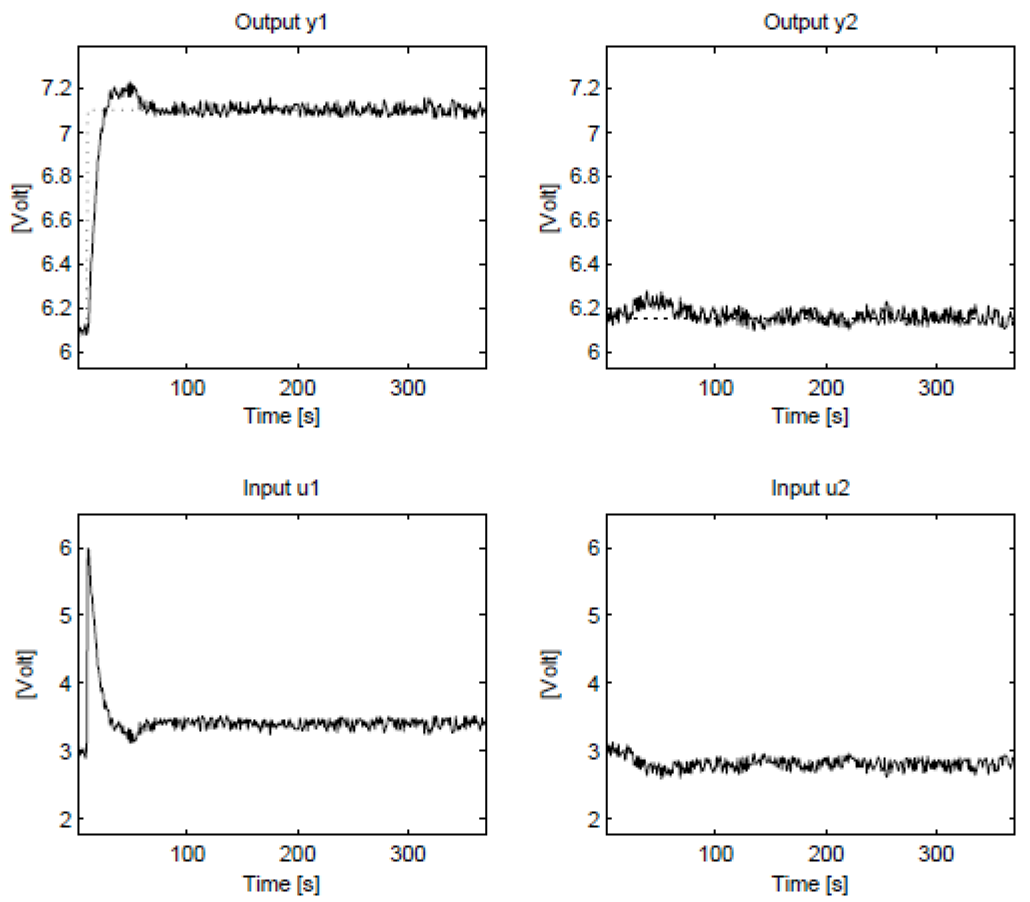


Fig. 4. Responses for decentralized PI control of the quadruple-tank process in minimum-phase setting. The input is a unit reference step in r_1.

Fundamental Limitation on Performance (Time domain)

