7.3 (or 4.7) Duality in Linear Programming

Introduction

Which one is easier to solve?

$$\max\ z = 5x_1 - 2x_2$$
$$2x_1 + x_2 \le 9$$
$$x_1 - 2x_2 \le 2$$
$$-3x_1 + 2x_2 \le 3$$
$$x_1, x_2 \ge 0$$

Primal :ابتدایی

or

$$\min\ f_d = 9y_1 + 2y_2 + 3y_3$$
$$2y_1 + y_2 - 3y_3 \ge 5$$
$$y_1 - 2y_2 + 2y_3 \ge -2$$
$$y_1, y_2, y_3 \ge 0$$

√

Dual :دوگان

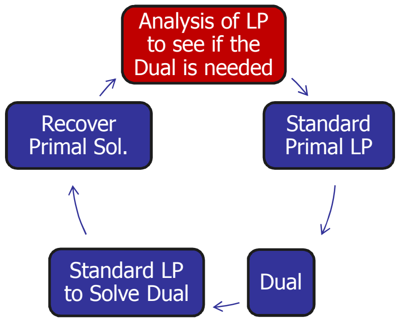
حجم محاسبات در LP با افزایش تعداد معادلات قیود افزایش می‌یابد. بنابراین هنگامی که تعداد معادلات قیود نسبت به تعداد متغیرهای طراحی زیاد است، توصیه می‌شود که به جای حل ابتدایی، دوگان مسئله حل شود. آنگاه از جدول نهایی دوگان، جواب مسئله ابتدایی قابل بازیابی است.

Introduction

Associated with every LP problem is another problem called the dual. The original LP is called the primal problem. Dual variables are related to Lagrange multipliers of the primal constraints.

Solution of the dual problem can be recovered from the final primal solution, and vice versa.

Therefore, only one of the two problems needs to be solved.



Standard Primal LP

Relations between the primal and dual problems:

No. of Design Variables=n

$$\max z_p = d_1 x_1 + \cdots + d_n x_n = \sum_{i=1}^{n} d_i x_i = d^T X$$

No. of constraints =m

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n \le e_1 \\ \cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \le e_m \end{cases} \qquad (AX \le e)$$

$$x_j \ge 0; \quad \text{j=1 to n}$$

Dual LP Problem

No. of Design Variables=m

$$\min f_d = e_1 y_1 + \cdots + e_m y_m = \sum_{i=1}^{m} e_i y_i = e^T y$$

No. of constraints=n

$$\begin{cases} a_{11}y_1 + \cdots + a_{m1}y_m \ge d_1 \\ \cdots \\ a_{1n}y_1 + \cdots + a_{mn}y_m \ge d_n \end{cases} \qquad (A^T Y \ge d)$$

$$y_i \ge 0; \quad \text{i=1 to } m$$

Dual of an LP Problem

Example 7.2 (۴.۲۲):

Write the dual of the problem:

$$\max \quad z = 5x_1 - 2x_2$$
$$2x_1 + x_2 \le 9$$
$$x_1 - 2x_2 \le 2$$
$$-3x_1 + 2x_2 \le 3$$
$$x_1, x_2 \ge 0$$

The problem is already in the standard primal form and we have:

$$d = \begin{bmatrix} 5 \\ -2 \end{bmatrix}, \quad e = \begin{bmatrix} 9 \\ 2 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ -3 & 2 \end{bmatrix}$$

The dual is:

$$\min \quad f_d = 9y_1 + 2y_2 + 3y_3$$
$$2y_1 + y_2 - 3y_3 \ge 5$$
$$y_1 - 2y_2 + 2y_3 \ge -2$$
$$y_1, y_2, y_3 \ge 0$$

Treatment of Equality Constraints

Each equality constraint can be replaced by a pair of inequalities.
For example: $2x_1+3x_2=5$
can be replaced by the pair
$2x_1+3x_2 \ge 5$ and
$2x_1+3x_2 \le 5$.
We can multiply the "≥ type" inequality by -1 to convert it into the standard primal form. Example 7.3 illustrates treatment of equality and "≥ type" constraints.

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Dual of an LP with Equality and "≥ Type" Constraints

Example 7.3 (۴.۲۳): Write the dual for the problem:

maximize $z_p=x_1+4x_2$

$x_1+2x_2\le 5$

$2x_1+x_2=4$

$x_1-x_2\ge 1$

$x_1, x_2\ge 0$

Solution.

$$2x_1+x_2=4 \rightarrow \begin{cases} 2x_1+x_2\le 4 \\ 2x_1+x_2\ge 4 \rightarrow -2x_1-x_2\le -4 \end{cases}$$

Thus, the standard primal of the above problem is:

$$\begin{aligned} \max\ z_p &= x_1+4x_2 \\ x_1+2x_2 &\le 5 \\ 2x_1+x_2 &\le 4 \\ -2x_1-x_2 &\le -4 \\ -x_1+x_2 &\le -1 \\ x_1,x_2 &\ge 0 \end{aligned}$$

