Classical Tools in Structural Optimization:
Optimization Using Variational Calculus

(Based on the "Elements of Structural Optimization")

Introduction to the Calculus of Variations

Consider the problem of determining a function $y(x)$ given at two points, $y(a)=y_a$ and $y(b)=y_b$, (kinematic boundary conditions) for which the integral

$$J=\int_a^b F(x,y,y')dx \qquad (2.2.1)$$

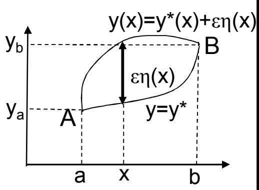
assumes a minimum or a maximum value.

Assuming $y^*(x)$ to be the function that minimizes our integral, consider another function $y(x)$ obtained by a small variation δy from $y^*(x)$,

$$y(x)=y^*(x)+\boxed{\delta y}=y^*(x)+\boxed{\varepsilon\eta(x)},\ y'=y^{*\prime}+\overbrace{\varepsilon\eta'}^{\delta y'} \qquad (2.2.2)$$

where ε is a small amplitude (perturbation) parameter and $\eta(x)$ a shape function.

From (2.2.2) we can evaluate $\eta(a)$:



$y(a)=y^*(a)+\varepsilon\eta(a)$, $\varepsilon\eta(a)=y(a)-y^*(a)=0$,

Therefore:

$\eta(a)=0$, and similarly $\eta(b)=0$, (2.2.3)

so that $y(a)$ and $y(b)$ will remain unchanged.

We substitute Eq. (2.2.2) into the integral (2.2.1), so that J becomes a function of only the perturbation parameter ε:

$$J(\varepsilon) = \int_a^b F(x, y^* + \varepsilon\eta, y^{*\prime} + \varepsilon\eta')dx \qquad (2.2.4)$$

Knowing that the value of the integral J attains an extremum for $\varepsilon=0$, one can use ordinary calculus to write the necessary condition

$$\left.\frac{dJ(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \int_a^b \left(\frac{\partial F}{\partial y}\frac{dy}{d\varepsilon} + \frac{\partial F}{\partial y'}\frac{dy'}{d\varepsilon}\right)dx = 0 \qquad (2.2.5)$$

Using Eq. (2.2.2) and defining ($\left.dJ/d\varepsilon\right|_{\varepsilon=0}$) to be the first variation of the functional J denoted by δJ we obtain

$$\delta = d/d\varepsilon \qquad \delta J = \int_a^b \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right)dx = 0 \qquad (2.2.6)$$

The property of interchangeability of the two operators

$$\varepsilon\eta' = \varepsilon\frac{d\eta}{dx} = \frac{d}{dx}\varepsilon\eta = \frac{d}{dx}\delta y = \delta\left(\frac{dy}{dx}\right) = \delta y' \qquad (2.2.7)$$

has been used in order to arrive at Eq. (2.2.6).

The variational operator δ is analogous to the differential operator in ordinary calculus, and the same rules that apply to the differential operator apply to the variational operator.

In the more general case F depends on more than one function and on higher order derivatives of these functions with respect to the independent variable x. For example, if

$$J = \int_a^b F(x, y_1, y_2, y_1', y_2', y_2'')dx \qquad (2.2.8)$$

then the condition that variation of the functional is zero may be written as

$$\delta J = \int_a^b \left(\frac{\partial F}{\partial y_1}\delta y_1 + \frac{\partial F}{\partial y_1'}\delta y_1' + \frac{\partial F}{\partial y_2}\delta y_2 + \frac{\partial F}{\partial y_2'}\delta y_2' + \frac{\partial F}{\partial y_2''}\delta y_2''\right)dx = 0. \qquad (2.2.9)$$

The necessary condition for extremum expressed in the form of Eq. (2.2.6) or (2.2.9) is usually not very useful. The terms that involve variation of derivatives can be integrated by parts in order to obtain more useful conditions. For example, integrating the second term of Eq. (2.2.6) $\delta J = \int_a^b \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right)dx = 0$ and rearranging we write (see the next slide)

$$\delta J = \frac{\partial F}{\partial y'}\delta y\Big|_a^b + \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y dx = 0. \qquad (2.2.10)$$

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$$\delta J = \int_a^b \left(\frac{\partial F}{\partial y}\delta y + \boxed{\frac{\partial F}{\partial y'}\delta y')dx}\right) = 0. \qquad (2.2.6)$$

$$\int_a^b v du = vu\Big|_a^b - \int_a^b u dv$$

$$\int_a^b \frac{\partial F}{\partial y'}\delta y' dx \xrightarrow[du = \delta y' dx \to u = \int \delta y' dx = \int \delta \frac{dy}{dx}dx = \delta y]{v = \frac{\partial F}{\partial y'} \to dv = \frac{d}{dx}\frac{\partial F}{\partial y'}dx}$$

$$= \frac{\partial F}{\partial y'}\delta y\Big|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y dx$$

$$\delta J = \frac{\partial F}{\partial y'}\delta y\Big|_a^b + \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y dx = 0. \qquad (2.2.10)$$

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For our problem the first term on the right hand side of Eq. (2.2.10) $(\frac{\partial F}{\partial y'}\delta y\Big|_a^b)$ vanishes due to the fact that the arbitrary function $\eta(x)$ satisfies the boundary conditions, $\eta(a)=\eta(b)=0$. By the definition of the variation it follows that

$$\delta y\,(a)=\frac{d}{d\,\varepsilon}y\,(a)=\frac{d}{d\,\varepsilon}[y\,^*(a)+\varepsilon\eta(a)]=\eta(a)=0$$

$$\delta y(a)=\delta y(b)=0. \qquad (2.2.11)$$

Thus, the necessary condition for the extremum of J reduces to

$$\delta J=\int_a^b\left[\frac{\partial F}{\partial y}-\frac{d}{dx}(\frac{\partial F}{\partial y'})\right]\delta y dx\;=0. \qquad (2.2.12)$$

Finally, since y is arbitrary(i.e. it is not constrained), we conclude that the coefficient of y in Eq. (2.2.12) must vanish identically over the interval of integration. Therefore, if $y(x)$ is to minimize (or maximize) J, it must satisfy the following condition, known as the Euler-Lagrange equation,

$$\frac{\partial F}{\partial y}-\frac{d}{dx}(\frac{\partial F}{\partial y'})=0. \qquad (2.2.13)$$

If the value of the unknown function is not specified at either or both ends, then the variation of $y(x)$ need not vanish at those points.
However, the first term on the right hand side of Eq. (2.2.10) must still vanish independently, in order for the relation to hold.
That is if $y(x)$ is not prescribed at the end points the following conditions, often called the natural boundary conditions, must be satisfied.

$$\left[\frac{\partial F}{\partial y'}\right]_{x=a}=0,\quad\text{and}\quad\left[\frac{\partial F}{\partial y'}\right]_{x=b}=0 \qquad (2.2.14)$$

بنابراین با این شرایط مرزی معادله $\frac{\partial F}{\partial y}-\frac{d}{dx}(\frac{\partial F}{\partial y'})=0$ همچنان جواب باقی خواهد ماند.

Example 2.2.1

روابط کتاب با فرض مرکز مختصات در مرکز نوشته شده است. در شکل کتاب مرکز مختصات در A است که باید تصحیح شود.

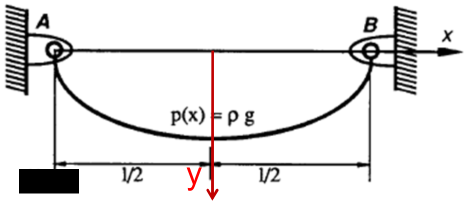


Figure 2.2.1 Supported cable under its own weight.

Consider the problem of determining the equilibrium configuration *y(x)* of a flexible, constant cross-section cable hanging under its own weight between two points, a distance l apart. This is a rather well-known fixed point problem of the calculus of variations.

The cable assumes a position that is consistent with its potential energy being a minimum. Hence, to determine the equilibrium shape *y(x)* we need to minimize the potential energy functional which can be expressed in terms of the unknown shape function as

$$J = \int \rho g y ds \qquad (2.2.15)$$

where ρg is the weight per unit length and ds is an element of arc length of the cable. Relating the arc length to the horizontal coordinate x, with the origin at the center, we rewrite Eq. (2.2.15) as

$$J = \rho g \int_{-l/2}^{l/2} y \sqrt{1+y'^2} dx \ . \qquad (2.2.16)$$

At this point one can either take the variation of Eq. (2.2.16) or, since this is a fixed-end-point problem, apply the Euler-Lagrange equation of Eq. (2.2.13)$\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0.\right)$ derived previously. The resulting necessary condition for the potential energy to be minimum reduces to the following ordinary differential equation

$$\sqrt{1+y'^2} - \frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right) = 0. \qquad (2.2.17)$$

$$\sqrt{1+y'^2}-\frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right)=0$$

$$\sqrt{1+y'^2}-\frac{(y'^2+yy'')\sqrt{1+y'^2}-\frac{\cancel{2}(y'y'')(yy')}{\cancel{2}\sqrt{1+y'^2}}}{1+y'^2}=0$$

$$\sqrt{1+y'^2}-\frac{(y'^2+yy'')(1+y'^2)-y'^2yy''}{(1+y'^2)\sqrt{1+y'^2}}=0$$

$$\frac{(1+y'^2)^2-[(y'^2+yy''+y'^4+\cancel{yy''y'^2})-\cancel{y'^2yy''}]}{(1+y'^2)\sqrt{1+y'^2}}=0$$

$$\frac{(1+\cancel{y'^4}+2y'^2)-(y'^2+yy''+\cancel{y'^4})}{(1+y'^2)\sqrt{1+y'^2}}=0$$

$$\frac{1+y'^2-yy''}{(1+y'^2)\sqrt{1+y'^2}}=0 \qquad yy''-y'^2-1=0 \qquad (2.2.18)$$

Introducing $dy/dx=t$ we can determine d^2y/dx^2 as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(t)=\frac{dt}{dy}\left(\frac{dy}{dx}\right)=t\frac{dt}{dy}$$

