Multivariable Control Systems

Ali Karimpour
Associate Professor
Ferdowsi University of Mashhad

Lecture 10

References are appeared in the last slide.

Uncertainty in Multivariable Systems and Quantitative feedback theory

Topics to be covered include:

- **Introduction**
- **Types of Uncertainty in Multivariable Systems**
- **Robust Stability of Uncertain Systems.**
- **Quantitative Feedback Theory**
- **QFT Design Procedure.**

Introduction

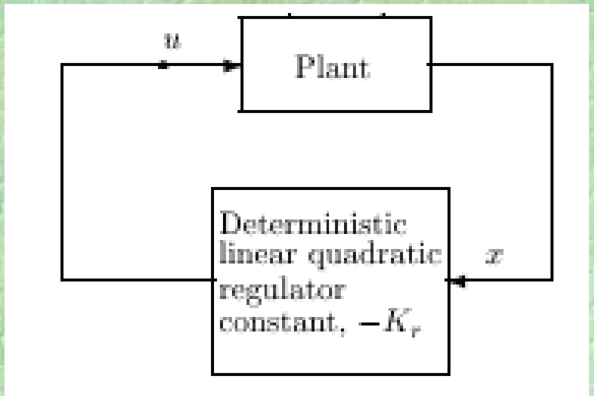
Reason of uncertainty:

- Linearization or ignoring some non-linear parts of system.
- Ignoring sensors or actuators dynamics in modeling.
- Ignoring high frequency behavior or model order reduction.
- Changing operating point.
- System exhaustion.
- Fault in some part of system or fatigue.

Why do one need to examine uncertainty in control systems?

Introduction

LQG Control: Optimal state feedback



$$J_r = \int_0^\infty \left(z^T Q z + u^T R u \right) dt$$

where $z = Mx,\quad Q = Q^T \geq 0$ and $R = R^T > 0$

The optimal solution for any initial state is

$$u(t) = -K_r x(t)$$

where

$$K_r = R^{-1} B^T X$$

Where $X = X^T \geq 0$ is the unique positive-semidefinite solution of the algebraic Riccati equation

$$A^T X + XA - XBR^{-1} B^T X + M^T QM = 0$$

Introduction

Robustness Properties

For an LQR-controlled system if the weight *R* is chosen to be diagonal, then

$$S = \left(I + K_r (sI - A)^{-1} B\right)^{-1} \quad \text{satisfies} \quad \bar{\sigma}\left(S(j\omega)\right) \le 1, \ \forall \omega$$

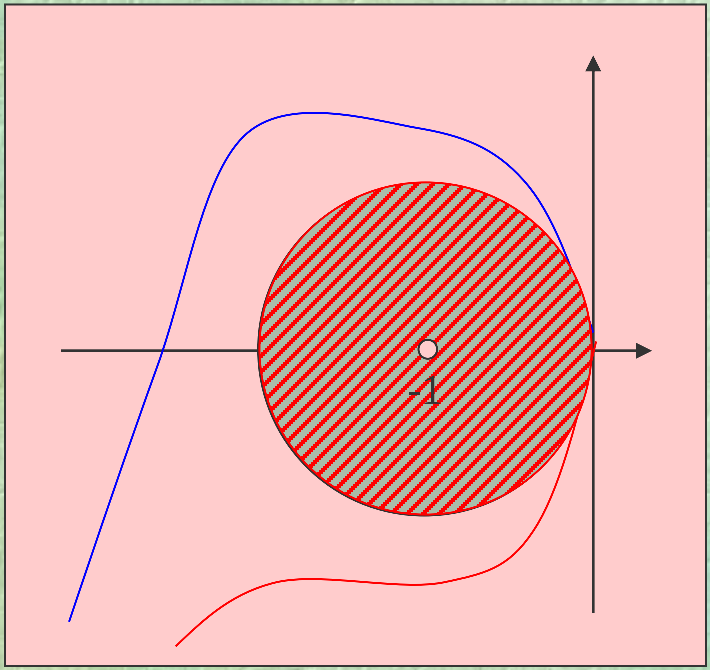
This means that in the LQR-controlled system $u = -K_r x$, a complex perturbation $diag\{k_i e^{j\theta_i}\}$ can be introduced at the plant inputs without causing instability provided

$$\theta_i = 0 \ and \ 0.5 \le k_i < \infty \ , \ i = 1, 2, \ldots, m$$

Or

$$k_i = 1 \ and \ |\theta_i| \le 60^\circ \ , \ i = 1, 2, \ldots, m$$

Nyquist plot in MIMO case



This was brought starkly to the attention of the control community by Doyle (1978) (in a paper entitled "Guaranteed Margins for LQR Regulators" with a very compact abstract which simply states "There are none").

Introduction

Example 10-1: **LQR design of a second order process.**

$$G(s) = \frac{2s+3}{s^2+3s+2} \quad \Rightarrow \quad \dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

The cost function to be minimized is

$$J_r = \int_0^\infty \left(y^2 + Ru^2\right) dt \qquad Let\ \ R = 0.0001$$

$$\Rightarrow \quad K_r = R^{-1} B^T X = [86.7008\ \text{-}75.3816]$$

$$\Rightarrow \quad \lambda\left(A - BK_r\right) = -7.1596 \pm 6.9828i$$

Stable for $\theta = 0\ and\ 0.5 \le k < \infty$

$\lambda\left(A - BK_r k\right)\ stable\ for\ all\ k \ge 0.5$

Let uncertainty in b as $b_\varepsilon = \begin{bmatrix} 1 \\ 1+\varepsilon \end{bmatrix}$

For $\varepsilon = 0.19$ system is unstable.

Exercise 10-1: Derive ε for R=0.0001.

Exercise 10-2: Derive curve of ε versus 1/R

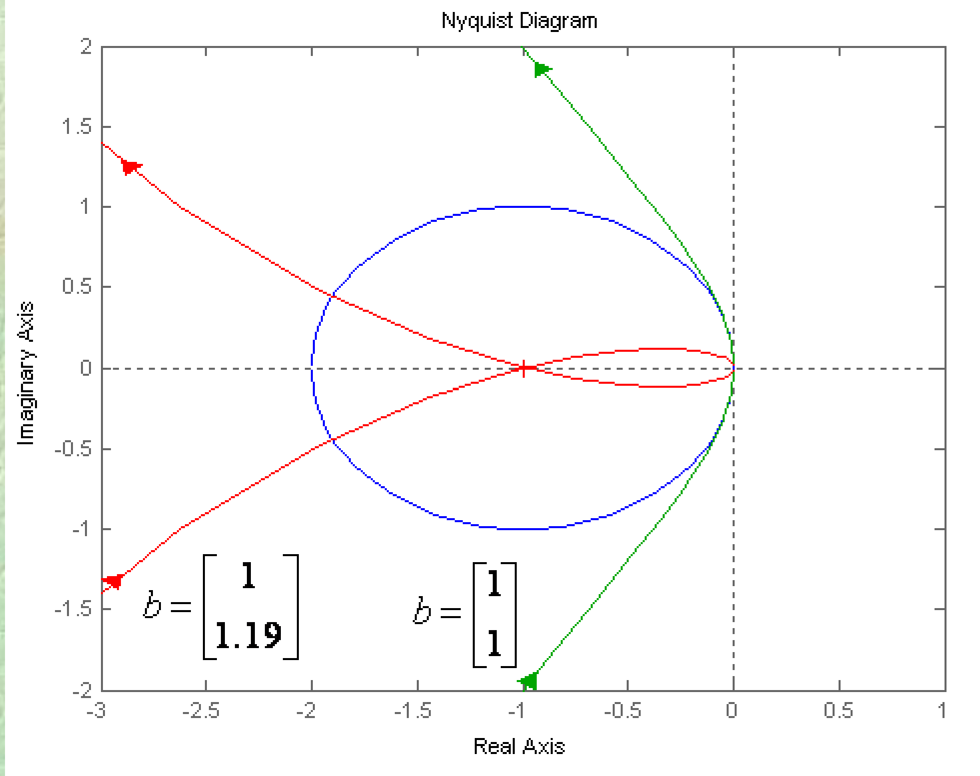
Introduction

Example 10-1: LQR design of a second order process.

$$G(s)=\frac{2s+3}{s^2+3s+2}$$

$$\dot{x}=\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}x+\begin{bmatrix}1\\1\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 1\end{bmatrix}x$$



Introduction

Example 10-2: Decoupling controller

$$G(s) = \frac{1}{s^2+3s+2}\begin{bmatrix} 2-47s & 56s \\ -42s & 50s+2 \end{bmatrix}$$

The pre compensator approach may be extended by introducing a post compensator

$$G_s(s) = W_{sp}(s)G(s)W_s(s) \Rightarrow \begin{bmatrix} 7 & -8 \\ -6 & 7 \end{bmatrix} G(s) \begin{bmatrix} 7 & 8 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{2}{s+2} \end{bmatrix} = G_d(s)$$

The overall controller is then

$$K(s) = W_s(s)K_s(s)W_{sp}(s) \Rightarrow K(s) = \begin{bmatrix} 7 & 8 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} 7 & -8 \\ -6 & 7 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

We have good stability margin in both channel.

Exercise 10-3: Derive stability margin for different value of δ if $K(s) = \begin{bmatrix} k-\delta & 0 \\ 0 & k+\delta \end{bmatrix}$

For k=1 so find the smallest δ that lead to instability. Repeat for k=2.

Uncertainty in Multivariable Systems and Quantitative feedback theory

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Types of Uncertainty in Multivariable Systems

Type of uncertainty

- Parametric (real) uncertainty. "structured uncertainty"

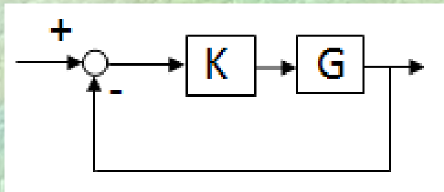
- Dynamic (frequency-dependent) uncertainty or nonparametric uncertainty. "unstructured uncertainty"

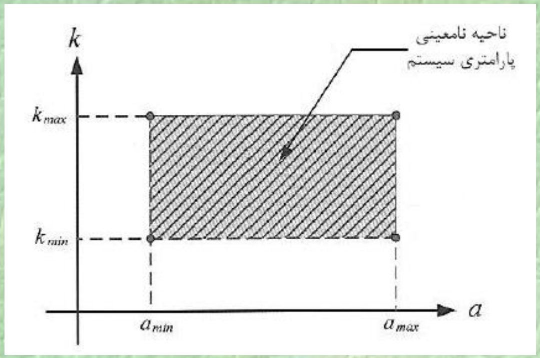
Types of Uncertainty in Multivariable Systems

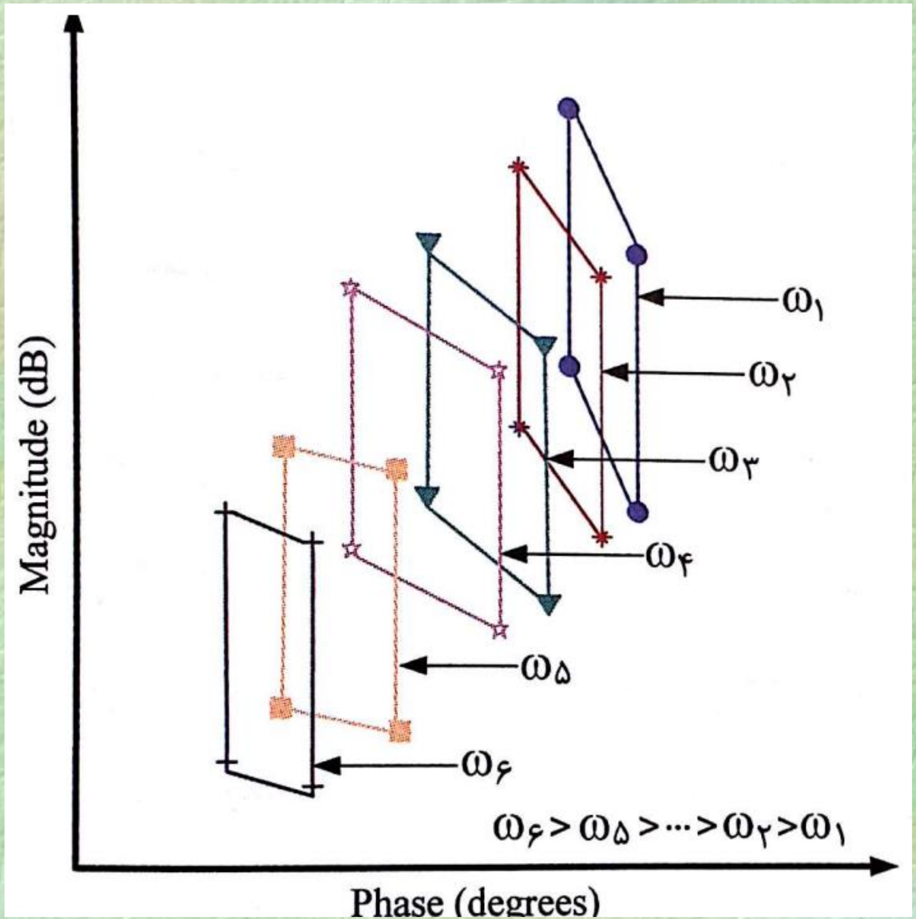
- Parametric (real) uncertainty.

"structured uncertainty"

$$G(s) = \frac{k}{s(s+a)}$$







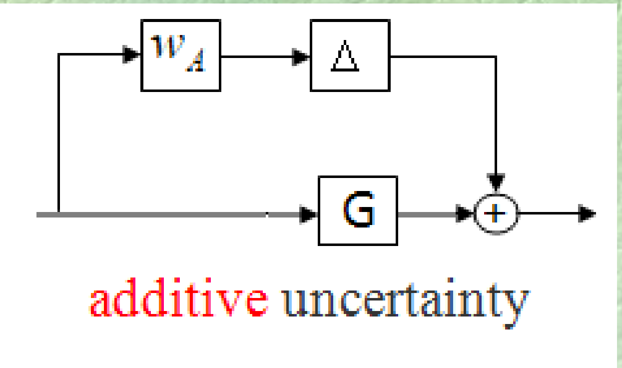
Types of Uncertainty in Multivariable Systems

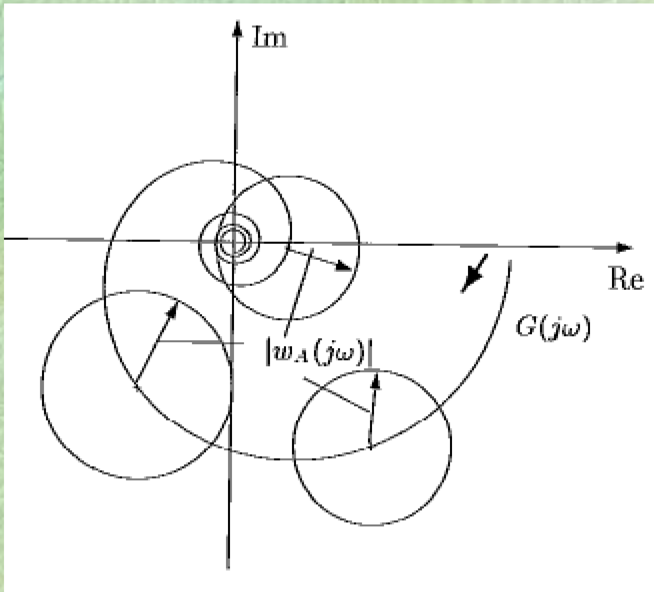
- Dynamic (frequency-dependent) uncertainty or nonparametric uncertainty.

“unstructured uncertainty”

Consider additive uncertainty as:

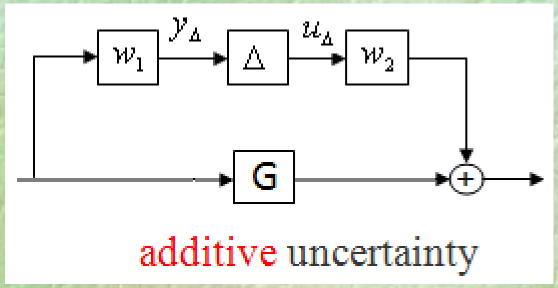
$$G_p(s) = G(s) + w_A(s)\Delta(s); \ \left|\Delta(j\omega)\right| \le 1, \forall \omega$$



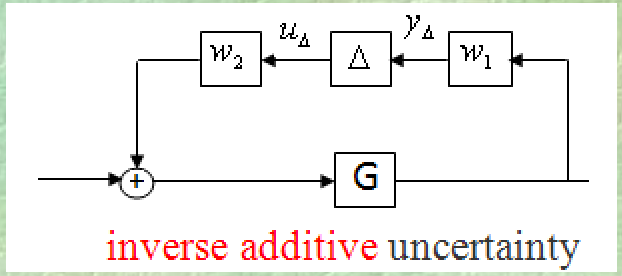


Types of Uncertainty in Multivariable Systems

Type of unstructured uncertainty

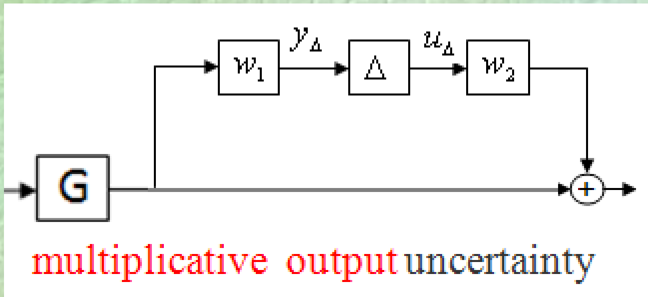


$$G_p(s) = G(s) + w_2(s)\Delta(s)w_1(s)$$

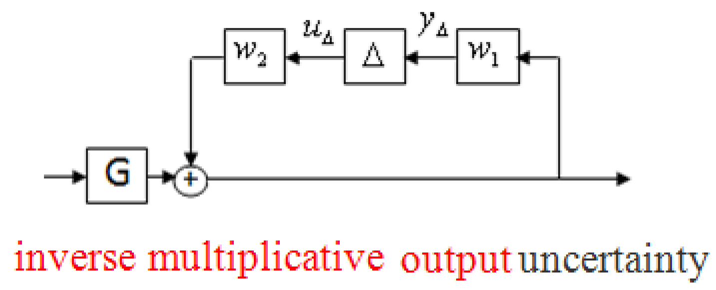


$$G_p(s) = G(s)\left(I - w_2(s)\Delta(s)w_1(s)G(s)\right)^{-1}$$

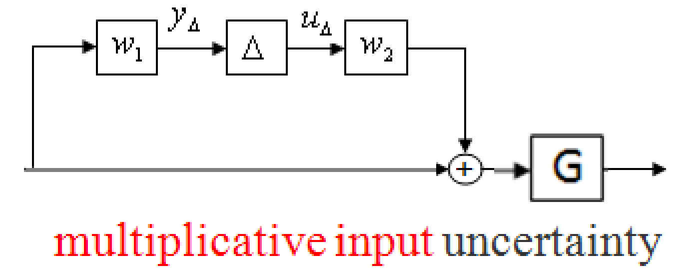
Types of Uncertainty in Multivariable Systems



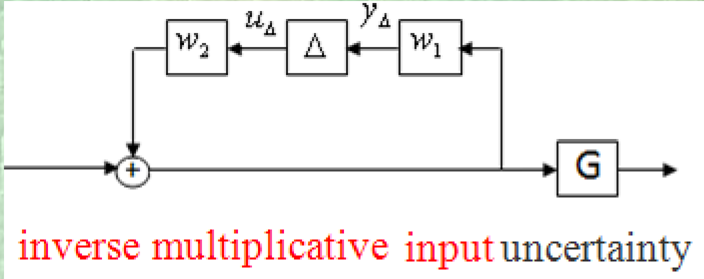
$$G_p(s) = \left(I + w_2(s)\Delta(s)w_1(s)\right)G(s)$$



$$G_p(s) = \left(I - w_2(s)\Delta(s)w_1(s)\right)^{-1}G(s)$$



$$G_p(s) = G(s)\left(I + w_2(s)\Delta(s)w_1(s)\right)$$



$$G_p(s) = G(s)\left(I - w_2(s)\Delta(s)w_1(s)\right)^{-1}$$

Types of Uncertainty in Multivariable Systems

Parametric uncertainty → Nonparametric uncertainty

Example 10-3: Consider a plant with parametric uncertainty

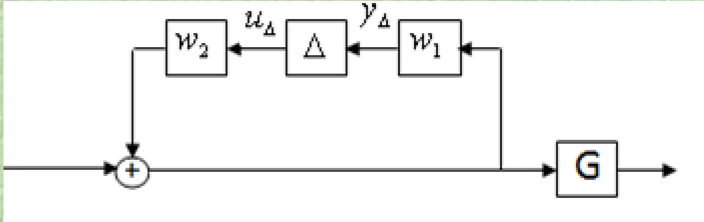
$$G_p(s) = \frac{1}{\tau_p s + 1} G_0(s) \qquad \tau_{\min} \le \tau_p \le \tau_{\max}$$

Now let

$$\tau_p = \bar{\tau}(1 - r_\tau \Delta) \qquad \bar{\tau} = \frac{1}{2}(\tau_{\min} + \tau_{\max}) \qquad r_\tau = \frac{(\tau_{\min} - \tau_{\max})/2}{\bar{\tau}}, \ |\Delta| < 1$$

$$G_p(s) = \frac{G_0(s)}{1 + \bar{\tau}s - r_\tau \bar{\tau} s \Delta} = \frac{G_0(s)}{1 + \bar{\tau}s} \frac{1}{1 - w_2(s)\Delta w_1(s)} \qquad w_2(s) = \frac{r_\tau \bar{\tau} s}{1 + \bar{\tau}s}, w_1(s) = 1$$

$$G_p(s) = G(s)\big(1 - w_2(s)\Delta w_1(s)\big)^{-1}$$



inverse multiplicative input uncertainty

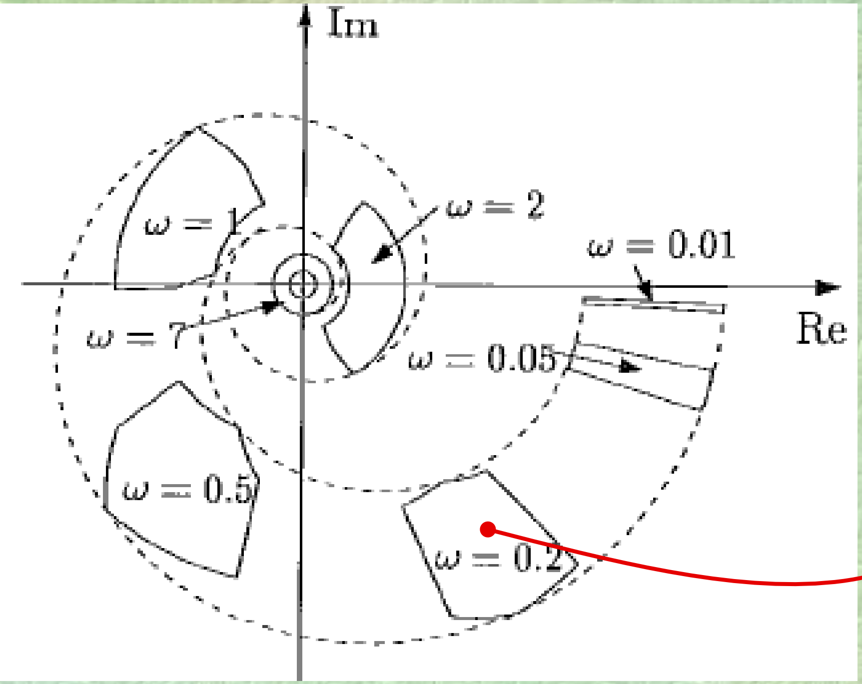
Nonparametric uncertainty has more conservativeness.

