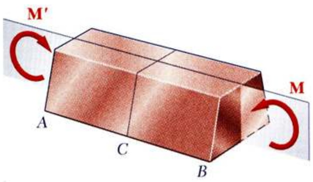
فصل ۱۰- لنگر ماند سطح

Second Moment of Inertia

گردآوری و تنظیم: محمدحسین ابوالبشری

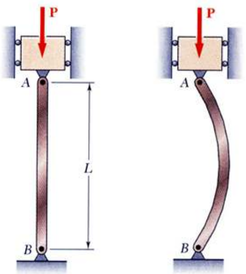
چند نکته:

- لنگرهای ماند سطح تنها یک خاصیت هندسی‌اند.
- لنگرهای ماند سطح واحدشان $(\text{length})^4$ ($[ft^4]$ or $[m^4]$) است.
- لنگرهای ماند سطح حول محورهای مار بر مرکزوار یک جسم یا شکل حداقل است.



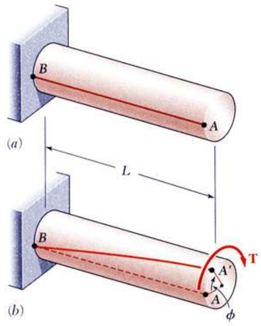
- لنگرهای ماند سطح چه کاربردهایی دارند؟

تحلیل تنش: خمش: $\sigma = My/I$



کمانش: $P_{cr} = \pi^2 EI/L^2$

چند نکته (ادامه):



•لنگرهای ماند سطح چه کاربردهایی دارند؟

تحلیل تنش:

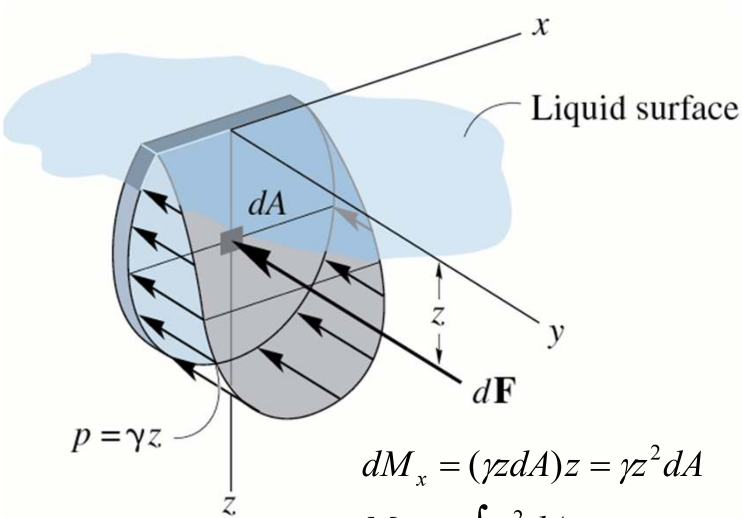
پیچش: $\tau = Tr/J$

دینامیک: $M = I_m \alpha$

که I_m عبارت است از ممان اینرسی جرمی $I_m \equiv \int r^2 dm$ [kg·m²]

Moments of Inertia

Second Moment About an Axis

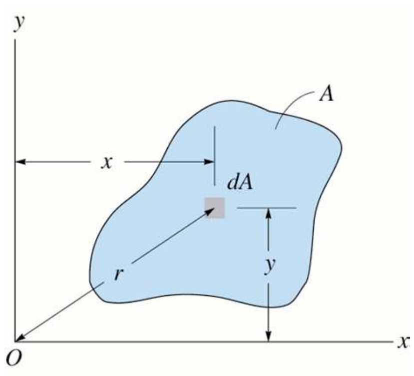


$$dM_x = (\gamma z dA) z = \gamma z^2 dA$$

$$M_x = \gamma \int z^2 dA$$

Moments of Inertia

Second Moment of an Area About an Axis



Moment of Inertia

$$I_x = \int_A dI_x = \int_A y^2 dA$$

$$I_y = \int_A dI_y = \int_A x^2 dA$$

Polar Moment of Inertia

$$J_o = \int_A dJ_o = \int_A r^2 dA$$

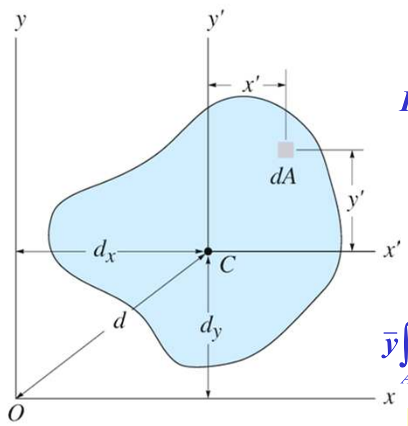
$$= \int_A \left(x^2 + y^2\right) dA = I_x + I_y$$

