Multivariable Control Systems

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Lecture 6

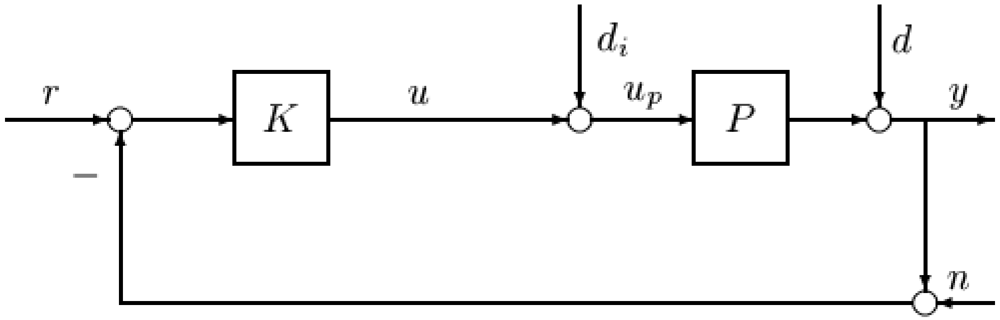
References are appeared in the last slide.

Stability of Multivariable Feedback Control Systems

Topics to be covered include:

- **Well - Posedness of Feedback Loop**
- **Internal Stability**
- **The Nyquist Stability Criterion**
  - **The Generalized Nyquist Stability Criterion**
  - **Nyquist arrays and Gershgorin bands**
- **Coprime Factorizations over Stable Transfer Functions**
- **Stabilizing Controllers**
- **Strong and Simultaneous Stabilization**

Well - Posedness of Feedback Loop



Assume that the plant P and the controller K are fixed real rational proper transfer matrices.

The first question one would ask is whether the feedback interconnection makes sense or is physically realizable.

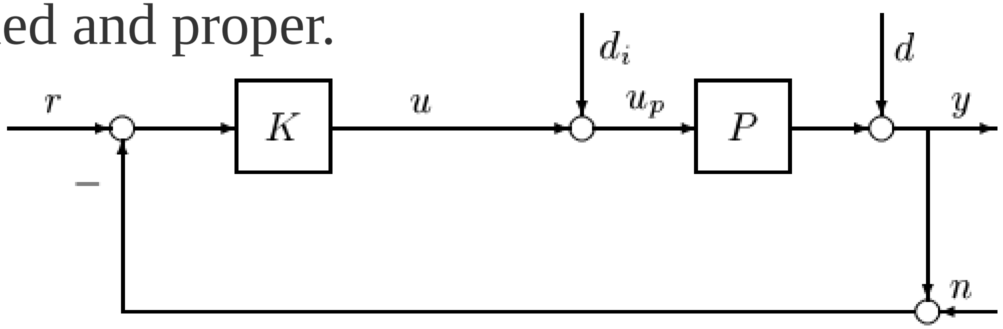
Let $P = -\dfrac{s-1}{s+2}, \quad K = 1$

$$u = \frac{s+2}{3}(r-n-d) + \frac{s-1}{3} d_i$$

Hence, the feedback system is not physically realizable!

Well - Posedness of Feedback Loop

Definition 6-1 A feedback system is said to be well-posed if all closed-loop transfer matrices are well-defined and proper.



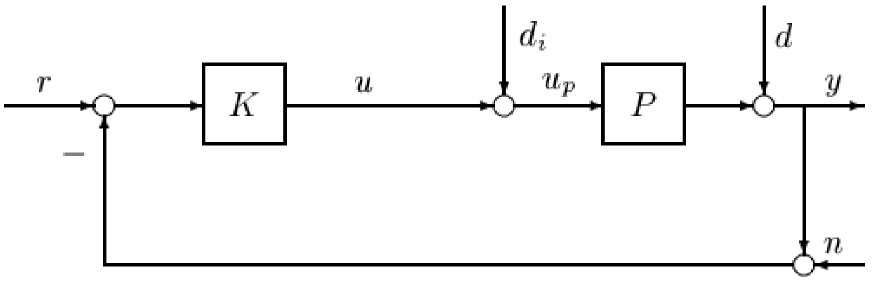
Now suppose that all transfer matrices from the signals *r*, *n*, *d* and d_i to *u* are respectively well-defined and proper.

Thus *y* and all other signals are also well-defined and the related transfer matrices are proper.

So the system is well-posed if and only if the transfer matrix from d_i and *d* **to** *u* **exists and is proper**.

Well - Posedness of Feedback Loop

So the system is well-posed if and only if the transfer matrix from d_i and d to u **exists and is proper**.



Theorem 6-1 The feedback system in Figure is well-posed if and only if

$$I + K(\infty)P(\infty) \quad \text{is invertible}$$

Proof

$$u = -(I + KP)^{-1}\begin{bmatrix} K & KP \end{bmatrix}\begin{bmatrix} d \\ d_i \end{bmatrix}$$

Thus well - posedness is equivalent to the condition that $(I + KP)^{-1}$ exist and is proper.

And this is equivalent to the condition that the constant term of the transfer matrix $I + K(\infty)P(\infty)$ is invertible. □

Well - Posedness of Feedback Loop

Transfer matrix $I + K(\infty)P(\infty)$ is invertible.

is equivalent to either one of the following two conditions:

$$\begin{bmatrix} I & K(\infty) \\ -P(\infty) & I \end{bmatrix} \text{ is invertible}$$

$$I + K(\infty)P(\infty) \text{ is invertible}$$

The well- posedness condition is simple to state in terms of state-space realizations. Introduce realizations of *P* and *K:*

$$P \cong \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad K \cong \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}$$

SO well- posedness is equivalent to the condition that

$$\begin{bmatrix} I & \hat{D} \\ -D & I \end{bmatrix} \text{ is invertible}$$

Stability of Multivariable Feedback Control Systems

- **Well - Posedness of Feedback Loop**
- **Internal Stability**
- **The Nyquist Stability Criterion**
  - **The Generalized Nyquist Stability Criterion**
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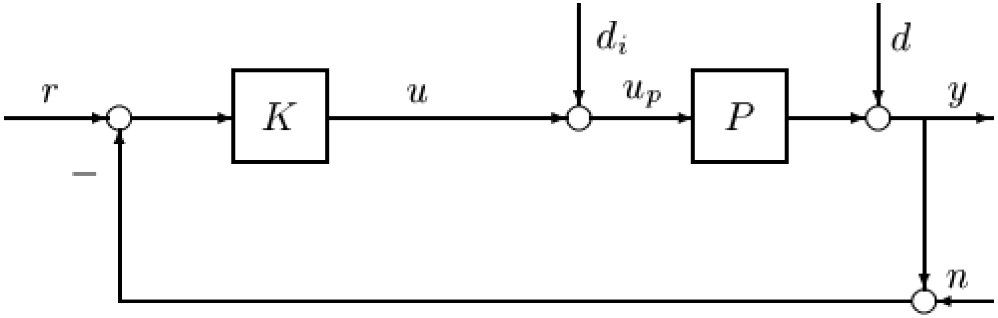
Internal Stability



Assume that the realizations for *P(s)* and *K(s)* are stabilizable and detectable. Let *x* and $\hat{x}$ denote the state vectors for *P* and *K*, respectively.

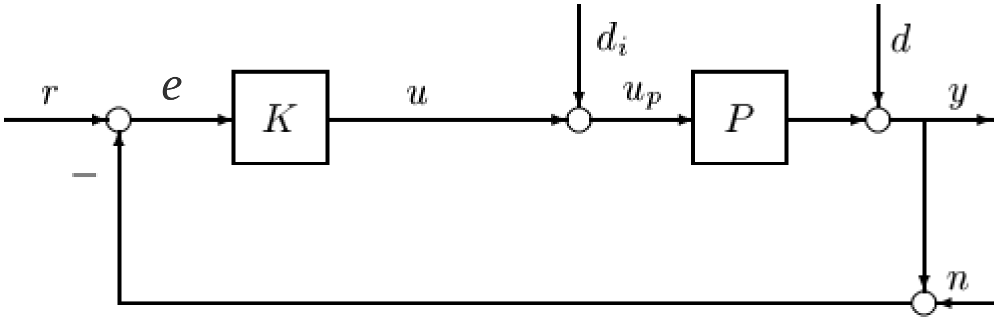
$$\dot{x} = Ax + Bu \qquad \dot{\hat{x}} = \hat{A}\hat{x} - \hat{B}y$$

$$y = Cx + Du \qquad u = \hat{C}\hat{x} - \hat{D}y$$

Definition 6-2

The system of Figure is said to be internally stable if the origin $(x,\hat{x}) = (0,0)$ is asymptotically stable i.e. the states $(x,\hat{x})$ go to zero from all initial states when $r = 0, d = 0, d_i = 0$ and $n = 0$