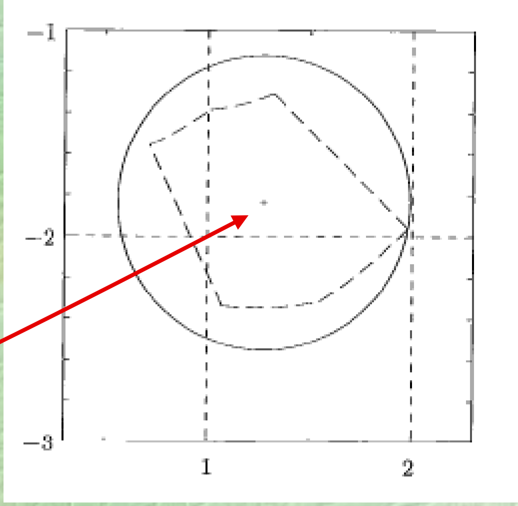
Types of Uncertainty in Multivariable Systems

Parametric uncertainty → Nonparametric uncertainty

Example 10-4: Consider a plant with two parametric uncertainty

$$G_p(s)=\frac{k}{\tau s+1}e^{-\theta s} \quad 2\le k,\tau,\theta\le 3$$





Types of Uncertainty in Multivariable Systems

Consider additive uncertainty as:

$$G_p(s) = G(s) + w_A(s)\Delta(s); \ \left|\Delta(j\omega)\right| \le 1, \forall \omega$$

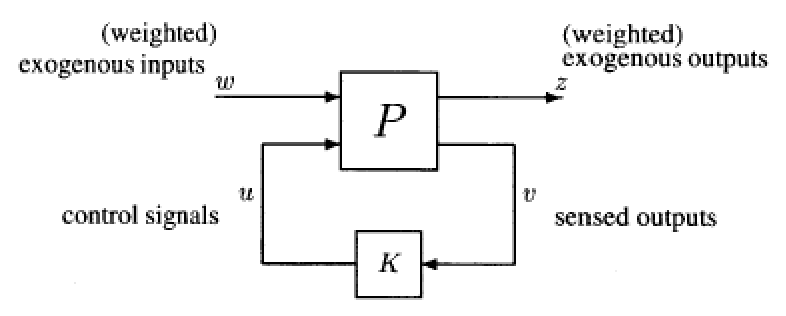
Additive uncertainty can be represent by multiplicative one:

$$G_p(s) = G(s)\left(1 + w_M(s)\Delta(s)\right); \ \left|\Delta(j\omega)\right| \le 1, \forall \omega$$

$$w_M(j\omega) = G(j\omega)^{-1} w_A(j\omega)$$

Types of Uncertainty in Multivariable Systems

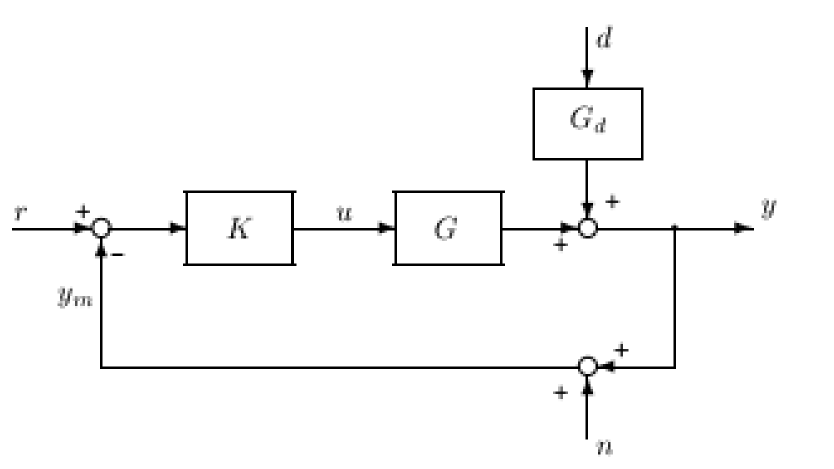
System without uncertainty



$$\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \qquad u = Kv$$

$$z = \left(P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\right)w = Nw$$

$$N = F_l(P, K)$$

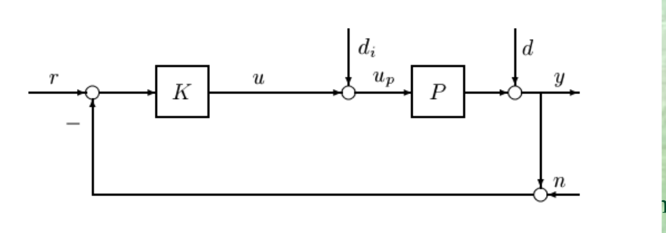


$$z = y \qquad v = r - n - y$$

$$w = [r \quad d \quad n]^T \qquad u = u$$

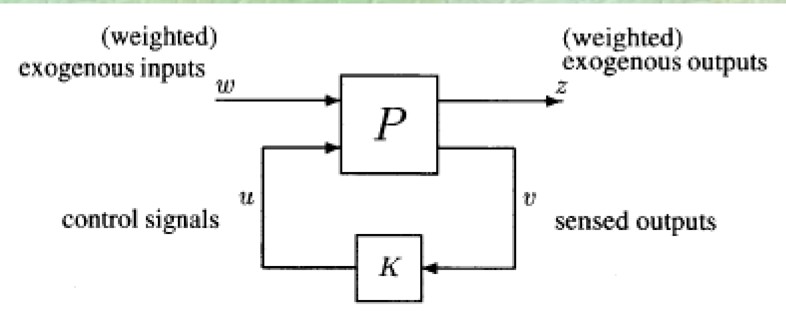
$$\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} y \\ r - n - y \end{bmatrix} = \left[\begin{array}{ccc|c} 0 & G_d & 0 & G \\ \hline I & -G_d & -I & -G \end{array}\right] \begin{bmatrix} r \\ d \\ n \\ u \end{bmatrix}$$

Exercise 10-4: Derive P for following system.



Types of Uncertainty in Multivariable Systems

System without uncertainty

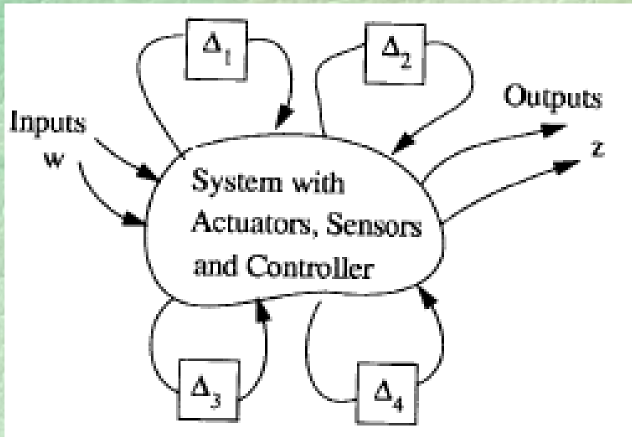


$$\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \qquad u = Kv$$

$$z = \left(P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\right)w = Nw$$

$$N = F_l(P, K)$$

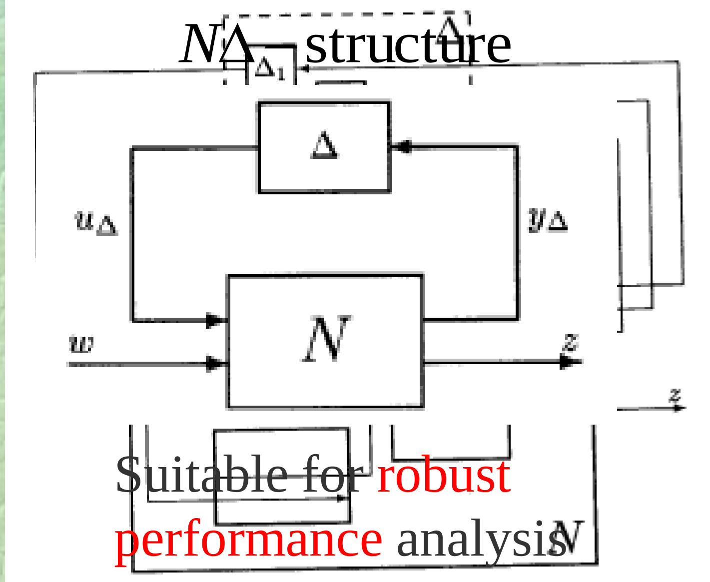
System with uncertainty



Pull out

uncertainty

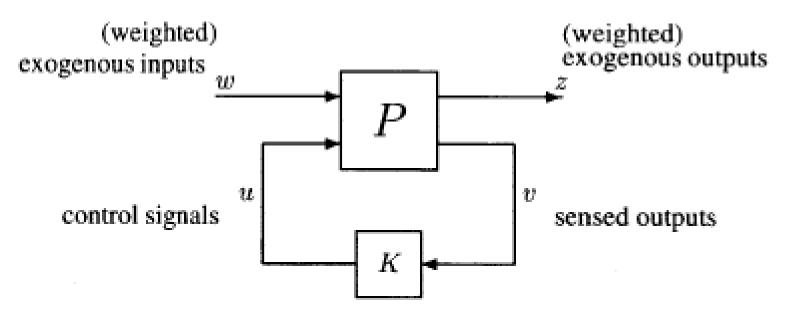
$N\Delta -$ structure



Suitable for robust performance analysis

Types of Uncertainty in Multivariable Systems

System without uncertainty

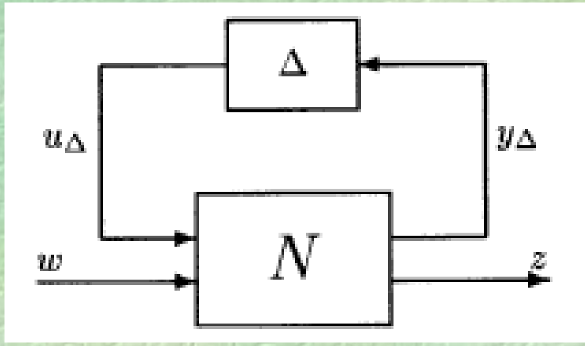


$$\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \qquad u = Kv$$

$$z = \left(P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\right)w = Nw$$

$$N = F_l(P, K)$$

System with uncertainty NΔ structure



$$\begin{bmatrix} y_\Delta \\ z \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} u_\Delta \\ w \end{bmatrix} \qquad u_\Delta = \Delta y_\Delta$$

$$z = \left(N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}\right)w = Fw$$

$$F = F_u(N, \Delta)$$

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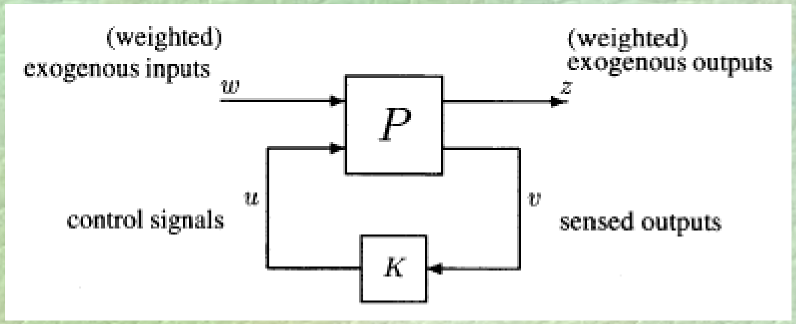
Robust Stability of Uncertain Systems

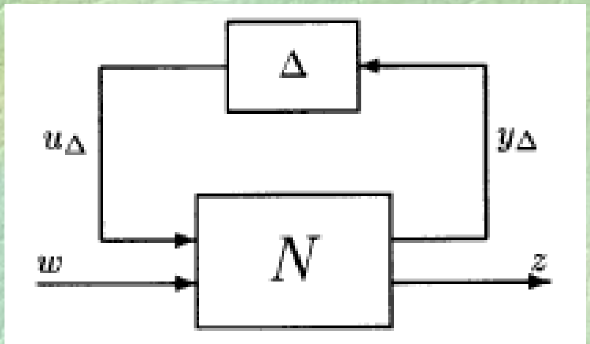
System without uncertainty



Suitable for nominal performance analysis

$$z = \left(P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\right)w = Nw$$

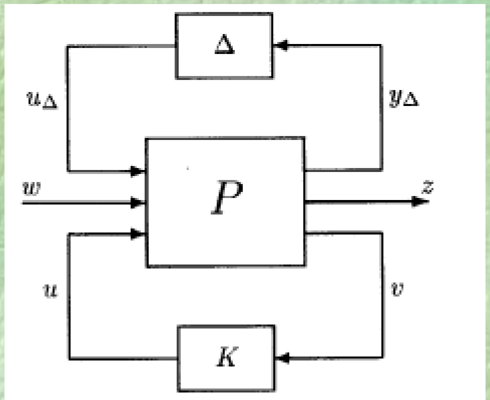
System with uncertainty $N\Delta-$structure



Suitable for robust performance analysis

$$z = \left(N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}\right)w = Fw$$

General Control Configuration

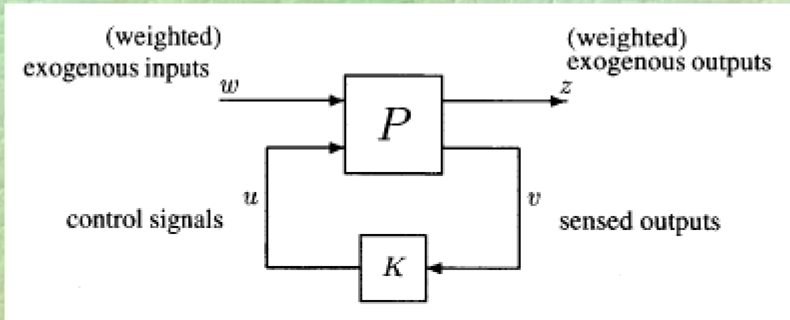


Suitable for controller design

Checking robust stability?

Robust Stability of Uncertain Systems

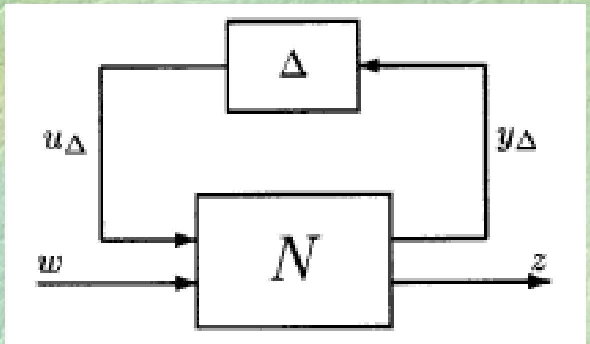
System without uncertainty



Suitable for nominal performance analysis

General Control Configuration

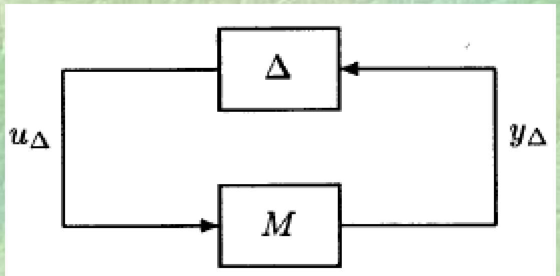
System with uncertainty $N\Delta-$structure



Suitable for robust performance analysis

$$z=\left(N_{22}+N_{21}\Delta(I-N_{11}\Delta)^{-1}N_{12}\right)w=Fw$$

$M\Delta-$structure

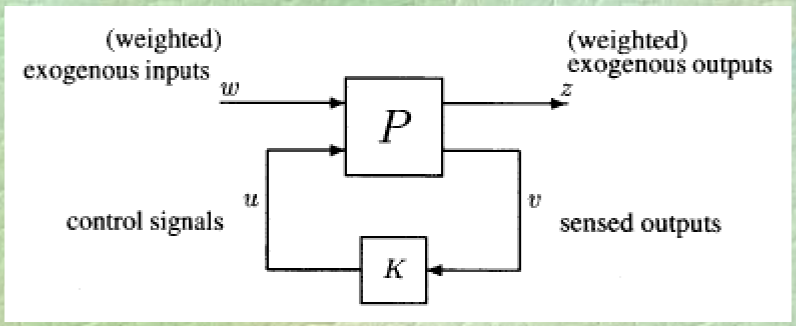


$$M=N_{11}$$

Suitable for robust stability analysis

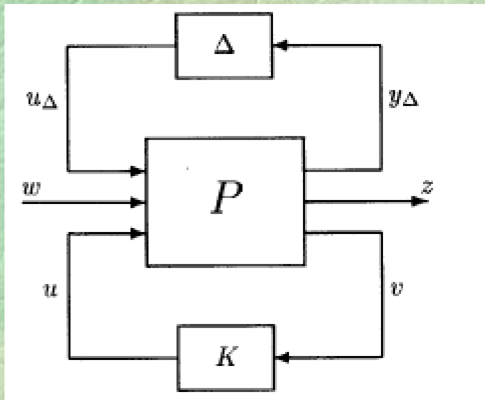
Robust Stability of Uncertain Systems

System without uncertainty



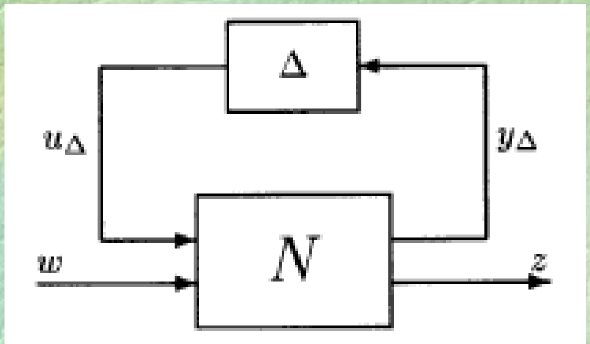
Suitable for nominal performance analysis

General Control Configuration



Suitable for controller design

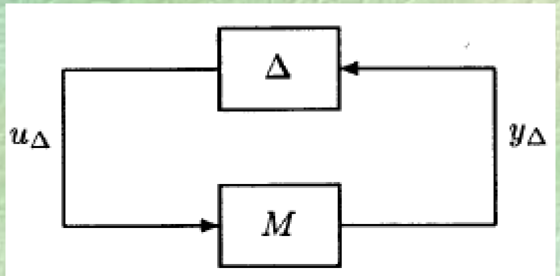
System with uncertainty $N\Delta-$structure



Suitable for robust performance analysis

$$z=\left(N_{22}+N_{21}\Delta(I-N_{11}\Delta)^{-1}N_{12}\right)w=Fw$$

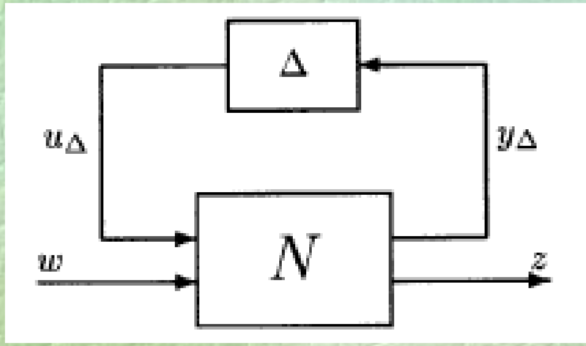
$M\Delta-$structure



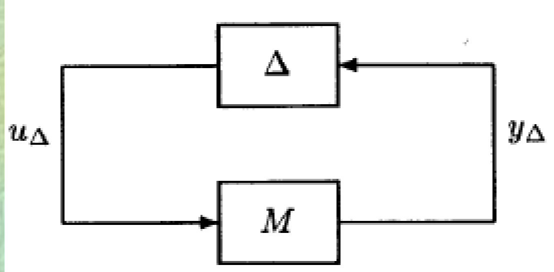
$$M=N_{11}$$

Suitable for robust stability analysis

Robust Stability of Uncertain Systems



$M\Delta-$structure



NS: *N* is internally stable

Suitable for robust stability analysis

RS: NS and F=F $_u$(N,Δ) is stable for any $\|\Delta\|_\infty \le 1$

Theorem 10-1: RS for unstructured("full") perturbation.

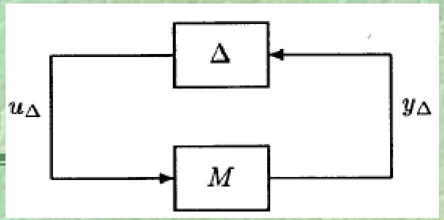
Assume that the nominal system *M*(s) is stable (NS) and that the perturbations Δ(s) are stable. Then

The MΔ-structure is stable for all Δ satisfying $\|\Delta\|_\infty \le \gamma$ $\Leftrightarrow \quad \|M\|_\infty < 1/\gamma \quad \Leftrightarrow \quad \bar{\sigma}(M(j\omega)) < 1/\gamma \;\; \forall \omega$

The MΔ-structure is stable $\Leftrightarrow \quad \|\Delta M\|_\infty < 1 \quad \Leftrightarrow \quad \bar{\sigma}(\Delta(j\omega)) < 1/\bar{\sigma}(M(j\omega)) \;\; \forall \omega$

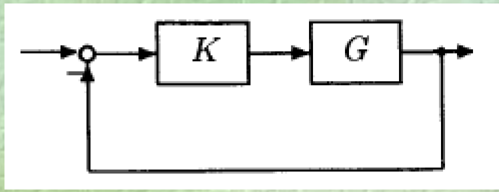
Robust Stability of Uncertain Systems

$M\Delta -$structure

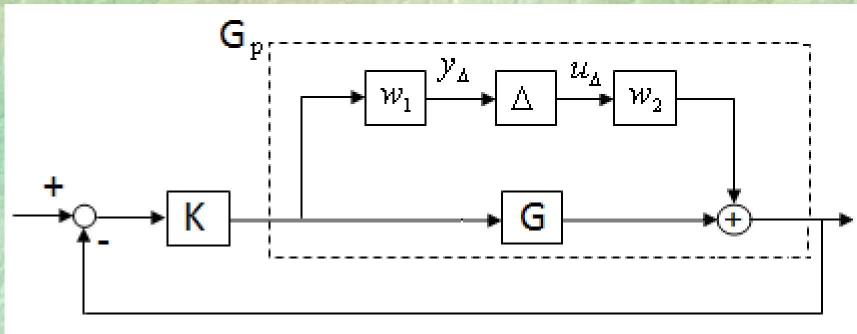


Suitable for robust stability analysis

System without uncertainty



System with additive uncertainty



$$G_p = G + w_2 \Delta w_1$$

$$y_\Delta = M u_\Delta$$

$$M = -w_1 K (I + GK)^{-1} w_2$$

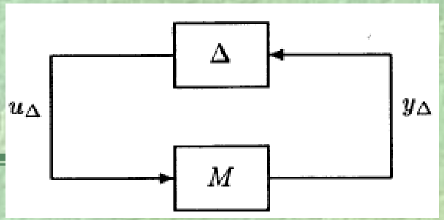
Robust stability condition: In the case of $\|\Delta\|_\infty \le 1$

$$\|M\|_\infty = \left\| w_1 K (I + GK)^{-1} w_2 \right\|_\infty < 1$$

In the case of free Δ $\quad \bar{\sigma}(\Delta M) < 1$

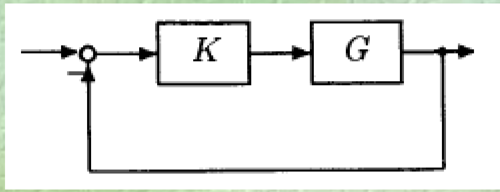
Robust Stability of Uncertain Systems

$M\Delta -$structure

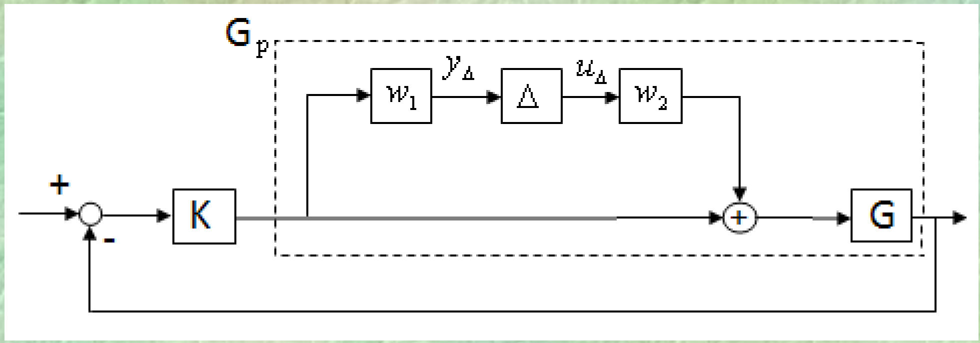


Suitable for robust stability analysis

System without uncertainty



System with multiplicative input uncertainty



$$G_p = G(I + w_2 \Delta w_1)$$

$$y_\Delta = M u_\Delta$$

$$M = w_1 K (I + GK)^{-1} G w_2$$

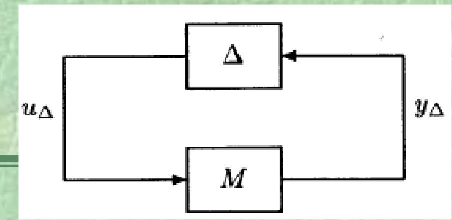
Robust stability condition: In the case of $\|\Delta\|_\infty \le 1$