Always Positive

Moments of Inertia

Parallel Axis Theorem for an Area

فرض کنید C مرکزوار سطح است.



$$I_x = \int_A y^2 dA = \int_A \left(y' + d_y\right)^2 dA$$

$$I_x = \int_A y'^2 dA + 2d_y \int_A y' dA + \int_A d_y^{\,2} dA$$

$$I_x = I_{x'} + 2d_y \int_A y' dA + d_y^{\,2} \int_A dA$$

$$I_x = \bar{I}_{x'} + 2d_y \bar{y} \int_A dA + A d_y^{\,2}$$

$$\bar{y} \int_A dA = 0 \quad since \ \bar{y} = 0$$

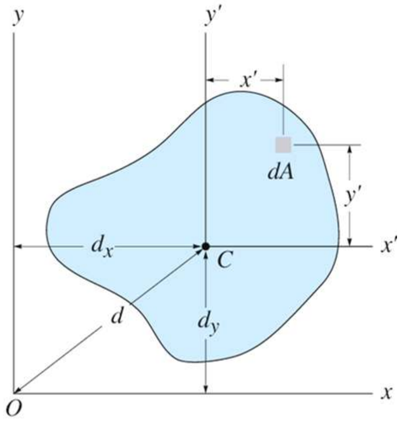
زیرا آن فاصله بین محور x' از مرکزوار است و می‌دانیم که محور از مرکز وار می‌گذرد و فاصله‌ای ندارد.

$$I_x = \bar{I}_{x'} + A d_y^{\,2}$$

Moments of Inertia

Parallel Axis Theorem for an Area

لنگر ماند یک سطح حول یک محور برابر است با لنگر ماند سطح حول محورهای موازی با آن که از مرکزوار سطح می‌گذرد به اضافه حاصلضرب مساحت در مجذور فاصله عمودی بین محورها



$$I_x = \bar{I}_{x'} + A d_y^{\,2}$$

$$I_y = \bar{I}_{y'} + A d_x^{\,2}$$

$$J_o = \bar{J}_C + A d^2$$

Moments of Inertia

Radius of Gyration of an Area

(The centroid of the second moment of an area)

$$\bar{x} W_R = \int \tilde{x}\, dW$$

$$I_x = k_x^2 A = \int_A y^2 dA$$

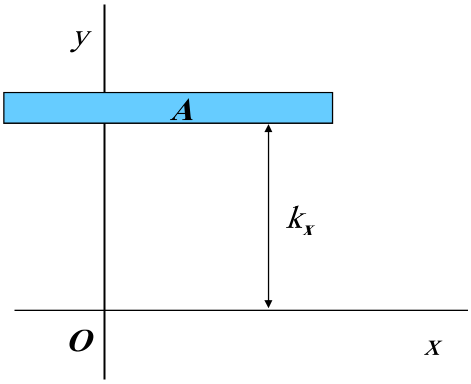
$$I_y = k_y^2 A = \int_A x^2 dA$$

$$J_0 = k_0^2 A = \int_A r^2 dA$$

$$k_x = \sqrt{\frac{I_x}{A}}$$

$$k_y = \sqrt{\frac{I_y}{A}}$$

$$k_0 = \sqrt{\frac{I_0}{A}}$$



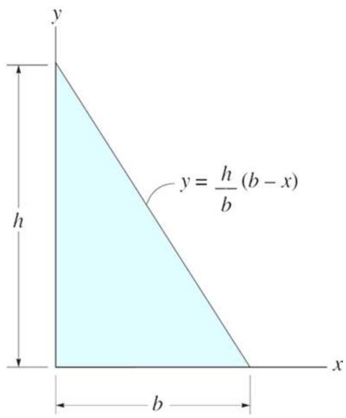
The ***radius of gyration of an area*** ***A*** with respect to the x axis is defined as the distance k_x, where $I_x = k_x^2 A$. With similar definitions for the radii of gyration of A with respect to the y axis and with respect to O, we have

$$k_x = \sqrt{\frac{I_x}{A}} \qquad k_y = \sqrt{\frac{I_y}{A}} \qquad k_O = \sqrt{\frac{J_O}{A}}$$