Internal Stability



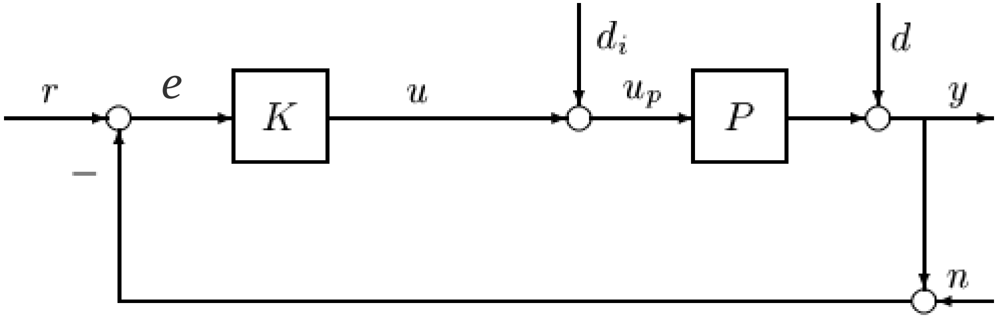
$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} I & \hat{D} \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \underbrace{\left(\begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -\hat{B} \end{bmatrix} \begin{bmatrix} I & \hat{D} \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix} \right)}_{\tilde{A}} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \tilde{A} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

Theorem 6-2

The system of above Figure with given stabilizable and detectable realizations for P and K is internally stable if and only if $\tilde{A}$ is a Hurwitz matrix (All eigenvalues are in open left half plane).

Internal Stability



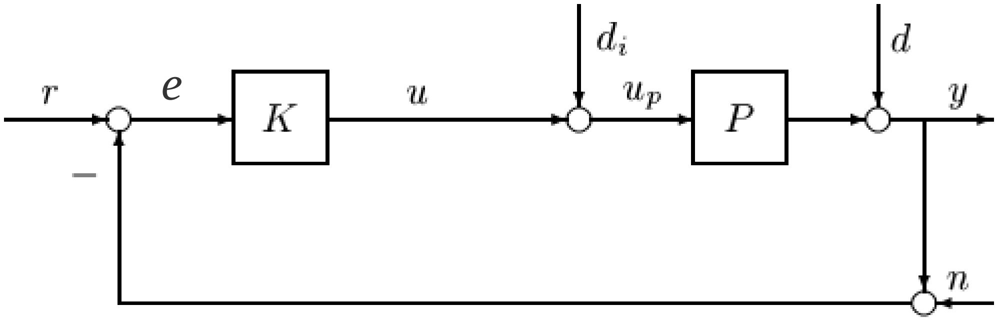
$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} I & \hat{D} \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \underbrace{\left(\begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -\hat{B} \end{bmatrix} \begin{bmatrix} I & \hat{D} \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix} \right)}_{\tilde{A}} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \tilde{A} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

What about stability in the sense of Lyapunov?

The system of above Figure with given stabilizable and detectable realizations for P and K is stable in the sense of Lyapunov if and only if all eigenvalues of $\tilde{A}$ be in ....

Internal Stability



$$\begin{bmatrix} I & K \\ -P & I \end{bmatrix}\begin{bmatrix} u_p \\ -e \end{bmatrix} = \begin{bmatrix} d_i \\ -r \end{bmatrix}$$

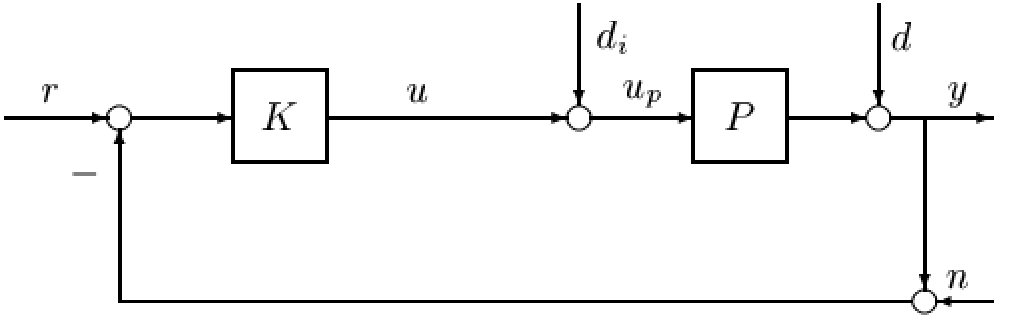
Theorem 6-3

The system in Figure is internally stable if and only if the transfer matrix

$$\begin{bmatrix} I & K \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} I - K(I+PK)^{-1}P & -K(I+PK)^{-1} \\ (I+PK)^{-1}P & (I+PK)^{-1} \end{bmatrix}$$

from (d_i , -r) to (u_p , - e) be a proper and stable transfer matrix.

Internal Stability



$$\begin{bmatrix} I & K \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} I - K(I+PK)^{-1}P & -K(I+PK)^{-1} \\ (I+PK)^{-1}P & (I+PK)^{-1} \end{bmatrix}$$

Note that to check internal stability, it is necessary (and sufficient) to test whether each of the four transfer matrices is stable.

Let $P = \dfrac{s-1}{s+1}, \quad K = \dfrac{1}{s-1}$

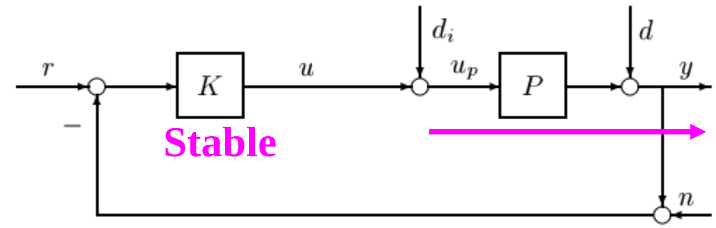
$$\begin{bmatrix} u_p \\ -e \end{bmatrix} = \begin{bmatrix} \dfrac{s+1}{s+2} & -\dfrac{s+1}{(s-1)(s+2)} \\ \dfrac{s-1}{s+2} & \dfrac{s+1}{s+2} \end{bmatrix} \begin{bmatrix} d_i \\ -r \end{bmatrix}$$

Stability cannot be concluded even if three of the four transfer matrices are stable.

Internal Stability

Theorem 6-4

Suppose K is stable. Then the system in the figure is internally stable if and only if $(I+PK)^{-1}P$ is stable.



Remember $\begin{bmatrix} I-K(I+PK)^{-1}P & -K(I+PK)^{-1} \\ (I+PK)^{-1}P & (I+PK)^{-1} \end{bmatrix}$

Proof: The necessity is obvious. To prove the sufficiency let

$$Q=(I+PK)^{-1}P$$

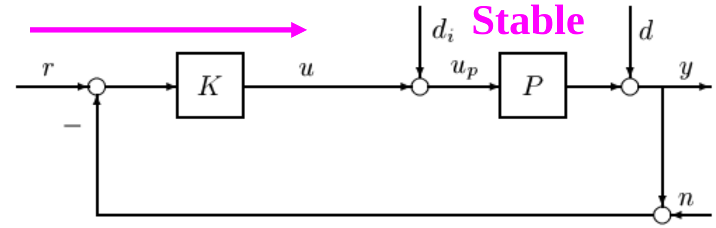
$I-K(I+PK)^{-1}P=I-KQ$ ✓ Stable

$-K(I+PK)^{-1}=-K(I-QK)$ ✓ Stable

$(I+PK)^{-1}P=Q$ ✓ Stable

$(I+PK)^{-1}=I+(I+PK)^{-1}-I=I-(I+PK)^{-1}PK=I-QK$ ✓ Stable

Internal Stability



Theorem 6-5

Suppose P is stable. Then the system in the figure is internally stable if and only if $K(I+PK)^{-1}$ is stable.