$$\|M\|_\infty = \left\| w_1 K (I + GK)^{-1} G w_2 \right\|_\infty < 1$$

In the case of free Δ $\qquad \bar{\sigma}(\Delta M) < 1$

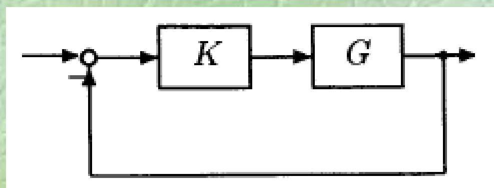
Robust Stability of Uncertain Systems

$M\Delta-$structure



Suitable for robust stability analysis

System without uncertainty

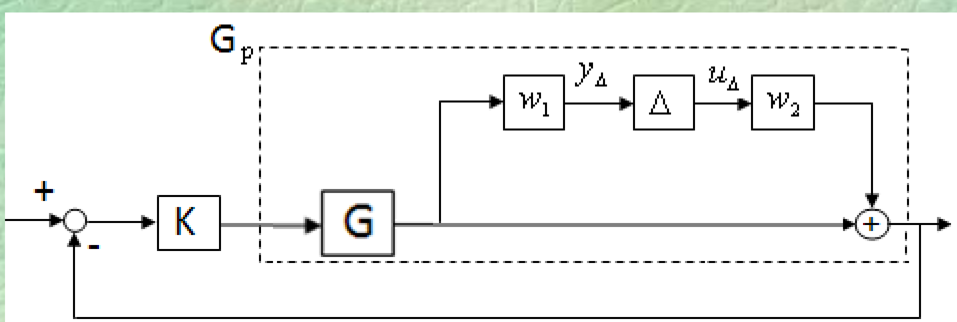


$$G_p = (I + w_2 \Delta w_1) G$$

System with multiplicative output uncertainty

$$y_\Delta = M u_\Delta$$

$$M = w_1 GK (I + GK)^{-1} w_2$$



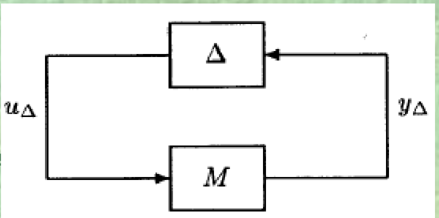
Robust stability condition: In the case of $\|\Delta\|_\infty \le 1$

$$\|M\|_\infty = \left\| w_1 GK (I + GK)^{-1} w_2 \right\|_\infty < 1$$

In the case of free Δ $\qquad \bar{\sigma}(\Delta M) < 1$

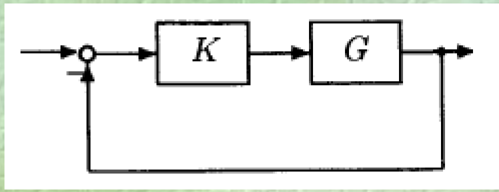
Robust Stability of Uncertain Systems

$M\Delta -$structure



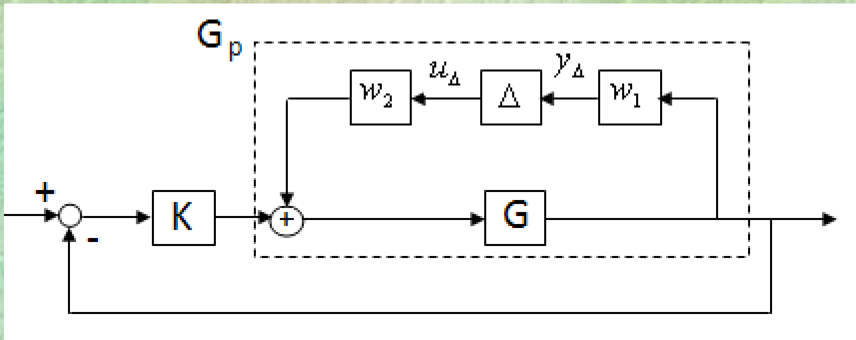
Suitable for robust stability analysis

System without uncertainty



$$G_p = G(I - w_2 \Delta w_1 G)^{-1}$$

System with inverse additive uncertainty



$$y_\Delta = M u_\Delta$$

$$M = w_1 G(I + KG)^{-1} w_2$$

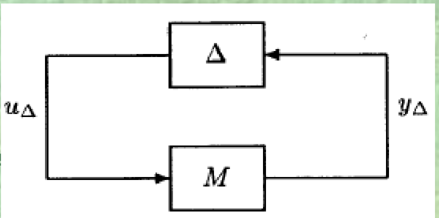
Robust stability condition: In the case of $\|\Delta\|_\infty \le 1$

$$\|M\|_\infty = \left\| w_1 G(I + KG)^{-1} w_2 \right\|_\infty < 1$$

In the case of free Δ $\qquad \bar{\sigma}(\Delta M) < 1$

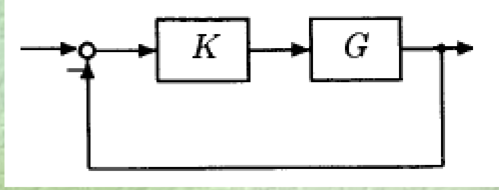
Robust Stability of Uncertain Systems

$M\Delta-$structure



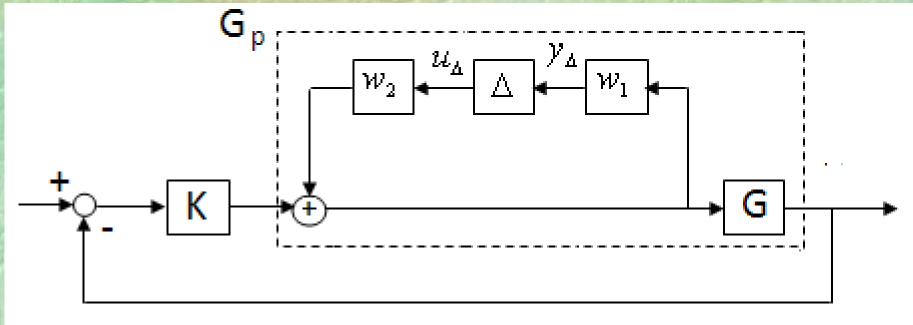
Suitable for robust stability analysis

System without uncertainty



$$G_p = G(I - w_2 \Delta w_1)^{-1} \qquad \|w_2 \Delta w_1\|_\infty < 1$$

System with inverse multiplicative input uncertainty



$$y_\Delta = M u_\Delta$$

$$M = w_1 (I + KG)^{-1} w_2$$

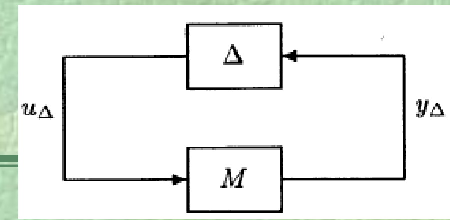
Robust stability condition: In the case of $\|\Delta\|_\infty \le 1$

$$\|M\|_\infty = \|w_1 (I + KG)^{-1} w_2\|_\infty < 1$$

In the case of free Δ $\qquad \bar{\sigma}(\Delta M) < 1$

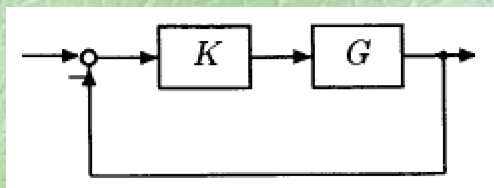
$M\Delta -$structure

Robust Stability of Uncertain Systems



Suitable for robust stability analysis

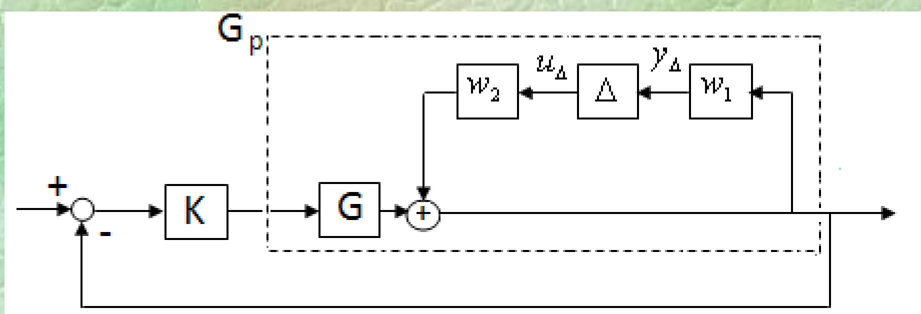
System without uncertainty



$$G_p = (I - w_2 \Delta w_1)^{-1} G \qquad \|w_2 \Delta w_1\|_\infty < 1$$

System with inverse multiplicative output uncertainty

$$M = w_1 (I + GK)^{-1} w_2$$



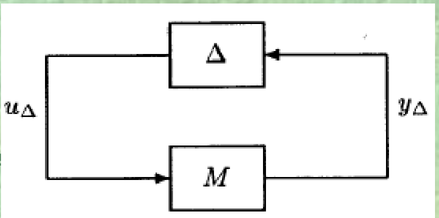
Robust stability condition: In the case of $\|\Delta\|_\infty \le 1$

$$\|M\|_\infty = \left\| w_1 (I + GK)^{-1} w_2 \right\|_\infty < 1$$

In the case of free Δ $\qquad \bar{\sigma}(\Delta M) < 1$

$M\Delta -$structure

Robust Stability of Uncertain Systems

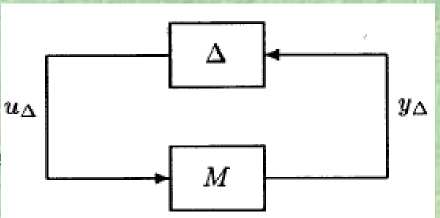


Suitable for robust stability analysis

Uncertainty	Perturbed Plant	M in MΔ-structure
Additive uncertainty	$G_p = G + w_2 \Delta w_1$	$M = w_1 K (I + GK)^{-1} w_2$
Multiplicative input uncertainty	$G_p = G(I + w_2 \Delta w_1)$	$M = w_1 K (I + GK)^{-1} G w_2$
Multiplicative output uncertainty	$G_p = (I + w_2 \Delta w_1) G$	$M = w_1 G K (I + GK)^{-1} w_2$
Inverse additive uncertainty	$G_p = G\left(I - w_2 \Delta w_1 G\right)^{-1}$	$M = w_1 G (I + KG)^{-1} w_2$
Inverse multiplicative input uncertainty	$G_p = G\left(I - w_2 \Delta w_1\right)^{-1}$	$M = w_1 (I + KG)^{-1} w_2$
Inverse multiplicative output uncertainty	$G_p = \left(I - w_2 \Delta w_1\right)^{-1} G$	$M = w_1 (I + GK)^{-1} w_2$

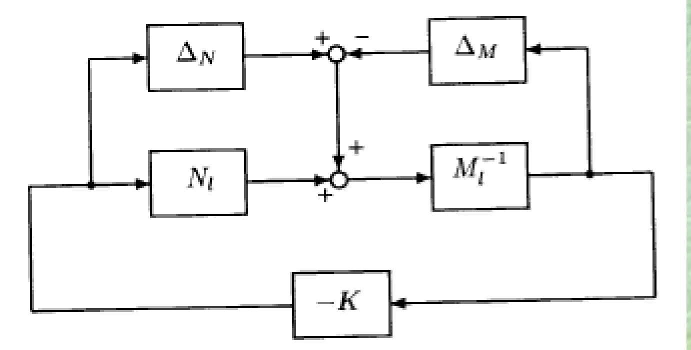
$M\Delta -$structure

Robust Stability of Uncertain Systems



Suitable for robust stability analysis

System with coprime factor uncertainty



$$G = M_l^{-1} N_l$$

$$G_p = (M_l + \Delta_M)^{-1}(N_l + \Delta_N)$$

$$\Delta = [\Delta_N \quad \Delta_M]$$

$$M = -\begin{bmatrix} K \\ I \end{bmatrix} (I + GK)^{-1} M_l^{-1}$$

Since there is no weight for uncertainty so the theorem is

$$RS: \quad \forall \left\| \Delta_N \quad \Delta_M \right\|_\infty \le \varepsilon \quad \Leftrightarrow \quad \left\| M \right\|_\infty < 1/\varepsilon$$

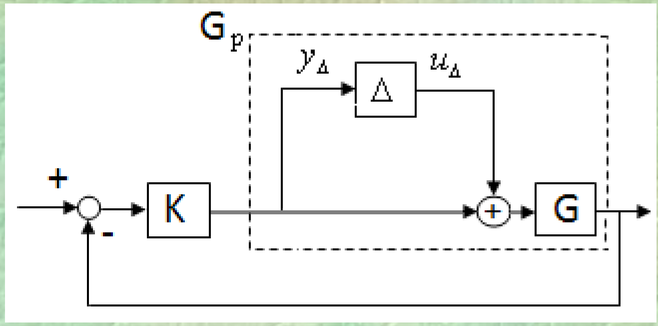
Robust Stability of Uncertain Systems

Remind Example 10-2: Decoupling controller

$$G(s)=\frac{1}{s^2+3s+2}\begin{bmatrix}2-47s & 56s\\ -42s & 50s+2\end{bmatrix} \Rightarrow K(s)=\begin{bmatrix}7 & 8\\ 6 & 7\end{bmatrix}\begin{bmatrix}k & 0\\ 0 & k\end{bmatrix}\begin{bmatrix}7 & -8\\ -6 & 7\end{bmatrix}=\begin{bmatrix}k & 0\\ 0 & k\end{bmatrix}$$

Consider system with multiplicative input uncertainty

$$G_p=G(I+\Delta)$$

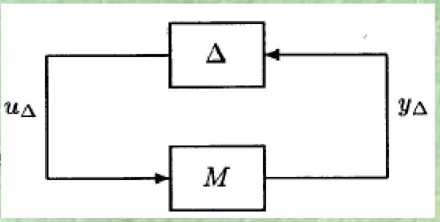


$$\bar{\sigma}\left(\Delta K(I+GK)^{-1}G\right)<1$$

$$\bar{\sigma}(\Delta)<1/\bar{\sigma}\left(K(I+GK)^{-1}G\right)$$

Robust Stability of Uncertain Systems

$M\Delta -$structure

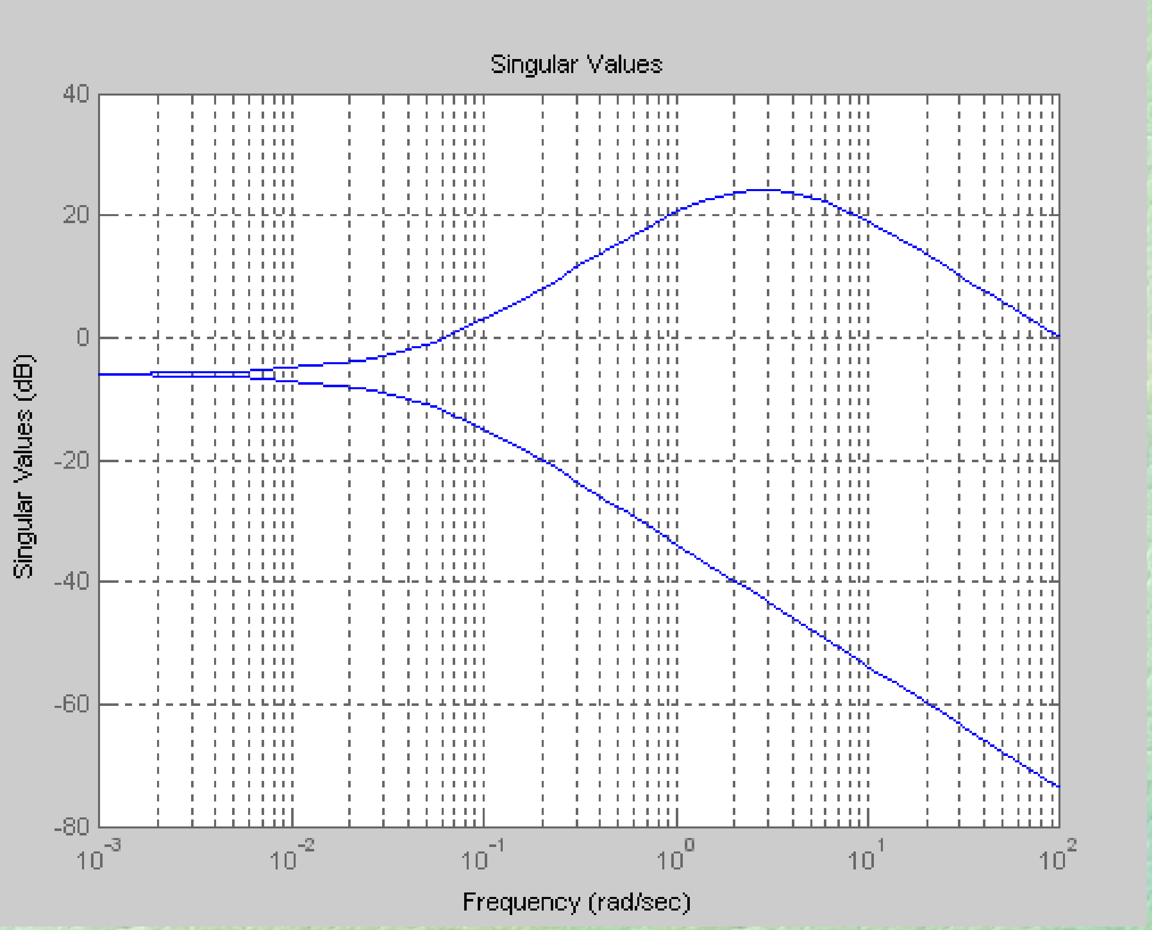


Suitable for robust stability analysis

$$\bar{\sigma}(\Delta) < 1/\bar{\sigma}\left(K(I+GK)^{-1}G\right)$$

$$\bar{\sigma}\left(K(I+GK)^{-1}G\right) = 24.2\ db = 15.85$$

$$\bar{\sigma}(\Delta) < 1/15.85 = 0.0631$$



Uncertainty in Multivariable Systems and Quantitative feedback theory

- **Introduction**
- **Types of Uncertainty in Multivariable Systems**
- **Robust Stability of Uncertain Systems.**
- **Quantitative Feedback Theory(QFT)**
- **QFT Design Procedure.**

Uncertainty Model and Plant Templates

The various origins of model uncertainty:

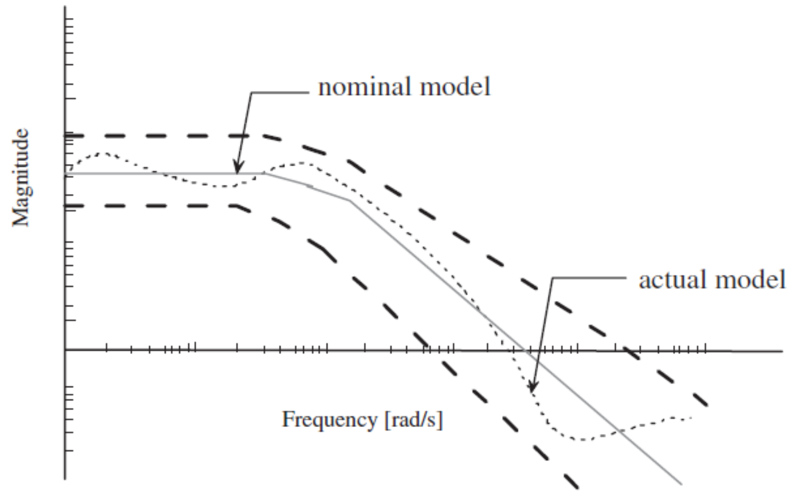
I) Parametric uncertainty

Parametric uncertainty implies specific knowledge of variations in parameters of the transfer function.

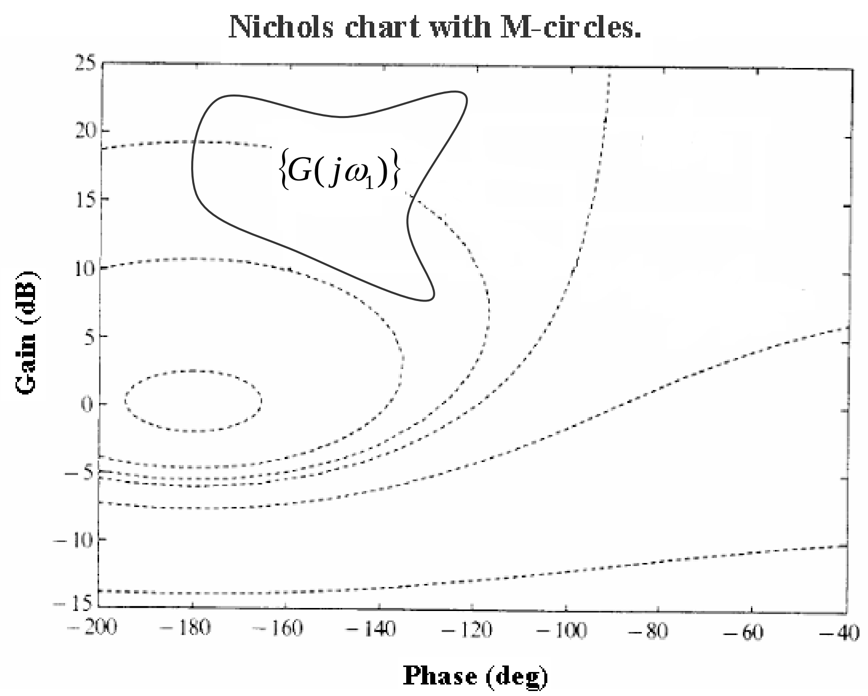
II) Non-parametric uncertainty

The main source of non-parametric uncertainty is error in the model.

$$a_p = a_o\,(1 + \sigma\Delta)\,, \qquad -1 \le \Delta \le 1$$



Typical behavior of plant uncertainty .



