Multivariable Control Systems

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Lecture 7

References are appeared in the last slide.

Controllability, Observability and Realization

Topics to be covered include:

- Controllability and Observability of Linear Dynamical Equations
- Output Controllability and Functional Controllability
- Realization of Proper Rational Transfer Function Matrices
- Model Order Reduction of Non-Minimal Representations
- Model Order Reduction of Minimal Representations

    Truncation Method

    Residualization Method

    Hankel Norm Approximation

Controllability and Observability of Linear Dynamical Equations

Definition 7-1

The state equation $\dot{x} = Ax + Bu$ or the pair (A, B) is said to be controllable if for any initial state x_0 and any final state x_1, there exists an input that transfers x_0 to x_1 in a finite time. Otherwise (A,B) is said to be uncontrollable.

Definition 7-2

The state equation

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

or the pair (A,C) is said to be observable if for any unknown initial state x_0 , there exists a finite time $t_1 > 0$ such that the knowledge of the input u and the output y over $[0,t_1]$ suffices to determine uniquely the initial state x_0.
Other wise, the equation is unobservable.

Controllability and Observability of Linear Dynamical Equations

Theorem 7-1: Controllability

The n-dimensional linear time-invariant state equation

$$\dot{x} = Ax + Bu$$

is controllable if and only if any of the following equivalent condition is satisfied.

1. The $n \times (np)$ controllability matrix

$$S = [B \quad AB \quad A^2B \quad \ldots\ldots \quad A^{n-1}B]$$

has rank n (full row rank).

2. The $n \times n$ controllability grammian

$$W_{ct} = \int_0^t e^{A\tau} BB^* e^{A^*\tau} d\tau$$

is nonsingular for any $t > 0$.

3. For every eigenvalue λ of A, the $n \times (n+p)$ complex matrix $[\lambda I - A \quad | \quad B]$ has rank n (full row rank).

Controllability and Observability of Linear Dynamical Equations

Controllability test

The n-dimensional linear time-invariant dynamical equation $\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Eu\end{aligned}$ is controllable if and only if the matrix $S = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$ has rank n (full row rank).

Is it necessary to calculate till $A^{n-1}B$? No! $A^{\mu-1}B$ or $A^{n-p}B$

p is rank of B and μ is controllability index and it is the maximum value in the set of controllability indices. $\mu = \max\{\mu_1, \mu_2, \dots\}$

Corollary 7-1: Controllability from controllability index.

The n-dimensional linear time-invariant dynamical equation $\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Eu\end{aligned}$ is controllable if and only if the matrix $[B \quad AB \quad A^2B \ \dots \ A^{\mu-1}B]$ has rank n (full row rank).

→ Partial controllability matrix.

How to derive controllability indices?

Controllability and Observability of Linear Dynamical Equations

Controllability indices?

1- Derive

$$S = [b_1 \;\; b_2 \ldots . \; Ab_1 \;\; Ab_2 \ldots . \; A^2 b_1 \;\; A^2 b_2 \ldots .. \; A^3 b_1 \;\; A^3 b_2 \ldots ..]$$

2- Choose the initial column of S which make it full rank

$$S = [\boxed{b_1 \;\; b_2 \ldots . \; Ab_1 \;\; Ab_2 \ldots . \; A^2 b_1 \;\; A^2 b_2 \ldots .. \; A^3 b_1 \;\; A^3 b_2 \ldots} ..]$$

μ_1 is number independent columns of S corresponding to b_1

μ_2 is number independent vectors of S corresponding to b_2 and also independent from b_1, Ab_1, … $A^{\mu_1 - 1} b_1$.

μ_3 is number independent vectors of S corresponding to b_3 and also independent from b_1, Ab_1, … $A^{\mu_1 - 1} b_1$ and b_2, Ab_2, … $A^{\mu_2 - 1} b_2$.

Note: $\mu_1 + \mu_2 + \cdots = n$

Controllability index?

$$\mu = \max\{\mu_1, \mu_2, \ldots\}$$

Controllability and Observability of Linear Dynamical Equations

Theorem 7-2: Observability

The n-dimensional linear time-invariant dynamical equation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Eu$$

is observable if and only if any of the following equivalent condition is satisfied.

1. The $(nq)\times n$ observability matrix

$$V = \begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ . \\ CA^{n-1} \end{bmatrix}$$

has rank n (full column rank).

2. The $n \times n$ observability grammian

$$W_{ot} = \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau$$

is nonsingular for any $t > 0$.

3. For every eigenvalue λ of A, the $(n+q)\times n$ complex matrix $\begin{bmatrix} \lambda I - A \\ ---- \\ C \end{bmatrix}$

has rank n (full column rank).

Controllability and Observability of Linear Dynamical Equations

The n-dimensional linear time-invariant dynamical equation $\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Eu\end{aligned}$ is observable if and only if the matrix $V = \begin{bmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{bmatrix}$ has rank n (full column rank).

Is it necessary to calculate till CA^{n-1} ? No! CA^{v-1} or CA^{n-q}

q is rank of C and v is observability index and it is the maximum value in the set of observability indices. $v = \max\{v_1, v_2, \dots\}$

Corollary 7-2: Observability from observability index.

The n-dimensional linear time-invariant dynamical equation $\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Eu\end{aligned}$ is observable if and only if the matrix $V = \begin{bmatrix} C \\ CA \\ \cdots \\ CA^{v-1} \end{bmatrix}$ has rank n (full column rank).

→ Partial observability matrix.

How to derive observability indices? Similar to controllability indices.

Controllability and Observability in Rosenbrock's system matrix

Rosenbrock's system matrix is:

$$\mathrm{P}(s) = \begin{bmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{bmatrix} \qquad G(s) = R(s)P(s)^{-1}Q(s) + W(s)$$

Now suppose one find the greatest left common factor of *P* and *Q* as

$$P(s) = L(s)\overline{P}(s) \qquad Q(s) = L(s)\overline{Q}(s)$$

Now if L(s) is not unimodular then there is i.d.z and so reduced order system is:

$$G(s) = R(s)P(s)^{-1}Q(s) + W(s) = R(s)\overline{P}(s)^{-1}\overline{Q}(s) + W(s)$$

How to derive i.d.z. ?

$$[P(s) \quad Q(s)] \quad \xrightarrow{\text{Smith form}} \quad [S(s) \quad 0]$$

Input decoupling zeros are roots of |S(s)|=0

Controllability and Observability in Rosenbrock's system matrix

Rosenbrock's system matrix is:

$$\mathrm{P}(s)=\begin{bmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{bmatrix} \qquad G(s)=R(s)P(s)^{-1}Q(s)+W(s)$$

Now suppose one find the greatest right common factor of *P* and *R* as

$$P(s)=\hat{P}(s)L(s) \qquad R(s)=\hat{R}(s)L(s)$$

Now if L(s) is not unimodular then there is o.d.z and so reduced order system is:

$$G(s)=R(s)P(s)^{-1}Q(s)+W(s)=\hat{R}(s)\hat{P}(s)^{-1}Q(s)+W(s)$$

How to derive o.d.z. ?

$$\begin{bmatrix} P(s) \\ R(s) \end{bmatrix} \quad \text{Smith} \longrightarrow \text{form} \quad \begin{bmatrix} S(s) \\ 0 \end{bmatrix}$$

Output decoupling zeros are roots of |S(s)|=0

Controllability and Observability of Linear Dynamical Equations

Example 7-1: Find the i.d.z. and o.d.z. of following system.

$$\mathrm{P}(s)=\left[\begin{array}{cc:c} s(s+1) & 0 & s \\ 0 & s(s+2) & s \\ \hdashline -1 & -1 & 0 \end{array}\right]$$

$$\begin{bmatrix} P(s) & Q(s) \end{bmatrix}=\begin{bmatrix} s(s+1) & 0 & s \\ 0 & s(s+2) & s \end{bmatrix} \xrightarrow{\text{Smith Form}} \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \end{bmatrix}$$