Moments of Inertia

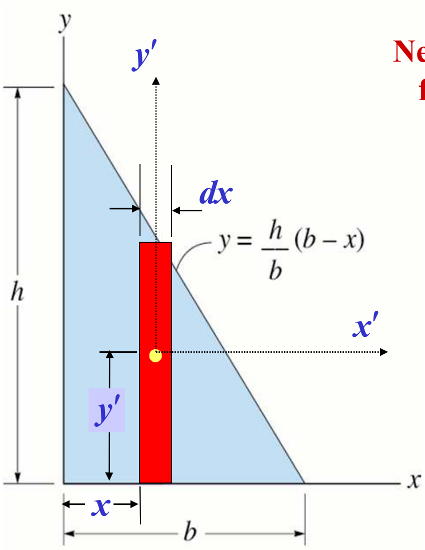
Sample Problem 10.1

Determine the moment of Inertia about the x-axis



Moments of Inertia

Sample Problem 10.1

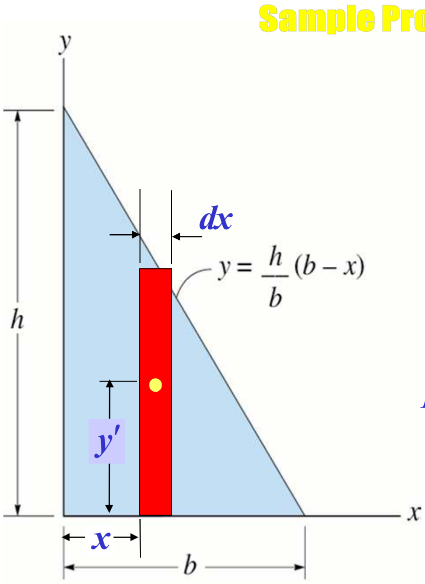


Need the parallel axis theorem for the differential element

$$dI_x = dI_{x'} + y'^2\, dA$$

Moments of Inertia

Sample Problem 10.1



با استفاده از ممان اینرسی مستطیل:

$$dI_x = dI_{x'} + y'^2\, dA$$

$$dI_x = \frac{1}{12}(dx)y^3 + \left(\frac{y}{2}\right)^2 y\,dx$$

$$dI_x = \left[\frac{1}{12}y^3 + \frac{y^3}{4}\right]dx = \frac{1}{3}y^3\,dx$$

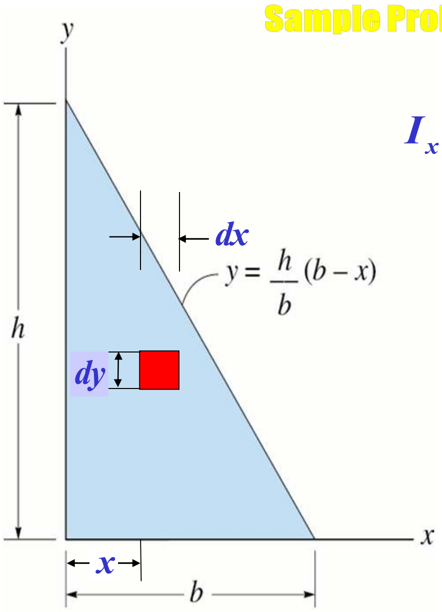
$$I_x = \int_0^b \left(\frac{y^3}{3}\right)dx = \frac{1}{3}\int_0^b \left(h - \frac{h}{b}x\right)^3 dx$$

$$I_x = \left(\frac{1}{3}\right)\left(-\frac{b}{4h}\right)(u)^4\Big|_h^0$$

$$I_x = \frac{bh^3}{12}$$

Moments of Inertia

Sample Problem 10.1



با استفاده از یک المان مربع مستطیل:

$$I_x = \int_A y^2 dA = \int_A y^2 dydx$$

$$I_x = \int_0^b \int_0^y y^2 dydx$$

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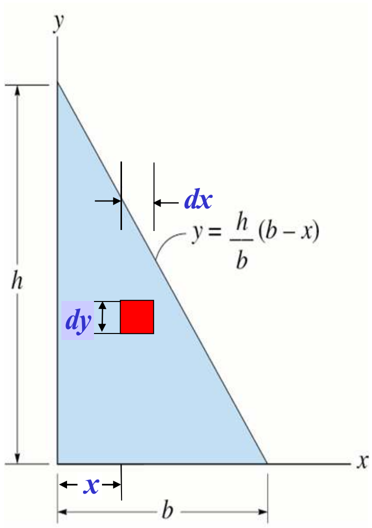
Moments of Inertia