Remember $\begin{bmatrix} I-K(I+PK)^{-1}P & -K(I+PK)^{-1} \\ (I+PK)^{-1}P & (I+PK)^{-1} \end{bmatrix}$

Proof: The necessity is obvious. To prove the sufficiency let

$$Q = K(I+PK)^{-1}$$

$I-K(I+PK)^{-1}P = I-QP$ ✓ Stable

$-K(I+PK)^{-1} = -Q$ ✓ Stable

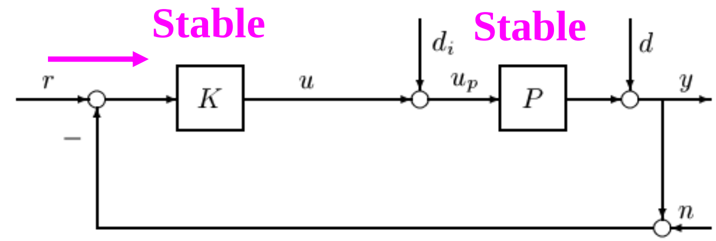
$(I+PK)^{-1}P = (I-PQ)P$ ✓ Stable

$(I+PK)^{-1} = I+(I+PK)^{-1}-I = I-PK(I+PK)^{-1} = I-PQ$ ✓ Stable

Internal Stability

Theorem 6-6

Suppose P and K are both stable. Then the system in the figure is internally stable if and only if $(I+PK)^{-1}$ is stable.



Remember $\begin{bmatrix} I-K(I+PK)^{-1}P & -K(I+PK)^{-1} \\ (I+PK)^{-1}P & (I+PK)^{-1} \end{bmatrix}$

Proof: The necessity is obvious. To prove the sufficiency let

$$Q=(I+PK)^{-1}$$

$$I-K(I+PK)^{-1}P=I-KQP \quad \checkmark \quad \text{Stable}$$

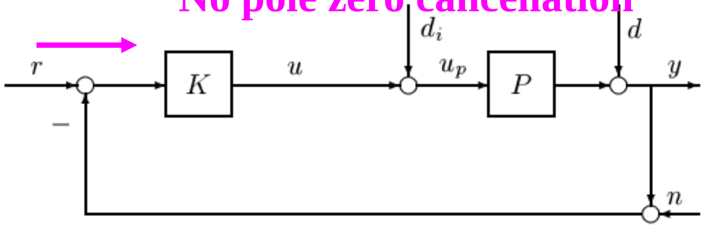
$$-K(I+PK)^{-1}=-KQ \quad \checkmark \quad \text{Stable}$$

$$(I+PK)^{-1}P=QP \quad \checkmark \quad \text{Stable}$$

$$(I+PK)^{-1}=Q \quad \checkmark \quad \text{Stable}$$

Internal Stability

No pole zero cancellation



Theorem 6-7

The system in the figure is internally stable if and only if

Remember $\begin{bmatrix} I-K(I+PK)^{-1}P & -K(I+PK)^{-1} \\ (I+PK)^{-1}P & (I+PK)^{-1} \end{bmatrix}$

i) The number of open RHP poles of $P(s)K(s) = n_k + n_P$

ii) $(I+PK)^{-1}$ is stable.

n_K is the number of RHP polse of K n_P is the number of RHP polse of P

Proof:

See: "Essentials of Robust control" written by Kemin Zhou

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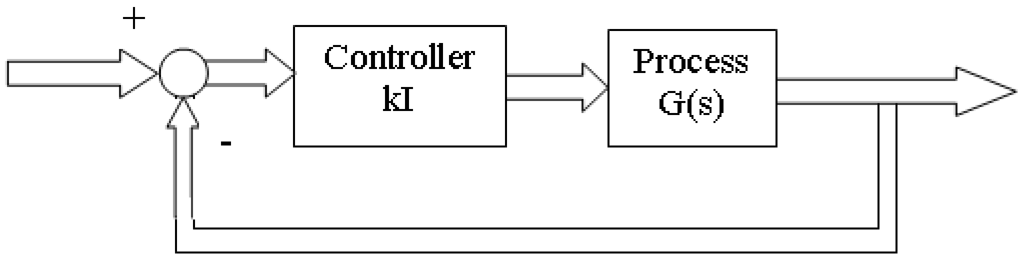
The Nyquist Stability Criterion



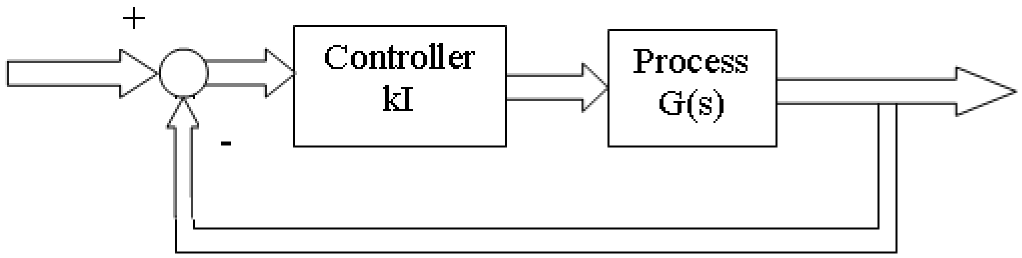
Now we are going to check the stability of the above system for various real values of k.

Let $\det[I + kG(s)]$ have P_o poles and P_c zeros in the RHP plane.

Then just as SISO systems, the Nyquist plot of $\phi(s) = \det(I + kG(s))$ encircles the origin, $P_c - P_o$ times.

However, we would have to draw the Nyquist locus of |I+kG(s)| for each value of k in which we were interested, whereas in the classical Nyquist criterion we draw a locus only once, and then infer stability properties for all values of k.

The Nyquist Stability Criterion



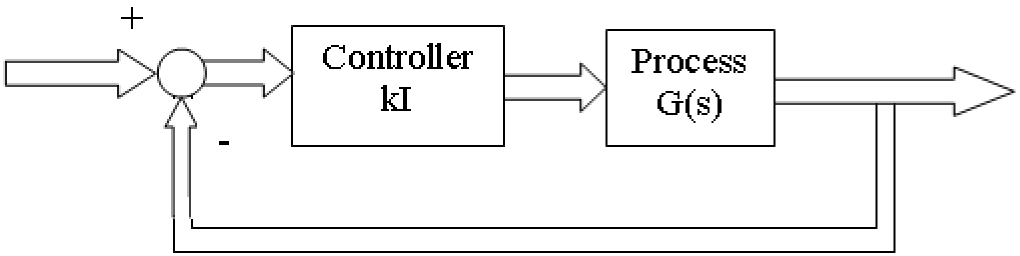
$$\det[I + kG(s)] = \prod_i [1 + k\lambda_i(s)]$$

Where $\lambda_i(s)$ is an eigenvalue of $G(s)$

$$\vartheta\{\det[I + kG(s)]\} = \sum_i \vartheta(1 + k\lambda_i(s))$$

$$0 \quad \rightarrow \quad -1$$

The Nyquist Stability Criterion

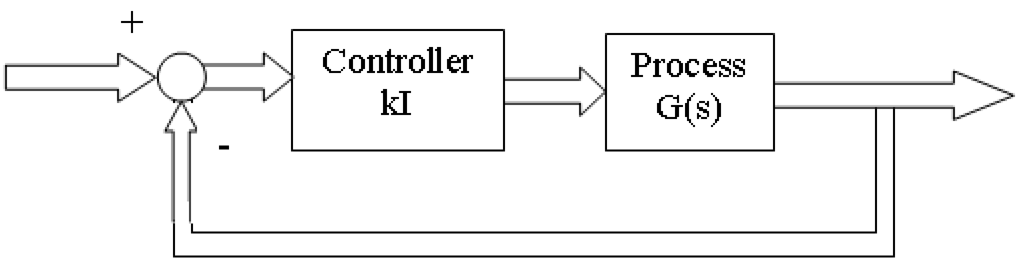


Theorem 6-8 (Generalized Nyquist theorem)

If $G(s)$ with no hidden unstable modes, has P_o unstable (Smith-McMillan) poles, then the closed-loop system with return ratio $-kG(s)$ is stable if and only if the characteristic loci of $kG(s)$, taken together, encircle the point -1, P_o times anticlockwise.

$$Z_{I+kG(s)} - P_{I+kG(s)} = N_{-1} \qquad \Rightarrow \qquad Z_{I+kG(s)} - P_{G(s)} = N_{-1}$$

The Nyquist Stability Criterion



Example 6-1: **Let**

$$G(s)=\begin{bmatrix} \frac{0.5(s+1)^2+1}{(s+1)^3} & \frac{-0.5(s-1)}{(s+1)} \\ \frac{1}{(s-1)(s+1)^3} & 0 \end{bmatrix}$$

Suppose that $G(s)$ has no hidden modes, check the stability of system for different values of k.