So intput decoupling zeros are: |S(s)|=0 → 0 and 0

$$\begin{bmatrix} P(s) \\ R(s) \end{bmatrix}=\begin{bmatrix} s(s+1) & 0 \\ 0 & s(s+2) \\ 1 & 1 \end{bmatrix} \xrightarrow{\text{Smith Form}} \begin{bmatrix} 1 & 0 \\ 0 & s \\ 0 & 0 \end{bmatrix}$$

So output decoupling zero is: |S(s)|=0 → 0

Controllability and Observability of Linear Dynamical Equations

Example 7-2: Reduce the following system if it is possible.

$$\mathrm{P}(s)=\left[\begin{array}{cc|c} s(s+1) & 0 & s \\ 0 & s(s+2) & s \\ \hline -1 & -1 & 0 \end{array}\right] \quad \text{Clearly order of system is 4.}$$

$$P(s)=\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}\begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix} \qquad Q(s)=\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So the reduced order system is:

$$\mathrm{P}(s)=\begin{bmatrix} s+1 & 0 & 1 \\ 0 & s+2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \quad \text{Clearly order of system is 2.}$$

Exercise 7-1: Find the i.d.z. and o.d.z. of following system and also check the controllability and observability of system.

$$\mathrm{P}(s)=\left[\begin{array}{c|c} (s+1)^2 & s^3 \\ \hline -1 & 2-s \end{array}\right]$$

Controllability, Observability and Realization

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Output Controllability

Definition 7-3: Output Controllability

Dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

is said to be output controllable if for every $y(0)$ and every vector $y_1 \in R^p$ there exist a finite time t_1 and control $u_1(t) \in R^m$, that transfers the output from $y(0)$ to $y_1 = y(t_1)$.

Dynamical system is output controllable if and only if

$$rank\begin{bmatrix} CB & CAB & \dots & CA^{n-1}B & D \end{bmatrix} = p$$

Functional Controllability

Definition 7-4: Functional controllability.

An m-input l-output system $G(s)$ is functionally controllable if the normal rank of $G(s)$, denoted r, is equal to the number of outputs; that is, if $G(s)$ has full row rank. A plant is functionally uncontrollable if $r < l$.

Remark 1: The minimal requirement for functional controllability is that we have at least many inputs as outputs, i.e. $m \geq l$

Remark 2: A plant is *functionally uncontrollable* if and only if

$$\sigma_l(G(j\omega)) = 0, \ \forall \omega$$

Remark 3: For SISO plants just $G(s)$=0 is functionally uncontrollable.

Remark 4: A MIMO plant is functionally uncontrollable if the gain is identically zero in some output direction at all frequencies.

Functional Controllability

Example 7-3: **a) A Functionally controllable system that is not state controllable.**

$$G(s)=\begin{bmatrix}\frac{1}{s+1} & \frac{2}{s+3}\\ \frac{1}{s+1} & \frac{1}{s+1}\end{bmatrix}$$

$$\dot{x}(t)=\begin{bmatrix}0 & -3 & 0 & 0\\ 1 & -4 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & -2\end{bmatrix}x(t)+\begin{bmatrix}3 & 2\\ 1 & 2\\ 1 & 1\\ 1 & 1\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}x(t)$$

b) A state and output controllable system that is not Functionally controllable.

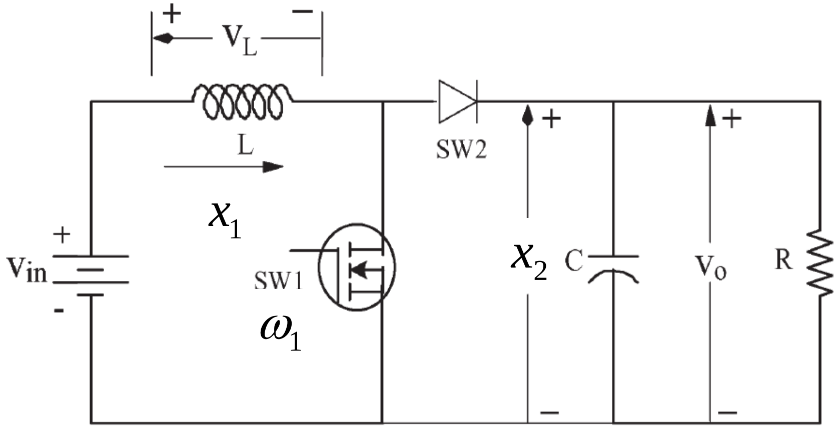
$$\dot{x}(t)=\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}x(t)+\begin{bmatrix}0 & 1\\ 1 & 0\\ 0 & 0\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}x(t)$$

$$G(s)=\begin{bmatrix}\frac{1}{s} & \frac{1}{s^2}\\ \frac{1}{s^2} & \frac{1}{s^3}\end{bmatrix}$$

Functional Controllability

Example 7-4: dc-dc boost converter



$$\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix}=\begin{bmatrix}\dfrac{2498s+3.049\times10^6}{s^2+609s+3.207\times10^5}\\ \dfrac{-2.5\times10^5 s+1.217\times10^8}{s^2+609s+3.207\times10^5}\end{bmatrix}\hat{\omega}_1$$

functionally uncontrollable

or new system design :

$$\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix}=\begin{bmatrix}\dfrac{2498s+3.049\times10^6}{s^2+609s+3.207\times10^5} & \dfrac{12.5\,s+7440}{s^2+609s+3.207\times10^5}\\ \dfrac{-2.5\times10^5 s+1.217\times10^8}{s^2+609s+3.207\times10^5} & \dfrac{6.25\times10^5}{s^2+609s+3.207\times10^5}\end{bmatrix}\begin{bmatrix}\hat{\omega}_1\\ \hat{v}_{in}\end{bmatrix}$$

functionally controllable

Functional Controllability

An m-input l-output system $G(s) = C(sI - A)^{-1} B$ is functionally uncontrollable if

1- The system is input deficient or $rank(B) < l$

2- The system is output deficient or $rank(C) < l$

3- The system has fewer states than outputs $rank(sI - A) < l$

If the plant is not functionally controllable, i.e. $r < l$ then there are l-r output directions, denoted y_i which cannot be affected.

$$y_i^H(j\omega)G(j\omega) = 0 \quad i = 1, \ldots, l - r$$

From an SVD of $G(j\omega)$ the uncontrollable output directions $y_i(j\omega)$ are the last l-r columns of $Y(j\omega)$.

Functional Controllability

Example 7-5:

$$G(s)=\begin{bmatrix}\dfrac{1}{s+1} & \dfrac{2}{s+1}\\ \dfrac{2}{s+2} & \dfrac{4}{s+2}\end{bmatrix}$$

This is easily seen since column 2 of $G(s)$ is two times column 1.

The uncontrollable output directions at low and high frequencies are, respectively,

$$y_0(0)=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}, \qquad y_0(\infty)=\frac{1}{\sqrt{5}}\begin{bmatrix}2\\ -1\end{bmatrix}$$

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Realization of Proper Rational Transfer Function Matrices

Dynamical equation
(state-space) description

$$\dot{x} = Ax + Bu$$
$$y = Cx + Eu$$

This transformation is unique. →

The input-output description
(transfer function matrix)

$$G(s) = C(sI - A)^{-1}B + E$$

The input-output description
(transfer function matrix)

$$G(s) = C(sI - A)^{-1}B + E$$

Realization is not unique →

Dynamical equation
(state-space) description

$$\dot{x} = Ax + Bu$$
$$y = Cx + Eu$$

Theorem 7-3

A transfer function matrix $G(s)$ is realizable by a finite dimensional linear time invariant dynamical equation if and only if $G(s)$ is a proper rational matrix.

Proof: See "Linear system theory and design" Chi-Tsong Chen

Irreducible realizations

Definition 7-6

A linear time-invariant dynamical equation is said to be reducible if and only if there exist a linear time-invariant dynamical equation of lesser dimension that has the same transfer function matrix. Otherwise, the equation is irreducible.

Theorem 7-5

A linear time invariant dynamical equation is irreducible if and only if it is controllable and observable.