Uncertainty Model and Plant Templates

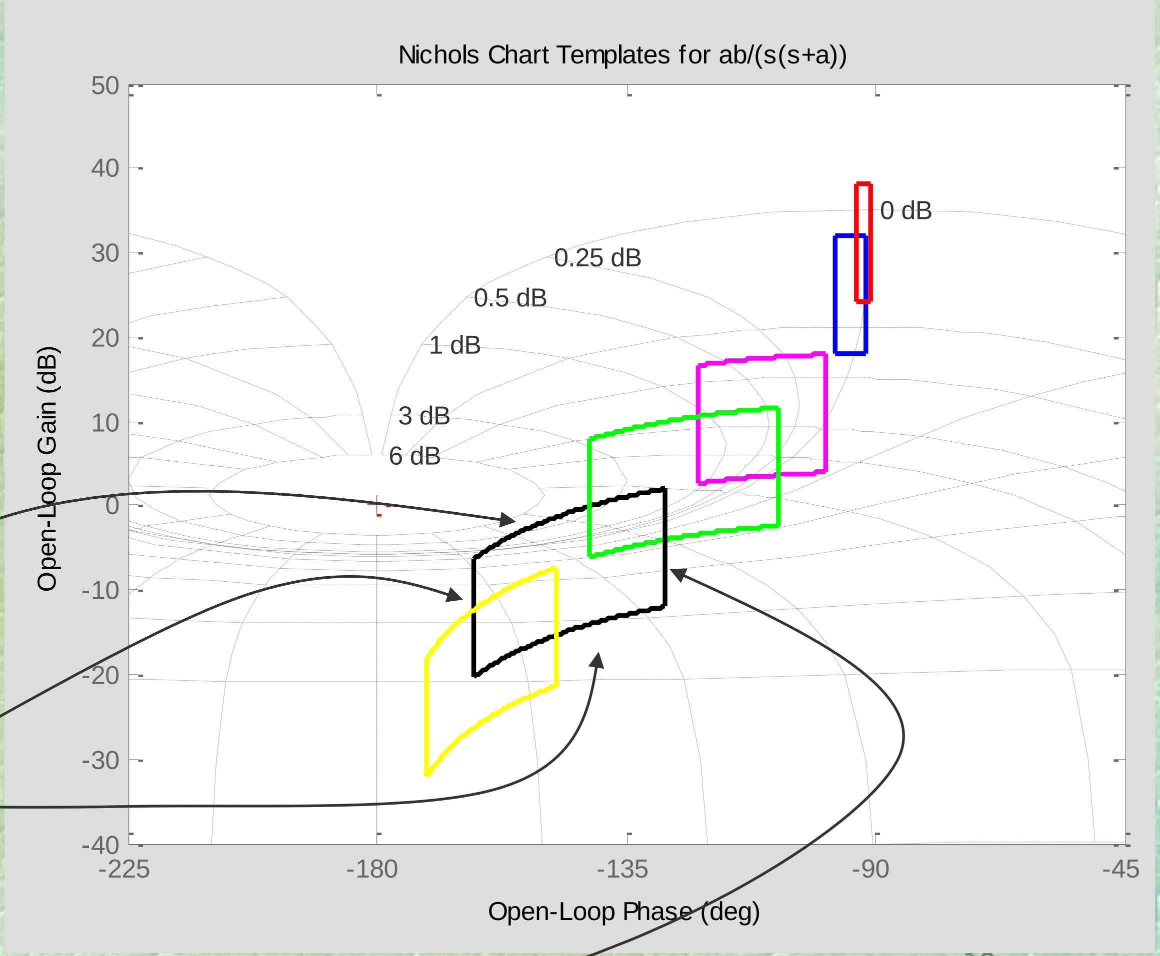
I) Parametric uncertainty

Nichols Chart for $G(s)=\dfrac{ab}{s(s+a)}$

$a\in[1 \quad 4] \quad b\in[1 \quad 5]$

$f\in\{0.01 \quad 0.02 \quad 0.1 \quad 0.2 \quad 0.5 \quad 1\}$

- $\omega=2\pi(0.01)$
- $\omega=2\pi(0.02)$
- $\omega=2\pi(0.10)$
- $\omega=2\pi(0.20)$
- $\omega=2\pi(0.50)$
- $\omega=2\pi(1.00)$



$a\in[1 \quad 4] \quad b=5$

$a\in[1 \quad 4] \quad b=1$

$a=1 \quad b\in[1 \quad 5]$

$a=4 \quad b\in[1 \quad 5]$

Uncertainty Model and Plant Templates

I) Parametric uncertainty

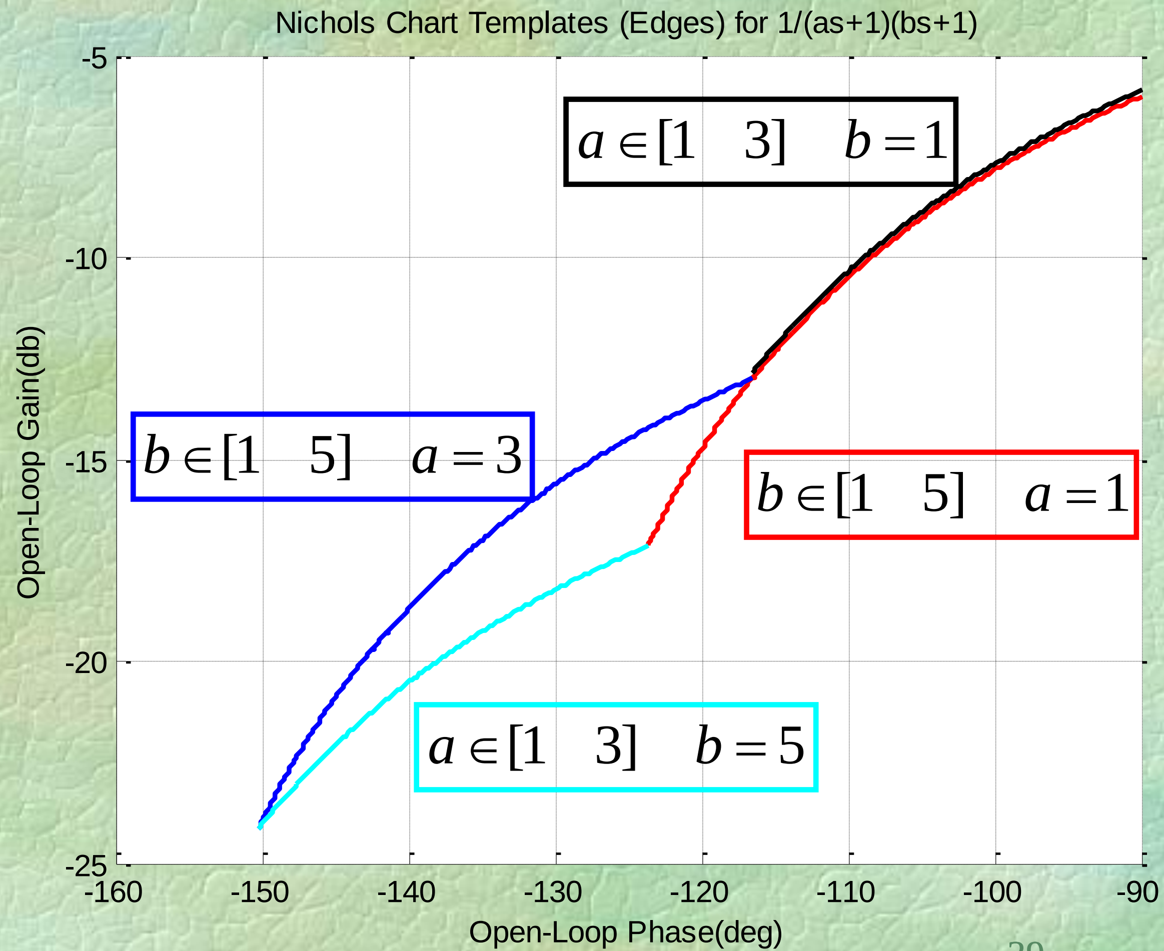
Sometimes edges are not ok!

Nichols Chart for

$$G(s)=\frac{1}{(as+1)(bs+1)}$$

$$a\in[1\quad 3]\quad b\in[1\quad 5]$$

$$\omega=1rad/\sec$$



Uncertainty Model and Plant Templates

I) Parametric uncertainty

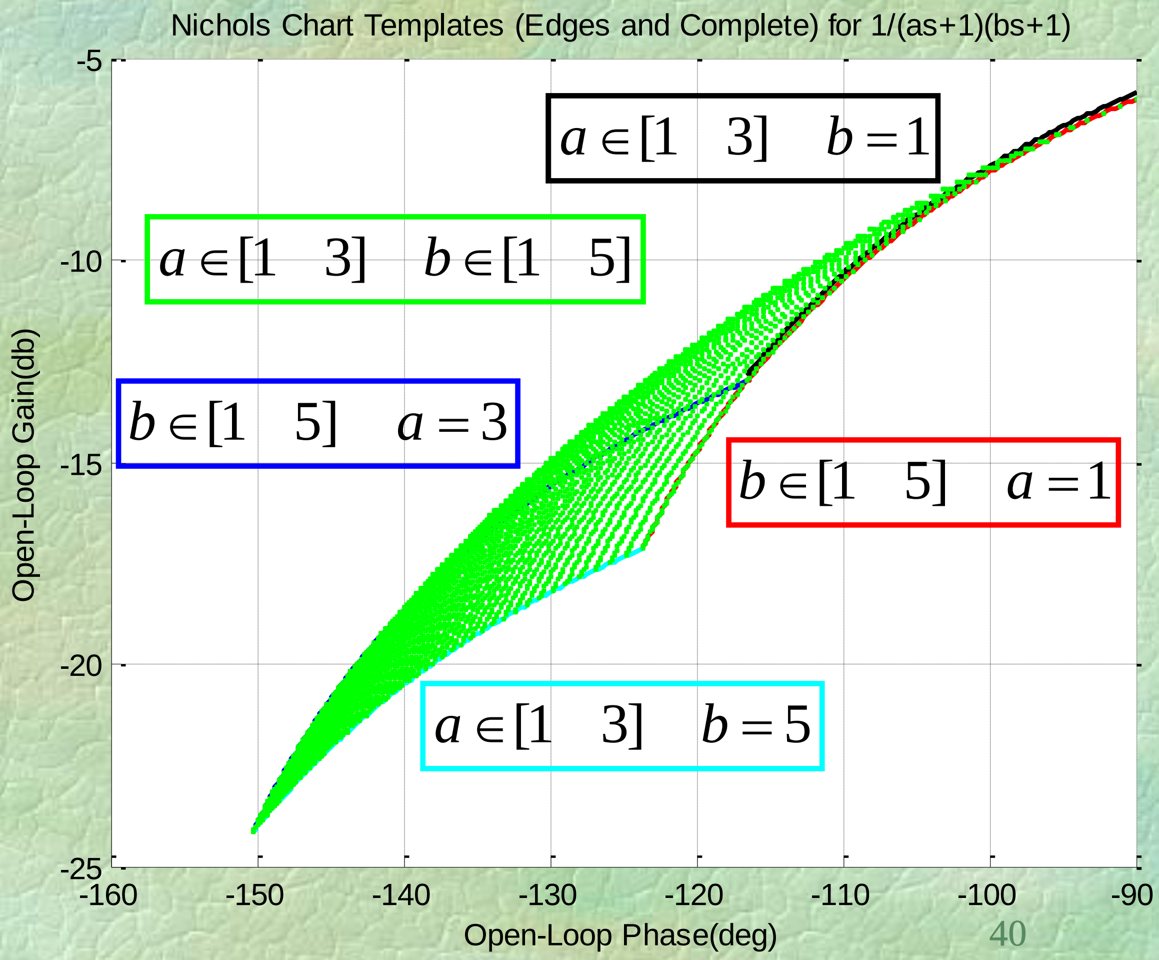
Be careful to use the edges.

Nichols Chart for

$$G(s)=\frac{1}{(as+1)(bs+1)}$$

$a\in[1\quad 3]\quad b\in[1\quad 5]$

$\omega=1 rad/\sec$



Uncertainty Model and Plant Templates

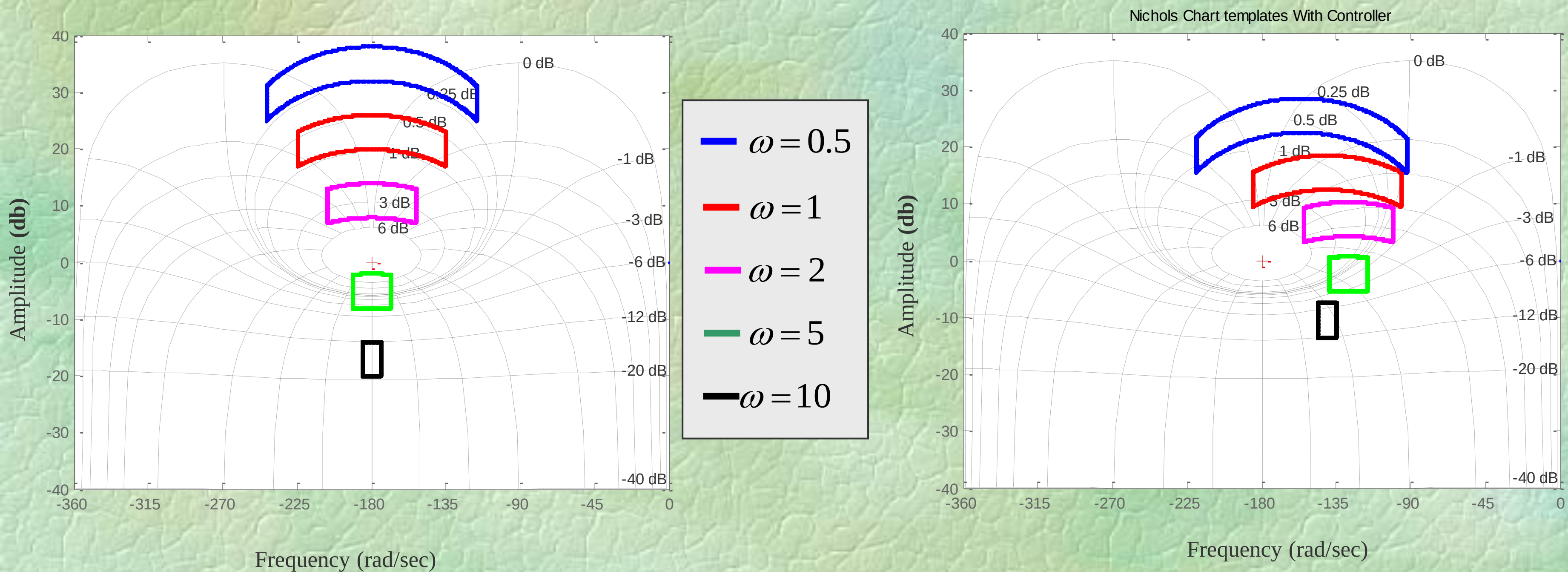
How a QFT controller works?

Nichols Chart for $G(s)=\dfrac{k}{s(s-a)}$

$$k\in[10\ \ 20]\qquad a\in[-1\ \ +1]$$

Nichols Chart for $G(s)=\dfrac{k}{s(s-a)}$

$$K(s)=3\frac{s+1}{s+10}$$



Dr. Ali Karimpour Mar 2021

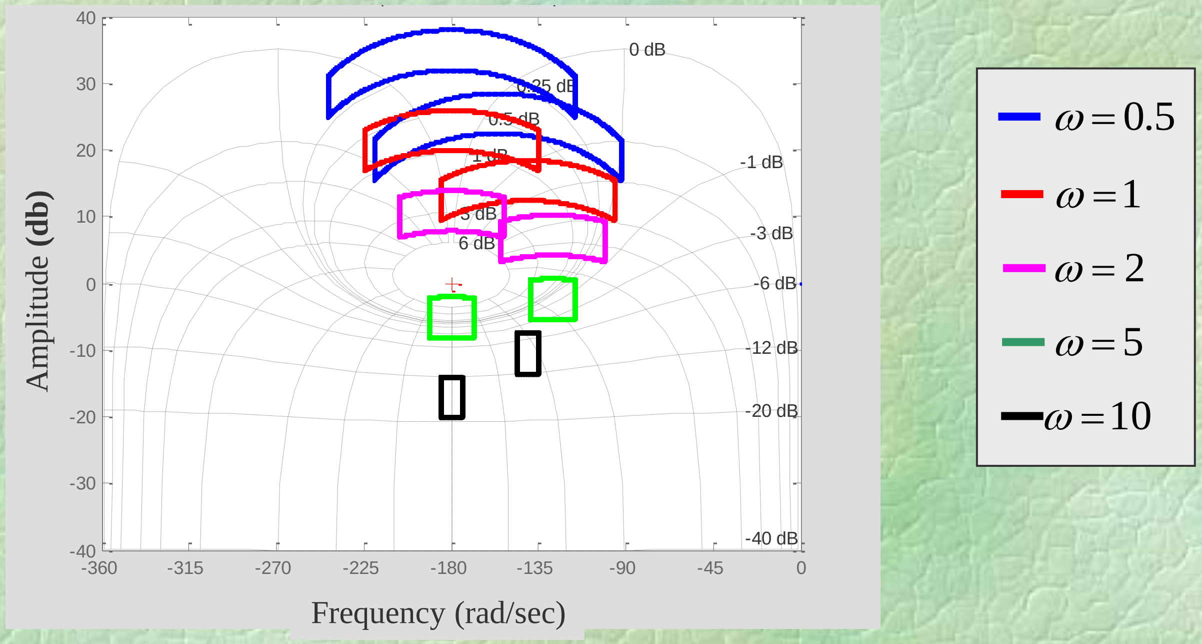
Uncertainty Model and Plant Templates

How a QFT controller works?

System with and without Controller

Nichols Chart for $G(s)=\dfrac{k}{s(s-a)}$ $\qquad K(s)=3\dfrac{s+1}{s+10}$

$$\omega \in \{0.5 \quad 1 \quad 2 \quad 5 \quad 10\}$$

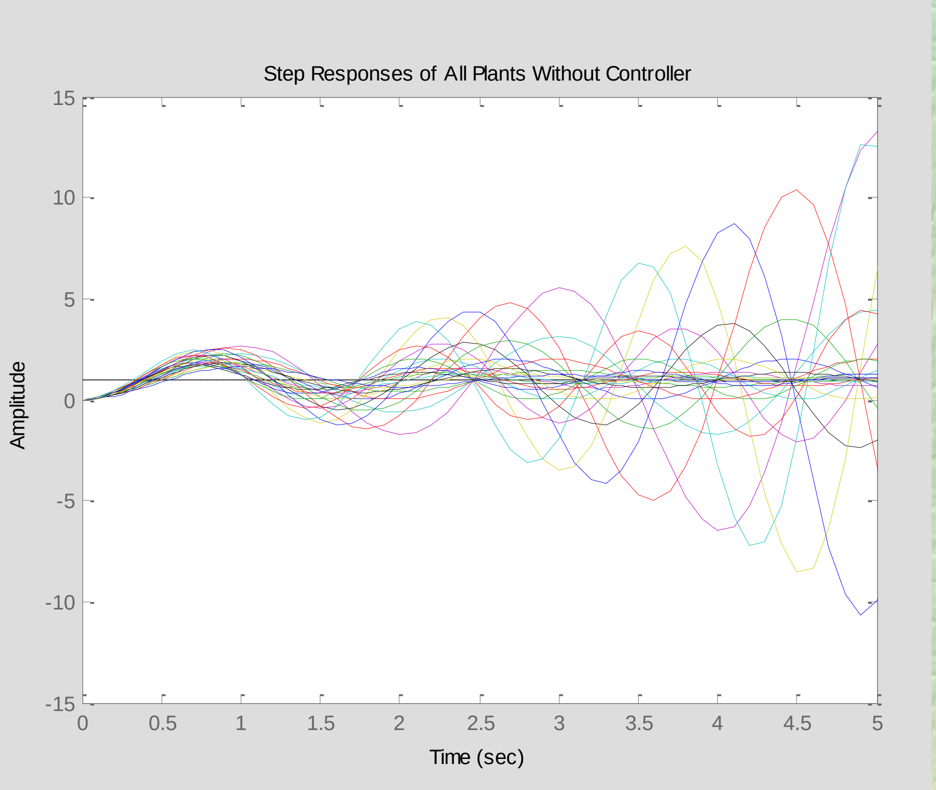


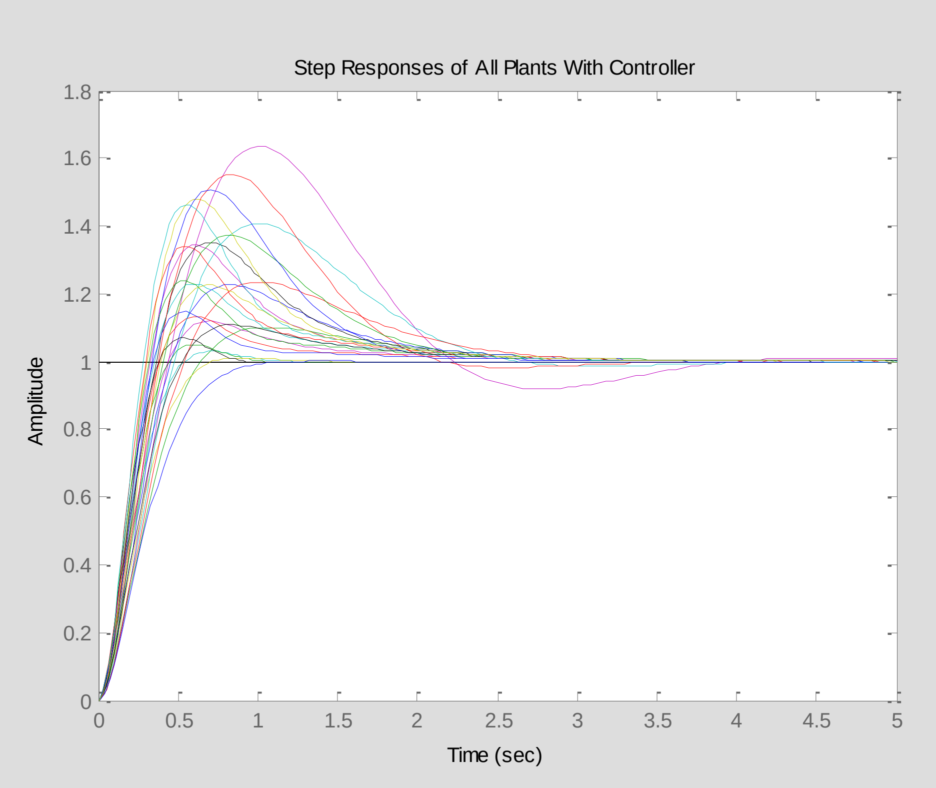
Uncertainty Model and Plant Templates

How a QFT controller works?

Step Response of System with and without Controller ($K(s)$=1)

$$G(s) = \frac{k}{s(s-a)} \qquad K(s) = 3\frac{s+1}{s+10}$$





QFT Design Procedure

Choice of Frequency Array

An appropriate frequency band for a computing templates and bounds has to be selected.

Choice of Nominal Plant

In order to compute bounds, it is necessary to choose a plant from the uncertainty set as the nominal plant. It is common practice to select a nominal plant which we think is most convenient for design.

QFT Bounds Computations

Given the plant templates, QFT converts closed-loop magnitude specifications into magnitude and phase constraints on a nominal open-loop function.

QFT Loop-shaping

The final step in a QFT design involves the design (loop shaping) of a nominal loop function that meets its bounds. The controller design then proceeds using the Nichols chart and classical loop-shaping ideas.

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QFT Design Procedure

The QFT approach assumes that the plant uncertainty is represented by a set of templates on the complex plane at some frequency ω_k.

It also assumes that the design specification is in the form of bounds on the magnitudes of the frequency-response transfer functions

$$|S(j\omega)| \le M_s \qquad\qquad a(\omega) \le |T(j\omega)| \le b(\omega)$$

The QFT technique leads to a design which satisfies these specifications for all permissible plant variations.

Horowitz and Sidi (1972) have obtained sufficient conditions on frequency-domain bounds which imply the satisfaction of time-domain bounds, the frequency-domain bounds obtained in this way do not appear to be unduly conservative.

QFT Design Procedure

Bounds for S

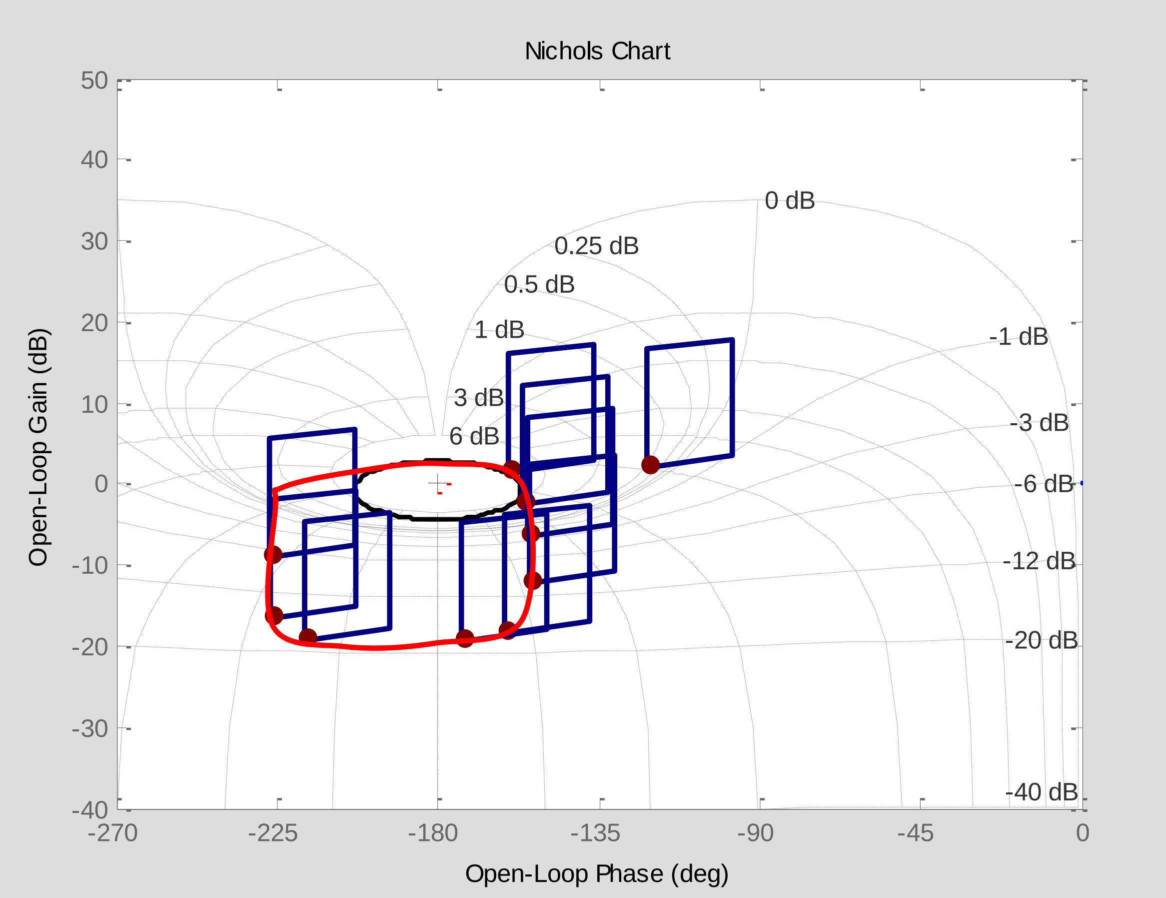
Let we need |s|<2.5 so: ⇒ L must be outside of black curve.