We rewrite Eq. (2.2.18) as

$$yy''-y'^2-1=y\left(t\frac{dt}{dy}\right)-t^2-1=0 \quad \text{or} \quad \frac{tdt}{t^2+1}=\frac{dy}{y} \qquad (2.2.19)$$

Integrating Eq. (2.2.19) once we obtain

$$\int\frac{xdx}{a+bx^2}=\frac{1}{2b}\ln\frac{a+bx^2}{b} \quad \therefore \int\frac{tdt}{t^2+1}=\int\frac{dy}{y}$$

$$\frac{1}{2}\ln\frac{1+t^2}{1}+C=\ln y \rightarrow \ln(1+t^2)=2\ln y-2C=\ln y^2-\ln c_1^{\,2}$$

$$1+t^2=\frac{y^2}{c_1^2}\rightarrow t^2=\frac{y^2}{c_1^2}-1 \quad \text{or} \quad t=\frac{dy}{dx}=\sqrt{\frac{y^2}{c_1^2}-1} \qquad (2.2.20)$$

Finally, one more integration yields

$$\frac{dy}{dx}=\sqrt{\frac{y^2}{c_1^2}-1}\rightarrow\frac{dy}{\sqrt{\frac{y^2}{c_1^2}-1}}=dx\,,\quad \text{if}\ \ \frac{y}{c_1}=u\ \ \text{then}\ dy=c_1du\ \ \text{and we have:}$$

$$\frac{c_1du}{\sqrt{u^2-1}}=dx\,.$$ We know: $$\int\frac{du}{\sqrt{u^2-a^2}}=\ln(u+\sqrt{u^2-a^2})$$, Therefore:

$$c_1\ln(u+\sqrt{u^2-1})=x+c_3\rightarrow\ln(u+\sqrt{u^2-1})=\frac{x}{c_1}+\frac{c_3}{c_1}$$

$$u+\sqrt{u^2-1}=e^{\frac{x}{c_1}+c_2}\rightarrow\sqrt{u^2-1}=e^{\frac{x}{c_1}+c_2}-u$$

$$\cancel{u^2}-1=e^{2(\frac{x}{c_1}+c_2)}+\cancel{u^2}-2ue^{(\frac{x}{c_1}+c_2)}\rightarrow\quad -1=e^{2(\frac{x}{c_1}+c_2)}-2ue^{(\frac{x}{c_1}+c_2)}$$

$$u=\frac{1}{2}(e^{(\frac{x}{c_1}+c_2)}+e^{-(\frac{x}{c_1}+c_2)})=\cosh(\frac{x}{c_1}+c_2)$$

$$y=c_1u=c_1\cosh(\frac{x}{c_1}+c_2),\qquad t=\frac{dy}{dx}=\sinh(\frac{x}{c_1}+c_2)\qquad (2.2.21)$$

$$y(x)=c_1\cosh(\frac{x}{c_1}+c_2)\,,\quad \boxed{t=\frac{dy}{dx}=\sinh(\frac{x}{c_1}+c_2)}\qquad (2.2.21)$$

The condition

$$\left.\frac{dy}{dx}\right|_0=0\qquad (2.2.22)$$

yields $c_2=0$, because: $\left.\frac{dy}{dx}\right|_0=\sinh(c_2)=0$

while c_1 can be determined from the condition

$$y(-l/2)=y(l/2)=c_1\cosh(\pm l/(2c_1))=0\qquad (2.2.23)$$

Equation (2.2.21) is the equation of a catenary.

2.3.2 Function Subjected to an Integral Constraint

For problems in which the unknown design variables are functions constrained by functionals, variational calculus also employs Lagrange multipliers.

Recall that for the supported cable problem the Euler-Lagrange equation was obtained by allowing the variation of the cable shape function δy to be arbitrary, or in other words by allowing *y(x)* to be completely unconstrained except for the kinematic boundary conditions.

However, if the function *y(x)* is required to satisfy a subsidiary integral equality constraint of the form

$$\int_a^b g\left[y(x)\right]dx = c \qquad (2.3.22)$$

then the extremum of the functional *J[y(x)]* can be determined by the use of the Lagrange multiplier technique.

In this case the necessary condition for an extremum is the vanishing of the first variation of an auxiliary functional

$$L = J + \lambda\left[\int_a^b g\left[y(x)\right]dx - c\right] \qquad (2.3.23)$$

In the following example we illustrate the use of this technique for determination of the cross-sectional area distribution of minimum weight beams for a specified displacement at a point along the span.

Example 2.3.2

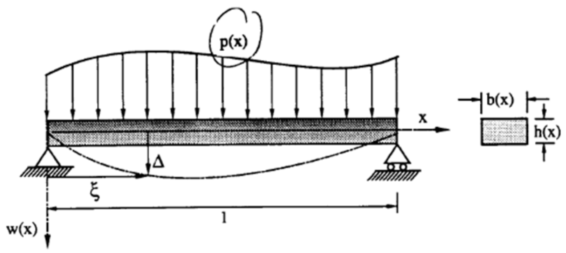


Figure 2.3.2 Design of beams for a specified displacement [1].

Consider a statically determinate beam of variable cross section *A(x)* loaded by a concentrated and/or distributed loads and moments which produce a moment distribution *M(x)* along the beam.

We want to minimize the volume *V* of the beam subject to the requirement that the displacement at a point $x=\xi$ is equal to a specified value Δ.

[1] Burnett, R.L., Minimum Weight Design of Beams for Deflection, J. EM Division ASCE, Vol EM1, 1961, pp. 75-95.

This problem, studied by Barnett [1], is formulated as

$$\text{Minimize} \qquad V = \int_0^l A(x)\,dx$$

$$\text{subject to} \qquad w(\xi) - \Delta = 0 \qquad (2.3.24)$$

A convenient expression for the displacement of a beam at a point $x=\xi$ is obtained again by using the method of virtual load discussed in the previous example problem. That is

$$w(\xi) = \int_0^l \frac{M(x)m(x)}{EI(x)}dx \qquad (2.3.25)$$

moment distribution

elastic modulus of the beam material

cross-sectional moment of inertia

moment distribution generated by a unit load applied at $x=\xi$

Since the cross-sectional area distribution function of the beam is the design variable, the moment of inertia term has to be expressed in terms of the area.

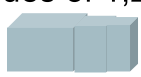
Commonly, the beam moment of inertia function is related to the cross-sectional area function as

$$I(x) = \alpha\left[A(x)\right]^n \qquad (2.3.26)$$

where α is a constant related to some physical dimension of the cross-section, and n is a constant that depends on the physical relation between the two functions.

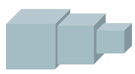
Here we limit the constant n to the integer values of 1,2, or 3.

$n=1$ is for a rectangular cross-section beam of constant depth whose width varies along the length. A plane-tapered beam.



$$I(x) = \alpha\left[A(x)\right]$$

$n=2$ is obtained when both the width and the depth of the cross-section vary without changing its aspect ratio.



$$I(x) = \alpha\left[A(x)\right]^2$$

$n=3$ is for a cross-section with a variable depth and a constant width. A depth-tapered beam.

$$I(x) = \alpha\left[A(x)\right]^3$$

The auxiliary functional for the minimization problem, Eq. (2.3.24) takes the following form.

$$L=\int_0^l A(x)dx+\lambda\left[\int_0^l \frac{M(x)m(x)}{EI(x)}dx-\Delta\right] \qquad (2.3.27)$$

The necessary condition for the constrained minimum is the vanishing of the first variation of this auxiliary functional. At this point we set $n=1$ in order to simplify the following derivation.
First variation of the second term is

$$\delta\int\frac{M(x)m(x)}{EI(x)}dx=\delta\int\frac{M(x)m(x)}{E\alpha A(x)}dx=-\int\frac{M(x)m(x)}{E\alpha A^2(x)}\delta A\,dx$$

Thus the first variation of Eq. (2.3.27) becomes

$$\delta L=\int_0^l\left[1-\lambda\frac{M(x)m(x)}{\alpha EA^2(x)}\right]\delta A\,dx=0 \qquad (2.3.28)$$

The corresponding Euler-Lagrange equation is

$$1-\lambda\frac{M(x)m(x)}{\alpha EA^2(x)}=0 \quad \text{or} \quad A(x)=\lambda^{\frac{1}{2}}\left(\frac{Mm}{\alpha E}\right)^{\frac{1}{2}} \qquad (2.3.29)$$

The unknown Lagrange multiplier in Eq. (2.3.29) must be determined from the displacement constraint in Eq. (2.3.24). That is, using Eqs. (2.3.25), (2.3.26), and (2.3.29) in Eq. (2.3.24) we can extract

$$w(\xi)=\int_0^l\frac{M(x)m(x)}{EI(x)}dx \qquad I(x)=\alpha\left[A(x)\right]^n \qquad w(\xi)-\Delta=0$$

$$\Delta=w(\xi)\rightarrow\Delta=\int_0^l\frac{M(x)m(x)}{EI(x)}dx=\int_0^l\frac{M(x)m(x)}{E\alpha A(x)}dx$$

$$=\int_0^l\frac{M(x)m(x)}{E\alpha\lambda^{\frac{1}{2}}\left(\frac{Mm}{\alpha E}\right)^{\frac{1}{2}}}dx=\frac{(\alpha E)^{1/2}}{\alpha E\,\lambda^{\frac{1}{2}}}\int[M(x)m(x)]^{1/2}dx\rightarrow$$

$$\lambda^{\frac{1}{2}}=\frac{1}{\Delta}\int_0^l\left(\frac{Mm}{\alpha E}\right)^{\frac{1}{2}}dx \qquad (2.3.30)$$