Theorem 7-6

Let the dynamical equation $\{A, B, C, E\}$ be an irreducible realization of a $p \times q$ proper rational matrix $G(s)$. Then $\{\bar{A}, \bar{B}, \bar{C}, \bar{E}\}$ is also an irreducible realization of $G(s)$ if and only if $\{A, B, C, E\}$ and $\{\bar{A}, \bar{B}, \bar{C}, \bar{E}\}$ are equivalent, that is, there exist a nonsingular constant matrix P such that $\bar{A} = PAP^{-1}$, $\bar{B} = PB$, $\bar{C} = CP^{-1}$ and $\bar{E} = E$

Irreducible realizations

Definition 7-5: Characteristic polynomial and degree of G(s)

Consider a proper rational matrix *G(s)* factored as $G(s) = D_l^{-1}(s)N_l(s) = N_r(s)D_r^{-1}(s)$. It is assumed that $D_l(s)$, $N_l(s)$, $D_r(s)$ and $N_r(s)$ are polynomials matrices. It is assumed that $D_l(s)$ and $N_l(s)$ are left coprime and $D_r(s)$ and $N_r(s)$ are right coprime(Irreducible LMFD and RMFD). Then the characteristic polynomial of *G(s)* is defined as

$$\det D_r(s) \qquad or \qquad \det D_l(s)$$

And the degree of *G(s)* is defined as

$$\deg G(s) = \deg\det D_r(s) = \deg\det D_l(s)$$

where deg det stands for the degree of determinant. Note that the polynomial $\det D_r(s)$ and $\det D_l(s)$ differ at most by a nonzero constant.

Theorem 7-4:

Let the multivariable linear time-invariant dynamical equation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Eu$$

be a realization of the proper rational matrix *G(s)*. Then the state space realization is irreducible (controllable and observable) if and only if

$$\det(sI - A) = k\,[\text{Characteristic polynomial of } G(s)]$$

Realization of proper rational transfer functions

$$g(s)=\frac{\hat\beta_0 s^n+\hat\beta_1 s^{n-1}+\dots\dots+\hat\beta_n}{\hat\alpha_0 s^n+\hat\alpha_1 s^{n-1}+\dots\dots+\hat\alpha_n},\quad \hat\alpha_0\neq 0 \quad\Rightarrow\quad g(s)=\frac{\beta_1 s^{n-1}+\beta_2 s^{n-2}+\dots\dots+\beta_n}{s^n+\alpha_1 s^{n-1}+\dots\dots+\alpha_n}+\frac{\hat\beta_0}{\hat\alpha_0}$$

There are different forms of realization

- Observable canonical form

$$\begin{bmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\\ .\\ \dot x_n\end{bmatrix}=\begin{bmatrix}0&0&\dots&0&-\alpha_n\\ 1&0&\dots&0&-\alpha_{n-1}\\ 0&1&\dots&0&-\alpha_{n-2}\\ .&.&\dots&.&.\\ 0&0&\dots&1&-\alpha_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ .\\ x_n\end{bmatrix}+\begin{bmatrix}\beta_n\\ \beta_{n-1}\\ \beta_{n-2}\\ .\\ \beta_1\end{bmatrix}u$$

$$y=\begin{bmatrix}0&0&\dots&0&1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ .\\ x_n\end{bmatrix}+\frac{\hat\beta_0}{\hat\alpha_0}u$$

- Controllable canonical form

$$\begin{bmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\\ .\\ \dot x_n\end{bmatrix}=\begin{bmatrix}0&1&0&\dots&0\\ 0&0&1&\dots&0\\ .&.&.&\dots&.\\ 0&0&0&\dots&1\\ -\alpha_n&-\alpha_{n-1}&-\alpha_{n-2}&\dots&-\alpha_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ .\\ x_n\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 0\\ .\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}\beta_n&\beta_{n-1}&\beta_{n-2}&\dots&\beta_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ .\\ x_n\end{bmatrix}+\frac{\hat\beta_0}{\hat\alpha_0}u$$

Irreducible realization of proper rational transfer functions

Realization from the Hankel matrix (Minimal)

$$g(s)=\frac{\beta_0 s^n+\beta_1 s^{n-1}+\ldots\ldots+\beta_n}{s^n+\alpha_1 s^{n-1}+\ldots\ldots+\alpha_n} \qquad g(s)=h(0)+h(1)s^{-1}+h(2)s^{-2}+h(3)s^{-3}+\ldots\ldots$$

The coefficients $h(i)$ will be called Markov parameters.

$$H(\alpha,\beta)=\begin{bmatrix} h(1) & h(2) & h(3) & \ldots & h(\beta) \\ h(2) & h(3) & h(4) & \ldots & h(\beta+1) \\ h(3) & h(4) & h(5) & \ldots & h(\beta+2) \\ . & . & . & \ldots & . \\ . & . & . & \ldots & . \\ h(\alpha) & h(\alpha+1) & h(\alpha+2) & \ldots & h(\alpha+\beta-1) \end{bmatrix}$$

It is called a Hankel matrix of order $\alpha\times\beta$. Note that the coefficient $h(0)$ is not involved in $H(\alpha,\beta)$.

Irreducible realization of proper rational transfer functions

Realization from the Hankel matrix

Theorem 7-7: Consider the proper transfer function g(s) as

$$g(s)=\frac{\beta_0 s^n+\beta_1 s^{n-1}+\beta_2 s^{n-2}+......+\beta_n}{s^n+\alpha_1 s^{n-1}+......+\alpha_n}$$

then g(s) has degree *m* if and only if

$$\rho\big(H(m,m)\big)=\rho\big(H(m+k,m+l)\big) \qquad \text{for every} \quad k,l=1,\,2,\,3,\,....$$

Irreducible realization of proper rational transfer functions

Now consider the dynamical equation

$$\dot{x} = Ax + bu$$
$$y = cx + eu$$

$$g(s) = c(sI - A)^{-1}b + e = s^{-1}c(I - s^{-1}A)^{-1}b + e$$
$$= e + cbs^{-1} + cAbs^{-2} + cA^2bs^{-3} + ...$$
$$h(i) = cA^{i-1}b \qquad i = 1,\ 2,\ 3, ...$$

$$H(n+1,n) = \begin{bmatrix} h(1) & h(2) & ..... & h(n) \\ h(2) & & ..... & h(n+1) \\ . & . & ..... & . \\ . & . & ..... & . \\ h(n) & h(n+1) & ..... & h(2n-1) \\ h(n+1) & h(n+2) & ..... & h(2n) \end{bmatrix}$$

Let the first m rows be linearly independent and the $(m+1)$ th row of $H(n+1,n)$ be linearly dependent on its previous rows. So

$$[a_1 \quad a_2 \quad ..... \quad a_m \quad 1 \quad 0 \quad ..... \quad 0]\, H(n+1,n) = 0$$

Irreducible realization of proper rational transfer functions

$$[a_1 \quad a_2 \quad ..... \quad a_m \quad 1 \quad 0 \quad ..... \quad 0]H(n+1,n)=0$$

We claim that the m -dimensional dynamical equation

$$\dot{x}=\begin{bmatrix} 0 & 1 & 0 & ..... & 0 & 0 \\ 0 & 0 & 1 & ..... & 0 & 0 \\ 0 & 0 & 0 & ..... & 0 & 0 \\ . & . & . & ..... & . & . \\ . & . & . & ..... & . & . \\ 0 & 0 & 0 & ..... & 0 & 1 \\ -a_1 & -a_2 & -a_3 & ..... & -a_{m-1} & -a_m \end{bmatrix}x+\begin{bmatrix} h(1) \\ h(2) \\ h(3) \\ . \\ . \\ h(m-1) \\ h(m) \end{bmatrix}u \qquad \text{(I)}$$

$$y=[1 \quad 0 \quad 0 \quad ..... \quad 0 \quad 0]x+h(0)u$$

is a controllable and observable (irreducible realization).