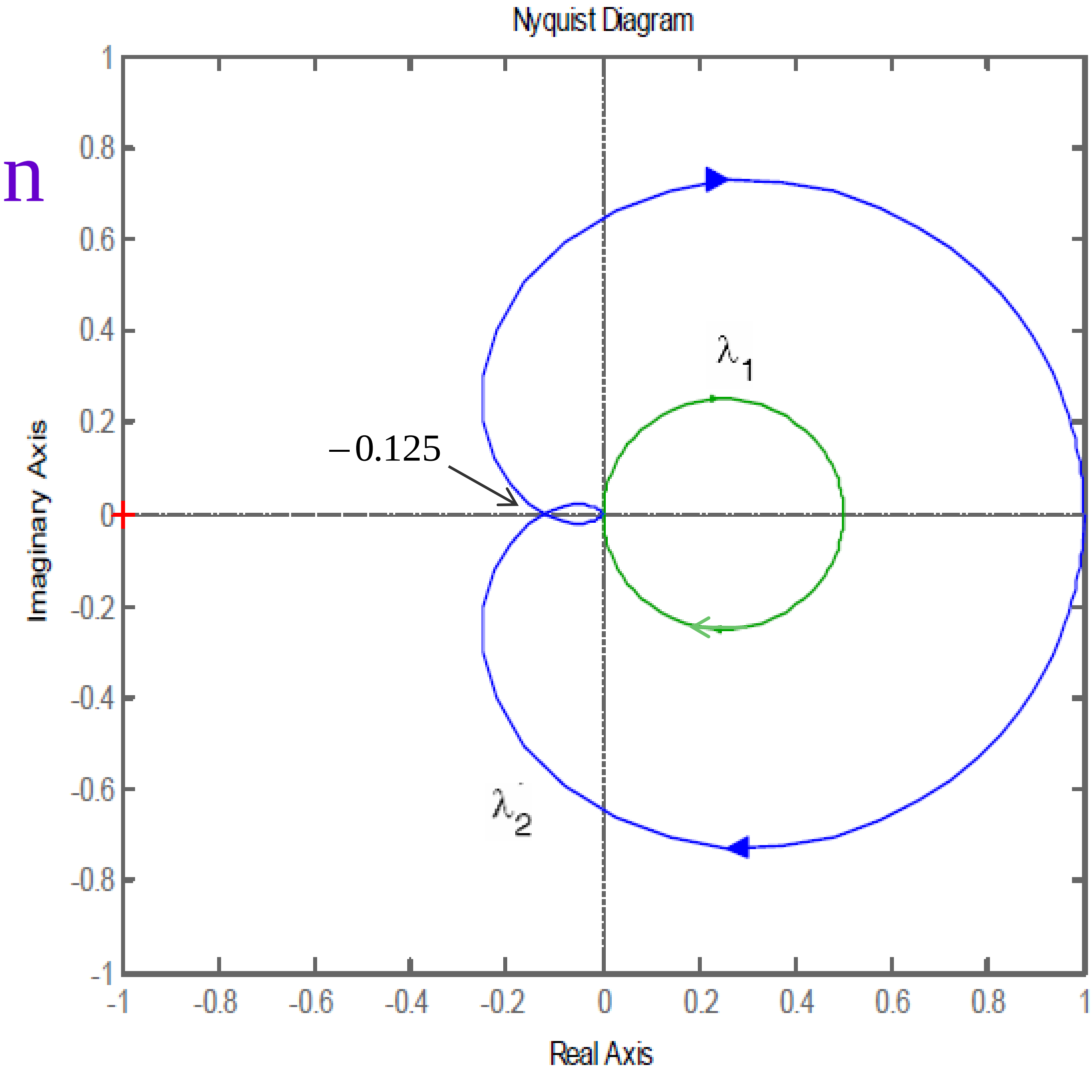
$$\left|\lambda I - G(s)\right| = 0$$

$$\lambda_1 = \frac{0.5}{(s+1)} \qquad \lambda_2 = \frac{1}{(s+1)^3}$$

The Nyquist Stability Criterion

$$\lambda_1 = \frac{0.5}{(s+1)}$$

$$\lambda_2 = \frac{1}{(s+1)^3}$$



Since $G(s)$ has one unstable poles, we will have closed loop stability if these loci give one net encirclements of -1/k (ccw) when a negative feedback kI is applied.

$-\infty < -1/k < -0.125$ no encirclement so there is one RHP closed loop pole.

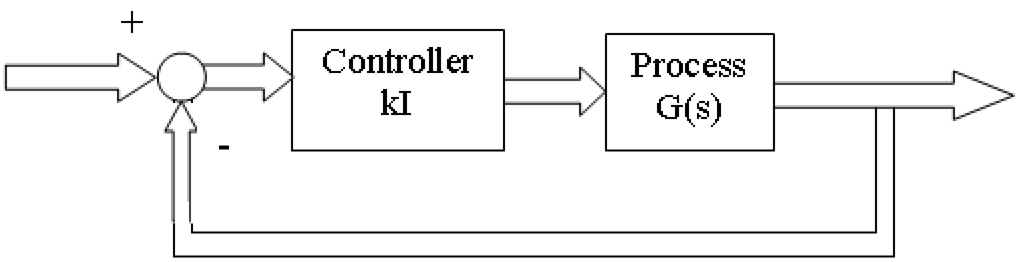
$-0.125 < -1/k < 0.5$ two encirclement so there is three RHP closed loop pole.

$0.5 < -1/k < 1$ one encirclement so there is two RHP closed loop pole.

$-1/k > 1$ no encirclement so there is one RHP closed loop pole.

The Nyquist Stability Criterion

Example 6-2: **Let**



$$G(s) = \frac{1}{1.25(s+1)(s+2)} \begin{bmatrix} s-1 & s \\ -6 & s-2 \end{bmatrix}$$

Suppose that $G(s)$ has no hidden modes, check the stability of system for different values of k.

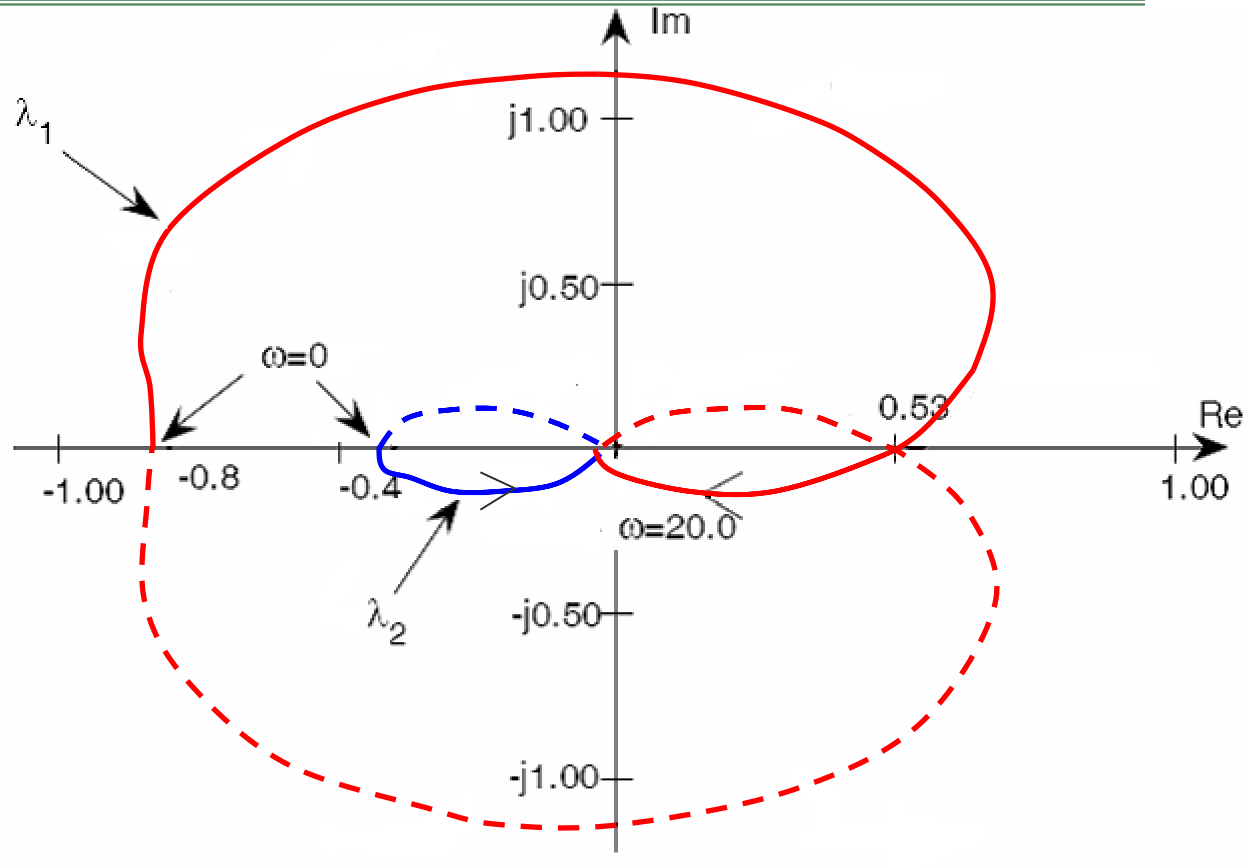
$$\left|\lambda I - G(s)\right| = 0$$

$$\lambda_1 = \frac{2s-3-\sqrt{1-24s}}{2.5(s+1)(s+2)} \qquad \lambda_2 = \frac{2s-3+\sqrt{1-24s}}{2.5(s+1)(s+2)}$$