Then, the optimal area distribution and the corresponding volume are given by

$$A(x)=\lambda^{\frac{1}{2}}\left(\frac{Mm}{\alpha E}\right)^{\frac{1}{2}} \rightarrow A^{*}=[\frac{1}{\Delta}\int_{0}^{l}\left(\frac{M(\eta)m(\eta)}{\alpha E}\right)^{\frac{1}{2}}d\eta]\left(\frac{M(x)m(x)}{\alpha E}\right)^{\frac{1}{2}}$$

$$A^{*}(x)=\frac{1}{\alpha E\Delta}\left[\int_{0}^{l}\left(M(\eta)m(\eta)\right)^{\frac{1}{2}}d\eta\right][M(x)m(x)]^{\frac{1}{2}} \qquad (2.3.31)$$

and

$$V^{*}=\int_{0}^{l}A^{*}(x)dx=\int_{0}^{l}\left\{\frac{1}{\alpha E\Delta}\underbrace{\left[\int_{0}^{l}\left(M(\eta)m(\eta)\right)^{\frac{1}{2}}d\eta\right]}_{\text{Constant}}[M(x)m(x)]^{\frac{1}{2}}\right\}dx$$

$$=\frac{1}{\alpha E\Delta}\left\{\int_{0}^{l}[M(\eta)m(\eta)]^{\frac{1}{2}}d\eta\right\}\int_{0}^{l}[M(x)m(x)]^{\frac{1}{2}}dx$$

$$=\frac{1}{\alpha E\Delta}\left\{\int_{0}^{l}[M(\eta)m(\eta)]^{\frac{1}{2}}d\eta\right\}\int_{0}^{l}[M(\eta)m(\eta)]^{\frac{1}{2}}d\eta$$

$$V^{*}=\frac{1}{\alpha E\Delta}\left[\int_{0}^{l}\left(M(\eta)m(\eta)\right)^{\frac{1}{2}}d\eta\right]^{2} \qquad (2.3.32)$$

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2.3.3 Finite Subsidiary Conditions قیدهای تکمیلی دیگر

The problems discussed in the previous section involve a rather simple integral constraint that require a constant Lagrange multiplier in the auxiliary functional.

In a more general case, as mentioned earlier, we are interested in extremizing functionals of several functions and their derivatives with respect to more than one independent variable [e.g. $J=\int_{a}^{b}F(x,y_1,y_2,y_1',y_2',y_2'')dx$].

In addition, there may be m finite subsidiary constraints of the form

$$h_i\left(x_1,\cdots,x_n,y_1,\cdots,y_p,\frac{\partial y_1}{\partial x_1},\cdots,\frac{\partial y_p}{\partial x_n}\right)=0, \qquad i=1,\cdots,m \qquad (2.3.33)$$

imposed on the problem.

These constraints may range from simple algebraic equations to highly complicated differential equations that must be satisfied at every point over the entire domain of the problem.

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The Lagrange multiplier method, in this case, still reduces to extremizing an auxiliary functional of the form

$$L = \int_v \left(f + \sum_{i=1}^{m} \lambda_i h_i \right) dv \qquad (2.3.34)$$

The Lagrange multipliers, however, are no longer constants but functions of the coordinates $x_1,\ldots,x_n$.

Example 2.3.3

Design a cantilever beams of prescribed volume and prescribed loads for minimum deflection. Except for a slight change of notation, this example is based upon Makky and Ghalib's solution [2].

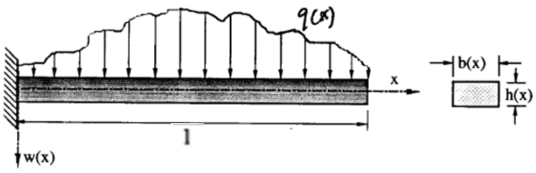

Figure 2.3.3 Optimum Design of a Beam for Minimum Deflection.

beam fixed at the end $x=0$, free at the end $x=l$,
acted upon by: transverse loading $q(x)$ per unit of length.
The objective is to minimize some norm of the transverse displacement of the beam for a given total volume, V_0.
The norm we choose is the integral of the transverse displacement w over the length of the beam. The loading $q(x)$ is restricted to be unidirectional in order to render(make) the norm appropriate.

[2] Makky, S.M. and Ghalib, M.A., Design for Minimum Deflection, Eng. Opt., pp 9-13, 1979.

The functional to be minimized: $\int_0^l w(x)dx$, w(x) is the displacement field which must satisfy the equation of equilibrium of the beam as well as the constraint on the total volume of material. The equation of equilibrium is expressed as

$$[s(x)w'']'' - q(x) = 0, \quad (2.3.35)$$

with boundary conditions

at x=0: $$w = 0 \quad \text{and} \quad w' = 0 \quad (2.3.36)$$

at x=l: $$\text{Moment: } sw'' = 0, \text{ and} \quad \text{Shear force}: (sw'')' = s'w'' + sw''' = 0 \quad (2.3.37)$$

s(x) being the bending stiffness of the beam that can be related, through Eq. (2.3.26), to the cross-sectional area of the beam by

$s(x)=EI(x)=\alpha EA^n(x)$, n=1,2, or 3. (2.3.38)

In addition to the subsidiary condition of Eq. (2.3.35), we must specify an integral constraint on the total volume, namely

$$\int_0^l A(x)dx = V_0 \quad (2.3.39)$$

$$[s(x)w'']'' = [s'w'' + sw''']' = [s''w'' + s'w''' + s'w''' + sw'''']$$
$$= [s''w'' + 2s'w''' + sw'''']$$

When the coordinate system and the direction of the load are downward:

$$-[s(x)w'']'' + q(x) = 0$$

The auxiliary functional is

$$L(w(x), s(x), A(x), \lambda_1, \lambda_2(x)) = \int_0^l w(x)dx + \lambda_1\left[\int_0^l A(x)dx - V_0\right]$$
$$-\int_0^l \lambda_2(x)\left[sw'''' + 2s'w''' + s''w'' - q\right]dx \quad (2.3.40)$$

which must be stationary with respect to the functions $w(x)$, $s(x)$, $A(x)$, $\lambda_2(x)$, and the parameter λ_1. (λ_1 is variable but not a function of x like $\lambda_2(x)$, and it comes out of integral. If it is in the integral, $\partial L/\partial\lambda_1=0$ requires that $V_0 = 0$, which is not acceptable.)

We note, however, that $A(x)$ depends on $s(x)$ through Eq. (2.3.38): [$s(x)=EI(x)=\alpha EA^n(x)$, n =1,2, or 3]. Hence,

$$\delta A = \left(\frac{dA}{ds}\right)\delta s \quad (2.3.41)$$

The first variation of *L*

$$\delta L = \int_0^l \delta w(x)dx + \lambda_1\left[\int_0^l \delta A(x)dx\right] + \delta\lambda_1\left[\int_0^l A dx - V_0\right]$$
$$- \int_0^l \delta\lambda_2(x)\left[s w'''' + 2s' w''' + s'' w'' - q\right]dx \qquad (2.3.42)$$
$$- \int_0^l \lambda_2(x)\left[\delta s w'''' + \underline{s\,\delta w''''} + 2\delta s' w''' + 2s'\delta w''' + \delta s'' w'' + s''\delta w''\right]dx = 0$$

can be simplified by several integrations by parts.

$$-\int_0^l \lambda_2 s\,\delta w'''' dx = -\lambda_2 s\,\delta w'''\Big|_0^l + \int_0^l \frac{d}{dx}(\lambda_2 s)\delta w''' dx$$
$$\frac{d}{dx}(\lambda_2 s)\delta w''\Big|_0^l - \int_0^l \frac{d^2}{dx^2}(\lambda_2 s)\delta w'' dx$$
$$-\frac{d^2}{dx^2}(\lambda_2 s)\delta w'\Big|_0^l + \int_0^l \frac{d^3}{dx^3}(\lambda_2 s)\delta w' dx$$
$$= -\lambda_2 s\,\delta w'''\Big|_0^l + \frac{d}{dx}(\lambda_2 s)\delta w''\Big|_0^l - \frac{d^2}{dx^2}(\lambda_2 s)\delta w'\Big|_0^l + \frac{d^3}{dx^3}(\lambda_2 s)\delta w\Big|_0^l - \int_0^l \frac{d^4}{dx^4}(\lambda_2 s)\delta w\, dx$$

The first variation of *L*

$$\delta L = \int_0^l \delta w(x)dx + \lambda_1\left[\int_0^l \delta A(x)dx\right] + \delta\lambda_1\left[\int_0^l A dx - V_0\right]$$
$$- \int_0^l \delta\lambda_2(x)\left[s w'''' + 2s' w''' + s'' w'' - q\right]dx \qquad (2.3.42)$$
$$- \int_0^l \lambda_2(x)\left[\delta s w'''' + s\,\delta w'''' + \underline{2\delta s' w'''} + 2s'\delta w''' + \delta s'' w'' + s''\delta w''\right]dx = 0$$

can be simplified by several integrations by parts.

$$-\int_0^l 2\lambda_2 w'''\delta s' dx = 2\left(-\lambda_2 w'''\delta s\Big|_0^l + \int_0^l \frac{d}{dx}(\lambda_2 w''')\delta s\, dx\right)$$

The first variation of *L*

$$\delta L=\int_0^l \delta w(x)dx+\lambda_1\left[\int_0^l \delta A(x)dx\right]+\delta\lambda_1\left[\int_0^l A dx-V_0\right]$$
$$-\int_0^l \delta\lambda_2(x)\left[sw''''+2s'w'''+s''w''-q\right]dx \qquad (2.3.42)$$
$$-\int_0^l \lambda_2(x)\left[\delta s w''''+s\,\delta w''''+2\delta s'w'''+\underline{2s'\delta w'''}+\delta s''w''+s''\delta w''\right]dx=0$$

can be simplified by several integrations by parts.

$$-\int_0^l 2\lambda_2 s'\delta w'''dx=2(-\lambda_2 s'\delta w''\Big|_0^l+\int_0^l \frac{d}{dx}(\lambda_2 s')\delta w''dx)$$
$$2(\frac{d}{dx}(\lambda_2 s')\delta w'\Big|_0^l-\int_0^l \frac{d^2}{dx^2}(\lambda_2 s')\delta w'dx)$$
$$2(-\frac{d^2}{dx^2}(\lambda_2 s')\delta w\Big|_0^l+\int_0^l \frac{d^3}{dx^3}(\lambda_2 s')\delta w\,dx)$$
$$=2(-\lambda_2 s'\delta w''\Big|_0^l+\frac{d}{dx}(\lambda_2 s')\delta w'\Big|_0^l-\frac{d^2}{dx^2}(\lambda_2 s')\delta w\Big|_0^l+\int_0^l \frac{d^3}{dx^3}(\lambda_2 s')\delta w\,dx)$$