Exercise 7-2: Show that (I) is a controllable and observable (irreducible realization) of

$$\dot{x}=Ax+bu$$

$$y=cx+eu$$

Irreducible realization of proper rational transfer functions

Example 7-6: Derive three different realization for following system.

$$g(s)=\frac{2s^3+18s^2+48s+32}{s^3+6s^2+11s+6}=\frac{6s^2+26s+20}{s^3+6s^2+11s+6}+2$$

Observable canonical form realization is:

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}0&0&-6\\1&0&-11\\0&1&-6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}20\\26\\6\end{bmatrix}u$$

$$y=\begin{bmatrix}0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+2u$$

Controllable canonical form realization is:

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-6&-11&-6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}u$$

$$y=\begin{bmatrix}20&26&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+2u$$

$$H(4,3)=\begin{bmatrix}6&-10&14\\-10&14&-10\\14&-10&-34\\-10&-34&230\end{bmatrix}$$

We can show that the rank of $H(4,3)$ is 2. So

$$\begin{bmatrix}6&5&1&0\end{bmatrix}H(4,3)=0$$

$$\dot{x}=\begin{bmatrix}0&1\\-6&-5\end{bmatrix}x+\begin{bmatrix}6\\-10\end{bmatrix}u$$

$$y=\begin{bmatrix}1&0\end{bmatrix}x+2u$$

Realization of Proper Rational Matrices

There are many approaches to find irreducible realizations for proper rational matrices.

1. One approach is to first find a reducible realization and then apply the reduction procedure to reduce it to an irreducible one.

2. In the second approach irreducible realization will yield directly.

Realization of Proper Rational Matrices

Method I: Gilbert diagonal representation.

Each element of G(s) has real distinct poles.

$$G(s) = C\ diag\left\{(s-\lambda_1)^{-1}, \cdots, (s-\lambda_r)^{-1}\right\} B + D = \sum_{k=1}^{r} \frac{G_k}{s-\lambda_k} + D$$

Reminders

$$G_k = C_k B_k \quad C_k \ is\ q \times \rho_k \quad B_k \ is\ \rho_k \times p$$

$$G_k = \lim_{s \to \lambda_k} (s-\lambda_k) G(s)$$

$$A = diag\left\{\lambda_1 I_{\rho_1}, \quad \ldots \quad , \quad \lambda_r I_{\rho_r}\right\}$$

$$C = \begin{bmatrix} C_1 & \ldots & C_r \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ \vdots \\ B_r \end{bmatrix}$$

Realization of Proper Rational Matrices

Example 7-7: Derive Gilbert diagonal representation for following system.

$$G(s)=\begin{bmatrix}\dfrac{1}{s+1} & \dfrac{2(s+4)}{(s+1)(s+4)} \\ \dfrac{-1}{(s+1)(s+2)} & \dfrac{1}{s+2}\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\ -1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad \begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} \qquad \begin{bmatrix}0\\ 0\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}$$

$$G(s)=\frac{1}{s+1}\begin{bmatrix}1 & 2\\ -1 & 0\end{bmatrix}+\frac{1}{s+2}\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}+\frac{1}{s+4}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\dot{x}=\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -2 & 0\\ 0 & 0 & 0 & -4\end{bmatrix}x+\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 1\\ 1 & 1\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 2 & 0 & 0\\ -1 & 0 & 1 & 0\end{bmatrix}x$$

$$\xrightarrow{\text{minimal realization}}$$

$$\dot{x}=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -2\end{bmatrix}x+\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 2 & 0\\ -1 & 0 & 1\end{bmatrix}x$$

Realization of Proper Rational Matrices

Method I: Gilbert diagonal representation.

Repetitive real poles.

$$\dot{x}=\begin{bmatrix}-2 & 1 & 0 & 0\\ 0 & -2 & 1 & 0\\ 0 & 0 & -2 & 1\\ 0 & 0 & 0 & -2\end{bmatrix}x+\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 2 & 3\end{bmatrix}u$$

$$y=\begin{bmatrix}a & d & g & k\\ b & e & h & l\\ c & f & i & m\end{bmatrix}x$$

$$G(s)=\begin{bmatrix}a & d & g & k\\ b & e & h & l\\ c & f & i & m\end{bmatrix}\begin{bmatrix}\frac{1}{s+2} & \frac{1}{(s+2)^2} & \frac{1}{(s+2)^3} & \frac{1}{(s+2)^4}\\ 0 & \frac{1}{s+2} & \frac{1}{(s+2)^2} & \frac{1}{(s+2)^3}\\ 0 & 0 & \frac{1}{s+2} & \frac{1}{(s+2)^2}\\ 0 & 0 & 0 & \frac{1}{s+2}\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 2 & 3\end{bmatrix}$$

$$G(s)=\frac{1}{(s+2)^4}\begin{bmatrix}a\\ b\\ c\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}+\frac{1}{(s+2)^3}\begin{bmatrix}d\\ e\\ f\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}+\frac{1}{(s+2)^2}\begin{bmatrix}g\\ h\\ i\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}+\frac{1}{s+2}\begin{bmatrix}k\\ l\\ m\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}$$

Realization of Proper Rational Matrices

Method I: Gilbert diagonal representation.

Repetitive real eigenvalues.

$$G(s)=\frac{1}{(s-\lambda)^3}M_1+\frac{1}{(s-\lambda)^2}M_2+\frac{1}{(s-\lambda)}M_3$$

$rank\,(M_1)=r_1$ → **r_1 Jordan block of order 3**

$rank\begin{bmatrix}M_1\\M_2\end{bmatrix}=r_2$ → **r_2 - r_1 Jordan block of order 2**

$rank\begin{bmatrix}M_1\\M_2\\M_3\end{bmatrix}=r_3$ → **$r_3 - r_2$ Jordan block of order 1**

$$r_1 \le r_2 \le r_3 \le m$$

Realization of Proper Rational Matrices

Example 7-8: Derive Gilbert diagonal representation for following system.

$$G(s)=\frac{1}{(s+1)^2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix} \quad \Rightarrow G(s)=\frac{1}{(s+1)^2}\underbrace{\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}}_{M_1}+\frac{1}{(s+1)}\underbrace{0}_{M_2}$$

$$\Rightarrow r(M_1)=2=r_1, \text{ and } M_1=\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix}$$

2 Jordan block of order 2

$$\Rightarrow r\left(\begin{bmatrix}M_1\\M_2\end{bmatrix}\right)=2=r_2,$$

0 Jordan block of order 1

$$\dot{x}=\begin{bmatrix}-1 & 1 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 1\\ 0 & 0 & 0 & -1\end{bmatrix}x+\begin{bmatrix}0 & 0\\ 1 & 1\\ 0 & 0\\ -1 & 1\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}x$$

Realization of Proper Rational Matrices

Example 7-9: Derive Gilbert diagonal representation for following system.

$$G(s)=\begin{bmatrix}\frac{1}{(s+1)^2} & \frac{1}{(s+2)}\\ \frac{1}{(s+1)} & \frac{1}{(s+2)}\end{bmatrix} \Rightarrow G(s)=\frac{1}{(s+1)^2}\underbrace{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}}_{M_1}+\frac{1}{(s+1)}\underbrace{\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}}_{M_2}+\frac{1}{(s+2)}\underbrace{\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}}_{M}$$

$$\Rightarrow r(M_1)=1=r_1,\ \text{and}\ M_1=\begin{bmatrix}1\\ 0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\ \text{one Jordan block of order 2.}$$

$$\Rightarrow r\left(\begin{bmatrix}M_1\\ M_2\end{bmatrix}\right)=1=r_2,\ \text{and}\ M_2=\begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\ r_2-r_1=\text{zero Jordan block of order 1.}$$

$$\Rightarrow r(M)=1,\ \text{and}\ M=\begin{bmatrix}1\\ 1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}$$

$$\dot{x}=\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 0\\ 0 & 0 & -2\end{bmatrix}x+\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}u \qquad y=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}x$$

Realization of Proper Rational Matrices

Example 7-10: Derive Gilbert diagonal representation for following system.