Now consider the templates at one frequency with nominal plant as blue curve.

$$G(s) = \frac{ab}{s(s+a)} \qquad a \in [1 \quad 4] \qquad b \in [1 \quad 5]$$

$$\omega = 0.1$$

$$G_{nominal}(s) = \frac{1}{s(s+1)} \qquad a = 1 \qquad b = 1$$



Bounds for template is red curve(for ω=0.1 rad/sec).

Effect of nominal plant?

QFT Design Procedure

Bounds for S

Let we need |s|<2.5 so: ⇒ L must be outside of black curve.

Changing Nominal plant

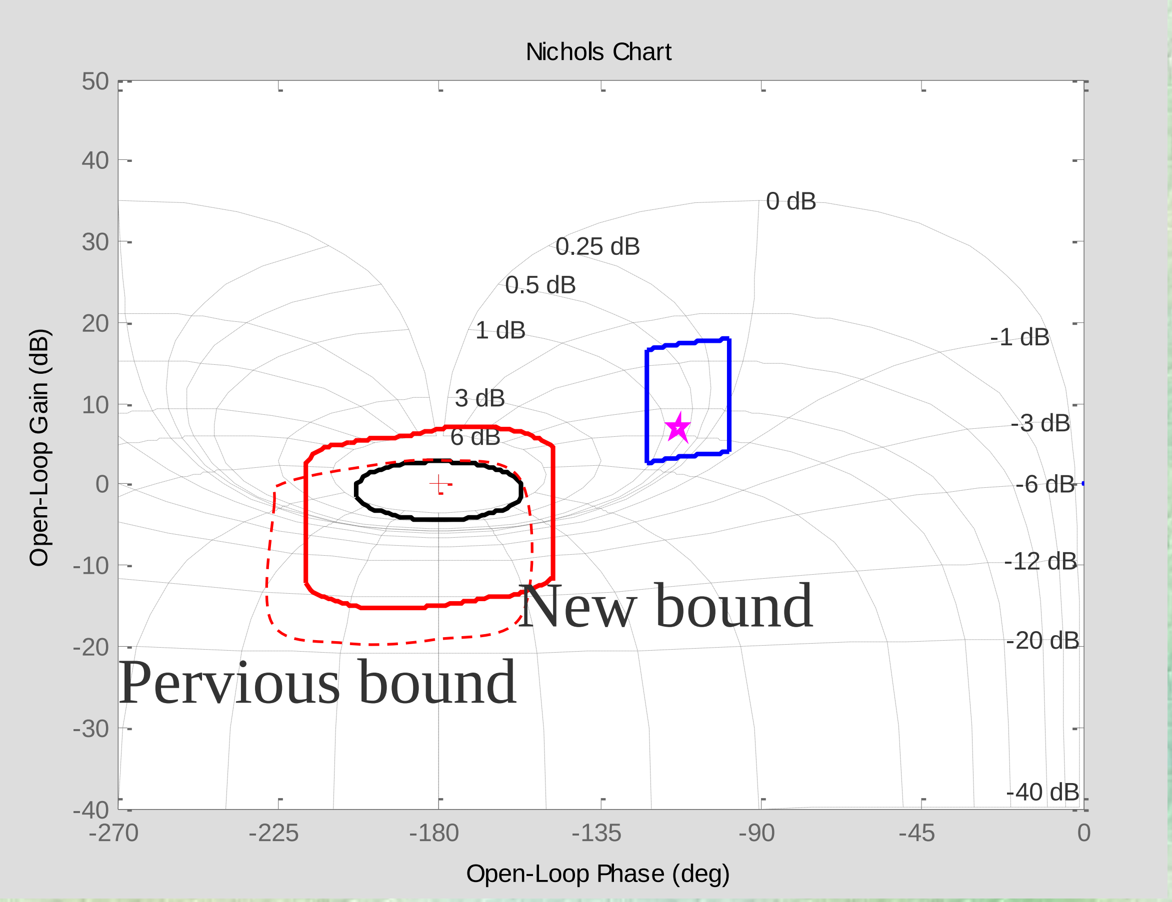
$$G(s)=\frac{ab}{s(s+a)} \qquad a\in[1 \quad 4] \qquad b\in[1 \quad 5]$$

$$G_{nominal}(s)=\frac{1}{s(s+1)} \qquad a=1 \qquad b=1$$

$$\omega=0.1$$

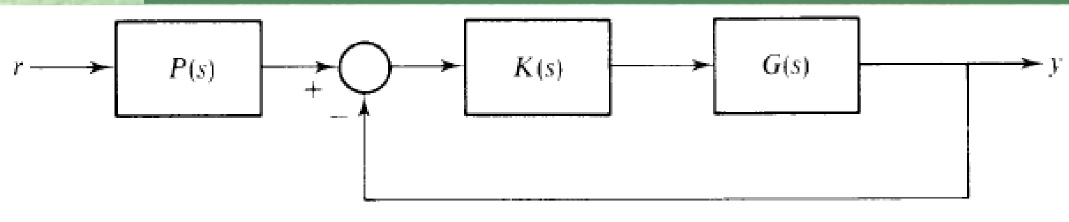
⇓

$$G_{nominal}(s)=\frac{2.25}{s(s+1.5)} \qquad a=1.5 \qquad b=1.5$$



QFT Design Procedure

Bounds for T



A feedback configurations with two degree of freedom.

We need:

$$a(\omega_1) \le |T(j\omega_1)| \le b(\omega_1)$$

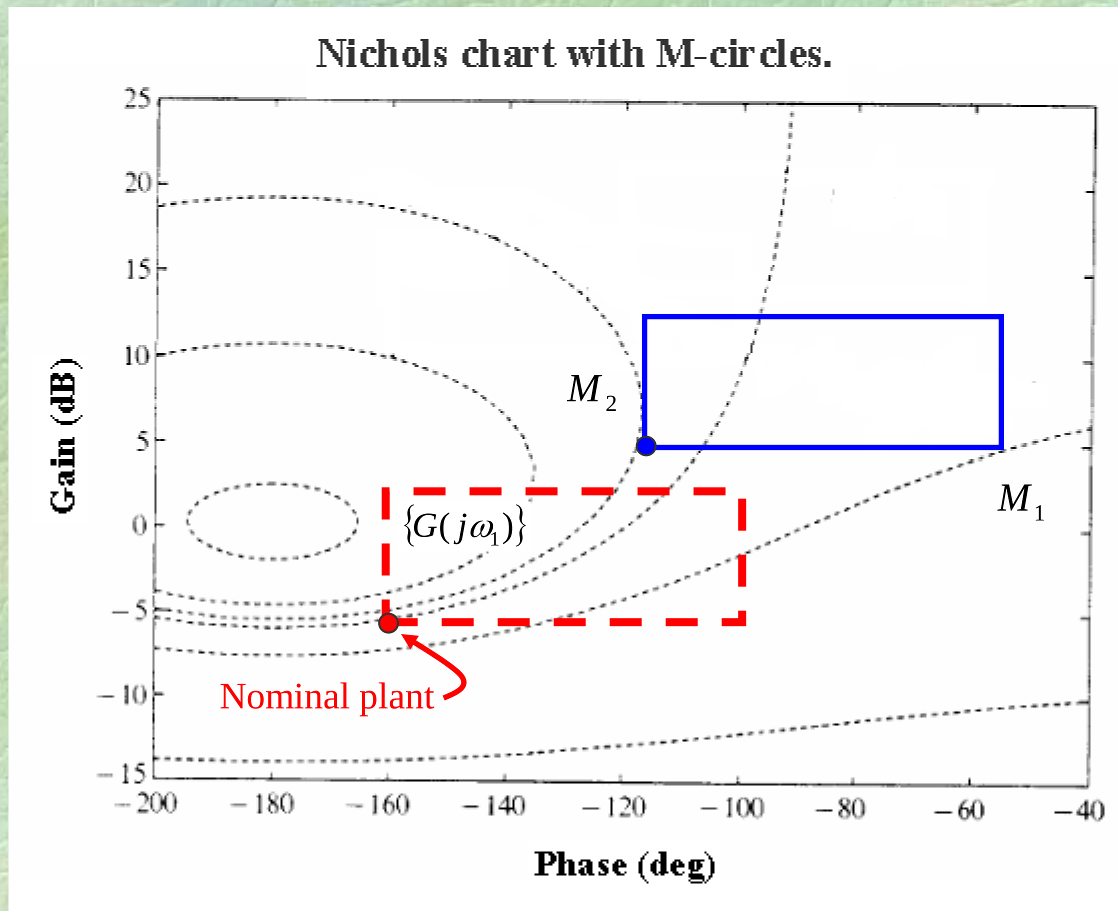
Since:

$$T(s) = L(s)\big(1+L(s)\big)^{-1} P(s)$$

If L would not intersect any pair of M-circles whose values differed by more than

$$|b(\omega_1)| / |a(\omega_1)| = M_2 / M_1$$

$$\Rightarrow \quad M_1 \le \frac{L(s)}{1+L(s)} \le M_2$$



QFT Design Procedure

Bounds for T

$$T(s) = L(s)\left(1+L(s)\right)^{-1}P(s)$$

If L would not intersect any pair of M-circles whose values differed by more than

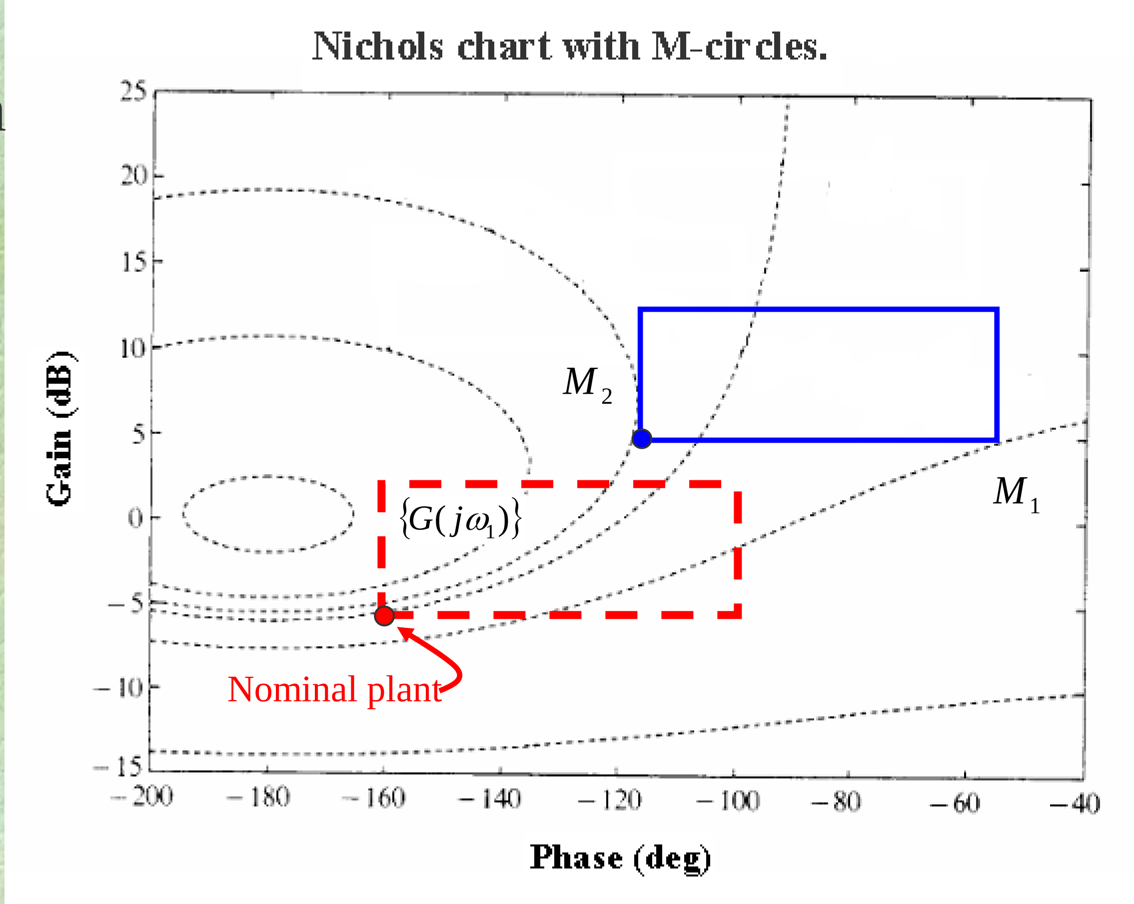
$$|b(\omega_1)|/|a(\omega_1)| = M_2/M_1$$

$$\Rightarrow \quad M_1 \le \frac{L(s)}{1+L(s)} \le M_2$$

$$\Rightarrow \quad a(\omega_1)c(\omega_1) \le \frac{L(s)}{1+L(s)} \le b(\omega_1)c(\omega_1)$$

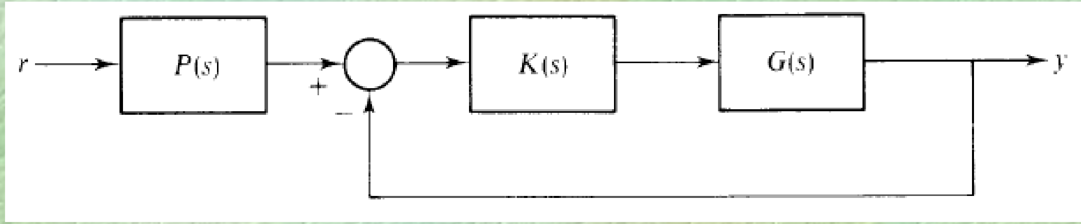
$P(\omega_1)$

$$\Rightarrow \quad a(\omega_1) \le \frac{1}{c(\omega_1)}\frac{L(s)}{1+L(s)} \le b(\omega_1)$$



QFT Design Procedure

Bounds for T



A feedback configurations with two degree of freedom.

Suppose that we know the template of all possible values of G at ω_1:

But we need:

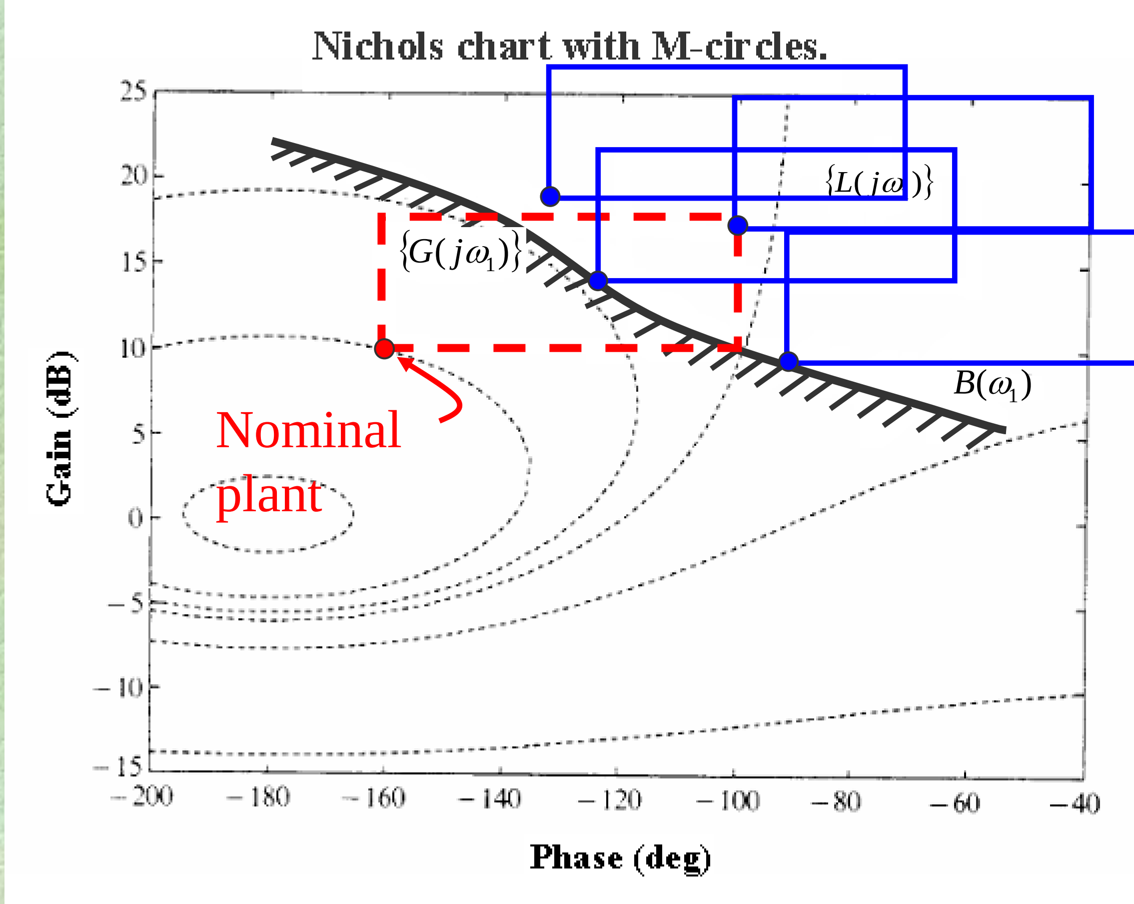
$$a(\omega_1) \le |T(j\omega_1)| \le b(\omega_1)$$

Since:

$$T(s) = L(s)\left(1 + L(s)\right)^{-1} P(s)$$

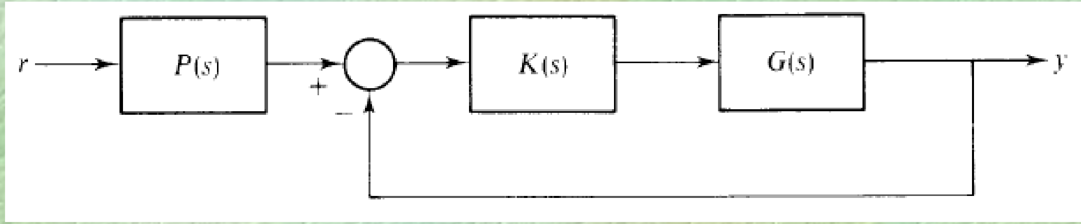
If L would not intersect any pair of M-circles whose values differed by more than

$$|b(\omega_1)| / |a(\omega_1)|$$



QFT Design Procedure

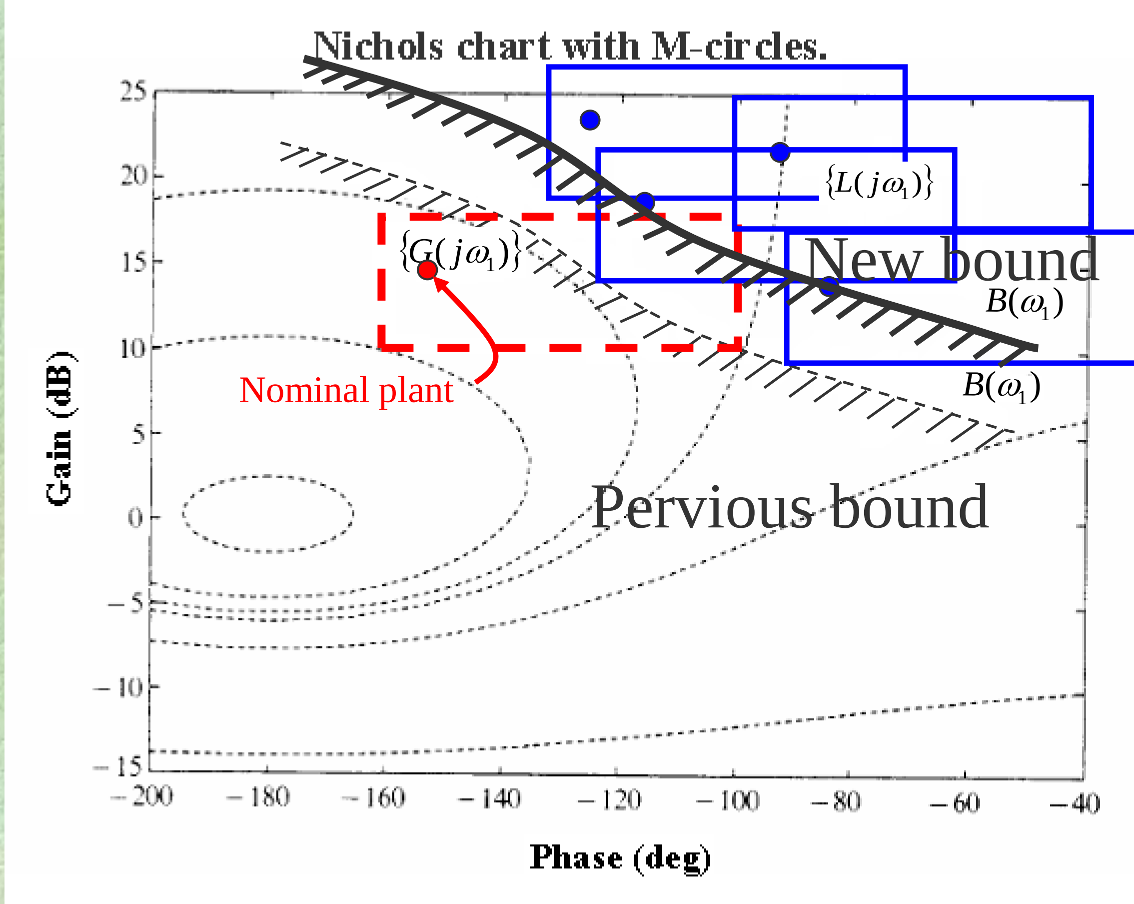
Bounds for T



A feedback configurations with two degree of freedom.

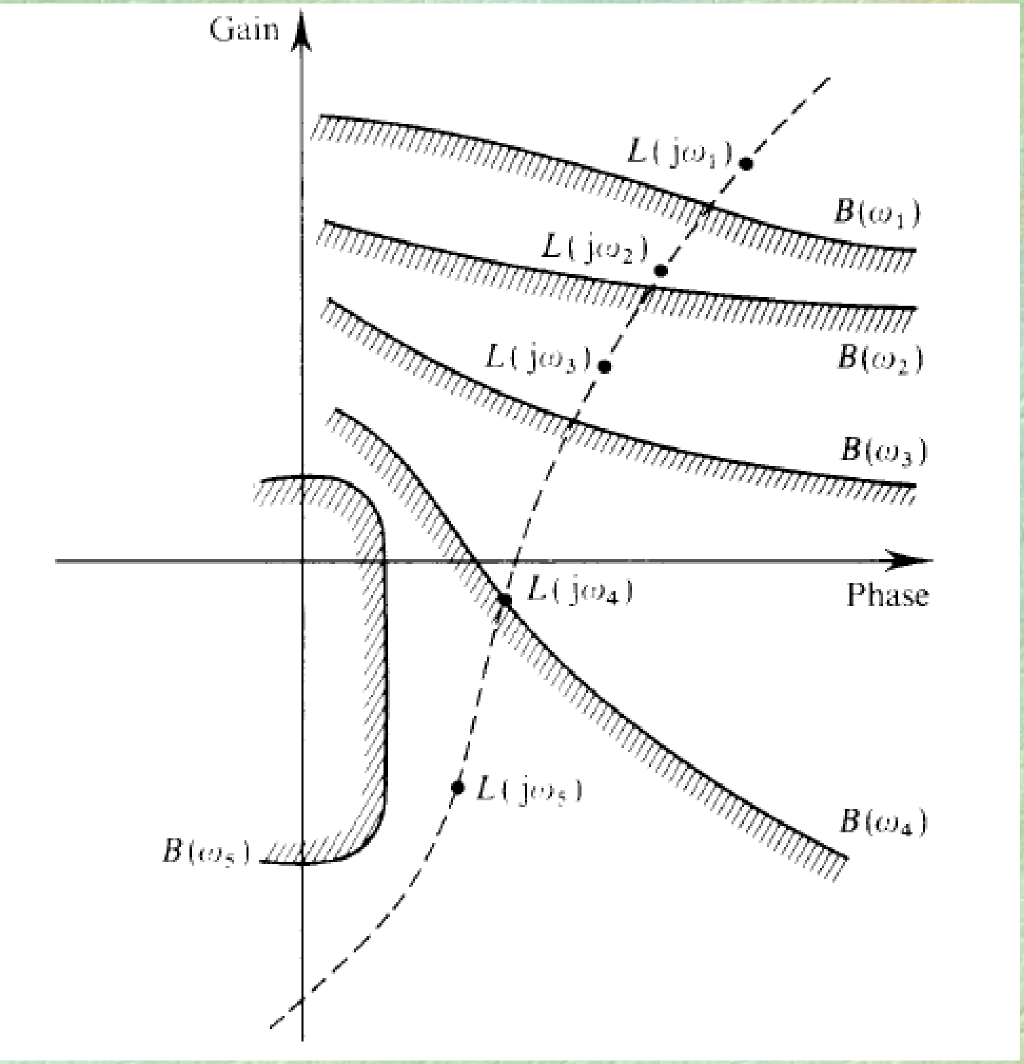
Changing Nominal plant

Leads Different Bounds



QFT Design Procedure

Bounds for T



$$a(\omega_i)c(\omega_i) \le \left| L(j\omega_i)\left(1+L(j\omega_i)\right)^{-1} \right| \le b(\omega_i)c(\omega_i)$$

In order to meet the design specifications

$$a(\omega_i) \le \left| T(j\omega_i) \right| \le b(\omega_i)$$

the pre-filter is chosen to have the gain

$$\left| P(j\omega_i) \right| \approx \left(c(\omega_i)\right)^{-1}$$

Similarly one can found bounds for *S*

QFT Design Procedure

QFT Algorithm:

1. Formulating of the closed-loop control performance specifications, i.e., stability margins, tracking and disturbance rejection.