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The first variation of *L*

$$\delta L=\int_0^l \delta w(x)dx+\lambda_1\left[\int_0^l \delta A(x)dx\right]+\delta\lambda_1\left[\int_0^l A dx-V_0\right]$$
$$-\int_0^l \delta\lambda_2(x)\left[sw''''+2s'w'''+s''w''-q\right]dx \qquad (2.3.42)$$
$$-\int_0^l \lambda_2(x)\left[\delta s w''''+s\,\delta w''''+2\delta s'w'''+2s'\delta w'''+\underline{\delta s''w''}+s''\delta w''\right]dx=0$$

can be simplified by several integrations by parts.

$$-\int_0^l \lambda_2 w''\delta s''dx=-\lambda_2 w''\delta s'\Big|_0^l+\int_0^l \frac{d}{dx}(\lambda_2 w'')\delta s'dx$$
$$\frac{d}{dx}(\lambda_2 w'')\delta s\Big|_0^l-\int_0^l \frac{d^2}{dx^2}(\lambda_2 w'')\delta s dx$$
$$=-\lambda_2 w''\delta s'\Big|_o^l+\frac{d}{dx}(\lambda_2 w'')\delta s\Big|_o^l-\int_0^l \frac{d^2}{dx^2}(\lambda_2 w'')\delta s dx$$

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The first variation of *L*

$$\delta L=\int_0^l \delta w(x)dx+\lambda_1\left[\int_0^l \delta A(x)dx\right]+\delta\lambda_1\left[\int_0^l A dx-V_0\right]$$
$$-\int_0^l \delta\lambda_2(x)\left[sw''''+2s'w'''+s''w''-q\right]dx \qquad (2.3.42)$$
$$-\int_0^l \lambda_2(x)\left[\delta s w''''+s\,\delta w''''+2\delta s' w'''+2s'\delta w'''+\delta s'' w''+\underline{s''\delta w''}\right]dx=0$$

can be simplified by several integrations by parts.

$$-\int_0^l \lambda_2 s''\delta w'' dx=-\lambda_2 s''\delta w'\Big|_0^l+\boxed{\int_0^l \frac{d}{dx}(\lambda_2 s'')\delta w' dx}$$
$$\boxed{\frac{d}{dx}(\lambda_2 s'')\delta w\Big|_0^l-\int_0^l \frac{d^2}{dx^2}(\lambda_2 s'')\delta w\, dx}$$
$$=-\lambda_2 s''\delta w'\Big|_0^l+\frac{d}{dx}(\lambda_2 s'')\delta w\Big|_0^l-\int_0^l \frac{d^2}{dx^2}(\lambda_2 s'')\delta w\, dx$$

Collecting the terms multiplied by arbitrary variations δw, δs, $\delta\lambda_1$, and $\delta\lambda_2$ and equating them to zero independently we obtain the following Euler-Lagrange equations:

$$\delta L=\int_0^l \underline{\delta w\,(x)dx}+\lambda_1\left[\int_0^l\left(\frac{dA}{ds}\right)\delta s\,dx\right]+\delta\lambda_1\left[\int_0^l A dx-V_0\right]$$

$$-\int_0^l \delta\lambda_2(x)\left[sw''''+2s'w'''+s''w''-q\right]dx-\int_0^l \lambda_2(\delta s w'''')dx$$

$$-\lambda_2 s\,\delta w'''\Big|_0^l+\frac{d}{dx}(\lambda_2 s)\delta w''\Big|_0^l-\frac{d^2}{dx^2}(\lambda_2 s)\delta w'\Big|_0^l+\frac{d^3}{dx^3}(\lambda_2 s)\delta w\Big|_0^l-\int_0^l\frac{d^4}{dx^4}(\lambda_2 s)\underline{\delta w}\,dx$$

$$+2\left(-\lambda_2 s'\delta w''\Big|_0^l+\frac{d}{dx}(\lambda_2 s')\delta w'\Big|_0^l-\frac{d^2}{dx^2}(\lambda_2 s')\delta w\Big|_0^l+\int_0^l\frac{d^3}{dx^3}(\lambda_2 s')\underline{\delta w}\,dx\right)$$

$$+2\left(-\lambda_2 w'''\,\delta s\Big|_0^l+\int_0^l\frac{d}{dx}(\lambda_2 w''')\delta s dx\right)$$

$$-\lambda_2 w''\,\delta s'\Big|_o^l+\frac{d}{dx}(\lambda_2 w'')\delta s\Big|_o^l-\int_0^l\frac{d^2}{dx^2}(\lambda_2 w'')\delta s dx$$

$$-\lambda_2 s''\delta w'\Big|_0^l+\frac{d}{dx}(\lambda_2 s'')\delta w\Big|_0^l-\int_0^l\frac{d^2}{dx^2}(\lambda_2 s'')\underline{\delta w}\,dx=0$$

$$\delta w=0:\quad 1-\frac{d^4}{dx^4}(\lambda_2 s)+2\frac{d^3}{dx^3}(\lambda_2 s')-\frac{d^2}{dx^2}(\lambda_2 s'')=0$$

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$$\delta w=0:\quad 1-\frac{d^4}{dx^4}(\lambda_2 s)+2\frac{d^3}{dx^3}(\lambda_2 s')-\frac{d^2}{dx^2}(\lambda_2 s'')=0$$

$$1-(\lambda_2''''s+\lambda_2 s''''+4\lambda_2'''s'+4\lambda_2's'''+6\lambda_2''s'')+$$
$$2(\lambda_2 s''''+3\lambda_2's'''+3\lambda_2''s''+\lambda_2'''s')-(\lambda_2''s''+2\lambda_2's'''+\lambda_2 s'''')=0$$

$$1-\lambda_2''''s-\cancel{\lambda_2 s''''}-4\lambda_2'''s'-\cancel{4\lambda_2's'''}-\underline{6\lambda_2''s''}+$$
$$\cancel{2\lambda_2 s''''}+\cancel{6\lambda_2's'''}+\underline{6\lambda_2''s''}+2\lambda_2'''s'-\underline{\lambda_2''s''}-\cancel{2\lambda_2's'''}-\cancel{\lambda_2 s''''}=0$$

$$1-\lambda_2''''s-2\lambda_2'''s'-\lambda_2''s''=1-(\lambda_2''s)''=0 \qquad (2.3.43)$$

$$\delta L=\int_0^l \delta w(x)dx+\lambda_1\left[\int_0^l\left(\frac{dA}{ds}\right)\delta s\,dx\right]+\delta\lambda_1\left[\int_0^l A dx-V_0\right]$$

$$-\int_0^l \delta\lambda_2(x)\left[sw''''+2s'w'''+s''w''-q\right]dx-\int_0^l \lambda_2(\delta s w'''')dx$$

$$-\lambda_2 s\,\delta w'''\Big|_0^l+\frac{d}{dx}(\lambda_2 s)\delta w''\Big|_0^l-\frac{d^2}{dx^2}(\lambda_2 s)\delta w'\Big|_0^l+\frac{d^3}{dx^3}(\lambda_2 s)\delta w\Big|_0^l-\int_0^l\frac{d^4}{dx^4}(\lambda_2 s)\delta w\,dx$$

$$+2\left(-\lambda_2 s'\delta w''\Big|_0^l+\frac{d}{dx}(\lambda_2 s')\delta w'\Big|_0^l-\frac{d^2}{dx^2}(\lambda_2 s')\delta w\Big|_0^l+\int_0^l\frac{d^3}{dx^3}(\lambda_2 s')\delta w\,dx\right)$$

$$+2\left(-\lambda_2 w'''\,\delta s\Big|_0^l+\int_0^l\frac{d}{dx}(\lambda_2 w''')\delta s\,dx\right)$$

$$-\lambda_2 w''\,\delta s'\Big|_o^l+\frac{d}{dx}(\lambda_2 w'')\,\delta s\Big|_o^l-\int_0^l\frac{d^2}{dx^2}(\lambda_2 w'')\delta s\,dx$$

$$-\lambda_2 s''\delta w'\Big|_0^l+\frac{d}{dx}(\lambda_2 s'')\delta w\Big|_0^l-\int_0^l\frac{d^2}{dx^2}(\lambda_2 s'')\delta w\,dx=0$$

$$\delta s=0:\quad \lambda_1\frac{dA}{ds}-\lambda_2 w''''+2\frac{d}{dx}(\lambda_2 w''')-\frac{d^2}{dx^2}(\lambda_2 w'')=0$$

$$\delta s=0:\quad \lambda_1\frac{dA}{ds}-\lambda_2 w''''+2\frac{d}{dx}(\lambda_2 w''')-\frac{d^2}{dx^2}(\lambda_2 w'')=0$$

$$\lambda_1\frac{dA}{ds}-\lambda_2 w''''+2(\lambda_2' w'''+\lambda_2 w'''')-(\lambda_2' w''+\lambda_2 w''')'=$$

$$\lambda_1\frac{dA}{ds}-\cancel{\lambda_2 w''''}+2(\cancel{\lambda_2' w'''}+\cancel{\lambda_2 w''''})-(\lambda_2'' w''+\cancel{\lambda_2' w'''}+\cancel{\lambda_2' w'''}+\cancel{\lambda_2 w''''})=0$$