$$G(s)=\frac{1}{s^4}\begin{bmatrix} s^3-s^2+1 & 1 & -s^3+s^2-2 \\ 1.5s+1 & s+1 & -1.5s-2 \\ s^3-9s^2-s+1 & -s^2+1 & s^3-s-2 \end{bmatrix}$$

$$\Rightarrow G(s)=\frac{1}{s^4}\underbrace{\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{bmatrix}}_{M_1}+\frac{1}{s^3}\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1.5 & 1 & -1.5 \\ -1 & 0 & -1 \end{bmatrix}}_{M_2}+\frac{1}{s^2}\underbrace{\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -9 & -1 & 0 \end{bmatrix}}_{M_3}+\frac{1}{s}\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{M_4}$$

$C(:,1)$ $B(4,:)$

$$\Rightarrow r(M_1)=1=r_1,\ \text{and}\ M_1=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -2 \end{bmatrix}\ \text{one Jordan block of order 4.}$$

$C(:,2)$ $C(:,5)$ $B(4,:)$

$$\Rightarrow r\left(\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}\right)=2=r_2,\ \text{and}\ M_2=\begin{bmatrix} 0 & 0 \\ 1 & -0.5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -2 \\ -1 & 0 & -1 \end{bmatrix}\ \text{and}\ r_2-r_1=\text{one Jordan block of order 3.}$$

$B(7,:)$

Realization of Proper Rational Matrices

Example 7-10: Derive Gilbert diagonal representation for following system.

$$M_3=\begin{bmatrix}-1&0&1\\0&0&0\\-9&-1&0\end{bmatrix}\Rightarrow r\left(\begin{bmatrix}M_1\\M_2\\M_3\end{bmatrix}\right)=3=r_3,\ \text{and } M_3=\begin{bmatrix}0&0&1\\0&0&0\\-1&5&3\end{bmatrix}\begin{bmatrix}1&1&-2\\-1&0&-1\\-1&0&1\end{bmatrix}$$

$C(:,3)$ $C(:,5)$ $C(:,8)$ $B(4,:)$ $B(7,:)$ $B(9,:)$

and $r_3-r_2=$ one Jordan block of order 2.

$$M_4=\begin{bmatrix}1&0&-1\\0&0&0\\1&0&1\end{bmatrix}\Rightarrow r\left(\begin{bmatrix}M_1\\M_2\\M_3\\M_4\end{bmatrix}\right)=3=r_4,\ \text{and } r_4-r_3=\text{zero Jordan block of order 1.}$$

$C(:,4)$ $C(:,7)$ $B(4,:)$

$$M_4=\begin{bmatrix}0&0&-1\\0&0&0\\0&-1&0\end{bmatrix}\begin{bmatrix}1&1&-2\\-1&0&-1\\-1&0&1\end{bmatrix}$$

$B(7,:)$ $C(:,9)$ $B(9,:)$

Realization of Proper Rational Matrices

Example 7-10: Derive Gilbert diagonal representation for following system.

$$
\dot{x} = \begin{bmatrix}
0 & 1 & 0 & 0 & & & & & \\
 & 0 & 1 & 0 & & & & 0 & \\
 & & 0 & 1 & & & & & \\
 & & & 0 & & & & & \\
 & & & & 0 & 1 & 0 & & \\
 & & & & & 0 & 1 & & \\
 & & & & & & 0 & & \\
 & 0 & & & & & & 0 & 1 \\
 & & & & & & & & 0
\end{bmatrix} x + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 1 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-1 & 0 & -1 \\
0 & 0 & 0 \\
-1 & 0 & 1
\end{bmatrix} u
$$

$$
y = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
1 & 1 & 0 & 0 & -0.5 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 1 & 5 & -1 & 3 & 0
\end{bmatrix} x
$$

The Plant
is
Controllable
but
Unobservable

Realization of Proper Rational Matrices

Method II: Hankel form realization of a proper G(s). Let

$$G(s) = H(0) + H(1)s^{-1} + H(2)s^{-2} + \dots\dots\dots .$$

Consider the monic least common denominator of *G(s)* as

$$\psi(s) = s^{m} + \alpha_1 s^{m-1} + \alpha_2 s^{m-2} + \dots + \alpha_m$$

Then after deriving *H(i)* one can simply show

$$H(m+i) = -\alpha_1 H(m+i-1) - \alpha_2 H(m+i-2) - \dots - \alpha_m H(i) \qquad i \geq 1 \quad (I)$$

Let {*A*, *B*, *C* and *E*} be a realization of *G*(*s*) then we have

$$G(s) = E + C(sI - A)^{-1}B = E + CBs^{-1} + CABs^{-2} + CA^2Bs^{-3} + \dots\dots\dots .$$

Then {*A*, *B*, *C* and *E* } be a realization of *G*(*s*) if and only if

$$E = H(0) \qquad H(i+1) = CA^{i}B \quad i = 0, 1, 2, \dots.$$

Exercise 7-3: Proof equation (I)(just PhD students)

Realization of proper rational transfer functions

Then {A, B, C and E } be a realization of $G(s)$ if and only if

$$E = H(0) \qquad H(i+1) = CA^iB \quad i = 0, 1, 2, \ldots.$$

There are different forms of realization

- Observable canonical form

$$\dot{x} = \begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix} x + \begin{bmatrix} H(1) \\ H(2) \\ .. \\ H(m-1) \\ H(m) \end{bmatrix} u$$

$$y = \begin{bmatrix} I_q & 0 & 0 & \ldots & 0 \end{bmatrix} x + H(0)u$$

$$M = \begin{bmatrix} 0_q & I_q & 0_q & \ldots & 0_q \\ 0_q & 0_q & I_q & \ldots & 0_q \\ . & . & . & \ldots & . \\ 0_q & 0_q & 0_q & \ldots & I_q \\ -\alpha_m I_q & -\alpha_{m-1} I_q & -\alpha_{m-2} I_q & \ldots & -\alpha_1 I_q \end{bmatrix}$$

- Controllable canonical form

$$\dot{x} = \begin{bmatrix} & & \\ & N & \\ & & \end{bmatrix} x + \begin{bmatrix} I_p \\ 0_p \\ .. \\ 0_p \\ 0_p \end{bmatrix} u$$

$$y = \begin{bmatrix} H(1) & H(2) & h(3) & \ldots & H(m) \end{bmatrix} x + H(0)u$$

$$N = \begin{bmatrix} 0_p & 0_p & \ldots & 0_p & -\alpha_m I_p \\ I_p & 0_p & \ldots & 0_p & -\alpha_{m-1} I_p \\ 0_p & I_p & \ldots & ..0_p. & -\alpha_{m-2} I_p \\ . & . & \ldots & \ldots & . \\ 0_p & 0_p & \ldots & I_p & -\alpha_1 I_p \end{bmatrix}$$

Proof ?

Irreducible realization of Proper Rational Matrices

Now we shall discuss in the following a method which will yield directly irreducible realizations. This method is based on the Hankel matrices.

We also define the two following Hankel matrices

$$T=\begin{bmatrix} H(1) & H(2) & & H(m) \\ H(2) & H(3) & & H(m+1) \\ . & . & & . \\ H(m) & H(m+1) & & H(2m-1) \end{bmatrix} \qquad \tilde{T}=\begin{bmatrix} H(2) & H(3) & & H(m+1) \\ H(3) & H(4) & & H(m+2) \\ . & . & & . \\ H(m+1) & H(m+2) & & H(2m) \end{bmatrix}$$

Derive SVD of T

$$T=Y\Sigma U^{H}$$

$$\Sigma=\begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}, \quad S=diag\{\sigma_1,\sigma_2,......,\sigma_r\} \text{ with } \sigma_1\ge\sigma_2\ge.......\ge\sigma_r>0$$

Irreducible realization of Proper Rational Matrices

Derive SVD of T

$$T = Y\Sigma U^H$$

$$\Sigma = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}, \quad S = diag\{\sigma_1, \sigma_2, \ldots\ldots, \sigma_r\} \ \text{ with } \sigma_1 \geq \sigma_2 \geq \ldots\ldots \geq \sigma_r > 0$$

Let Y_r and U_r be the first r column of Y and U, then we can write T as

$$T = Y_r S U_r^H = Y_r S^{1/2} S^{1/2} U_r^H = \hat{Y}\hat{U}$$

Define the pseudo inverse of $\hat{Y}$ and $\hat{U}$ as

$$\hat{Y}^{\dagger} = S^{-1/2} Y_r^H \quad \text{and} \quad \hat{U}^{\dagger} = U_r S^{-1/2}$$

Theorem 7-8

Consider a $q \times p$ proper rational matrix $G(s)$ expanded as $G(s) = \sum_{i=0}^{\infty} H(i) s^{-i}$, we form T and factor T as $T = \hat{Y}\hat{U}$, by singular value decomposition. Then the $\{A, B, C, E\}$ defined by

$$A = \hat{Y}^{\dagger}\tilde{T}\hat{U}^{\dagger} \qquad B = \hat{U} I_{p,pm}^T \ (\text{first } p \text{ columns of } \hat{U})$$

$$C = I_{q,qm}\hat{Y} \ (\text{first } q \text{ rows of } \hat{Y}) \qquad E = H(0)$$

leads to an irreducible realization.