The Nyquist Stability Criterion

$$\lambda_1 = \frac{2s-3-\sqrt{1-24s}}{2.5(s+1)(s+2)}$$

$$\lambda_2 = \frac{2s-3+\sqrt{1-24s}}{2.5(s+1)(s+2)}$$



Since $G(s)$ has no unstable poles, we will have closed loop stability if these loci give zero net encirclements of -1/*k* when a negative feedback *kI* is applied.

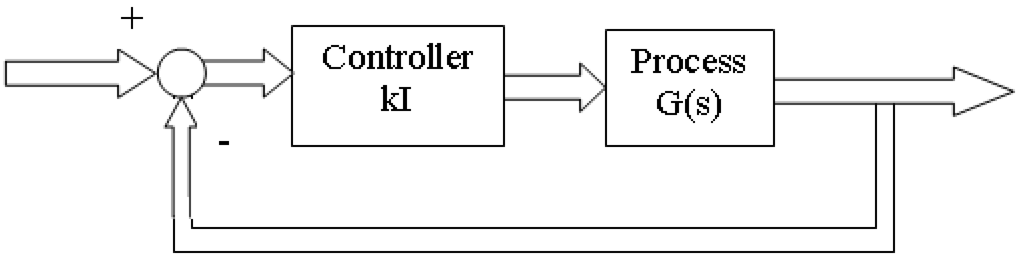
$-\infty < -1/k < -0.8$, $-0.4 < -1/k < 0$ and $0.53 < -1/k < \infty$ no encirclement.

$-0.8 < -1/k < -0.4$ one encirclement so one RHP pole in closed loop system.

$0 < -1/k < 0.53$ two encirclements so two RHP pole in closed loop system.

The Nyquist Stability Criterion

Example 6-3: a)Check the stability of system.
b) derive the closed loop system for k=2 and check the part a.



$$G(s) = \begin{bmatrix} 0 & 1 \\ \dfrac{s-1}{s+1} & 0 \end{bmatrix}$$

$Ans: It\ is\ stable\ for\ |k| < 1$

Nyquist arrays and Gershgorin bands

The **Nyquist array** of $G(s)$ is an array of graphs (not necessarily closed curve), the ij^{th} graph being the Nyquist locus of $g_{ij}(s)$.

Theorem 7-10 (Gershgorin's theorem)

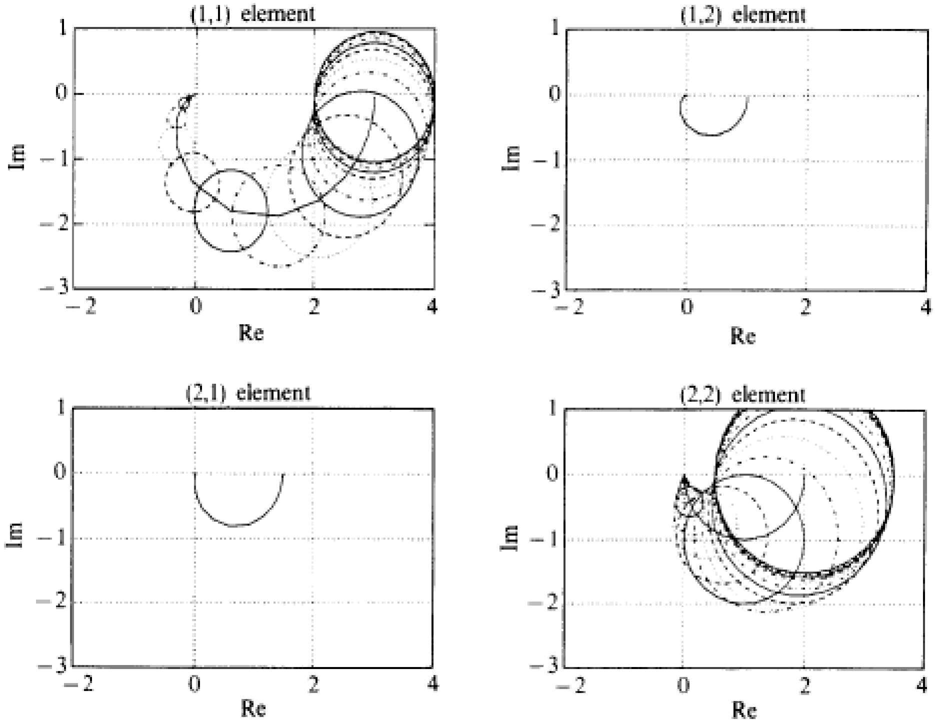
Let Z be a complex matrix of dimensions $m \times m$. The eigenvalues λ_i of Z are contained in two unions of circles around the diagonal elements as follows:

$$\left|\lambda_i - z_{ii}\right| < \sum_{\substack{j=1 \\ j \neq i}}^{m} \left|z_{ij}\right|, \qquad i = 1,\ 2,\ ....,m \qquad \left|\lambda_i - z_{ii}\right| < \sum_{\substack{j=1 \\ j \neq i}}^{m} \left|z_{ji}\right|, \qquad i = 1,\ 2,\ ....,m$$

The 'bands' obtained in this way are called **Gershgorin bands**, each is composed of **Gershgorin circles**.

Nyquist arrays and Gershgorin bands

Nyquist array, with Gershgorin bands for a sample system



If all the Gershgorin bands exclude the point -1, then we can assess closed-loop stability by counting the encirclements of -1 by the Gershgorin bands, since this tells us the number of encirclements made by the characteristic loci.

If the Gershgorin bands of $G(s)$ exclude the origin, then we say that $G(s)$ is **diagonally dominant** (row dominant or column dominant).

The greater the **degree of dominant** (of $G(s)$ or $I+G(s)$) – that is, the narrower the Gershgorin bands- the more closely does $G(s)$ resembles m non-interacting SISO transfer function.

Nyquist arrays and Gershgorin bands

And in general:

Suppose that $G(s)$ is square, that $K = \mathrm{diag}\{k_1, \ldots, k_m\}$ and that (Rosenbrock, 1970):

$$\left| g_{ii}(s) + \frac{1}{k_i} \right| > \sum_{j \neq i} |g_{ij}(s)|$$

for each i and for all s on the Nyquist contour; and let the ith Gershgorin band of $G(s)$ encircle the point $-1/k_i$, N_i times anticlock wise. Then the negative feedback system with return ratio $-G(s)K$ is stable if and only if

$$\sum_i N_i = P_0$$

where P_0 is the number of unstable poles of $G(s)$, and there are no hidden unstable modes.

Nyquist arrays and Gershgorin bands

Diagonal dominance of a $m \times m$ matrix $G(s)$:

Row Diagonal dominance: If for all s on the Nyquist contour,

$$\left| g_{ii}(s) \right| > \sum_{\substack{j=1 \\ j \neq i}}^{m} \left| g_{ij}(s) \right| \qquad i = 1, 2, \ldots, m$$

Column Diagonal dominance: If for all s on the Nyquist contour,

$$\left| g_{ii}(s) \right| > \sum_{\substack{j=1 \\ j \neq i}}^{m} \left| g_{ji}(s) \right| \qquad i = 1, 2, \ldots, m$$

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Coprime Factorizations over Stable Transfer Functions

Two polynomials $m(s)$ and $n(s)$, with real coefficients, are said to be coprime if

- their greatest common divisor is a constant number or
- they have no common zeros or
- there exist polynomials $x(s)$ and $y(s)$ such that $x(s)m(s)+y(s)n(s)=1$

Exercise6-1 : Let $n(s)=s^2+5s+6$ and $m(s)=s$ find $x(s)$ and $y(s)$ if n and m are coprime.