$$\lambda_1\frac{dA}{ds}-\lambda_2'' w''=0 \qquad (2.3.44)$$

$$\delta L=\int_0^l \delta w(x)dx+\lambda_1\left[\int_0^l\left(\frac{dA}{ds}\right)\delta s\,dx\right]+\underline{\delta\lambda_1}\left[\int_0^l A dx-V_0\right]$$
$$-\int_0^l \delta\lambda_2(x)\left[sw''''+2s'w'''+s''w''-q\right]dx-\int_0^l \lambda_2(\delta s w'''')dx$$
$$-\lambda_2 s\,\delta w'''\Big|_0^l+\frac{d}{dx}(\lambda_2 s)\delta w''\Big|_0^l-\frac{d^2}{dx^2}(\lambda_2 s)\delta w'\Big|_0^l+\frac{d^3}{dx^3}(\lambda_2 s)\delta w\Big|_0^l-\int_0^l\frac{d^4}{dx^4}(\lambda_2 s)\delta w\,dx$$
$$+2(-\lambda_2 s'\delta w''\Big|_0^l+\frac{d}{dx}(\lambda_2 s')\delta w'\Big|_0^l-\frac{d^2}{dx^2}(\lambda_2 s')\delta w\Big|_0^l+\int_0^l\frac{d^3}{dx^3}(\lambda_2 s')\delta w\,dx)$$
$$+2(-\lambda_2 w'''\delta s\Big|_0^l+\int_0^l\frac{d}{dx}(\lambda_2 w''')\delta s dx)$$
$$-\lambda_2 w''\delta s'\Big|_o^l+\frac{d}{dx}(\lambda_2 w'')\delta s\Big|_o^l-\int_0^l\frac{d^2}{dx^2}(\lambda_2 w'')\delta s dx$$
$$-\lambda_2 s''\delta w'\Big|_0^l+\frac{d}{dx}(\lambda_2 s'')\delta w\Big|_0^l-\int_0^l\frac{d^2}{dx^2}(\lambda_2 s'')\delta w\,dx=0$$
$$\delta\lambda_1=0:\quad \int_0^l A dx-V_0=0 \qquad (2.3.45)$$

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$$\delta L=\int_0^l \delta w(x)dx+\lambda_1\left[\int_0^l\left(\frac{dA}{ds}\right)\delta s\,dx\right]+\delta\lambda_1\left[\int_0^l A dx-V_0\right]$$
$$-\int_0^l \underline{\delta\lambda_2(x)}\left[sw''''+2s'w'''+s''w''-q\right]dx-\int_0^l \lambda_2(\delta s w'''')dx$$
$$-\lambda_2 s\,\delta w'''\Big|_0^l+\frac{d}{dx}(\lambda_2 s)\delta w''\Big|_0^l-\frac{d^2}{dx^2}(\lambda_2 s)\delta w'\Big|_0^l+\frac{d^3}{dx^3}(\lambda_2 s)\delta w\Big|_0^l-\int_0^l\frac{d^4}{dx^4}(\lambda_2 s)\delta w\,dx$$
$$+2(-\lambda_2 s'\delta w''\Big|_0^l+\frac{d}{dx}(\lambda_2 s')\delta w'\Big|_0^l-\frac{d^2}{dx^2}(\lambda_2 s')\delta w\Big|_0^l+\int_0^l\frac{d^3}{dx^3}(\lambda_2 s')\delta w\,dx)$$
$$+2(-\lambda_2 w'''\delta s\Big|_0^l+\int_0^l\frac{d}{dx}(\lambda_2 w''')\delta s dx)$$
$$-\lambda_2 w''\delta s'\Big|_o^l+\frac{d}{dx}(\lambda_2 w'')\delta s\Big|_o^l-\int_0^l\frac{d^2}{dx^2}(\lambda_2 w'')\delta s dx$$
$$-\lambda_2 s''\delta w'\Big|_0^l+\frac{d}{dx}(\lambda_2 s'')\delta w\Big|_0^l-\int_0^l\frac{d^2}{dx^2}(\lambda_2 s'')\delta w\,dx=0$$
$$\delta\lambda_2=0:\quad sw''''+2s'w'''+s''w''-q=0 \qquad (2.3.46)$$

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$$\delta L = \int_0^l \delta w(x)\,dx + \lambda_1\left[\int_0^l\left(\frac{dA}{ds}\right)\delta s\,dx\right] + \delta\lambda_1\left[\int_0^l A\,dx - V_0\right]$$

$$-\int_0^l \delta\lambda_2(x)\left[sw'''' + 2s'w''' + s''w'' - q\right]dx - \int_0^l \lambda_2(\delta s w'''')dx$$

$$-\lambda_2 s\,\delta w'''\Big|_0^l + \frac{d}{dx}(\lambda_2 s)\delta w''\Big|_0^l - \frac{d^2}{dx^2}(\lambda_2 s)\delta w'\Big|_0^l + \frac{d^3}{dx^3}(\lambda_2 s)\delta w\Big|_0^l - \int_0^l \frac{d^4}{dx^4}(\lambda_2 s)\delta w\,dx$$

$$+2\left(-\lambda_2 s'\delta w''\Big|_0^l + \frac{d}{dx}(\lambda_2 s')\delta w'\Big|_0^l - \frac{d^2}{dx^2}(\lambda_2 s')\delta w\Big|_0^l + \int_0^l \frac{d^3}{dx^3}(\lambda_2 s')\delta w\,dx\right)$$

$$+2\left(-\lambda_2 w'''\underline{\delta s}\Big|_0^l + \int_0^l \frac{d}{dx}(\lambda_2 w''')\delta s\,dx\right)$$

$$-\lambda_2 w''\delta s'\Big|_o^l + \frac{d}{dx}(\lambda_2 w'')\underline{\delta s}\Big|_o^l - \int_0^l \frac{d^2}{dx^2}(\lambda_2 w'')\delta s\,dx$$

$$-\lambda_2 s''\delta w'\Big|_0^l + \frac{d}{dx}(\lambda_2 s'')\delta w\Big|_0^l - \int_0^l \frac{d^2}{dx^2}(\lambda_2 s'')\delta w\,dx = 0$$

$$\delta s = 0 \quad \text{or} \quad -2\lambda_2 w''' + \frac{d}{dx}(\lambda_2 w'') = -2\lambda_2 w''' + (\lambda_2' w'' + \lambda_2 w''') = \lambda_2' w'' - \lambda_2 w''' = 0$$

(2.3.47)

$$\delta L = \int_0^l \delta w(x)\,dx + \lambda_1\left[\int_0^l\left(\frac{dA}{ds}\right)\delta s\,dx\right] + \delta\lambda_1\left[\int_0^l A\,dx - V_0\right]$$

$$-\int_0^l \delta\lambda_2(x)\left[sw'''' + 2s'w''' + s''w'' - q\right]dx - \int_0^l \lambda_2(\delta s w'''')dx$$

$$-\lambda_2 s\,\delta w'''\Big|_0^l + \frac{d}{dx}(\lambda_2 s)\delta w''\Big|_0^l - \frac{d^2}{dx^2}(\lambda_2 s)\delta w'\Big|_0^l + \frac{d^3}{dx^3}(\lambda_2 s)\delta w\Big|_0^l - \int_0^l \frac{d^4}{dx^4}(\lambda_2 s)\delta w\,dx$$

$$+2\left(-\lambda_2 s'\delta w''\Big|_0^l + \frac{d}{dx}(\lambda_2 s')\delta w'\Big|_0^l - \frac{d^2}{dx^2}(\lambda_2 s')\delta w\Big|_0^l + \int_0^l \frac{d^3}{dx^3}(\lambda_2 s')\delta w\,dx\right)$$

$$+2\left(-\lambda_2 w'''\delta s\Big|_0^l + \int_0^l \frac{d}{dx}(\lambda_2 w''')\delta s\,dx\right)$$

$$-\lambda_2 w''\underline{\delta s'}\Big|_o^l + \frac{d}{dx}(\lambda_2 w'')\delta s\Big|_o^l - \int_0^l \frac{d^2}{dx^2}(\lambda_2 w'')\delta s\,dx$$

$$-\lambda_2 s''\delta w'\Big|_0^l + \frac{d}{dx}(\lambda_2 s'')\delta w\Big|_0^l - \int_0^l \frac{d^2}{dx^2}(\lambda_2 s'')\delta w\,dx = 0$$

$$\delta s' = 0 \quad \text{or} \quad \lambda_2 w'' = 0$$

(2.3.48)

$$\delta L=\int_0^l \delta w(x)dx+\lambda_1\left[\int_0^l\left(\frac{dA}{ds}\right)\delta s\,dx\right]+\delta\lambda_1\left[\int_0^l A dx-V_0\right]$$

$$-\int_0^l \delta\lambda_2(x)\left[sw''''+2s'w'''+s''w''-q\right]dx-\int_0^l \lambda_2(\delta s w'''')dx$$

$$-\lambda_2 s\,\delta w'''\Big|_0^l+\frac{d}{dx}(\lambda_2 s)\delta w''\Big|_0^l-\frac{d^2}{dx^2}(\lambda_2 s)\delta w'\Big|_0^l+\frac{d^3}{dx^3}(\lambda_2 s)\underline{\delta w}\Big|_0^l-\int_0^l\frac{d^4}{dx^4}(\lambda_2 s)\delta w\,dx$$

$$+2(-\lambda_2 s'\delta w''\Big|_0^l+\frac{d}{dx}(\lambda_2 s')\delta w'\Big|_0^l-\frac{d^2}{dx^2}(\lambda_2 s')\underline{\delta w}\Big|_0^l+\int_0^l\frac{d^3}{dx^3}(\lambda_2 s')\delta w\,dx)$$

$$+2(-\lambda_2 w'''\delta s\Big|_0^l+\int_0^l\frac{d}{dx}(\lambda_2 w''')\delta s dx)$$

$$-\lambda_2 w''\delta s'\Big|_o^l+\frac{d}{dx}(\lambda_2 w'')\delta s\Big|_o^l-\int_0^l\frac{d^2}{dx^2}(\lambda_2 w'')\delta s dx$$

$$-\lambda_2 s''\delta w'\Big|_0^l+\frac{d}{dx}(\lambda_2 s'')\underline{\delta w}\Big|_0^l-\int_0^l\frac{d^2}{dx^2}(\lambda_2 s'')\delta w\,dx=0$$

$$\delta w=0\quad\text{or}\quad\frac{d^3}{dx^3}(\lambda_2 s)-2\frac{d^2}{dx^2}(\lambda_2 s')+\frac{d}{dx}(\lambda_2 s'')=0$$