Irreducible realization of Proper Rational Matrices

Derive SVD of T

$$T = Y \Sigma U^H$$

$$\Sigma = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}, \quad S = diag\{\sigma_1, \sigma_2, \ldots\ldots, \sigma_r\} \text{ with } \sigma_1 \geq \sigma_2 \geq \ldots\ldots \geq \sigma_r > 0$$

Let Y_r and U_r be the first r column of Y and U, then we can write T as

$$T = Y_r S U_r^H = Y_r S^{1/2} S^{1/2} U_r^H = \hat{Y}\hat{U}$$

Define the pseudo inverse of $\hat{Y}$ and $\hat{U}$ as

$$\hat{Y}^{\dagger} = S^{-1/2} Y_r^H \quad \text{and} \quad \hat{U}^{\dagger} = U_r S^{-1/2}$$

Theorem 7-8

Consider a $q \times p$ proper rational matrix $G(s)$ expanded as $G(s) = \sum_{i=0}^{\infty} H(i) s^{-i}$, we form T and factor T as $T = \hat{Y}\hat{U}$, by singular value decomposition. Then the $\{A, B, C, E\}$ defined by

$$A = \hat{Y}^{\dagger} \tilde{T} \hat{U}^{\dagger} \qquad B = \hat{U} I_{p,pm}^T \ (\textit{first } p \textit{ columns of } \hat{U})$$

$$C = I_{q,qm} \hat{Y} \ (\textit{first } q \textit{ rows of } \hat{Y}) \qquad E = H(0)$$

leads to an irreducible realization.

Exercise 7-4: Proof theorem 7-8.

Irreducible realization of Proper Rational Matrices

Example 7-11: Derive an irreducible realization for the following proper rational function.

$$G(s)=\begin{bmatrix}\dfrac{-2s^2-3s-2}{(s+1)^2} & \dfrac{1}{s}\\ \dfrac{4s+5}{s+1} & \dfrac{-3s-5}{s+1}\end{bmatrix}$$

Least common denominator of $G(s)$, is

$$\psi(s)=s(s+1)^2$$

$$G(s)=\begin{bmatrix}-2 & 0\\ 4 & -3\end{bmatrix}+\begin{bmatrix}1 & 1\\ 1 & -2\end{bmatrix}s^{-1}+\begin{bmatrix}-2 & 0\\ -1 & 2\end{bmatrix}s^{-2}+\begin{bmatrix}3 & 0\\ 1 & -2\end{bmatrix}s^{-3}+\begin{bmatrix}-4 & 0\\ -1 & 2\end{bmatrix}s^{-4}+\begin{bmatrix}5 & 0\\ 1 & -2\end{bmatrix}s^{-5}+\begin{bmatrix}-6 & 0\\ -1 & 2\end{bmatrix}s^{-6}.....$$

$$T=\begin{bmatrix}H(1) & H(2) & H(3)\\ H(2) & H(3) & H(4)\\ H(3) & H(4) & H(5)\end{bmatrix}=\begin{bmatrix}1 & 1 & -2 & 0 & 3 & 0\\ 1 & -2 & -1 & 2 & 1 & -2\\ -2 & 0 & 3 & 0 & -4 & 0\\ -1 & 2 & 1 & -2 & -1 & 2\\ 3 & 0 & -4 & 0 & 5 & 0\\ 1 & -2 & -1 & 2 & 1 & -2\end{bmatrix}$$

Non-zero singular values of T are 10.23, 5.79, 0.90 and 0.23.

So, $r = 4$.

Irreducible Realization of Proper Rational Matrices

$$Y_r = \begin{bmatrix} -0.3413 & 0.2545 & -0.8902 & -0.1621 \\ -0.2357 & -0.5238 & -0.0581 & -0.0071 \\ 0.5127 & -0.2078 & -0.1054 & -0.8264 \\ 0.2357 & 05238 & 0.0581 & 0.0071 \\ -0.6738 & 0.2627 & 0.4316 & -0.5392 \\ -0.2357 & -0.5238 & -0.0581 & -0.0071 \end{bmatrix} \qquad U_r = \begin{bmatrix} -0.4003 & -0.0196 & 0.4905 & -0.6574 \\ 0.1049 & 0.5872 & -0.6022 & -0.5306 \\ 0.5496 & -0.1057 & -0.0978 & 0.1026 \\ -0.1382 & -0.5432 & -0.3875 & -0.1888 \\ -0.6989 & 0.2311 & -0.2949 & 0.4522 \\ 0.1382 & 0.5432 & 0.3875 & 0.1888 \end{bmatrix}$$

$$\hat{Y} = Y_r S^{1/2} = \begin{bmatrix} -1.0915 & 0.6121 & -0.8443 & -0.0770 \\ -0.7539 & -1.2598 & -0.0551 & -0.0034 \\ 1.6398 & -0.4999 & -0.1000 & -0.3923 \\ 0.7539 & 1.2598 & 0.0551 & 0.0034 \\ -2.1553 & 0.6317 & 0.4093 & -0.2560 \\ -0.7539 & -1.2598 & -0.0551 & -0.0034 \end{bmatrix}$$

$$\hat{U} = S^{1/2} U_r^{\,H} = \begin{bmatrix} -1.2803 & 0.3355 & 1.7579 & -0.4421 & -2.2356 & 0.4421 \\ -0.0471 & 1.4124 & -0.2543 & -1.3066 & 0.5557 & 1.3066 \\ 0.4652 & -0.5711 & -0.0927 & -0.3675 & -0.2797 & 0.3675 \\ -0.3121 & -0.2519 & 0.0487 & -0.0896 & 0.2147 & 0.0896 \end{bmatrix}$$

Irreducible Realization of Proper Rational Matrices

$$\hat{Y}^{\dagger} = S^{-1/2} Y_r^{H} = \begin{bmatrix} -0.1067 & -0.0737 & 0.1603 & 0.0737 & -0.2107 & -0.0737 \\ 0.1058 & -0.2178 & -0.0864 & 0.2178 & 0.1092 & -0.2178 \\ -0.9386 & -0.0613 & -0.1112 & 0.0613 & 0.4551 & -0.0613 \\ -0.3415 & -0.0149 & -1.7406 & 0.0149 & -1.1356 & -0.0149 \end{bmatrix}$$

$$\hat{U}^{\dagger} = U_r S^{-1/2} = \begin{bmatrix} -0.1251 & -0.0081 & 0.5171 & -1.3847 \\ 0.0328 & 0.2441 & -0.6349 & -1.1176 \\ 0.1718 & -0.0440 & -0.1031 & 0.2161 \\ -0.0432 & -0.2259 & -0.4086 & -0.3976 \\ -0.2185 & 0.0961 & -0.3109 & 0.9525 \\ 0.0432 & 0.2259 & 0.4086 & 0.3976 \end{bmatrix}$$

$$A = \hat{Y}^{\dagger} \tilde{T} \hat{U}^{\dagger} = \begin{bmatrix} -1.2497 & 0.0369 & 0.2155 & -0.1904 \\ 0.1588 & -1.0139 & -0.1604 & 0.0772 \\ -0.2227 & -0.1800 & -0.2888 & 0.8076 \\ 0.1246 & -0.1181 & 0.1354 & -0.4476 \end{bmatrix}$$

$$B = \hat{U} I_{p,pm}^{T} = \begin{bmatrix} -1.2803 & 0.3355 \\ -0.0471 & 1.4124 \\ 0.4652 & -0.5711 \\ -0.3121 & -0.2519 \end{bmatrix}$$

$$C = I_{q,qm} \hat{Y} = \begin{bmatrix} -1.0915 & 0.6121 & -0.8443 & -0.0770 \\ -0.7539 & -1.2598 & -0.0551 & -0.0034 \end{bmatrix}$$

$$E = H(0) = \begin{bmatrix} -2 & 0 \\ 4 & -3 \end{bmatrix}$$

Exercise 7-5: Derive state space model of g(s) by theorem 7-8.