2. Generating templates. For a given uncertain plant $P(s) \in \{P\}$,select a series of frequency points ω_i, $i = 1, 2, ..., m$ according to the plant characteristics and the specifications.

3. Computation of QFT bounds. Find the intersection of bounds. An arbitrary member in the plant set is chosen as the nominal case.

4. Loop shaping for QFT controllers. The design of the QFT controller, K(s) is accomplished on the Nichols Chart. The QFT bounds at all frequencies must be satisfied and the closed-loop nominal system is stable;

5. Design of prefilters P(s). The Final step in QFT is to design the prefilter, P(s), such that the performance specifications are satisfied.

QFT Design Procedure Example

Example 10-5: Suppose that the set of our plants is:

$$G(s)=\frac{ab}{s(s+a)} \qquad a\in[1 \quad 4] \quad b\in[1 \quad 5]$$

Performance specification are:

a) |S|<2.5. (for robust stability) — Constrain on S ✓

b) Zero steady state error to step input.

c) Less than 50% overshoot to step input.

(b and c: Constrain on T)

d) Output must be above 90% after 1 second when step input applied.

QFT Design Procedure Example

a) |S|<2.5. (for robust stability) — Constrain on S

b) Zero steady state error to step input.

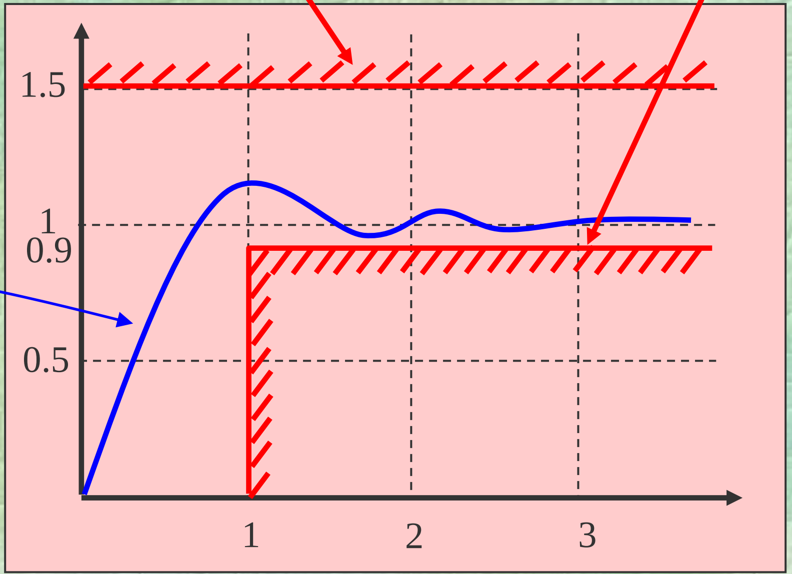
c) Less than 50% overshoot to step input.

d) Output must be above 90% after 1 second when step input applied.

Constrain on T

$$T(s) = \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

Acceptable step response

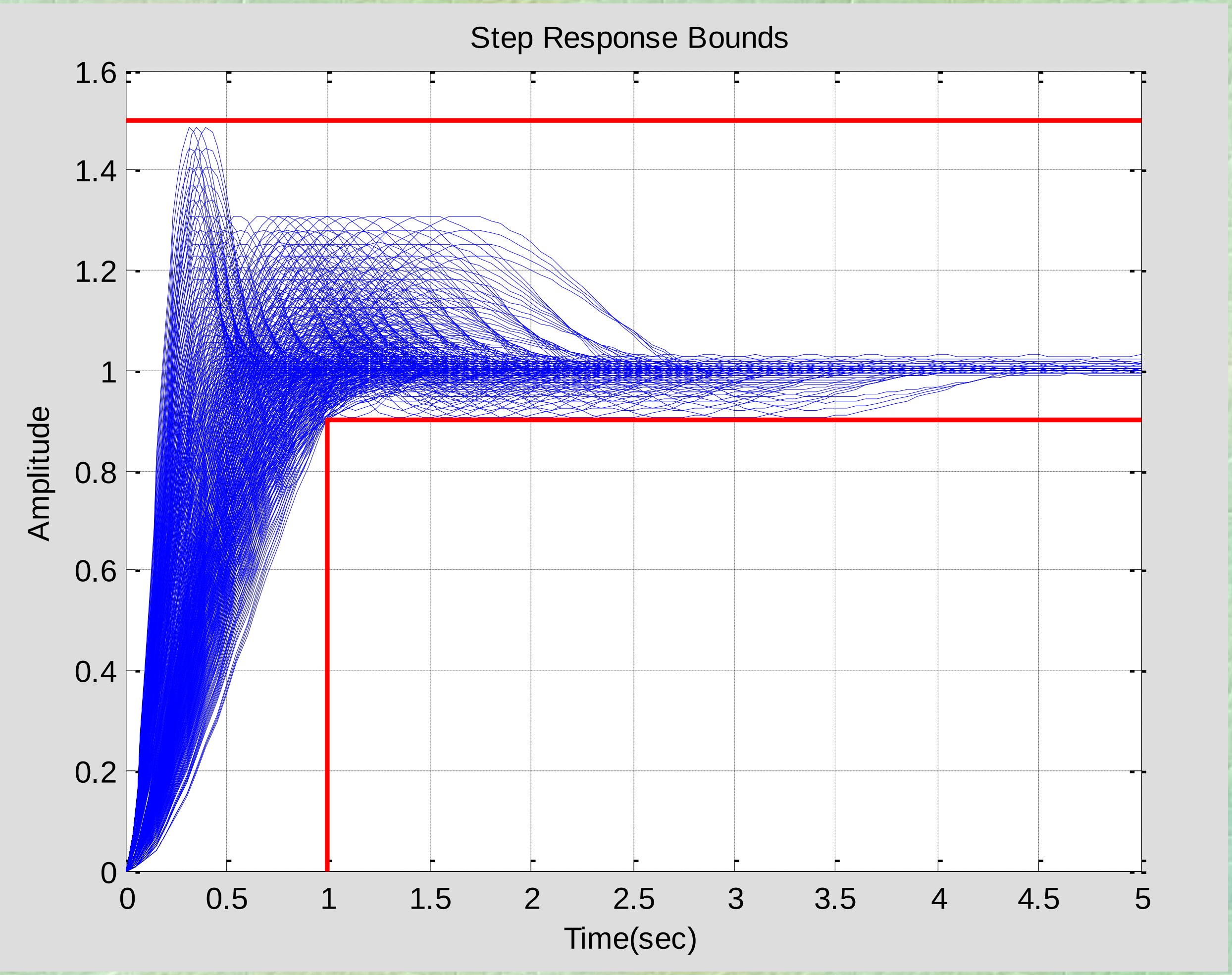


QFT Design Procedure Example

$$T(s) = \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

$$\omega_n \in \left[\omega_{\min} \quad \omega_{\max}\right]$$

$$\zeta \in \left[\zeta_{\min} \quad \zeta_{\max}\right]$$



QFT Design Procedure Example

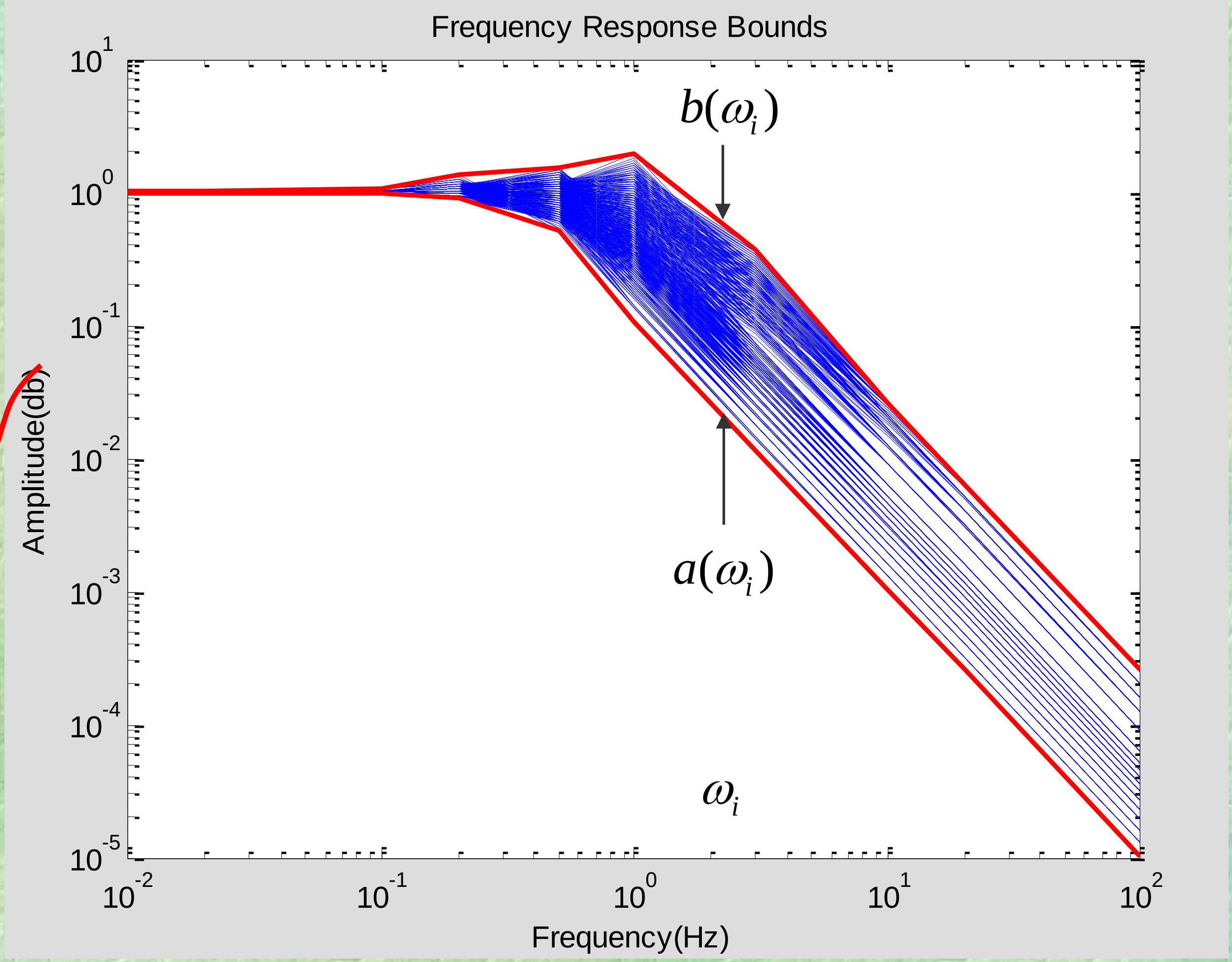
$$T(s)=\frac{\omega_n^{\ 2}}{s^2+2\zeta\omega_n s+\omega_n^{\ 2}}$$

$$\omega_n \in [\omega_{\min} \quad \omega_{\max}]$$

$$\zeta \in [\zeta_{\min} \quad \zeta_{\max}]$$

$$a(\omega_i) \le |T(j\omega_i)| \le b(\omega_i)$$

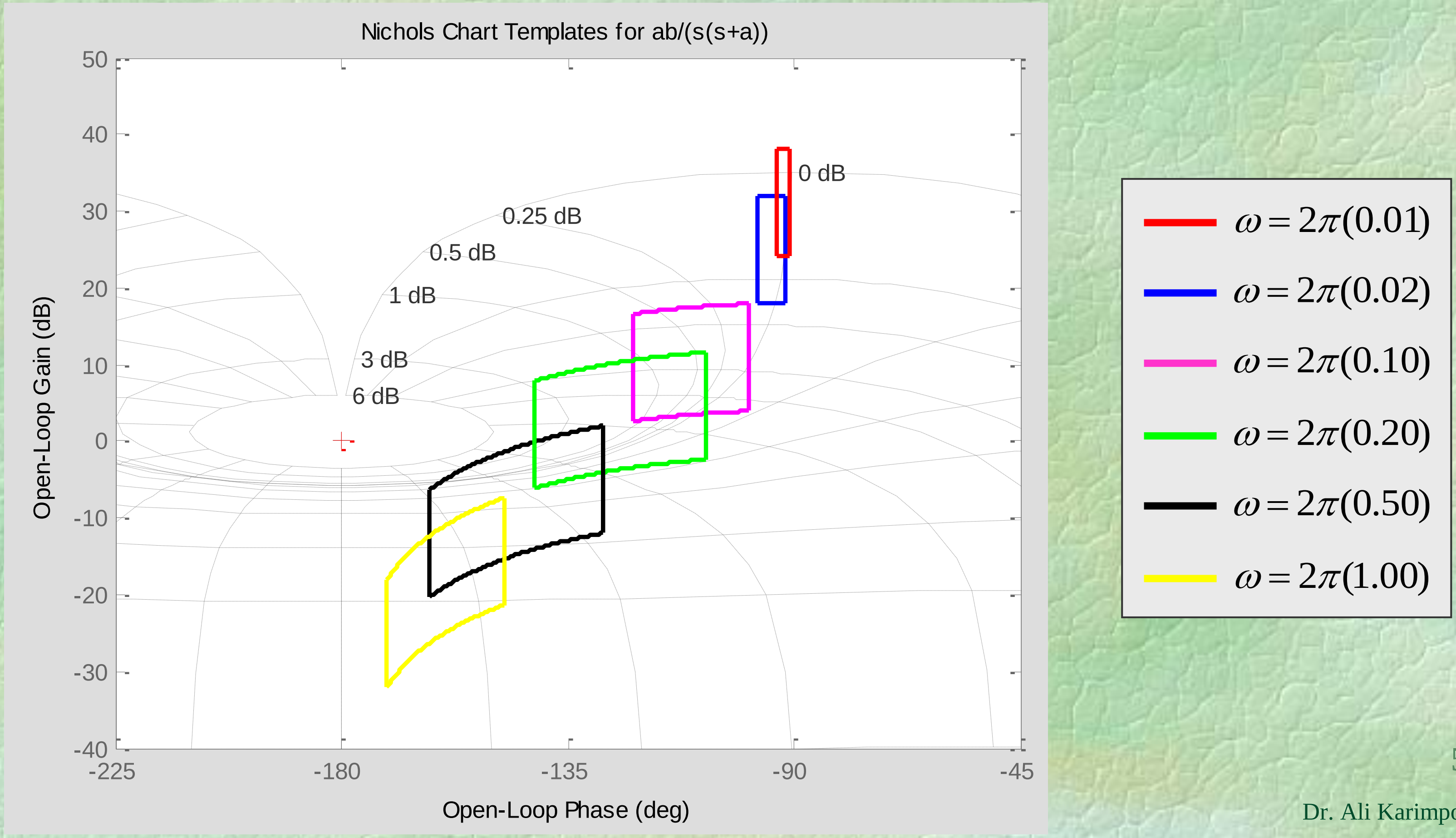
f	b/a
0.01	1.0011
0.02	1.0044
0.1	1.1141
0.2	1.5083
0.5	3.0155
1.0	17.770



QFT Design Procedure Example

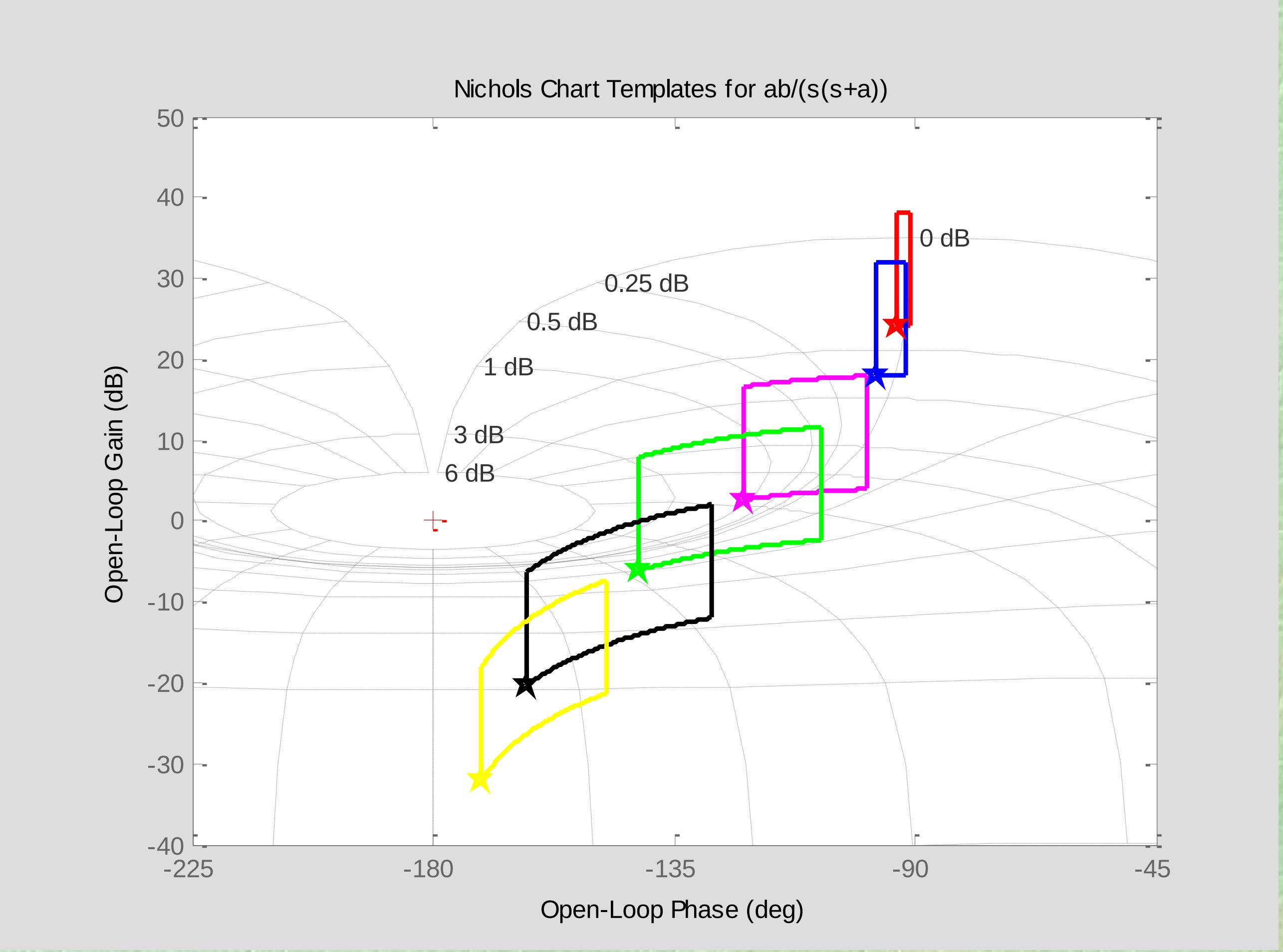
Templates are: $$G(s)=\frac{ab}{s(s+a)} \qquad a\in[1 \quad 4] \quad b\in[1 \quad 5]$$

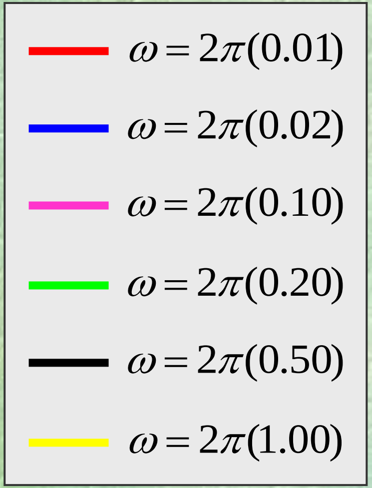
$$f\in\{0.01 \quad 0.02 \quad 0.1 \quad 0.2 \quad 0.5 \quad 1\}$$



QFT Design Procedure Example

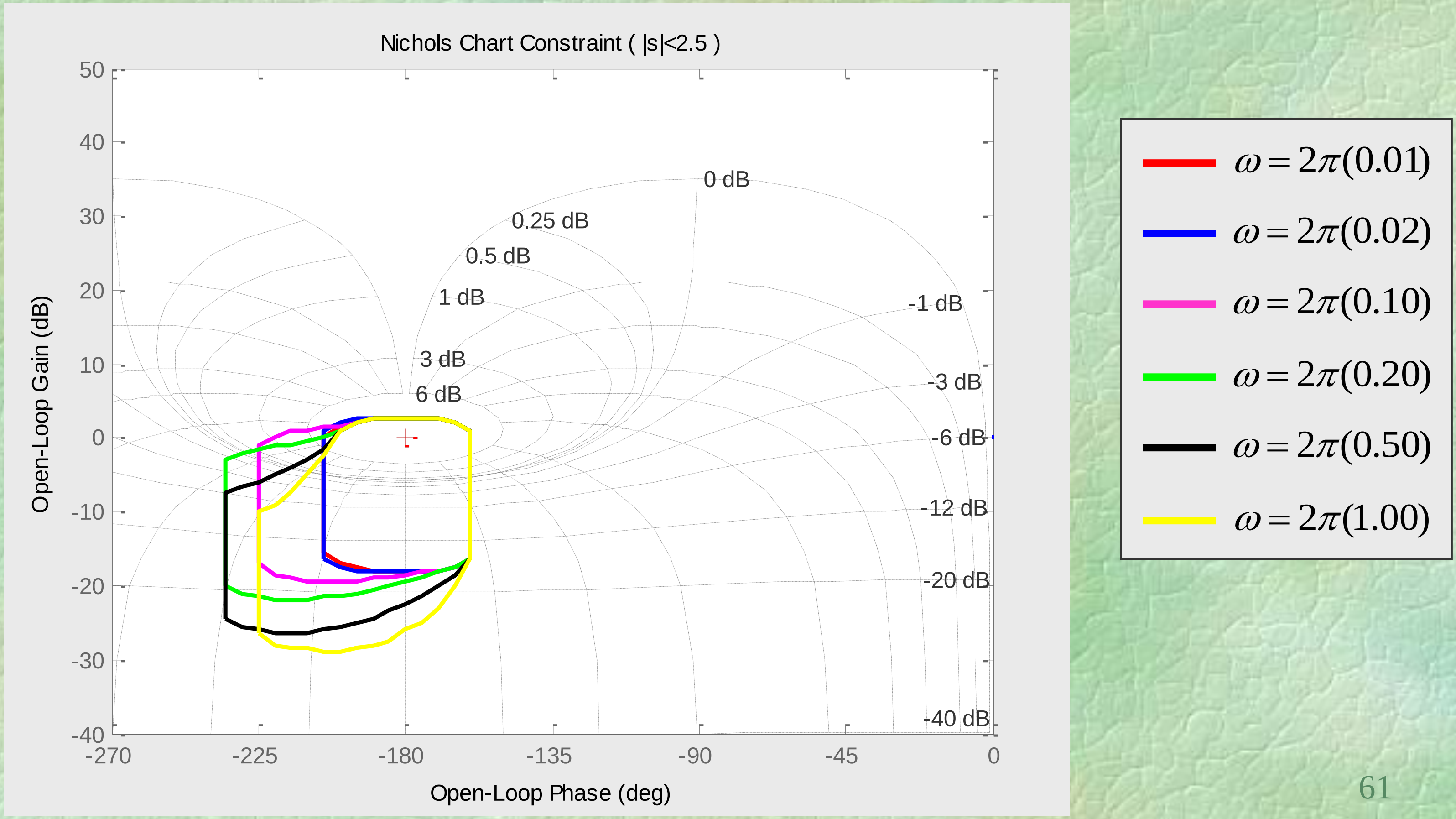
Nominal Plant is: $G(s) = \dfrac{ab}{s(s+a)}$ a=1 and b=1