Sample Problem 10.1



$$I_x = \int_A y^2 dA = \int_A y^2 dydx$$

$$I_x = \int_0^b \int_0^y y^2 dydx$$

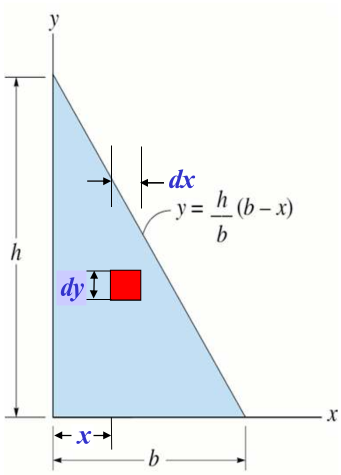
$$I_x = \int_0^b \left(\frac{1}{3}y^3\right)dx$$

$$I_x = \int_0^b \left(\frac{1}{3}\right)\left(\left[h - \frac{h}{b}x\right]^3\right)dx$$

گردآوری و تنظیم: محمدحسین ابوالبشری

Moments of Inertia

Sample Problem 10.1



$$Let\ u = h - \frac{h}{b}x,\ \ then$$

$$du = \left(-\frac{h}{b}\right)dx\ \ and\ \ dx = \left(-\frac{b}{h}\right)du$$

$$also\ u = h\ @\ x = 0\ \ and$$

$$u = 0\ @\ x = b$$

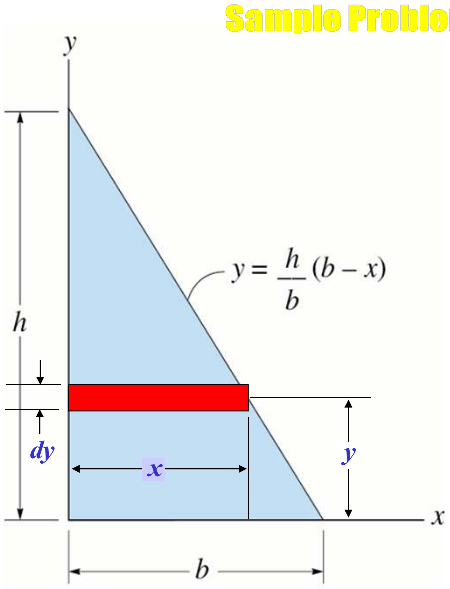
$$I_x = \left(\frac{1}{3}\right)\left(-\frac{b}{h}\right)\int_h^0 u^3\,du$$

$$I_x = \left(\frac{1}{3}\right)\left(-\frac{b}{h}\right)\int_h^0 u^3\,du = \left(\frac{1}{3}\right)\left(-\frac{b}{h}\right)\left(\frac{u^4}{4}\right)\Bigg|_h^0$$

$$I_x = \left(\frac{1}{12}\right)bh^3$$

Moments of Inertia

Sample Problem 10.1



$$I_x = \int_A y^2\,dA$$

$$dA = x\,dy = \left(b - \frac{b}{h}y\right)dy$$

$$I_x = \int_0^h y^2\left(b - \frac{b}{h}y\right)dy$$

$$I_x = \int_0^h \left(by^2 - \frac{b}{h}y^3\right)dy$$

$$I_x = \left[b\frac{y^3}{3} - \frac{b}{h}\frac{y^4}{4}\right]_0^h$$

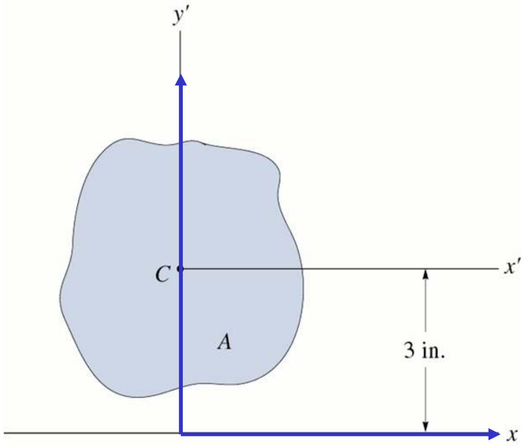
$$I_x = bh^3\left[\frac{1}{3} - \frac{1}{4}\right]$$