Exercise 6-2 : Let $n(s)=s^2+5s+6$ and $m(s)=s+2$ show that one cannot find $x(s)$ and $y(s)$ in $x(s)m(s)+y(s)n(s)=1$

Similarly, two transfer functions $m(s)$ and $n(s)$ in the set of stable transfer functions are said to be coprime over stable transfer functions if there exists $x(s)$ and $y(s)$ in the set of stable transfer functions such that

$$x(s)m(s)+y(s)n(s)=1$$

Bezout identities

Coprime Factorizations over Stable Transfer Functions

Definition 6-3 Two matrices *M* and *N* in the set of stable transfer matrices are right coprime over the set of stable transfer matrices if they have the same number of columns and if there exist matrices X_r and Y_r in the set of stable transfer matrices s.t.

$$\begin{bmatrix} X_r & Y_r \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = X_r M + Y_r N = I$$

Similarly, two matrices $\tilde{M}$ and $\tilde{N}$ in the set of stable transfer matrices are left coprime over the set of stable transfer matrices if they have the same number of rows and if there exist two matrices X_l and Y_l in the set of stable transfer matrices such that

$$\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} \begin{bmatrix} X_l \\ Y_l \end{bmatrix} = \tilde{M} X_l + \tilde{N} Y_l = I$$

Coprime Factorizations over Stable Transfer Functions

Now let *P* be a proper real-rational matrix. A right-coprime factorization (rcf) of *P* is a factorization of the form

$$P = NM^{-1}$$

where *N* and *M* are right-coprime in the set of stable transfer matrices.

Similarly, a left-coprime factorization (lcf) of *P* has the form

$$P = \tilde{M}^{-1}\tilde{N}$$

In lcf (rcf) the pair $\tilde{M}, \tilde{N}$ (M, N) are coprime in the set of stable transfer function matrices

Coprime Factorizations over Stable Transfer Functions

Remember: Matrix Fraction Description (MFD)

- Right matrix fraction description (RMFD)
- Left matrix fraction description (LMFD)

Let $G(s)$ is a $m \times m$ matrix and its the Smith McMillan is $\tilde{G}(s)$

Let define: $N(s) \overset{\Delta}{=} diag\left(\varepsilon_1(s), \dots, \varepsilon_r(s), 0, \dots, 0\right)$ $D(s) \overset{\Delta}{=} diag\left(\delta_1(s), \dots, \delta_r(s), 1, \dots, 1\right)$

$$\tilde{G}(s) = N(s)D(s)^{-1} \quad \text{or} \quad \tilde{G}(s) = D(s)^{-1}N(s)$$

In Matrix Fraction Description D(s) and N(s) are polynomial matrices

Coprime Factorizations over Stable Transfer Functions

Theorem 6-11 Suppose $P(s)$ is a proper real-rational matrix and

$$P(s) \cong \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is a stabilizable and detectable realization. Let F and L be such that $A+BF$ and $A+LC$ are both stable, and define

$$\begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} \cong \left[\begin{array}{c|cc} A+BF & B & -L \\ \hline F & I & 0 \\ C+DF & D & I \end{array}\right] \qquad \begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} \cong \left[\begin{array}{c|cc} A+LC & -(B+LD) & L \\ \hline F & I & 0 \\ C & -D & I \end{array}\right]$$

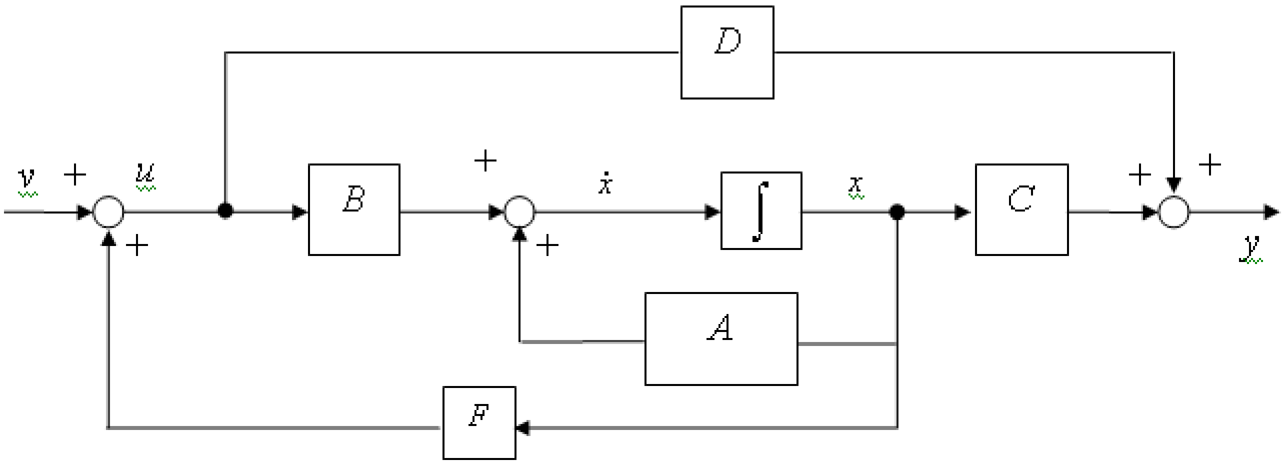
Then rcf and lcf of P are:

$$\boxed{P = NM^{-1}} \qquad \boxed{P = \tilde{M}^{-1}\tilde{N}}$$

Exercise : Let $P(s)=(s+1)/(s+2)$ find two different rcf for P.

Coprime Factorizations over Stable Transfer Functions

The right coprime factorization of a transfer matrix can be given a feedback control interpretation.



$$P$$

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$u = v + Fx$$

$$\dot{x} = (A + BF)x + Bv$$

$$u = Fx + v$$

$$y = (C + DF)x + Dv$$

the transfer matrix from *v to u is*

$$M(s) \cong \begin{bmatrix} A+BF & B \\ F & I \end{bmatrix}$$

and that from *v to y is*

$$N(s) \cong \begin{bmatrix} A+BF & B \\ C+DF & D \end{bmatrix}$$

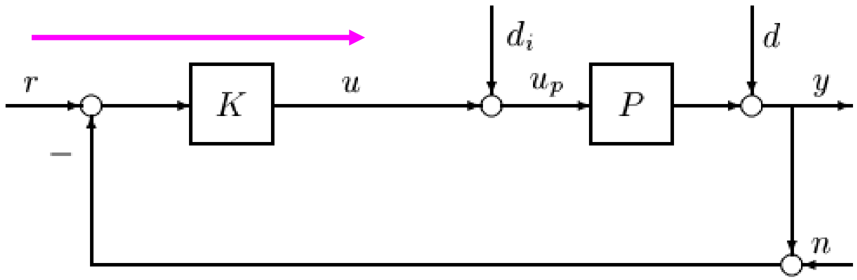
$$y(s) = N(s)v(s) \;= N(s)M^{-1}(s)u(s) \;= P(s)u(s) \quad \Rightarrow \quad P(s) = N(s)M^{-1}(s)$$

Exercise6-4 : Derive a similar interpretation for left coprime factorization.

Stability of Multivariable Feedback Control Systems

- **Well - Posedness of Feedback Loop**
- **Internal Stability**
- **The Nyquist Stability Criterion**
  - **The Generalized Nyquist Stability Criterion**
  - **Nyquist arrays and Gershgorin bands**
- **Coprime Factorizations over Stable Transfer Functions**
- **Stabilizing Controllers**
- **Strong and Simultaneous Stabilization**

Stabilizing Controllers



Theorem 6-12 Suppose P is stable.

Then the set of all stabilizing controllers in Figure can be described as

$$K = Q(I - PQ)^{-1}$$

for any Q in the set of stable transfer matrices and $I - P(\infty)Q(\infty)$ non singular.

Proof:

$$K = Q(I - PQ)^{-1} \quad \Rightarrow \quad K(I - PQ) = Q \quad \Rightarrow \quad Q = K(I + PK)^{-1}$$

✓ **System** is stable.