QFT Design Procedure Example

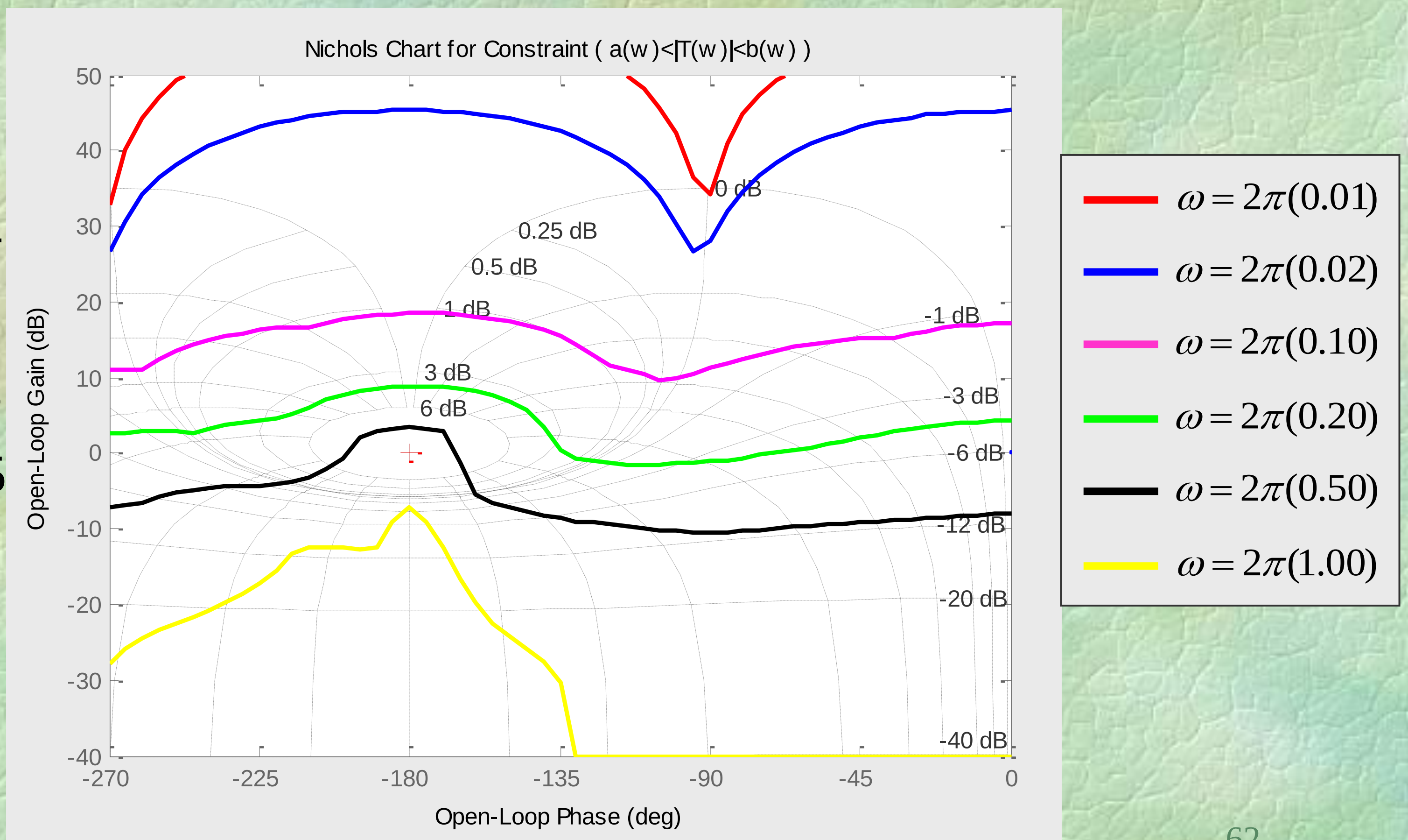
Bounds for $|S| < 2.5$



QFT Design Procedure Example

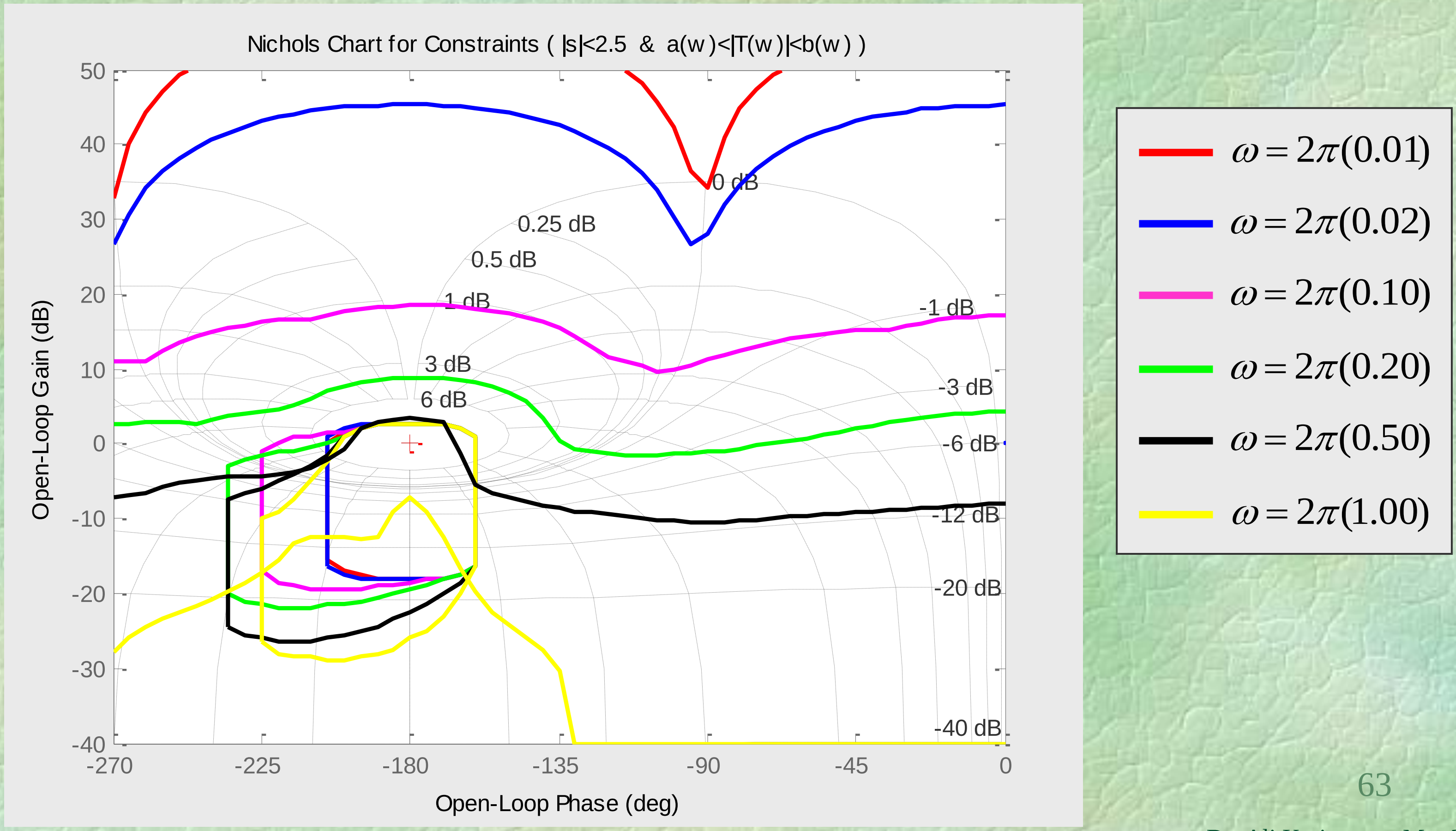
$$\text{Bounds for } a(\omega) < |T(\omega)| < b(\omega)$$

f	b/a
0.01	1.0011
0.02	1.0044
0.1	1.1141
0.2	1.5083
0.5	3.0155
1.0	17.770



QFT Design Procedure Example

Intersection of Bounds for $a(\omega) < |T(\omega)| < b(\omega)$ and $|S(\omega)| < 2.5$



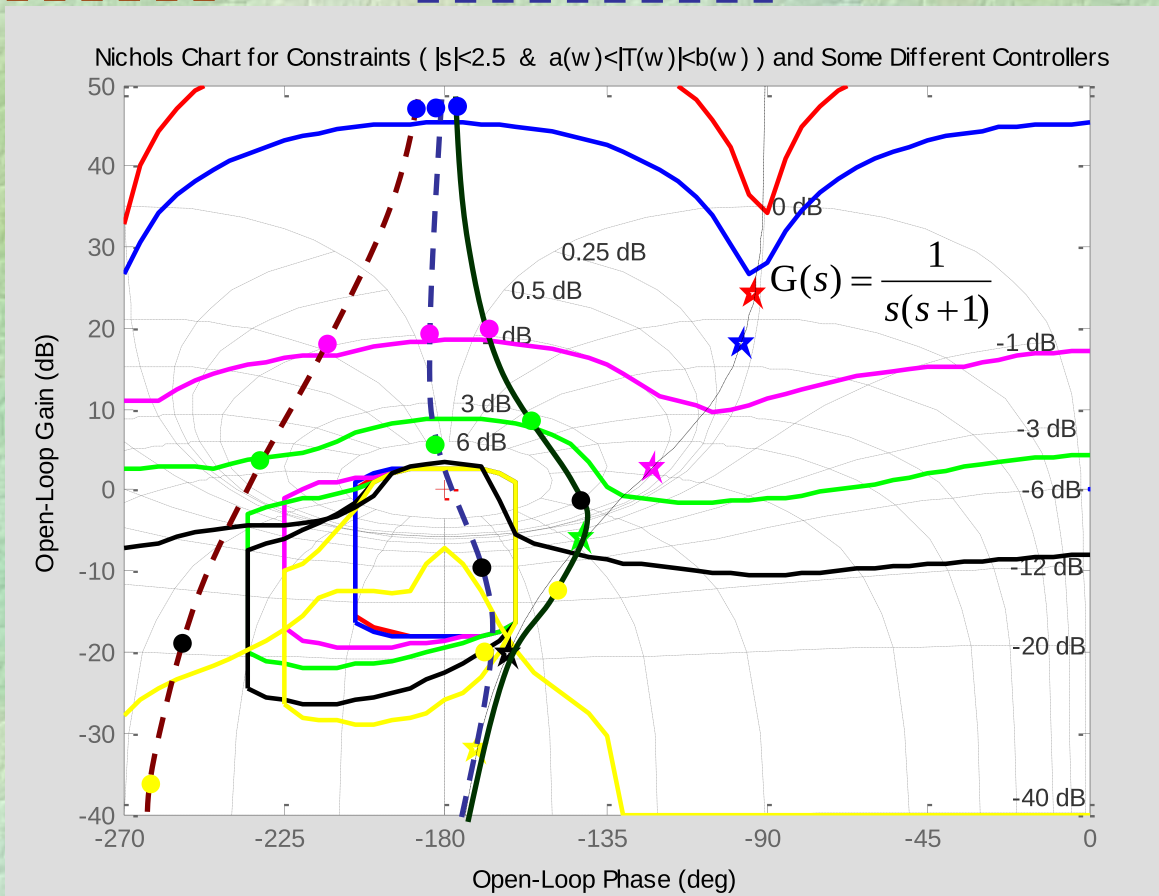
QFT Design Procedure Example

Three different controllers applied

$$K_1(s)=\frac{4}{s}$$

$$K_2(s)=\frac{5(s+2)^2}{s(s+5)}$$

$$K_3(s)=\frac{15(s+1)(s^2+3.6s+4)}{s(s+3)(s+5)}$$



$\omega=2\pi(0.01)$

$\omega=2\pi(0.02)$

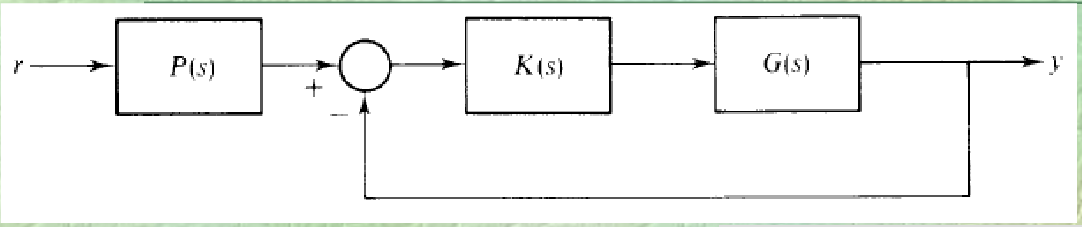
$\omega=2\pi(0.10)$

$\omega=2\pi(0.20)$

$\omega=2\pi(0.50)$

$\omega=2\pi(1.00)$

QFT Design Procedure Example



$$G(s) = \frac{ab}{s(s+a)}$$

$$a \in [1 \quad 4] \quad b \in [1 \quad 5]$$

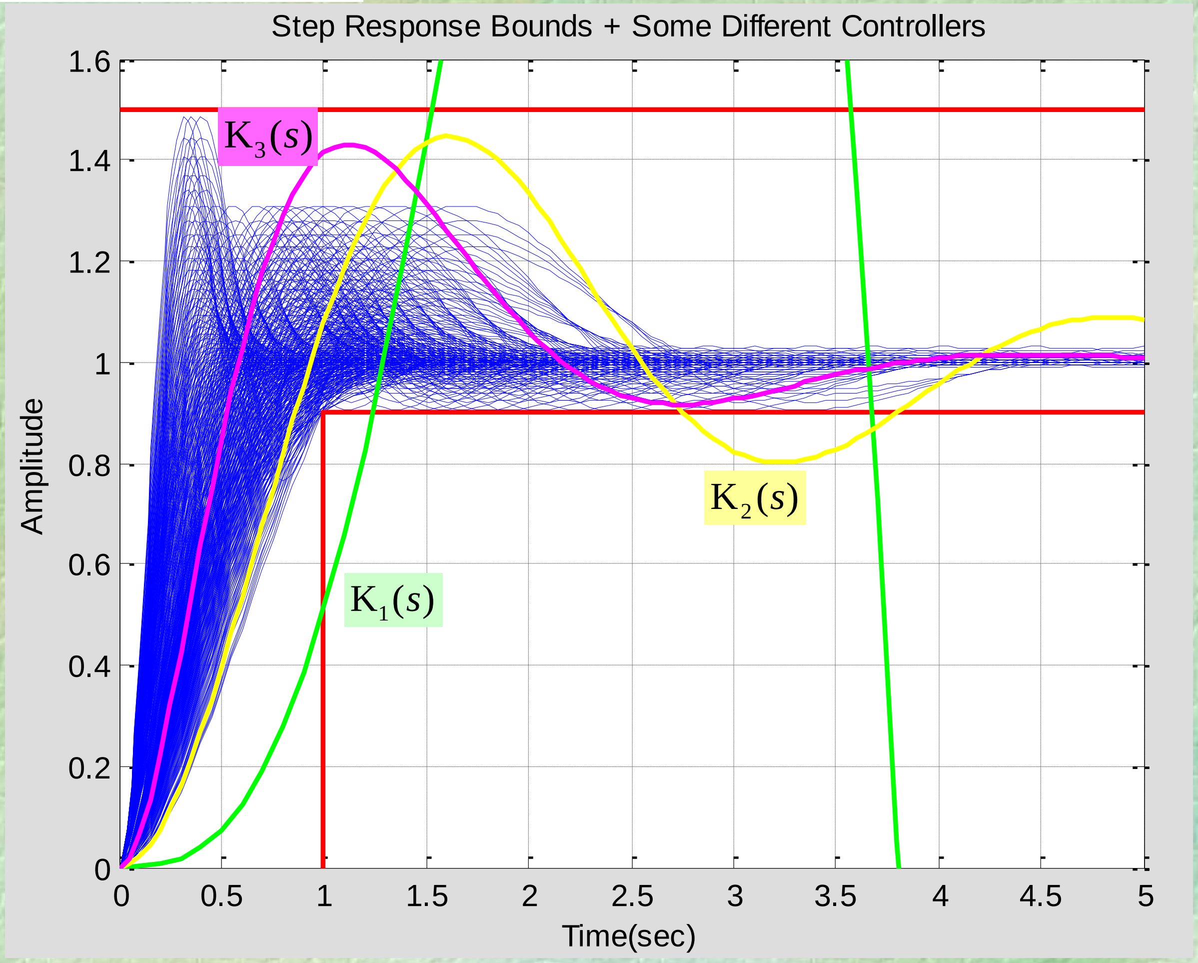
Three Different Controllers

$$\mathrm{P}(s) = 1$$

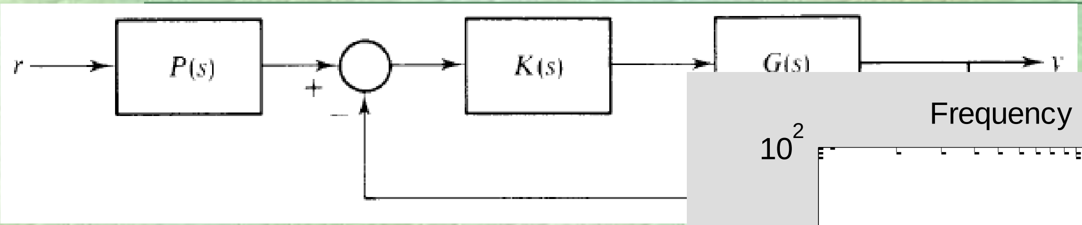
$$\mathrm{K}_1(s) = \frac{4}{s}$$

$$\mathrm{K}_2(s) = \frac{5(s+2)^2}{s(s+5)}$$

$$\mathrm{K}_3(s) = \frac{15(s+1)(s^2+3.6s+4)}{s(s+3)(s+5)}$$



QFT Design Procedure Example



$$G(s)=\frac{ab}{s(s+a)}$$

$$a\in[1\quad 4]\qquad b\in[1\quad 5]$$

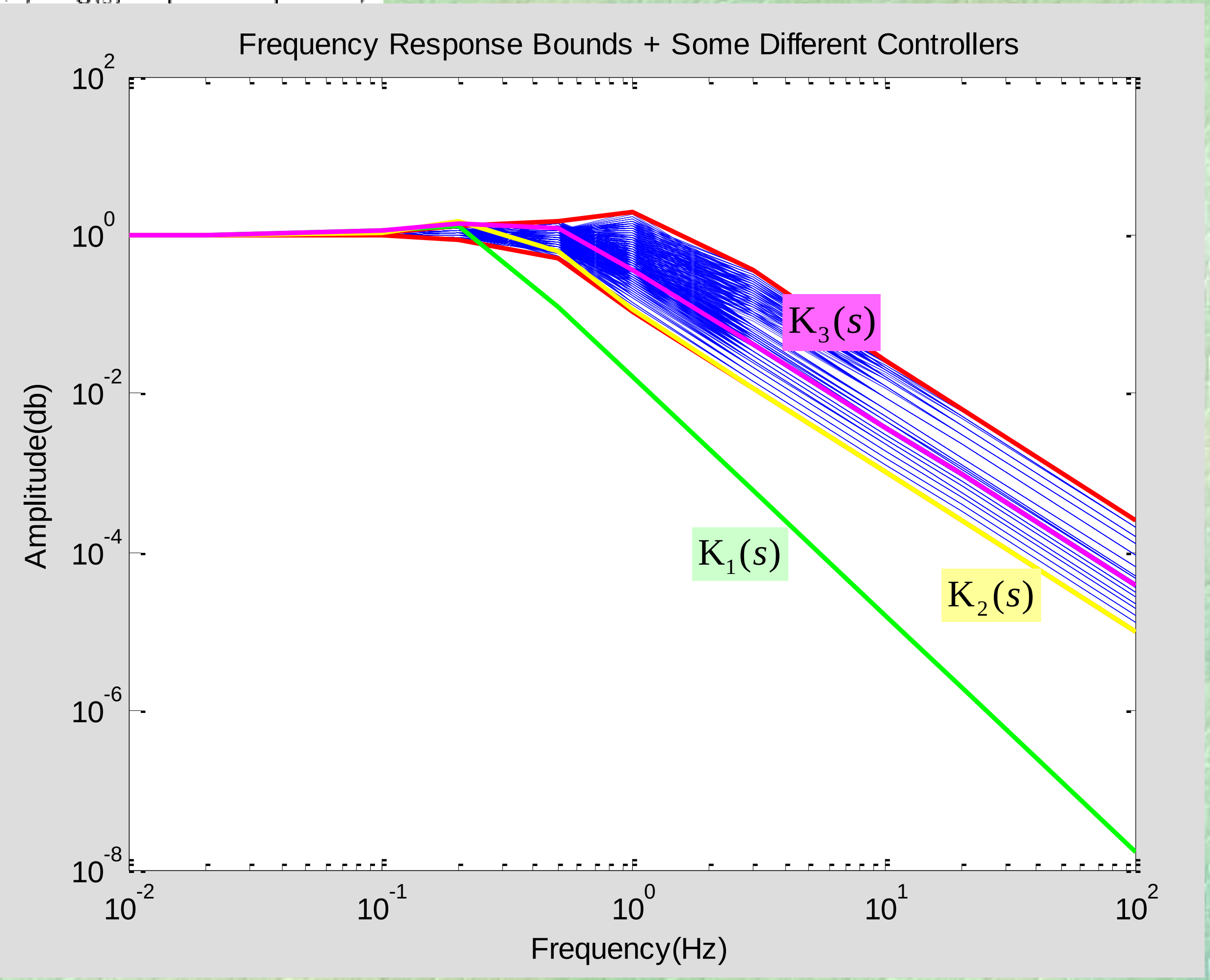
Three Different Controllers

$$\mathrm{P}(s)=1$$

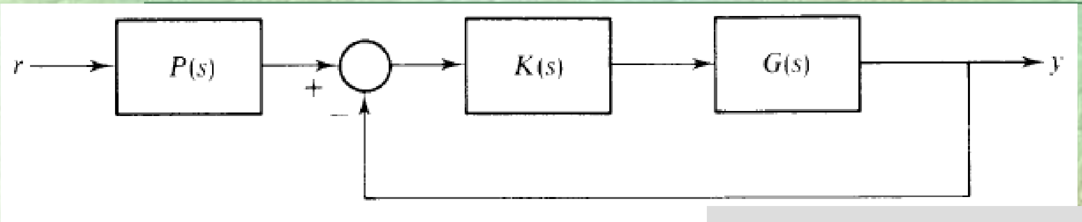
$$\mathrm{K}_1(s)=\frac{4}{s}$$

$$\mathrm{K}_2(s)=\frac{5(s+2)^2}{s(s+5)}$$

$$\mathrm{K}_3(s)=\frac{15(s+1)(s^2+3.6s+4)}{s(s+3)(s+5)}$$



QFT Design Procedure Example



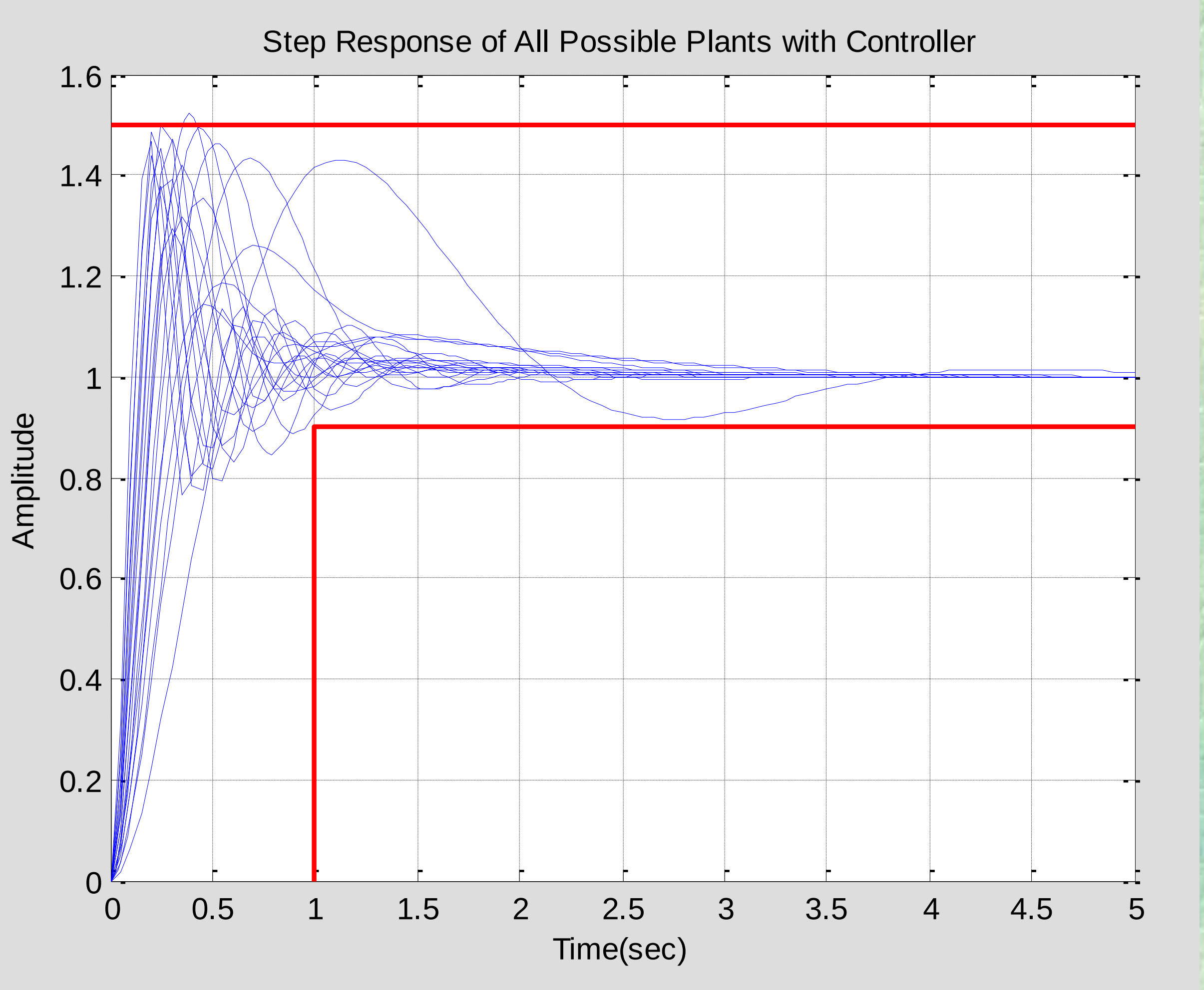
$$G(s) = \frac{ab}{s(s+a)}$$

$$a \in [1 \quad 4] \quad b \in [1 \quad 5]$$

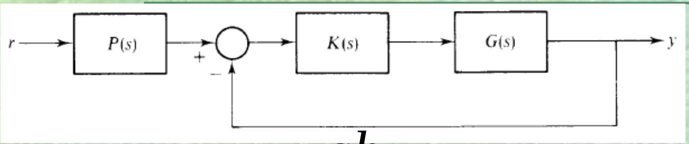
With Controller

$$\mathrm{P}(s) = 1$$

$$\mathrm{K}_3(s) = \frac{15(s+1)(s^2+3.6s+4)}{s(s+3)(s+5)}$$



QFT Design Procedure Example



$$G(s) = \frac{ab}{s(s+a)}$$

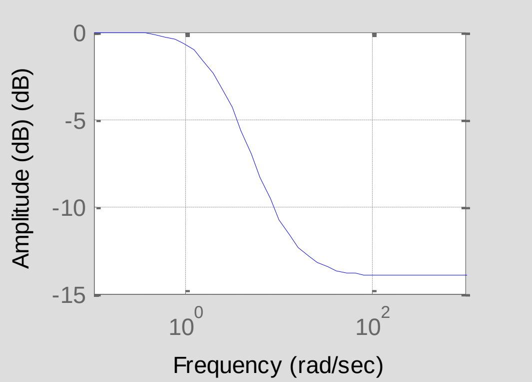
$$a \in [1 \quad 4] \quad b \in [1 \quad 5]$$