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{hb^3}{12}$$

Moments of Inertia

Sample Problem 10.3

The polar moment of inertia of the area is J_C=23 in^4 about the z' axis passing through the centroid C. If the moment of inertia about the y' axis is 5 in^4, the moment of inertia about the x axis is 40 in^4, determine the area A.



Moments of Inertia

Sample Problem 10.3

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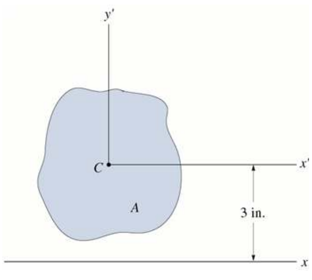


$$J_C = I_{x'} + I_{y'}$$

and

$$I_{x'} = J_C - I_{y'}$$

$$I_{x'} = 23 - 5 = 18 \quad in^4$$

using the parallel axis theorem

$$I_x = I_{x'} + Ad_y^2$$

$$A = \frac{(I_x - I_{x'})}{d_y^2}$$

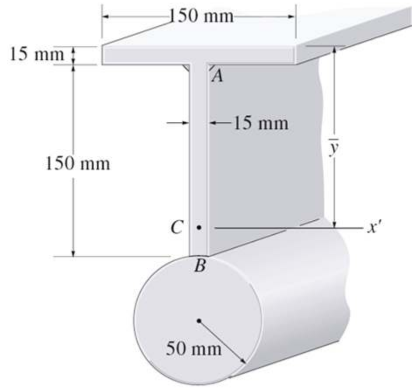
$$A = \frac{(40 - 18)}{3^2}$$

$$A = 2.44 \quad in^2$$

Moments of Inertia

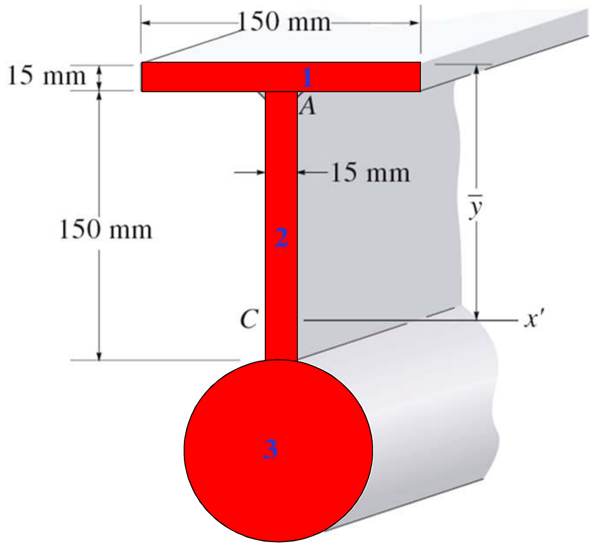
Sample Problem 10.4

Determine the moment of inertia of the beam's cross-sectional area about the x' axis. Neglect the size of the corner welds at A and B for the calculation , y=154.4 mm



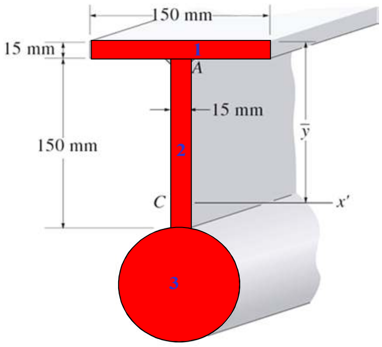
Moments of Inertia

Sample Problem 10.4



Moments of Inertia

Sample Problem 10.4



با استفاده از قضیه محورهای موازی:

$$I_{x'} = \bar{I} + A d_y^{\,2}$$

برای مجموع المان‌ها:

$$I_{x'} = \sum (I_{x'})_i = \sum \left(\bar{I} + A d_y^{\,2}\right)_i$$

$$I_{x'} = 95.9\,(10^6)\ \text{mm}^4$$

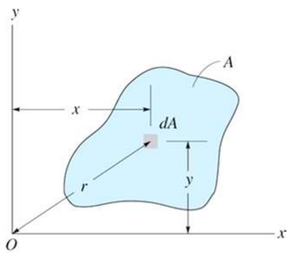
Segment	A (mm^2)	d_y (mm)	$I_{x'}$ (mm^4)	Ad_y^2 (mm^4)	I_x (mm^4)
1	150(15)	146.9	$150(15^3)/12$	$48.554(10^6)$	$48.596(10^6)$
2	15(150)	64.4	$15(150^3)/12$	$9.332(10^6)$	$13.550(10^6)$
3	$\pi(50)^2$	60.6	$\pi(50^4)/4$	$28.843(10^6)$	$33.751(10^6)$

Moments of Inertia

Product of Inertia

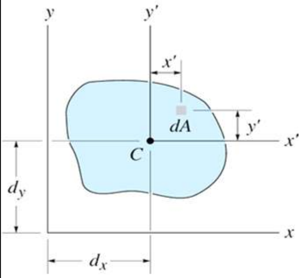
لنگر ماند حاصلضرب به شکل زیر تعریف می‌شود:

$$I_{xy} = \int_A xy\,dA$$

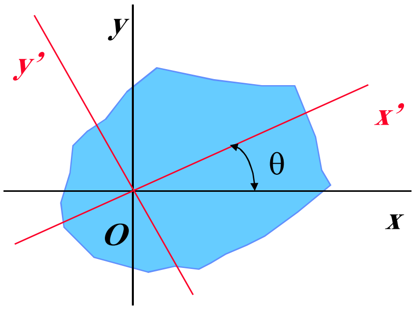


لنگر ماند حاصلضرب هر شکل متقارن حول محورهای تقارن صفر است.

قضیه محورهای موازی:



$$I_{xy} = \bar{I}_{x'y'} + A d_x d_y$$



The ***product of inertia of an area A*** is defined as

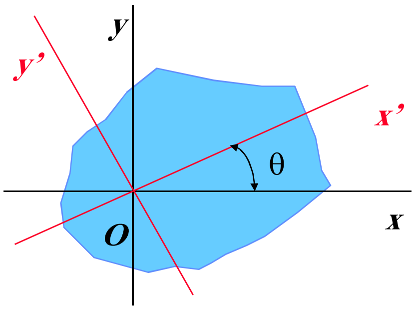
$$I_{xy} = \int xy\, dA$$

$I_{xy} = 0$ if the area A is symmetrical with respect to either or both coordinate axes.

The ***parallel-axis theorem for products of inertia*** is

$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\bar{y}A$$

where $\bar{I}_{x'y'}$ is the product of inertia of the area with respect to the centroidal axes x' and y' which are parallel to the x and y axes and $\bar{x}$ and $\bar{y}$ are the coordinates of the centroid of the area.

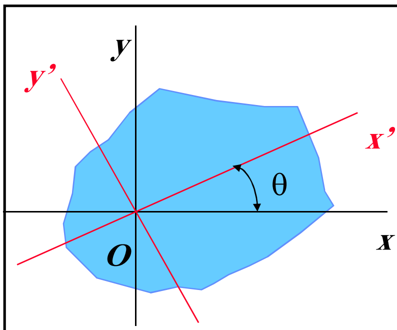


The relations between the moments and products of inertia in the primed and un-primed coordinate systems (assuming the coordinate axes are rotated counterclockwise through an angle θ) are