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$$\delta w=0\quad\text{or}\quad\frac{d^3}{dx^3}(\lambda_2 s)-2\frac{d^2}{dx^2}(\lambda_2 s')+\frac{d}{dx}(\lambda_2 s'')=0$$

$$(\lambda_2'''s+\lambda_2 s'''+3\lambda_2''s'+3\lambda_2's'')-2(\lambda_2's'+\lambda_2 s'')'+(\lambda_2's''+\lambda_2 s''')=$$

$$(\lambda_2'''s+\cancel{\lambda_2 s'''}+3\lambda_2''s'+\cancel{3\lambda_2's''})-2(\lambda_2''s'+\cancel{\lambda_2's''}+\cancel{\lambda_2's''}+\cancel{\lambda_2 s'''})+$$

$$(\cancel{\lambda_2's''}+\cancel{\lambda_2 s'''})=\lambda_2'''s+\lambda_2''s'=0$$

$$\lambda_2'''s+\lambda_2''s'=0 \qquad (2.3.49)$$

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$$\delta L = \int_0^l \delta w(x)dx + \lambda_1\left[\int_0^l \left(\frac{dA}{ds}\right)\delta s\, dx\right] + \delta\lambda_1\left[\int_0^l A dx - V_0\right]$$

$$-\int_0^l \delta\lambda_2(x)\left[sw'''' + 2s'w''' + s''w'' - q\right]dx - \int_0^l \lambda_2(\delta s w'''')dx$$

$$-\lambda_2 s\,\delta w'''\Big|_0^l + \frac{d}{dx}(\lambda_2 s)\delta w''\Big|_0^l - \frac{d^2}{dx^2}(\lambda_2 s)\delta w'\Big|_0^l + \frac{d^3}{dx^3}(\lambda_2 s)\delta w\Big|_0^l - \int_0^l \frac{d^4}{dx^4}(\lambda_2 s)\delta w\, dx$$

$$+2\left(-\lambda_2 s'\delta w''\Big|_0^l + \frac{d}{dx}(\lambda_2 s')\delta w'\Big|_0^l - \frac{d^2}{dx^2}(\lambda_2 s')\delta w\Big|_0^l + \int_0^l \frac{d^3}{dx^3}(\lambda_2 s')\delta w\, dx\right)$$

$$+2\left(-\lambda_2 w'''\,\delta s\Big|_0^l + \int_0^l \frac{d}{dx}(\lambda_2 w''')\delta s\, dx\right)$$

$$-\lambda_2 w''\,\delta s'\Big|_o^l + \frac{d}{dx}(\lambda_2 w'')\delta s\Big|_o^l - \int_0^l \frac{d^2}{dx^2}(\lambda_2 w'')\delta s\, dx$$

$$-\lambda_2 s''\delta w'\Big|_0^l + \frac{d}{dx}(\lambda_2 s'')\delta w\Big|_0^l - \int_0^l \frac{d^2}{dx^2}(\lambda_2 s'')\delta w\, dx = 0$$

$$\delta w' = 0 \quad \text{or} \quad -\frac{d^2}{dx^2}(\lambda_2 s) + 2\frac{d}{dx}(\lambda_2 s') - \lambda_2 s'' = 0$$

$$\delta w' = 0 \quad \text{or} \quad -\frac{d^2}{dx^2}(\lambda_2 s) + 2\frac{d}{dx}(\lambda_2 s') - \lambda_2 s'' = 0$$

$$-(\lambda_2' s + \lambda_2 s')' + 2(\lambda_2' s' + \lambda_2 s'') - \lambda_2 s'' =$$

$$-(\lambda_2'' s + \cancel{\lambda_2' s'} + \cancel{\lambda_2' s'} + \cancel{\lambda_2 s''}) + 2(\cancel{\lambda_2' s'} + \cancel{\lambda_2 s''}) - \cancel{\lambda_2 s''} = 0$$

$$\lambda_2'' s = 0 \qquad (2.3.50)$$

$$\delta L = \int_0^l \delta w(x)dx + \lambda_1\left[\int_0^l \left(\frac{dA}{ds}\right)\delta s\,dx\right] + \delta\lambda_1\left[\int_0^l A dx - V_0\right]$$

$$-\int_0^l \delta\lambda_2(x)\left[sw'''' + 2s'w''' + s''w'' - q\right]dx - \int_0^l \lambda_2(\delta s w'''')dx$$

$$-\lambda_2 s\,\delta w'''\Big|_0^l + \frac{d}{dx}(\lambda_2 s)\delta w''\Big|_0^l - \frac{d^2}{dx^2}(\lambda_2 s)\delta w'\Big|_0^l + \frac{d^3}{dx^3}(\lambda_2 s)\delta w\Big|_0^l - \int_0^l \frac{d^4}{dx^4}(\lambda_2 s)\delta w\,dx$$

$$+2(-\lambda_2 s'\delta w''\Big|_0^l + \frac{d}{dx}(\lambda_2 s')\delta w'\Big|_0^l - \frac{d^2}{dx^2}(\lambda_2 s')\delta w\Big|_0^l + \int_0^l \frac{d^3}{dx^3}(\lambda_2 s')\delta w\,dx)$$

$$+2(-\lambda_2 w'''\delta s\Big|_0^l + \int_0^l \frac{d}{dx}(\lambda_2 w''')\delta s dx)$$

$$-\lambda_2 w''\delta s'\Big|_o^l + \frac{d}{dx}(\lambda_2 w'')\delta s\Big|_o^l - \int_0^l \frac{d^2}{dx^2}(\lambda_2 w'')\delta s dx$$

$$-\lambda_2 s''\delta w'\Big|_0^l + \frac{d}{dx}(\lambda_2 s'')\delta w\Big|_0^l - \int_0^l \frac{d^2}{dx^2}(\lambda_2 s'')\delta w\,dx = 0$$

$$\delta w'' = 0 \quad \text{or} \quad \frac{d}{dx}(\lambda_2 s) - 2\lambda_2 s' = (\lambda_2' s + \lambda_2 s') - 2\lambda_2 s' = \lambda_2' s - \lambda_2 s' = 0 \qquad (2.3.51)$$

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$$\delta L = \int_0^l \delta w(x)dx + \lambda_1\left[\int_0^l \left(\frac{dA}{ds}\right)\delta s\,dx\right] + \delta\lambda_1\left[\int_0^l A dx - V_0\right]$$

$$-\int_0^l \delta\lambda_2(x)\left[sw'''' + 2s'w''' + s''w'' - q\right]dx - \int_0^l \lambda_2(\delta s w'''')dx$$

$$-\lambda_2 s\,\delta w'''\Big|_0^l + \frac{d}{dx}(\lambda_2 s)\delta w''\Big|_0^l - \frac{d^2}{dx^2}(\lambda_2 s)\delta w'\Big|_0^l + \frac{d^3}{dx^3}(\lambda_2 s)\delta w\Big|_0^l - \int_0^l \frac{d^4}{dx^4}(\lambda_2 s)\delta w\,dx$$

$$+2(-\lambda_2 s'\delta w''\Big|_0^l + \frac{d}{dx}(\lambda_2 s')\delta w'\Big|_0^l - \frac{d^2}{dx^2}(\lambda_2 s')\delta w\Big|_0^l + \int_0^l \frac{d^3}{dx^3}(\lambda_2 s')\delta w\,dx)$$

$$+2(-\lambda_2 w'''\delta s\Big|_0^l + \int_0^l \frac{d}{dx}(\lambda_2 w''')\delta s dx)$$

$$-\lambda_2 w''\delta s'\Big|_o^l + \frac{d}{dx}(\lambda_2 w'')\delta s\Big|_o^l - \int_0^l \frac{d^2}{dx^2}(\lambda_2 w'')\delta s dx$$

$$-\lambda_2 s''\delta w'\Big|_0^l + \frac{d}{dx}(\lambda_2 s'')\delta w\Big|_0^l - \int_0^l \frac{d^2}{dx^2}(\lambda_2 s'')\delta w\,dx = 0$$

$$\delta w''' = 0 \quad \text{or} \quad \lambda_2 s = 0 \qquad (2.3.52)$$

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Collecting the terms multiplied by arbitrary variations δw, δs, $\delta \lambda_1$, and $\delta \lambda_2$ and equating them to zero independently we obtain the following Euler-Lagrange equations

$$\delta w: \quad 1-(\lambda_2'' s)''=0 \tag{2.3.43}$$

$$\delta s: \quad \lambda_1 \frac{dA}{ds}-\lambda_2'' w''=0 \tag{2.3.44}$$

$$\delta \lambda_1: \quad \int_0^l A(x)dx - V_0=0 \tag{2.3.45}$$

$$\delta \lambda_2: \quad sw''''+2s'w'''+s''w''-q(x)=0 \tag{2.3.46}$$

together with the associated boundary conditions at x=0 and x=l

Either or

$$\delta s=0, \quad \lambda_2 w'''-\lambda_2' w''=0 \tag{2.3.47}$$

$$\delta s'=0, \quad \lambda_2 w''=0 \tag{2.3.48}$$

$$\delta w=0, \quad \lambda_2''' s+\lambda_2'' s'=0 \tag{2.3.49}$$

$$\delta w'=0, \quad \lambda_2'' s=0 \tag{2.3.50}$$

$$\delta w''=0, \quad -\lambda_2 s'+\lambda_2' s=0 \tag{2.3.51}$$

$$\delta w'''=0, \quad \lambda_2 s=0 \tag{2.3.52}$$

Equations (2.3.43) through (2.3.46) together with the associated boundary conditions are general enough that they apply to simply supported as well as to clamped beams.
For the cantilever beam the boundary conditions are Eqs. (2.3.36) and (2.3.37). Since the bending moment and the shear force at x=0 cannot vanish because of the unidirectional nature of the applied loading, the above conditions reduce to

(2.3.47) $\lambda_2(0)\,\mathrm{w}'''(0) - \lambda_2'(0)\,w''(0) = 0 \quad \rightarrow \lambda_2'(0)(\neq 0) = 0 \rightarrow \lambda_2'(0) = 0$

$$\lambda_2(0)\,\mathrm{w}''(0) = 0 \rightarrow \lambda_2(0)(\neq 0) = 0 \rightarrow \lambda_2(0) = 0$$