The dual for the primal is:

$$\begin{aligned} \min f_d &= 5y_1+4(y_2-y_3)-y_4 \\ y_1+2(y_2-y_3)-y_4 &\ge 1 \\ 2y_1+(y_2-y_3)+y_4 &\ge 4 \\ y_1,y_2,y_3,y_4 &\ge 0 \end{aligned}$$

Or, alternate treatment:

$$\begin{aligned} \min f_d &= 5y_1+4y_5-y_4 \\ y_1+2y_5-y_4 &\ge 1 \\ 2y_1+y_5+y_4 &\ge 4 \\ y_1,y_4 &\ge 0 \\ y_5 &= y_2-y_3 \quad \text{,unrestricted in sign} \end{aligned}$$

If the *ith* primal constraint is left as an equality, the *ith* dual variable is unrestricted in sign.

In a similar manner, we can show that if the primal variable is unrestricted in sign, then the *ith* dual constraint is an equality.

Recovery of Primal Formulation from Dual Formulation

To see this, let us convert the preceding dual problem into standard primal form:

$$\max \ z_p = -5y_1 - 4y_5 + y_4$$
$$-y_1 - 2y_5 + y_4 \le -1$$
$$-2y_1 - y_5 - y_4 \le -4$$
$$y_1, y_4 \ge 0;$$
$$y_5 = y_2 - y_3, \text{unrestricted in sign}$$

Dual of the problem

$$\min \ f_d = -x_1 - 4x_2$$
$$-x_1 - 2x_2 \ge -5$$
$$-2x_1 - x_2 = -4$$
$$x_1 - x_2 \ge 1$$
$$x_1, x_2 \ge 0$$

which is the same as the original problem (Example 7.3). Note that in the preceding dual problem, the second constraint is an equality because the second primal variable (y_5) is unrestricted in sign. Theorem 7.1 states this result.

Theorem 7.1 Dual of Dual: The dual of the dual problem is the primal problem.

Determination of Primal Solution from Dual Solution

Theorem 7.2 Relationship Between Primal and Dual

Let **x** and **y** be in the feasible sets of primal and dual problems, respectively. Then the following conditions hold:

1. $f_d(y) \ge z_p(x)$
2. If $f_d = z_p$, then **x** and **y** are solutions for the primal and dual problems, respectively.
3. If the primal is unbounded, the corresponding dual is infeasible, and vice versa.

Primal	Dual
unbounded	infeasible
infeasible	unbounded

4. If the primal is feasible and the dual is infeasible, then the primal is unbounded and vice versa.

Primal	Dual	Primal	Dual
feasible	infeasible	unbounded	
infeasible	feasible		unbounded

Theorem 7.3 Primal and Dual Solutions: Let both the primal and dual have feasible points. Then both have optimum solution in **x** and **y** respectively, and $f_d(y) = z_p(x)$

Theorem 7.4 Solution of Primal from Dual: If the *ith* dual constraint is a strict inequality at optimum, then the corresponding *ith* primal variable is nonbasic, i.e. it vanishes. Also, if the *ith* dual variable is basic, then the *ith* primal constraint is satisfied at equality. The conditions of Theorem 7.4 can be written as

$$\text{if } \sum_{i=1}^{m} a_{ij} y_i > d_j, \text{then } x_j = 0$$

(if the *jth* dual constraint is a strict inequality, then the *jth* primal variable is nonbasic)

$$\text{if } y_i > 0, \text{ then } \sum_{j=1}^{n} a_{ij} x_j = e_i$$

(if the *ith* dual variable is basic, the *ith* primal constraint is an equality).

These conditions can be used to obtain primal variables using the dual variables.

However, this is not necessary as the final dual tableau can be used directly to obtain primal variables.

Use of Dual Tableau to Recover Primal Solution

Theorem 7.5 Recovery of Primal Solution from Dual Tableau

The value of the ith primal variable =
The reduced cost coefficient of the slack or surplus variable associated with the ith dual constraint in the final dual tableau.

If a dual variable is nonbasic (i.e., has zero value), then:
Its reduced cost coefficient =
The value of slack or surplus variable for the corresponding primal constraint.

While using Theorem 7.5, the following additional points should be noted:

1. When the final primal tableau is used to recover the dual solution, the dual variables correspond to the primal constraints expressed in the "$\leq$" form only.

However, the primal constraints must be converted to standard Simplex form while solving the problem. Recall that all the right sides of constraints must be nonnegative for the Simplex method. The dual variables are nonnegative only for the constraints written in the "$\leq$" form.