$$g(s) = \frac{2s^3 + 18s^2 + 48s + 32}{s^3 + 6s^2 + 11s + 6}$$

Controllability, Observability and Realization

- Controllability and Observability of Linear Dynamical Equations
- Output Controllability and Functional Controllability
- Realization of Proper Rational Transfer Function Matrices
- Model Order Reduction of Non-Minimal Representations
- Model Order Reduction of Minimal Representations
  - Truncation Method
  - Residualization Method
  - Hankel Norm Approximation

Model Order Reduction of Non-Minimal Representations

Theorem 7-9 The controllability and observability of a linear time-invariant dynamical equation are invariant under any similarity transformation.

Theorem 7-10

Consider the *n*-dimensional linear time –invariant dynamical equation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Eu$$

If the controllability matrix of the dynamical equation has rank n_1 (where $n_1 < n$), then there exists an equivalence transformation

$$\bar{x} = Px$$

which transform the dynamical equation to

$$\begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u \qquad y = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + Eu$$

and the n_1-dimensional sub-equation

$$\dot{\bar{x}}_c = \bar{A}_c \bar{x}_c + \bar{B}_c u$$
$$y = \bar{C}_c \bar{x}_c + Eu$$

is controllable and has the same transfer function matrix as the first system.

Model Order Reduction of Non-Minimal Representations

Theorem 7-11

Consider the n-dimensional linear time –invariant dynamical equation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Eu$$

If the observability matrix of the dynamical equation has rank n_2 (where $n_2 < n$), then there exists an equivalence transformation

$$\bar{x} = Px$$

which transform the dynamical equation to

$$\begin{bmatrix} \dot{\bar{x}}_o \\ \dot{\bar{x}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{bmatrix} u \qquad y = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix} + Eu$$

and the n_2-dimensional sub-equation

$$\dot{\bar{x}}_o = \bar{A}_o \bar{x}_o + \bar{B}_o u$$
$$y = \bar{C}_o \bar{x}_o + Eu$$

is observable and has the same transfer function matrix as the first system.

Model Order Reduction of Non-Minimal Representations

Theorem 7-12 (Canonical decomposition theorem)

Consider the n-dimensional linear time –invariant dynamical equation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Eu$$

There exists an equivalence transformation

$$\bar{x} = Px$$

which transform the dynamical equation to

$$\begin{bmatrix} \dot{\bar{x}}_{c\bar{o}} \\ \dot{\bar{x}}_{co} \\ \dot{\bar{x}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{A}_{c\bar{o}} & \bar{A}_{12} & \bar{A}_{13} \\ 0 & \bar{A}_{co} & \bar{A}_{23} \\ 0 & 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_{c\bar{o}} \\ \bar{x}_{co} \\ \bar{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{B}_{c\bar{o}} \\ \bar{B}_{co} \\ 0 \end{bmatrix} u \qquad y = \begin{bmatrix} 0 & \bar{C}_{co} & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_{c\bar{o}} \\ \bar{x}_{co} \\ \bar{x}_{\bar{c}} \end{bmatrix} + Eu$$

and the reduced dimensional sub-equation

$$\dot{\bar{x}}_{co} = \bar{A}_{co}\bar{x}_{co} + \bar{B}_{co}u$$
$$y = \bar{C}_{co}\bar{x}_{co} + Eu$$

is observable and controllable and has the same transfer function matrix as the first system.

Controllability, Observability and Realization

- Controllability and Observability of Linear Dynamical Equations
- Output Controllability and Functional Controllability
- Realization of Proper Rational Transfer Function Matrices
- Model Order Reduction of Non-Minimal Representations
- Model Order Reduction of Minimal Representations

  Truncation Method

  Residualization Method

  Hankel Norm Approximation

Model Order Reduction of Minimal Representations

Consider following system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$\Rightarrow$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Eu$$

There are several model order reduction procedure:

- Truncation Method.
- Residualization Method (Singular Perturbation).
- Hankel norm truncation Method.
- Hankel norm residualization Method (Singular Perturbation).
- …………………..

Model Order Reduction of Minimal Representations
Truncation Method

Consider following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Eu$$

- Truncation Method. Let $x_2=0$

$$\dot{x}_1 = A_{11}x_1 + B_1 u$$

$$y = C_1 x_1 + Eu$$

High frequency response is not changed by truncation method.

$$G(\infty) = G_r(\infty) = E$$

- Residualization Method (Singular Perturbation). Let $\dot{x}_2 = 0$

$$\dot{x}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u$$

$$y = (C_1 - C_2A_{22}^{-1}A_{21})x_1 + (E - C_2A_{22}^{-1}B_2)u$$

Exercise 7-6: Show that steady state behavior is not changed by residualization method

$$G(0) = G_r(0)$$

Model Order Reduction of Minimal Representations
Truncation Method

Truncation procedure

$$\dot{x} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} x + \begin{bmatrix} b_1^T \\ b_2^T \\ \dots \\ b_n^T \end{bmatrix} u \quad \xrightarrow{\text{Truncation Method}} \quad \dot{x}_r = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_k \end{bmatrix} x_r + \begin{bmatrix} b_1^T \\ b_2^T \\ \dots \\ b_k^T \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix} x + Eu \qquad y = \begin{bmatrix} c_1 & c_2 & \dots & c_k \end{bmatrix} x + Eu$$

$\lambda_1, \lambda_2, \dots, \lambda_k$ are dominant poles and others are insignificant.

$$G(s) - G_r(s) = \sum_{i=k+1}^{n} \frac{c_i b_i^T}{s - \lambda_i}$$

Error is

$$\left\| G(s) - G_r(s) \right\|_\infty \le \sum_{i=k+1}^{n} \frac{\bar{\sigma}(c_i b_i^T)}{\left| \mathrm{Re}(\lambda_i) \right|}$$

Error value related to:

Model Order Reduction of Minimal Representations
Hankel Norm Approximation

Consider following system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Eu$$

Controllability gramians and observability gramians are:

$$P = \int_0^\infty e^{At} BB^T e^{A^T t} dt$$

$$Q = \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

Minimum energy required to steer the state of system from 0 to x_r is:

$$\|u\|^2 = x_r^T P^{-1} x_r$$

Maximum energy produced by observing the output of the system with initial state x_0 is:

$$\|y\|^2 = x_0^T Q x_0$$

Model Order Reduction of Minimal Representations
Hankel Norm Approximation

Consider following system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Eu$$

Controllability gramians and observability gramians are changed by similarity transformation.

A balanced realization is a realization with following property.

$$P = Q = \Sigma = diag\{\sigma_1 \quad \sigma_2 \quad \ldots \quad \sigma_n\} \qquad \sigma_i \geq \sigma_{i+1}$$ Hankel singular values

If $\sigma_k >> \sigma_{k+1}$ k is suitable value for reduced order realization.

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

A balanced realization →

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Eu$$

Hankel norm truncation method.