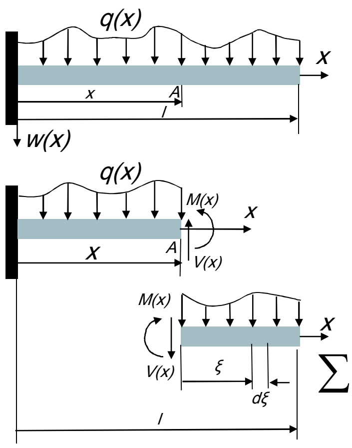
$$\lambda_2(0) = 0, \qquad \lambda_2'(0) = 0 \tag{2.3.53}$$

(2.3.49) $\lambda_2'''(\mathrm{l})\,\mathrm{s}(\mathrm{l}) + \lambda_2''(\mathrm{l})\,\mathrm{s}'(\mathrm{l}) = 0$

(2.3.50) $\lambda_2''(l)s(l) = 0$

$$\lambda_2''(l)s(l) = 0, \qquad \lambda_2'''(l)s(l) + \lambda_2''(l)s'(l) = 0 \tag{2.3.54}$$

We can integrate Eqs. (2.3.43) and (2.3.46) twice and make use of both boundary conditions of Eqs. (2.3.37) and (2.3.54) to get



$$\sum F_y = 0 \rightarrow v(x) = -\int_x^l \mathrm{q}(\zeta)\,\mathrm{d}\,\zeta = \int_l^x \mathrm{q}(\zeta)\,\mathrm{d}\,\zeta$$

$$v(x) = -\frac{dM(x)}{dx} \rightarrow M(x) = -\int_x^l \left(\int_l^x \mathrm{q}(\zeta)\,\mathrm{d}\,\zeta\right)dx = \int_l^x \left(\int_l^x \mathrm{q}(\zeta)\,\mathrm{d}\,\zeta\right)dx$$

We can integrate Eqs. (2.3.43) and (2.3.46) twice and make use of both boundary conditions of Eqs. (2.3.37) and (2.3.54) to get

در هر بار انتگرال گرفتن باید انتگرال نامعین گرفت. مثلاً از x تا l یا l تا x.

(2.3.43) $1-(\lambda_2''s)''=0 \rightarrow \int_l^x (\lambda_2''s)''d\zeta = \int_l^x d\zeta \rightarrow (\lambda_2''s)'=(x-l)$

$$(\lambda_2''s)'=(x-l) \rightarrow \int_l^x (\lambda_2''s)'dx = \int_l^x (x-l)dx \rightarrow \lambda_2''s = \frac{(x-l)^2}{2}$$

$$s\lambda_2'' = \frac{1}{2}(x-l)^2 \qquad (2.3.55)$$

(2.3.46) $sw'''' + 2s'w''' + s''w'' - q(x) = 0 \rightarrow \int_l^x (sw'')''\,d\zeta = \int_l^x q(\zeta)\,d\zeta$

$$(sw'')' = \int_l^x q(\zeta)\,d\zeta \rightarrow \int_l^x (sw'')'dx = \int_l^x \left(\int_l^x q(\zeta)\,d\zeta\right)dx$$

$$sw'' = \int_l^x \left(\int_l^x q(\zeta)\,d\zeta\right)dx \equiv p(x) \qquad (2.3.56)$$

(2.3.55) $s\lambda_2'' = \frac{1}{2}(x-l)^2 \rightarrow \lambda_2'' = \frac{1}{2s}(x-l)^2$ $\longrightarrow w''\lambda_2'' = \frac{p(x)}{2s^2}(x-l)^2$

(2.3.56) $sw'' = \int_l^x \left(\int_l^x q(\zeta)\,d\zeta\right)dx \equiv p(x) \rightarrow w'' = \frac{p(x)}{s}$

$$\lambda_2''w'' = \frac{1}{2}\frac{(x-l)^2 p(x)}{[s(x)]^2} \qquad (2.3.57)$$

Combining the last equation with the second Euler-Lagrange equation (2.3.44), we obtain

(2.3.44) $\lambda_1 \frac{dA}{ds} - \lambda_2''w'' = 0 \rightarrow \lambda_1 \frac{dA}{ds} = \frac{1}{2}\frac{(x-l)^2 p(x)}{[s(x)]^2}$

$$s^2(x)\frac{dA}{ds} = \frac{(x-l)^2 p(x)}{2\lambda_1} \qquad (2.3.58)$$

Specialization to Plane-Tapered Beams. The remainder of this problem will be specialized to plane-tapered beams, n=1, under a uniformly distributed load of intensity $q(x)=q_0$. Evaluating the distribution of $p(x)$, we find that Eq. (2.3.56) becomes

$$sw'' = p(x) = \int_l^x \left(\int_l^x q(\xi)d\xi\right)dx = \int_l^x \left(\int_l^x q_0 d\xi\right)dx = \int_l^x q_0(x-l)dx$$

$$p(x)=\int_l^x q_0(x-l)dx = q_0\left(\frac{x^2}{2}-lx\right)\Big|_l^x = q_0\left(\frac{x^2}{2}-lx-\frac{l^2}{2}+l^2\right)$$

$$=\frac{q_0}{2}(x^2-2lx+l^2)=\frac{q_0}{2}(l-x)^2$$

$$sw''=q_0\frac{(l-x)^2}{2} \quad (2.3.59)$$

Also, for a plane-tapered beam, *n* =1 and from (2.3.38): *s(x)=EI(x)=αEA^n(x)=αEA(x)*

$$A(x)=\frac{s(x)}{\alpha E},\ \frac{dA}{ds}=\frac{1}{\alpha E}=c^2=\text{constant} \quad (2.3.60)$$

Hence, Eq. (2.3.58) becomes

$$s^2(x)\frac{dA}{ds}=\frac{(x-l)^2 p(x)}{2\lambda_1}\rightarrow s^2(x)c^2=\frac{(x-l)^2\frac{q_0}{2}(l-x)^2}{2\lambda_1}$$

$$s^2(x)=\frac{(x-l)^4}{4}\frac{q_0}{\lambda_1 c^2}\rightarrow s(x)=\frac{(x-l)^2}{2c}\sqrt{\frac{q_0}{\lambda_1}} \quad (2.3.61)$$

and, therefore, the optimum distribution of the cross-sectional area is

$$A^*(x)=\frac{s(x)}{\alpha E}=\frac{(x-l)^2}{2c\alpha E}\sqrt{\frac{q_0}{\lambda_1}}=\frac{(x-l)^2}{2c(1/c^2)}\sqrt{\frac{q_0}{\lambda_1}}=\frac{c(x-l)^2}{2}\sqrt{\frac{q_0}{\lambda_1}} \quad (2.3.62)$$

The unknown Lagrange multiplier can be evaluated from the volume constraint of Eq. (2.3.45) to be

$$V_0=\int_0^l A^*(x)dx=\int_0^l \frac{c(x-l)^2}{2}\sqrt{\frac{q_0}{\lambda_1}}dx=\frac{c}{6}\sqrt{\frac{q_0}{\lambda_1}}(x-l)^3\Big|_0^l=\frac{cl^3}{6}\sqrt{\frac{q_0}{\lambda_1}}$$

$$\lambda_1=\frac{c^2 q_0 l^6}{36V_0^2} \quad (2.3.63)$$

The resulting optimal area and bending stiffness distributions are

$$A^*=\frac{c(x-l)^2}{2}\sqrt{\frac{q_0}{\lambda_1}}=\frac{c(x-l)^2}{2}\sqrt{\frac{q_0}{c^2q_0l^6/36v_0^2}}=\frac{3(x-l)^2v_0}{l^3}$$

$$s^*(x)=\frac{(x-l)^2}{2c}\sqrt{\frac{q_0}{c^2q_0l^6/36v_0^2}}=\frac{(x-l)^2}{c}\frac{3v_0}{cl^3}=\frac{3v_0(x-l)^2}{c^2l^3} \quad (2.3.64)$$

Substituting Eq. (2.3.64) into Eq. (2.3.59) and integrating it twice, we obtain the deflection function corresponding to the optimal beam as

$$sw'' = \frac{q_0}{2}(l-x) \rightarrow w'' = \frac{q_0}{2}(l-x)\frac{c^2 l^3}{3v_0(x-l)^2} = \frac{q_0 l^3 c^2}{6v_0}(l-x)^{-1}$$

$$w' = \frac{q_0 l^3 c^2}{6v_0}(1)\Big|_0^x = \frac{q_0 l^3 c^2}{6v_0}x$$

$$w(x) = \frac{c^2 q_0 l^3}{12V_0}x^2 \tag{2.3.65}$$

where the boundary conditions in Eq.(2.3.36) were used. The constant *c* for a rectangular plane-tapered beam with constant thickness *h* and varying width *b(x)* is

$$c^2 = \frac{1}{\alpha E} = \frac{1}{\frac{I}{A}E} = \frac{1}{\frac{bh^3/12}{bh}E} = \frac{12}{Eh^2} \tag{2.3.66}$$

The resulting deflection function is

$$w(x) = \frac{c^2 q_0 l^3}{12V_0}x^2 = \frac{(12/Eh^2)q_0 l^3}{12V_0}x^2 = \frac{q_0 l^3}{Eh^2 V_0}x^2 \tag{2.3.67}$$

For comparison, consider an equivalent uniform beam of the same total volume V_0, length l, constant thickness h, but a constant width

$$b_0 = \frac{V_0}{hl} \tag{2.3.68}$$

Its deflection $w_0(x)$ can be obtained by for times integrating of (2.3.46)

$$sw'''' - q_0 = 0 \rightarrow w'''' = q_0/s \rightarrow w'''' = q_0/EI$$

$$w''' = \frac{q_0}{EI}x + d_1 \rightarrow w'' = \frac{q_0}{EI}\frac{x^2}{2} + d_1 x + d_2$$

$$w' = \frac{q_0}{EI}\frac{x^3}{6} + \frac{x^2}{2}d_1 + d_2 x + d_3 \rightarrow w = \frac{q_0}{EI}\frac{x^4}{24} + \frac{x^3}{6}d_1 + \frac{x^2}{2}d_2 + d_3 x + d_4$$

The constants d_1 to d_4 can be evaluated from boundary conditions.