2. When a primal constraint is an equality, it is treated in the Simplex method by adding an artificial variable in Phase I. There is no slack or surplus variable associated with an equality. We also know from the previous discussion that the dual variable associated with the equality constraint is unrestricted in sign. Then the question is how to recover the dual variable for the equality constraint from the final primal tableau? There are a couple of ways of doing this.

   The first procedure is to convert the equality constraint into a pair of inequalities.

   The second way of recovering the dual variable for the equality constraint is to carry along its artificial variable column in Phase II of the Simplex method. Then the dual variable for the constraint is the reduced cost coefficient in the artificial variable column in the final primal tableau.

Use of Final Primal Tableau to Recover Dual Solutions

Example 7.6 (۴.۲۶): Solve the following LP problem and recover its dual solution from the final primal tableau:

Standard simplex

$$\max\ z_p = x_1 + 4x_2$$
$$x_1 + 2x_2 \le 5$$
$$2x_1 + x_2 = 4 \rightarrow \begin{cases} 2x_1 + x_2 \le 4 \\ 2x_1 + x_2 \ge 4 \end{cases}$$
$$x_1 - x_2 \ge 1$$
$$x_1, x_2 \ge 0$$

$$\begin{cases} \min\ f = -x_1 - 4x_2 \\ x_1 + 2x_2 + x_3 = 5 \\ 2x_1 + x_2 + x_4 = 4 \\ 2x_1 + x_2 - x_5 + x_7 = 4 \\ x_1 - x_2 - x_6 + x_8 = 1 \\ x_i \ge 0 \quad i = 1 \text{ to } 8 \end{cases}$$

Solution. The final tableau for the problem is given in Table 6-17. Using Theorem 7.5, the dual variables for the preceding four constraints are

1. $x_1+2x_2\le 5$: $y_1=0$, reduced cost coefficient of x_3, the slack variable
2. $2x_1+x_2\le 4$: $y_2=5/3$, reduced cost coefficient of x_4, the slack variable
3. $-2x_1-x_2\le -4$: $y_3=0$, reduced cost coefficient of x_5, the surplus variable
4. $-x_1+x_2\le -1$: $y_4=7/3$, reduced cost coefficient of x_6, the surplus variable

Thus, from the above discussion, the dual variable for the equality constraint $2x_1+x_2=4$ is $y_2-y_3=5/3$. Note also that $y_4=7/3$ is the dual variable for the fourth constraint written as $-x_1+x_2\le -1$ and not for the constraint $x_1-x_2\ge 1$.

Table 6-17: Final tableau: End of Phase II.

Basic↓	x_1	x_2	x_3 (Slack)	x_4 (Slack)	x_5 (surplus)	x_6 (surplus)	x_7 (Artificial)	x_8 (Artificial)	b
x_3	0	0	1	~~1~~ 0	0	−1	0	1	2
x_5	0	0	0	1	1	0	−1	0	0
x_2	0	1	0	$\frac{1}{3}$	0	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{2}{3}$
x_1	1	0	0	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{5}{3}$
Cost	0	0	0	$\frac{5}{3}$	0	$\frac{7}{3}$	0	$-\frac{7}{3}$	$f+\frac{13}{3}$

Now, let us re-solve the same problem with the equality constraint as it is.

TABLE 6-19-Final tableau Slack surplus Artificial

Basic↓	x_1	x_2	x_3	x_4	x_5	x_6	b
x_3	0	0	1	−1	−1	1	2
x_2	0	1	0	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$
x_1	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{3}$
Cost	0	0	0	$\frac{7}{3}$	$\frac{5}{3}$	$-\frac{7}{3}$	$f+\frac{13}{3}$
	(c_1')	(c_2')	(c_3')	(c_4')	(c_5')	(c_6')	
Artificial	0	0	0	0	1	1	$w-0$

Using Theorem 7.5 and the preceding discussion, the dual variables for the given three constraints are

1. $x_1+2x_2\le5$: $y_1=0$, reduced cost coefficient of x_3, the slack variable
2. $2x_1+x_2=4$: $y_2=5/3$, reduced cost coefficient of x_5, the artificial variable
3. $-x_1+x_2\le-1$: $y_4=7/3$, reduced cost coefficient of x_4, the surplus variable

We see that the two solutions are the same. Therefore, we do not have to replace an equality constraint by two inequalities in the standard Simplex method.

The reduced cost coefficient corresponding to the artificial variable associated with the equality constraint gives the value of the dual variable for the constraint.

Dual Variables as Lagrange Multipliers

Theorem 7.6

Let **x** and **y** be optimal solutions for the primal and dual problems, respectively. Then:

The dual variables **y** are also Lagrange multipliers for primal constraints.

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مسائل زیر را حل کرده و تا دو هفته دیگر تحویل فرمایید:

4) 145, 155

تمرین‌های قبلی:

4) 1,2,3, 9, 16, 20, 21, 26, 34,37, 38, 39, 41, 59, 68, 71,73