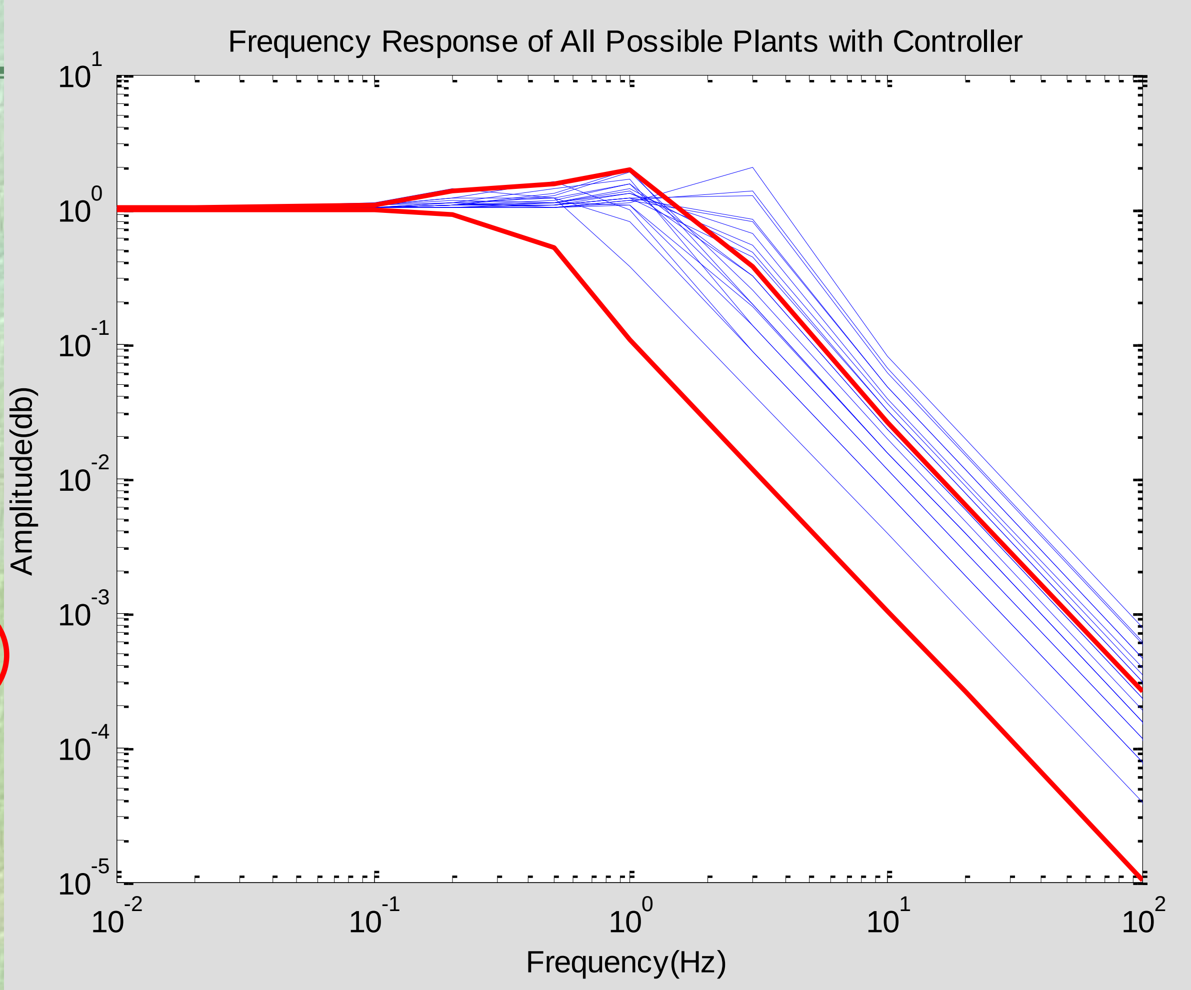
With Controller

$$K_3(s) = \frac{15(s+1)(s^2+3.6s+4)}{s(s+3)(s+5)} \qquad P(s) = 1$$

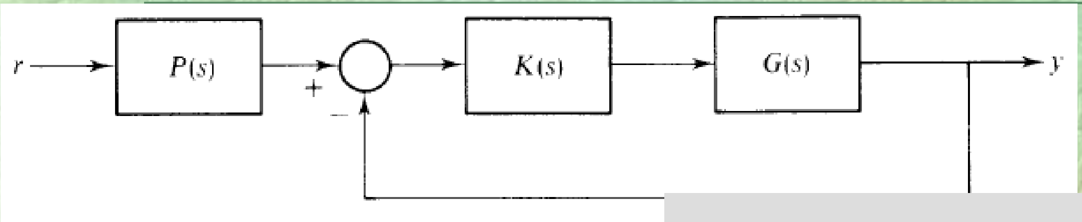
Let

$$P(s) = \frac{0.09s+1}{0.45s+1} \; ?$$





QFT Design Procedure Example



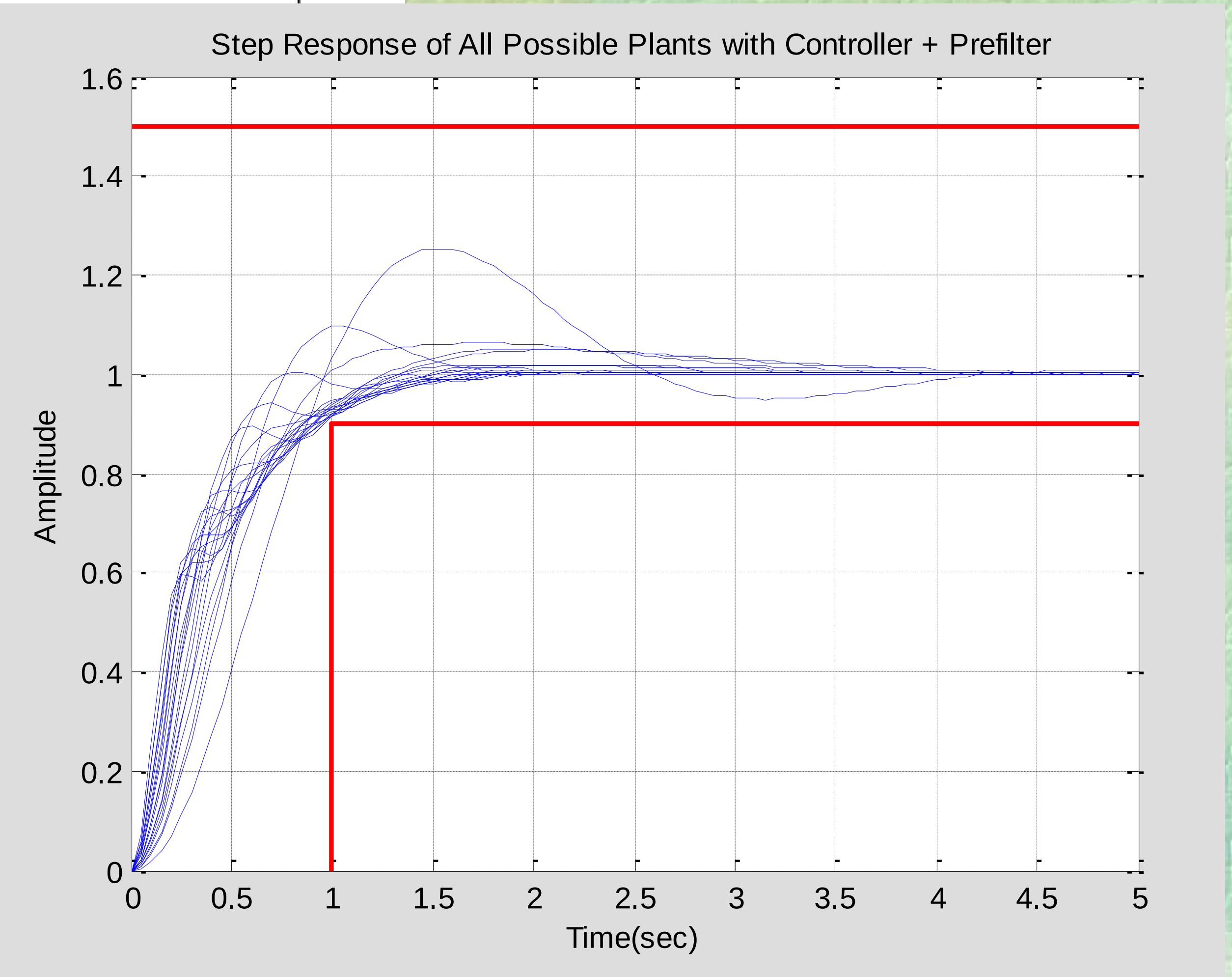
$$G(s) = \frac{ab}{s(s+a)}$$

$$a \in [1 \quad 4] \quad b \in [1 \quad 5]$$

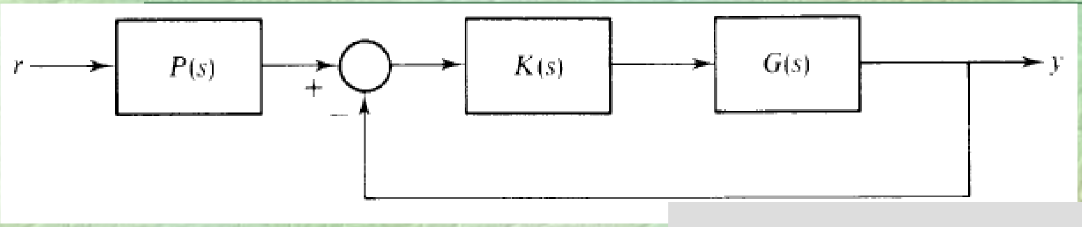
With Controller

$$\mathrm{P}(s) = \frac{0.09s+1}{0.45s+1}$$

$$\mathrm{K}_3(s) = \frac{15(s+1)(s^2+3.6s+4)}{s(s+3)(s+5)}$$



QFT Design Procedure Example



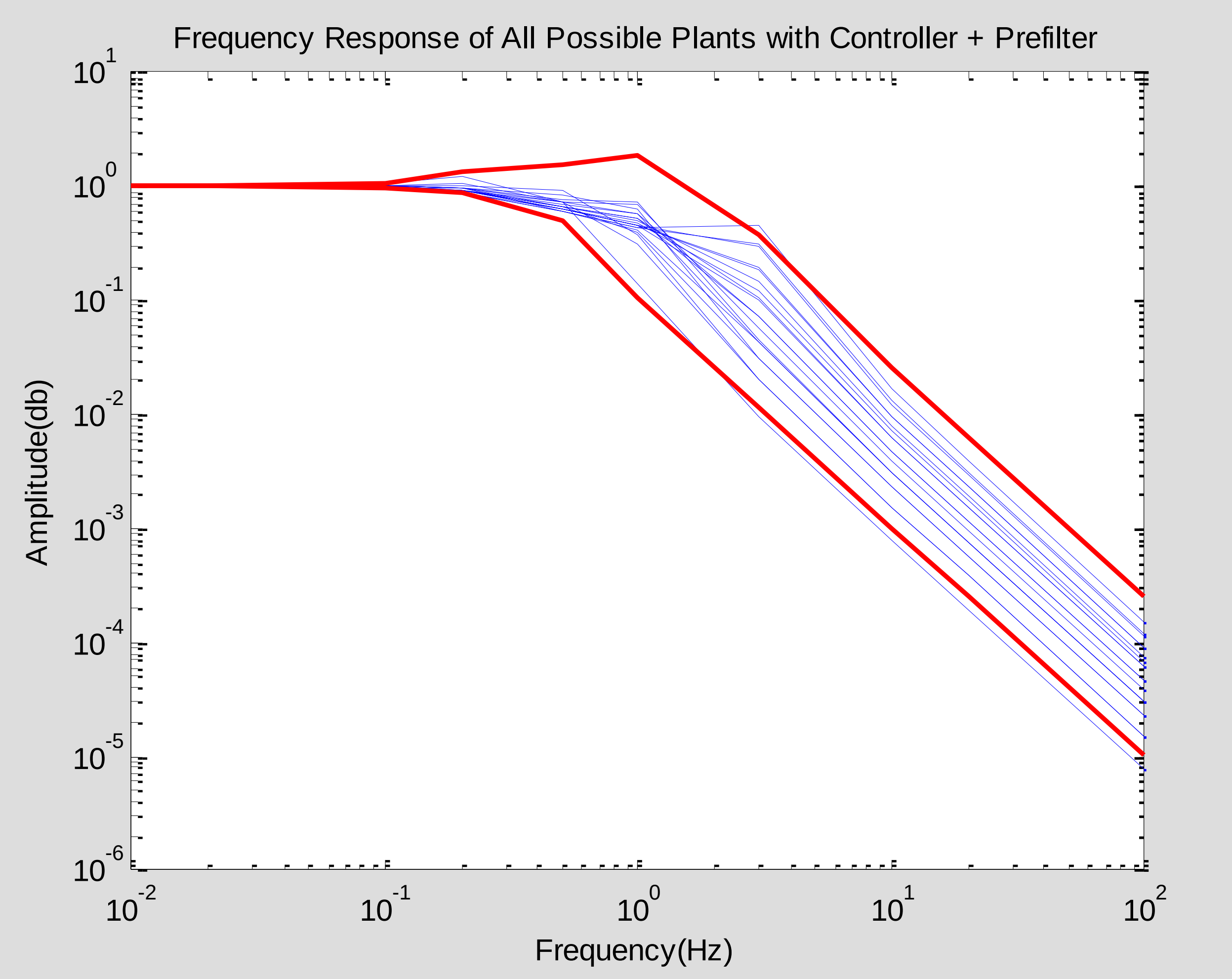
$$G(s) = \frac{ab}{s(s+a)}$$

$$a \in [1 \quad 4] \quad b \in [1 \quad 5]$$

With Controller

$$\mathrm{P}(s) = \frac{0.09s+1}{0.45s+1}$$

$$\mathrm{K}_3(s) = \frac{15(s+1)(s^2+3.6s+4)}{s(s+3)(s+5)}$$



Extension to MIMO systems

This design method is extended to multivariable problems as follows.

We are going to find t_{uv} (u^{th} output to v^{th} input)

$$(I + GK)y = GKP\,r$$

$$(\hat{G} + K)y = KP\,r$$

$$y_l = t_{lv} r_v$$

$$((\hat{G} + K)y)_u = \sum_l (\hat{g}_{ul} + k_{ul}) t_{lv} r_v$$

If $\quad r_j = 0 \quad$ for $\quad j \neq v$

$$(KP\,r)_u = (KP)_{uv} r_v = \sum_l k_{ul} p_{lv} r_v$$

Let $\quad k_{ij} = 0$

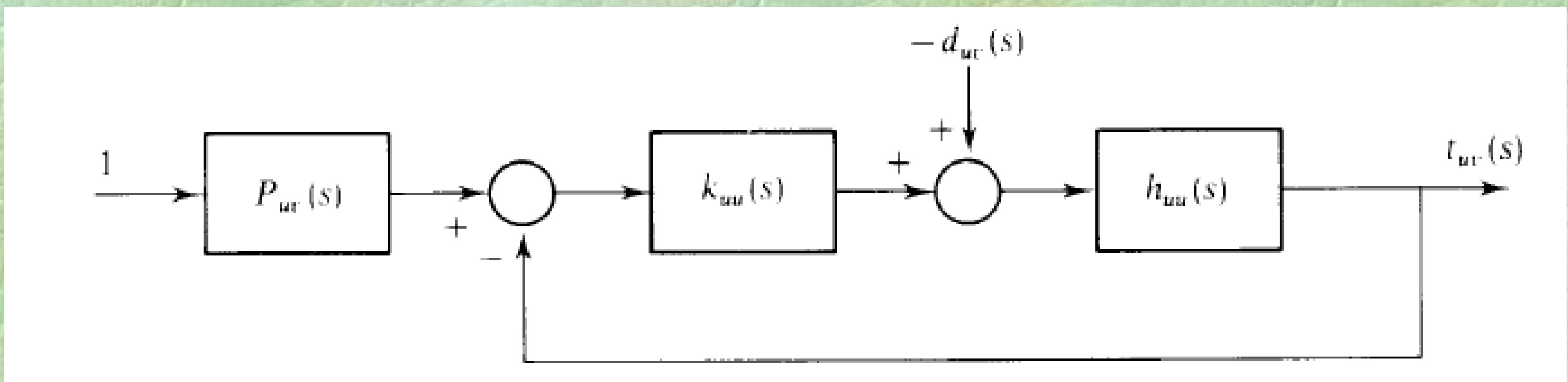
$$r_v \sum_{l \neq u} \hat{g}_{ul} t_{lv} + r_v (\hat{g}_{uu} + k_{uu}) t_{uv} = k_{uu} p_{uv} r_v$$

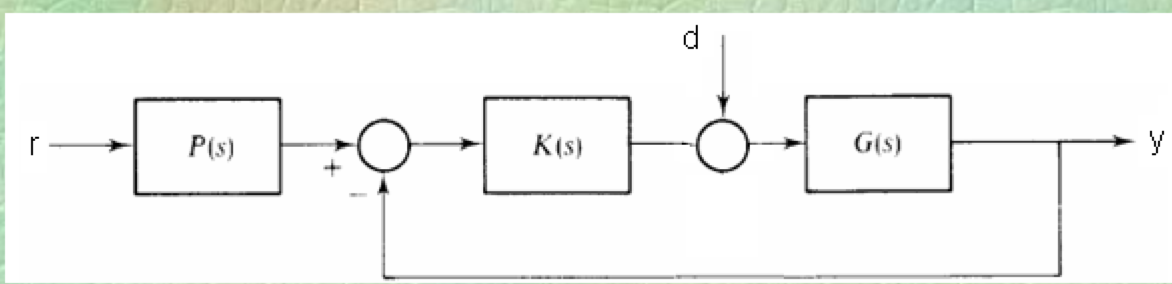
Let $\quad h_{ij} = 1/\hat{g}_{ij}$

$$t_{uv} = \frac{h_{uu} k_{uu} p_{uv}}{1 + h_{uu} k_{uu}} - \frac{h_{uu} d_{uv}}{1 + h_{uu} k_{uu}} \qquad d_{uv} = \sum_{l \neq u} \frac{t_{lv}}{h_{ul}}$$

Extension to MIMO systems

$$t_{uv} = \frac{h_{uu} k_{uu} p_{uv}}{1 + h_{uu} k_{uu}} - \frac{h_{uu} d_{uv}}{1 + h_{uu} k_{uu}}$$





$$y = \frac{GKP}{1+GK} r + \frac{G}{1+GK} d$$

$$t_{uv} = \frac{h_{uu} k_{uu} p_{uv}}{1 + h_{uu} k_{uu}} 1 - \frac{h_{uu}}{1 + h_{uu} k_{uu}} d_{uv}$$

$$d_{uv} = \sum_{l \neq u} \frac{t_{lv}}{h_{ul}}$$

Exercises

Exercise 10-1: Mentioned in the lecture.

Exercise 10-2: Mentioned in the lecture.

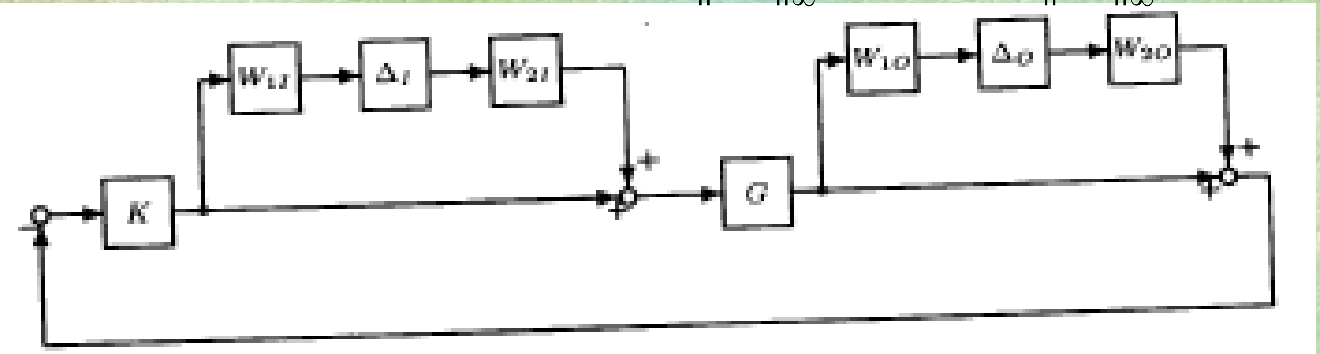
Exercise 10-3: Mentioned in the lecture.

Exercise 10-4: Mentioned in the lecture.

Exercise 10-5: Consider following block diagram. We have both input and output uncertainty.

a) Find the set of possible plants(G_p)

b) Find M and derive robust stability condition. ($\|\Delta_o\|_\infty \le 1$, and $\|\Delta_i\|_\infty \le 1$)



Exercise 10-6: Assume we have derived the following detailed model:

$$G_{detail}(s) = \frac{3(-0.5s+1)}{(2s+1)(0.1s+1)^2}$$

Suppose we chose G(s)=3/(2s+1) with multiplicative uncertainty. Derive suitable scaling Matrix.

References

References

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