(2.3.36) $\longrightarrow$
$$w\big|_{x=0} = 0 \rightarrow d_4 = 0$$
$$w'\big|_{x=0} = 0 \rightarrow d_3 = 0$$

(2.3.37) $\longrightarrow$
$$sw''\big|_{x=l} = 0 \rightarrow \frac{q_0}{EI}\frac{l^2}{2} + d_1 l + d_2 = 0 \qquad \rightarrow d_2 = \frac{q_0}{EI}\frac{l^2}{2}$$
$$sw'''\big|_{x=l} = 0 \rightarrow \frac{q_0}{EI} l + d_1 = 0 \rightarrow d_1 = -\frac{q_0}{EI} l$$

Hence: $w_0 = \frac{q_0}{EI}\frac{x^4}{24} - \frac{x^3}{6}\frac{q_0}{EI} l + \frac{x^2}{2}\frac{q_0}{EI}\frac{l^2}{2} = \frac{q_0}{24EI}(x^4 - 4lx^3 + 6l^2x^2)$

$$\int_0^l w_0 dx = \int_0^l \frac{q_0}{24EI}(x^4 - 4lx^3 + 6l^2x^2)dx = \frac{q_0}{24EI}\left[\frac{x^5}{5} - 4l\frac{x^4}{4} + 6l^2\frac{x^3}{3}\right]\Bigg|_0^l$$
$$= \frac{q_0}{24EI}\frac{6l^5}{5} = \frac{q_0 l^5}{20EI}$$

It is interesting to calculate the following ratio:

$$\frac{\int_0^l w(x)dx}{\int_0^l w_0(x)dx} = \frac{\frac{q_0 l^3}{Ev_0 h^2}\int_0^l x^2 dx}{[q_0 l^5 / 20E(b_0 h^3/12)]} = \frac{\frac{q_0 l^3}{Ev_0 h^2}\frac{l^3}{3}}{\frac{q_0 l^5}{20E(b_0 h^3/12)}} = \frac{\frac{q_0 l^3}{Ev_0 h^2}\frac{l^3}{3}}{\frac{q_0 l^5}{20E(\mathrm{v}_0 h^2/12l)}} = \frac{5}{9}$$
(2.3.69)

That is, the optimal beam is 1.8 times stiffer than the uniform beam of the same volume.
Several other cases of loading with different types of beams, *n*=1,2, and 3 may be found in Ref. 7. Some of these cases form a part of the exercises at the end of the chapter.

A.1.1 Integral Involving One Independent Variable

A special case of an integral which involves one independent variable (*x*) three dependent variables (*b*, *s* and *w* and derivatives of the first order and second order of *s* and derivatives of the second third and fourth order of *w* is considered. A typical integral of this form can be expressed by

$$\text{minimize} \quad J(b, s, w) = \int_{x_1}^{x_2} F(b, s, s', s'', w, w'', w''', w^{IV}, x)\, dx \,. \qquad (A.1)$$

$$\int_{x_1+kU_1}^{x_2+kU_2} F(x, b+kB, s+kS, s'+kS', s''+kS'', w+kW, w''+kW'',$$

$$w'''+kW''', w^{IV}+kW^{IV}, x)\, dx \qquad (A.2)$$

$$k \int_{x_1}^{x_2} \left\{ B\frac{\partial F}{\partial b} + \left(S\frac{\partial F}{\partial s} + S'\frac{\partial F}{\partial s'} + S''\frac{\partial F}{\partial s''} \right) + \right.$$

$$\left. \left(W\frac{\partial F}{\partial w} + W''\frac{\partial F}{\partial w''} + W'''\frac{\partial F}{\partial w'''} + W^{IV}\frac{\partial F}{\partial w^{IV}} \right) \right\} dx + k(U_2F_2 - U_1F_1) + K_2 \qquad (A.3)$$

$$\int_{x_1}^{x_2} S'\frac{\partial F}{\partial s'}\,dx = \left[S\frac{\partial F}{\partial s'}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} S\frac{d}{dx}(\frac{\partial F}{\partial s'})\,dx\ , \tag{A.4}$$

$$\int_{x_1}^{x_2} S''\frac{\partial F}{\partial s''}\,dx = \left[S'\frac{\partial F}{\partial s''} - S\frac{d}{dx}(\frac{\partial F}{\partial s''})\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} S\frac{d^2}{dx^2}(\frac{\partial F}{\partial s''})\,dx \tag{A.5}$$

$$\int_{x_1}^{x_2} W''\frac{\partial F}{\partial w''}\,dx = \left[W'\frac{\partial F}{\partial w''} - W\frac{d}{dx}(\frac{\partial F}{\partial w''})\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} W\frac{d^2}{dx^2}(\frac{\partial F}{\partial w''})\,dx \tag{A.6}$$

$$\int_{x_1}^{x_2} W'''\frac{\partial F}{\partial w'''}\,dx = \left[W''\frac{\partial F}{\partial w'''} - W'\frac{d}{dx}(\frac{\partial F}{\partial w'''}) + W\frac{d^2}{dx^2}(\frac{\partial F}{\partial w'''})\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} W\frac{d^3}{dx^3}(\frac{\partial F}{\partial w'''})\,dx \tag{A.7}$$

$$\int_{x_1}^{x_2} W^{IV}\frac{\partial F}{\partial w^{IV}}\,dx = \left[W'''\frac{\partial F}{\partial w^{IV}} - W''\frac{d}{dx}(\frac{\partial F}{\partial w^{IV}}) + W'\frac{d^2}{dx^2}(\frac{\partial F}{\partial w^{IV}}) - W\frac{d^3}{dx^3}(\frac{\partial F}{\partial w^{IV}})\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} W\frac{d^{IV}}{dx^{IV}}(\frac{\partial F}{\partial w^{IV}})\,dx\ . \tag{A.8}$$

$$k\int_{x_1}^{x_2}\left[B\frac{\partial F}{\partial b} + S\left\{\frac{\partial F}{\partial s} - \frac{d}{dx}(\frac{\partial F}{\partial s'}) + \frac{d^2}{dx^2}(\frac{\partial F}{\partial s''})\right\} + W\left\{\frac{\partial F}{\partial w} + \frac{d^2}{dx^2}(\frac{\partial F}{\partial w''}) - \frac{d^3}{dx^3}(\frac{\partial F}{\partial w'''}) + \frac{d^{IV}}{dx^{IV}}(\frac{\partial F}{\partial w^{IV}})\right\}\right]dx +$$
$$k\left[UF + S\frac{\partial F}{\partial s'} + S'\frac{\partial F}{\partial s''} - S\frac{d}{dx}(\frac{\partial F}{\partial s''}) + W'\frac{\partial F}{\partial w''} - W\frac{d}{dx}(\frac{\partial F}{\partial w''}) + W''\frac{\partial F}{\partial w'''} - W'\frac{d}{dx}(\frac{\partial F}{\partial w'''}) + W\frac{d^2}{dx^2}(\frac{\partial F}{\partial w'''}) + W'''\frac{\partial F}{\partial w^{IV}} - W''\frac{d}{dx}(\frac{\partial F}{\partial w^{IV}}) + W'\frac{d^2}{dx^2}(\frac{\partial F}{\partial w^{IV}}) - W\frac{d^3}{dx^3}(\frac{\partial F}{\partial w^{IV}})\right]_{x_1}^{x_2} + K_2 \tag{A.9}$$

$$\int_{x_1}^{x_2}\left[B\frac{\partial F}{\partial b} + S\left\{\frac{\partial F}{\partial s} - \frac{d}{dx}(\frac{\partial F}{\partial s'}) + \frac{d^2}{dx^2}(\frac{\partial F}{\partial s''})\right\} + W\left\{\frac{\partial F}{\partial w} + \frac{d^2}{dx^2}(\frac{\partial F}{\partial w''}) - \frac{d^3}{dx^3}(\frac{\partial F}{\partial w'''}) + \frac{d^{IV}}{dx^{IV}}(\frac{\partial F}{\partial w^{IV}})\right\}\right]dx = 0\ ; \tag{A.10}$$

$$\frac{\partial F}{\partial b} = 0 \, , \tag{A.11}$$

$$\frac{\partial F}{\partial s} - \frac{d}{dx}(\frac{\partial F}{\partial s'}) + \frac{d^2}{dx^2}(\frac{\partial F}{\partial s''}) = 0 \, , \tag{A.12}$$

$$\frac{\partial F}{\partial w} + \frac{d^2}{dx^2}(\frac{\partial F}{\partial w''}) - \frac{d^3}{dx^3}(\frac{\partial F}{\partial w'''}) + \frac{d^{IV}}{dx^{IV}}(\frac{\partial F}{\partial w^{IV}}) = 0. \tag{A.13}$$

$$\Bigg[UF + S\left\{\frac{\partial F}{\partial s'} - \frac{d}{dx}(\frac{\partial F}{\partial s''})\right\} + S'\frac{\partial F}{\partial s''} +$$

$$W\left\{-\frac{d}{dx}(\frac{\partial F}{\partial w''}) + \frac{d^2}{dx^2}(\frac{\partial F}{\partial w'''}) - \frac{d^3}{dx^3}(\frac{\partial F}{\partial w^{IV}})\right\} +$$

$$W'\left\{\frac{\partial F}{\partial w''} - \frac{d}{dx}(\frac{\partial F}{\partial w'''}) + \frac{d^2}{dx^2}(\frac{\partial F}{\partial w^{IV}})\right\} +$$

$$W''\left\{\frac{\partial F}{\partial w'''} - \frac{d}{dx}(\frac{\partial F}{\partial w^{IV}})\right\} + W'''\frac{\partial F}{\partial w^{IV}}\Bigg]_{x_1}^{x_2} = 0 \, . \tag{A.14}$$

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