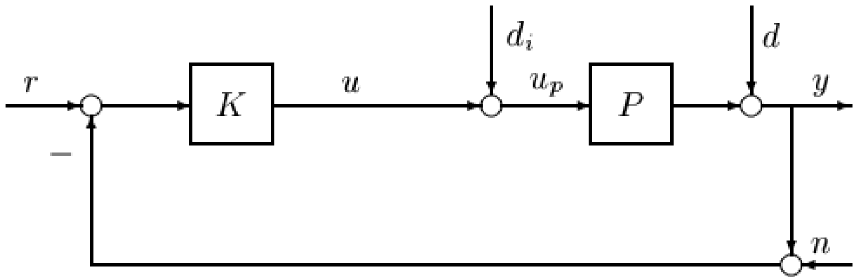
Now suppose the system is stable, so $K(I + PK)^{-1}$ is stable, then define

$$Q = K(I + PK)^{-1} = (I + KP)^{-1}K \quad \Rightarrow \quad Q + KPQ = K$$

$I - P(\infty)Q(\infty)$ is nonsingular so $\quad K = Q(I - PQ)^{-1}$

Stabilizing Controllers

Example 6-4 **For the plant**



$$P(s)=\frac{1}{(s+1)(s+2)}$$

Suppose that it is desired to find an internally stabilizing controller so that *y* asymptotically tracks a ramp input.

Solution: Since the plant is stable the set of all stabilizing controller is derived from

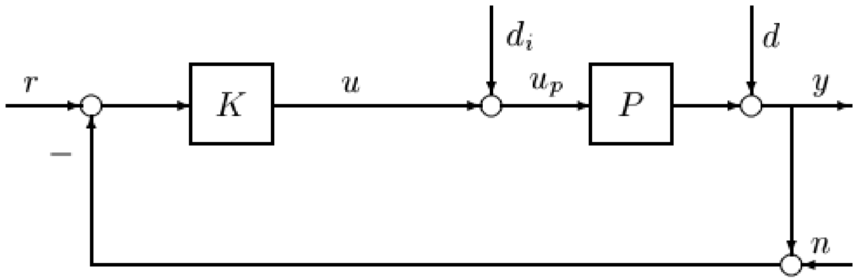
$K=Q(I-PQ)^{-1}$ for any stable Q such that $I-P(\infty)Q(\infty)$ is nonsingular, so let

$$Q=\frac{as+b}{s+3}$$

$$S=1-T=1-PK(I+PK)^{-1}=1-PQ=1-\frac{as+b}{(s+1)(s+2)(s+3)}=\frac{(s+1)(s+2)(s+3)-(as+b)}{(s+1)(s+2)(s+3)}$$

$$Q=\frac{11s+6}{s+3}$$

Stabilizing Controllers



Theorem 6-13

Let P be a proper real-rational matrix and $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$.

Then there exists a stabilizing controller

$$K = Y_l X_l^{-1} = X_r^{-1} Y_r$$

Where
$$\begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} = I$$

Proof. See “Multivariable Feedback Design By Maciejowski”

Stabilizing Controllers

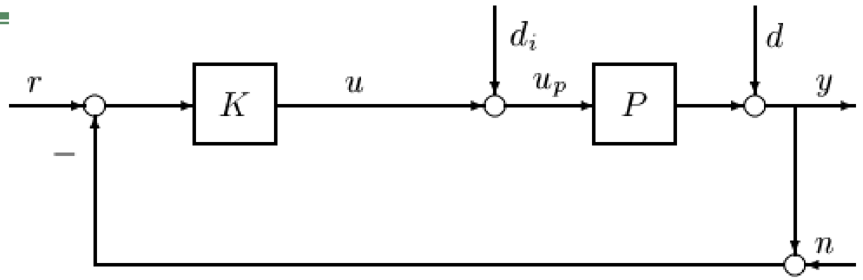
Theorem 6-11(remember) Suppose $P(s)$ is a proper real-rational matrix and

$$P(s) \cong \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is a stabilizable and detectable realization. Let F and L be such that $A+BF$ and $A+LC$ are both stable, and define

$$\begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} \cong \left[\begin{array}{c|cc} A+BF & B & -L \\ \hline F & I & 0 \\ C+DF & D & I \end{array}\right] \qquad \begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} \cong \left[\begin{array}{c|cc} A+LC & -(B+LD) & L \\ \hline F & I & 0 \\ C & -D & I \end{array}\right]$$

Stabilizing Controllers



Theorem 6-14

Let P be a proper real-rational matrix and $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ be corresponding rcf and lcf over the set of stable transfer matrices. Then the set of all stabilizing controllers in Figure 4-1 can be described as

$K = (X_r - Q_r\tilde{N})^{-1}(Y_r + Q_r\tilde{M})$	4-28

or

$K = (Y_l + MQ_l)(X_l - NQ_l)^{-1}$	4-29

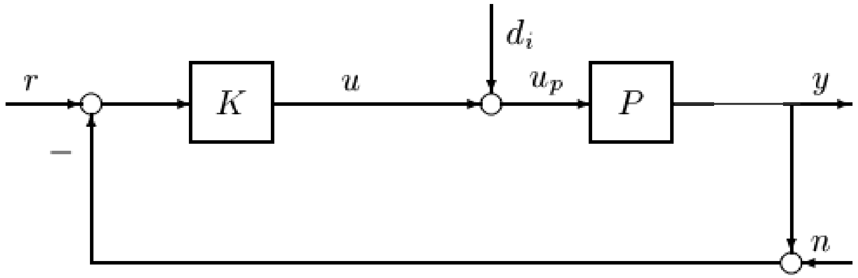
where Q_r is any stable transfer matrices and $X_r(\infty) - Q_r(\infty)\tilde{N}(\infty)$ is nonsingular or Q_l is any stable transfer matrices and $X_l(\infty) - N(\infty)Q_l(\infty)$ is nonsingular too.

Proof. See "Multivariable Feedback Design By Maciejowski"

Stabilizing Controllers

Example 6-5 For the plant

$$P(s)=\frac{1}{(s-1)(s-2)}$$



The problem is to find a controller that

1. The feedback system is internally stable.
2. The final value of y equals 1 when r is a unit step and d=0.
3. The final value of y equals zero when d is a sinusoid of 10 rad/s and r=0.

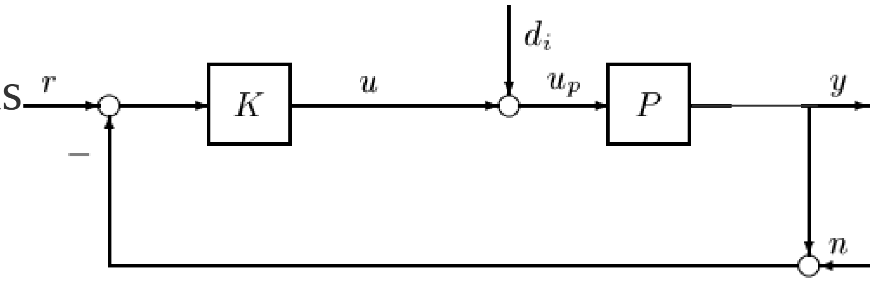
Clearly $A=\begin{bmatrix}0 & 1\\ -2 & 3\end{bmatrix}$ $\quad B=\begin{bmatrix}0\\ 1\end{bmatrix}$ $\quad C=[1 \quad 0]$ $\quad D=0$

To derive coprime factorization let F = [1 -5] and L = [-7 -23]^T
clearly $A+BF$ and $A+LC$ are stable.

$$\begin{bmatrix}M & -Y_l\\ N & X_l\end{bmatrix}\cong\left[\begin{array}{c|cc}A+BF & B & -L\\ \hline F & I & 0\\ C+DF & D & I\end{array}\right] \qquad \begin{bmatrix}X_r & Y_r\\ -\tilde{N} & \tilde{M}\end{bmatrix}\cong\left[\begin{array}{c|cc}A+LC & -(B+LD) & L\\ \hline F & I & 0\\ C & -D & I\end{array}\right]$$

Stabilizing Controllers

Solution: The set of all stabilizing controller is



$$K = (X_r - Q_r\tilde{N})^{-1}(Y_r + Q_r\tilde{M})$$

$$\begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} \cong \left[\begin{array}{c|cc} A+BF & B & -L \\ \hline F & I & 0 \\ C+DF & D & I \end{array}\right] \qquad \begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} \cong \left[\begin{array}{c|cc} A+LC & -(B+LD) & L \\ \hline F & I & 0 \\ C & -D & I \end{array}\right]$$