Hankel norm residualization Method

Model Order Reduction of Minimal Representations
Hankel Norm Approximation

Example 7-12: Consider following system.

$$G(s)=\frac{1}{s+1}+\frac{1}{s^2+s+4}$$

a) Derive a reduced 1st order system by Hankel truncation method.
b) Derive a reduced 1st order system by Hankel residualization method.
c) Draw Bode plot of real system and all reduced orders in the same plot.
d) Draw step response of real system and all reduced orders in the same plot.

$$\dot{x}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -4 & -5 & -2\end{bmatrix}x+\begin{bmatrix}0\\0\\1\end{bmatrix}u \qquad \Rightarrow \dot{x}=\begin{bmatrix}-0.8741 & -1.1929 & 0.3438\\ 1.1929 & -0.8161 & 1.5679\\ 0.3438 & -1.5679 & -0.3098\end{bmatrix}x+\begin{bmatrix}1.1176\\ -0.5585\\ -0.2510\end{bmatrix}u$$

$$y=\begin{bmatrix}5 & 2 & 1\end{bmatrix}x \qquad y=\begin{bmatrix}1.1176 & 0.5585 & -0.2510\end{bmatrix}x$$

Matlab: system=pck(A,B,C,D); sysbal(system)

Hankel truncation method

$$\dot{x}=-0.8741x+1.1176u \qquad y=1.1176x \qquad \Rightarrow g_{ht}(s)=\frac{1.249}{s+0.8741}$$

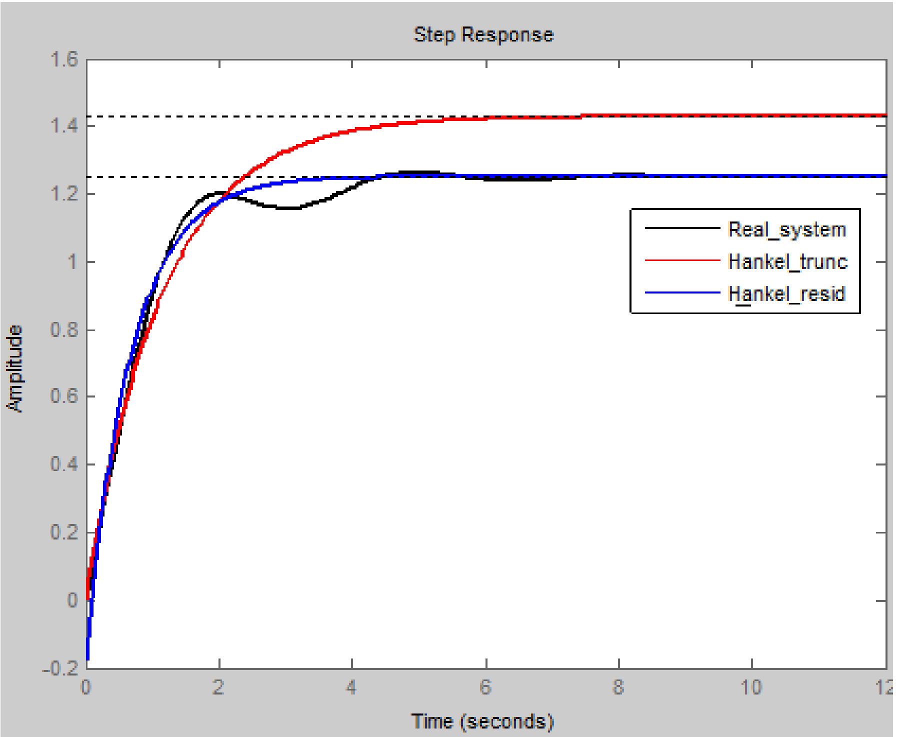
Hankel residualization method

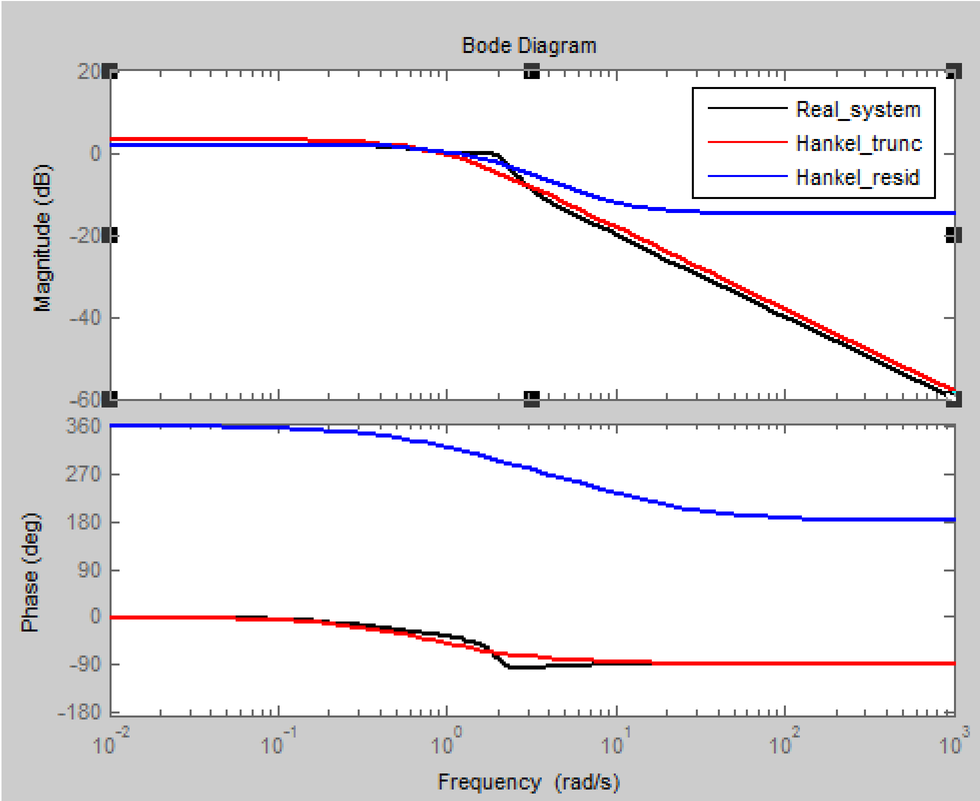
$$\dot{x}=-1.4754x+1.452u \qquad y=1.452x-0.1789u \qquad \Rightarrow g_{hr}(s)=\frac{-0.1789s+1.844}{s+1.4754}$$

Model Order Reduction of Minimal Representations
Hankel Norm Approximation

Example 7-13: Consider following system.

$$G(s) = \frac{1}{s+1} + \frac{1}{s^2+s+4} \quad g_{ht}(s) = \frac{1.249}{s+0.8741} \quad g_{hr}(s) = \frac{-0.1789s + 1.844}{s+1.4754}$$





Exercises

Exercise 7-1: Mentioned in the lecture. Exercise 7-2: Mentioned in the lecture.

Exercise 7-3: Mentioned in the lecture(just for PhD student).

Exercise 7-4: Mentioned in the lecture. Exercise 7-5: Mentioned in the lecture.

Exercise 7-6: Mentioned in the lecture.

Exercise 7-7: Check the contollability and observability of following systems.

a. $$P(s) = \left[\begin{array}{cc|cc} s^2+3s+2 & s+2 & 0 & 3s+6 \\ s+1 & s+2 & 1 & 0 \\ \hline 0 & s+2 & 0 & 0 \\ s+1 & 0 & 0 & 0 \end{array}\right]$$

b. $$\dot{x} = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} x + \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} -1 & 3 & 0 \end{bmatrix} x$$

Exercise 7-8: Find irreducible realization for following systems.

a. $$\begin{bmatrix} \dfrac{2s}{(s+2)(s+1)(s+3)} \\ \dfrac{s^2+2s+2}{s(s+1)^2(s+4)} \end{bmatrix}$$

b. $$\begin{bmatrix} \dfrac{2s+3}{(s+1)^2(s+2)} & \dfrac{s^2+2s+2}{s(s+1)^3} \end{bmatrix}$$

Exercises

Exercise 7-9: Find a reduced order(2^{nd} order) for following System:

$$G(s) = \frac{10}{(s+1)(s^2+s+10)}$$

a) By Hankel truncation method.
b) By Hankel residualization method.
c) Draw Bode plot of real system and all reduced orders in the same plot.
d) Draw step response of real system and all reduced orders in the same plot.

References

References

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