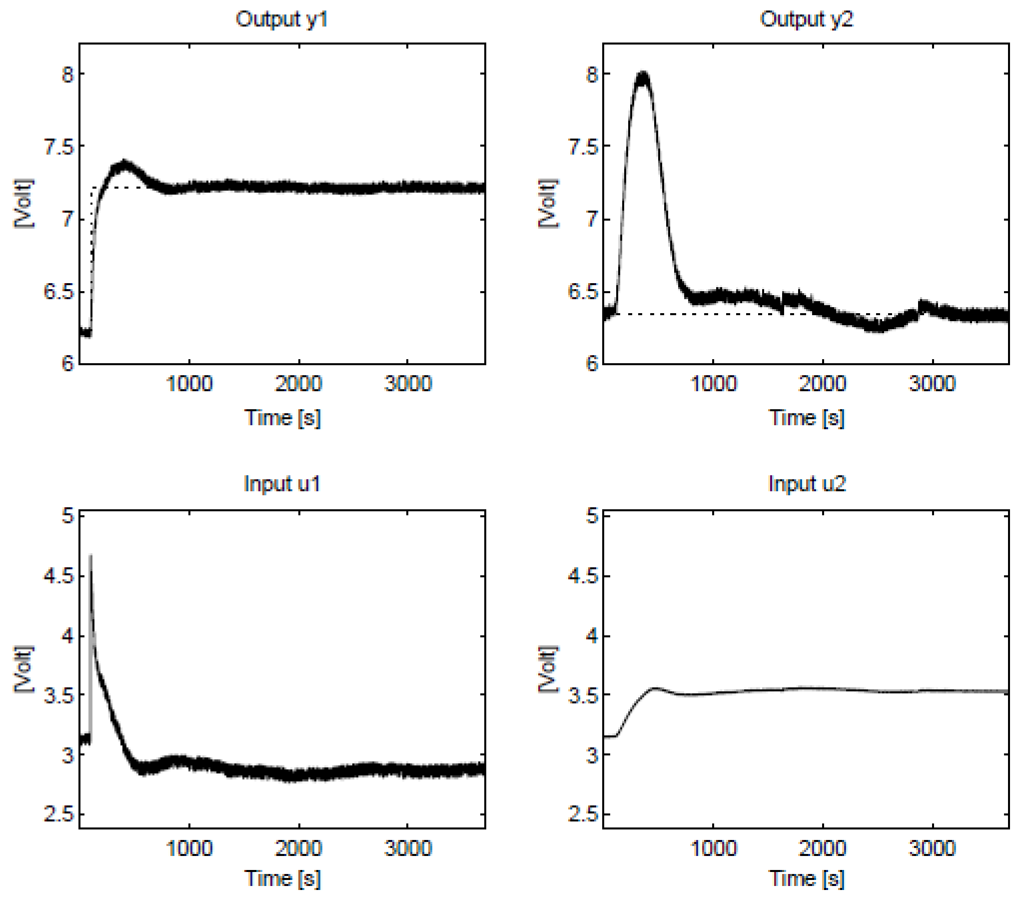


Fig. 5. Responses for decentralized PI control of the quadruple-tank process in non minimum-phase setting. The input is a unit reference step in r_1. Note that the settling time is about 10 times longer than for the minimum-phase setting shown in Fig. 4.

Exercises

5-1 Mentioned in the lecture.

5-2 Mentioned in the lecture.

5-3 Consider the following weight with f>1.

$$w_P(s) = \frac{s + M\omega_B^*}{s} \frac{s + fM\omega_B^*}{s + fM^2\omega_B^*}$$

Plot the weight for $f = 10$ and $M = 2$. Derive an upper bound on ω_B^* for the case with $f = 10$ and $M = 2$.

5-4 Consider the weight

$$w_P(s) = \frac{1}{M} + \left(\frac{\omega_B^*}{s}\right)^n$$

which requires $|S|$ to have a slope of n at low frequencies and requires its low-frequency asymptote to cross 1 at a frequency ω_B^*. Derive an upper bound on ω_B^* when the plant has a RHP- zero at z.

Exercises (Continue)

5-5 Consider the plant

$$G(s) = \begin{bmatrix} \alpha & 1 \\ \frac{1}{s+1} & \alpha \end{bmatrix}$$

a) Find the zero and its output direction. (Answer $z = \frac{1}{\alpha^2} - 1$ and $y_z = \begin{bmatrix} -\alpha \\ 1 \end{bmatrix}$)

b) Which values of α yield a RHP-zero, and which of these values is best/worst in terms of achievable performance? (Answer: We have a RHP-zero for $|\alpha| < 1$. Best for $\alpha = 0$ with zero at infinity; if control at steady-state required then worst for $\alpha = 1$ with zero at $s = 0$.)

c) Suppose $\alpha = 0.1$. Which output is the most difficult to control? Illustrate your conclusion using suitable Theorem (Answer: Output ۲ is the most difficult since the zero is mainly in that direction; we get interaction $\beta = 20$ if we want to control y_2 perfectly.

5-6 Repeat 5-5 for following plant.

$$G(s) = \frac{1}{s+1} \begin{bmatrix} s-\alpha & 1 \\ (\alpha+2)^2 & s-\alpha \end{bmatrix}$$

References

References

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