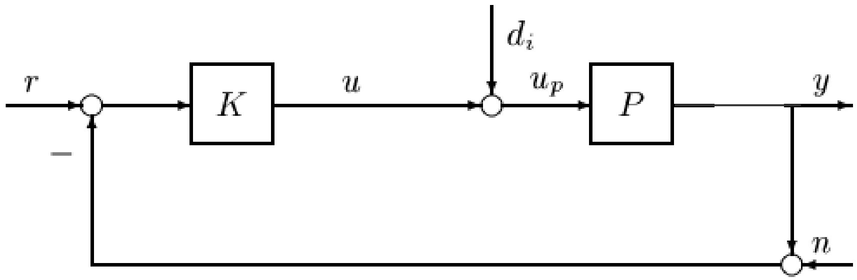
$$N = \frac{1}{(s+1)^2},\ M = \frac{(s-2)(s-1)}{(s+1)^2},\ Y_l = \frac{108s-72}{(s+1)^2},\ X_l = \frac{s^2+9s+38}{(s+1)^2}$$

$$\tilde{N} = \frac{1}{(s+2)^2},\ \tilde{M} = \frac{(s-2)(s-1)}{(s+2)^2},\ Y_r = \frac{108s-72}{(s+2)^2},\ X_r = \frac{s^2+9s+38}{(s+2)^2}$$

Stabilizing Controllers

Solution: The set of all stabilizing controller is:

$$K=(X_r-Q_r\tilde{N})^{-1}(Y_r+Q_r\tilde{M})$$



Clearly for any stable Q_r the condition 1 satisfied

To met condition 2 the transfer function from r to y must satisfy

$$y(s)=N(Y_r+Q_r\tilde{M})r(s) \quad \rightarrow \quad N(0)(Y_r(0)+Q_r(0)\tilde{M}(0))=1 \quad \Rightarrow \quad Q_r(0)=36.5$$

To met condition 3 the transfer function from d_i to y must satisfy

$$y(s)=N(X_r-Q_r\tilde{N})d_i(s) \quad \rightarrow \quad N(10j)(X_r(10j)-Q_r(10j)\tilde{N}(10j))=0 \quad \Rightarrow \quad Q_r(10j)=-62+90j$$

Now define

$$Q_r(s)=x_1+x_2\frac{1}{s+1}+x_3\frac{1}{(s+1)^2}$$

- **Exercise 6-5: Derive transfer function from r to y.**
- **Exercise 6-6: Derive transfer function from d_i to y.**

Exercise 6-7: Derive Q_r

Exercise 6-8: Simulate example 6-5

Stability of Multivariable Feedback Control Systems

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- **Strong and Simultaneous Stabilization**

Strong and Simultaneous

Practical control engineers are reluctant to use unstable controllers, especially when the plant itself is stable.

If the plant itself is unstable, the argument against using an unstable controller is less compelling.

However, knowledge of when a plant is or is not stabilizable with a stable controller is useful for another problem namely, simultaneous stabilization, meaning stabilization of several plants by the same controller.

Simultaneous stabilization of two plants can also be viewed as an example of a problem involving highly structured uncertainty.

A plant is strongly stabilizable if internal stabilization can be achieved with a controller itself is a stable transfer matrix.

Strong and Simultaneous

Theorem 6-15: P is strongly stabilizable if and only if it has an even number of real poles between every pairs of real RHP zeros(including zeros at infinity).

Proof. See "Linear feedback control By Doyle".

Example 6-6: Which of the following plant is strongly stabilizable?

$$P_1(s) = \frac{s-1}{s(s-2)} \qquad\qquad P_2(s) = \frac{(s-1)^2(s^2-s+1)}{(s-2)^2(s+1)^3}$$

Solution: P_1 is not strongly stabilizable since it has one pole between z=1 and z=∞
But P_2 is strongly stabilizable since it has two poles between z=1 and z=∞

Exercises

6-1 Mentioned in the lecture.

6-2 Mentioned in the lecture.

6-3 Mentioned in the lecture.

6-4 Mentioned in the lecture.

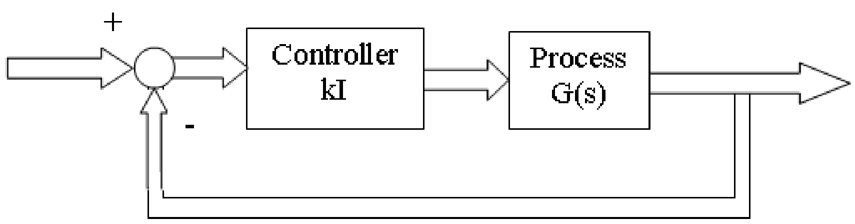
6-5 Mentioned in the lecture.

6-6 Mentioned in the lecture.

6-7 Mentioned in the lecture.

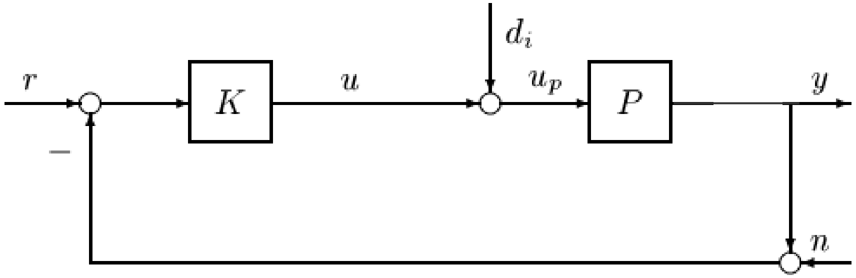
6-8 Mentioned in the lecture.

6-9 Check the stability of following system versus different values of k(Final).



$$G(s)=\begin{bmatrix}\frac{0.5}{s+1} & \frac{s}{s+3}\\ 0 & \frac{1}{(s+1)^3}\end{bmatrix}$$

6-10 Check the stability of following system versus different values of a and b(Final).



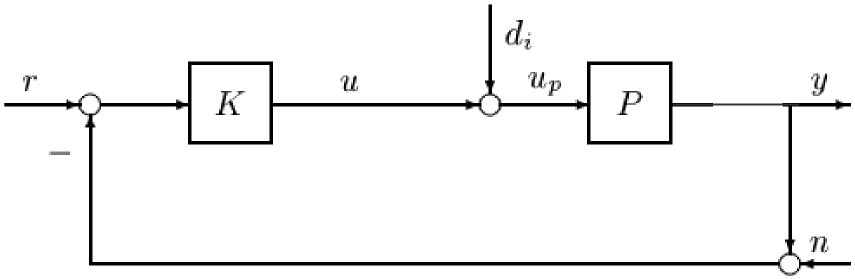
$$K=\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix} \qquad P=\begin{bmatrix}\frac{s+1}{(s+2)(s+3)} & \frac{s+1}{(s+2)^2}\\ \frac{s+4}{(s+3)^2} & \frac{s+4}{(s+2)(s+3)}\end{bmatrix}$$

$$ans: Stability\ condition \begin{cases} a+b>-5 \\ a+4b>-6 \end{cases}$$

Exercises

6-11 Check the stability of following system.

$$P(s)=\begin{bmatrix}\frac{1}{s-1} & \frac{1}{s-1}\\ \frac{1}{s+1} & \frac{2}{s+1}\end{bmatrix}\quad K(s)=\begin{bmatrix}\frac{2(s-1)}{s+1} & -1\\ \frac{1-s}{s+1} & 1\end{bmatrix}$$



$ans: It\ is\ not\ stable$

6-12 Find two different lcf's for the following transfer function matrix.

$$G(s)=\begin{bmatrix}\frac{s-1}{s+1} & \frac{s-2}{s+2}\end{bmatrix}$$

6-13 Find a lcf's and a rcf's for the following transfer matrix.

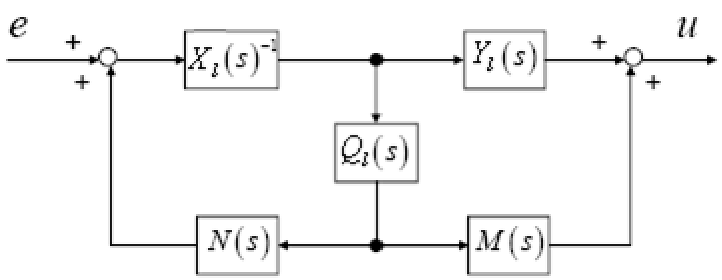
$$G(s)=\frac{1}{s+2}\begin{bmatrix}s-1 & 4\\ 4.5 & 2(s-1)\end{bmatrix}$$

6-14 Find a lcf's and a rcf's for the following transfer matrix.

$$G(s)=\frac{1}{(s+2)(s-3)}\begin{bmatrix}s-1 & 4\\ 4.5 & 2(s-1)\end{bmatrix}$$

Exercises

6-15 By use of MIMO rule derive the transfer matrix of following system.



Ans: $u = (Y_l + MQ_l)(X_l - NQ_l)^{-1}\ e$

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