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

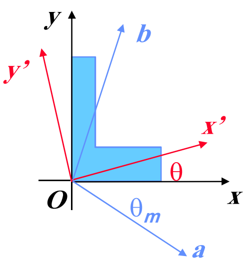


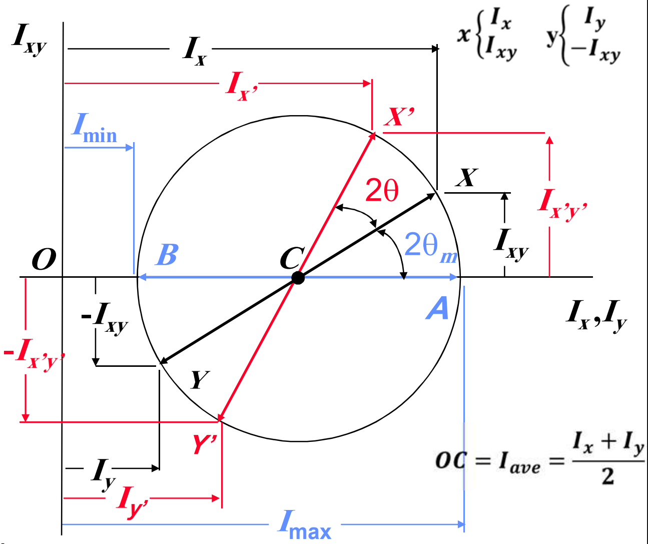
The ***principal axes of the area about O*** are the two axes perpendicular to each other, with respect to which the moments of inertia are maximum and minimum. The angles θ at which these occur are denoted as θ_m, obtained from

$$\tan 2\theta_m = -\frac{2\,I_{xy}}{I_x - I_y}$$

The corresponding maximum and minimum values of I are called the ***principal moments of inertia*** of the area about *O*. They are given by

$$I_{\text{max, min}} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$



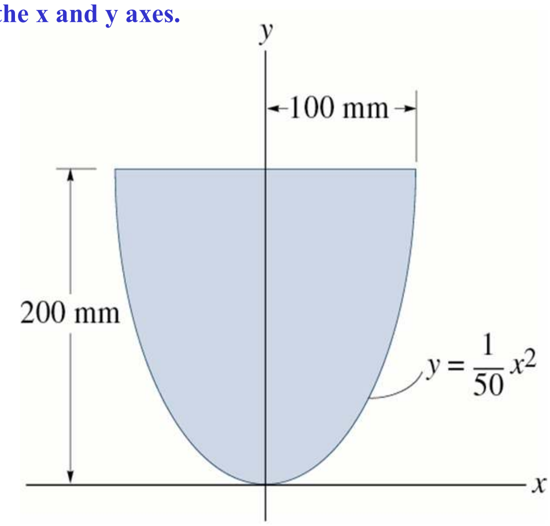


Transformation of the moments and products of inertia of an area under a rotation of axes can be represented graphically by drawing ***Mohr's circle***. An important property of Mohr's circle is that an angle θ on the cross section being considered becomes 2θ on Mohr's circle.

Moments of Inertia

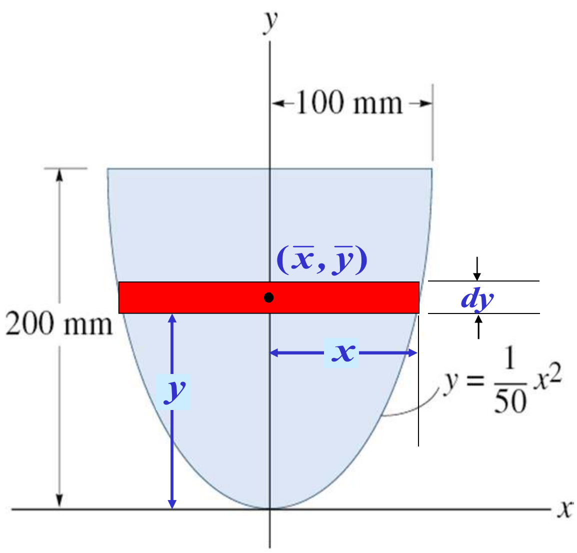
Sample Problem 10.5

Determine the product of inertia of the shaded portion of the parabola with respect to the x and y axes.



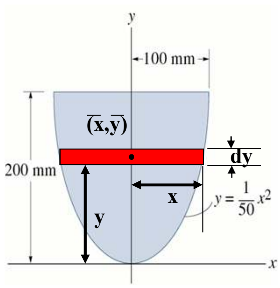
Moments of Inertia

Sample Problem 10.5



Moments of Inertia

Sample Problem 10.5



$$x = \sqrt{50}\, y^{\frac{1}{2}} \quad and$$

$$dA = 2xdy = 2\sqrt{50}\, y^{\frac{1}{2}} dy$$

$$\bar{x} = 0$$

$$\bar{y} = y$$

$$dI_{xy} = dI_{x'y'} + dA\, \bar{x}\, \bar{y}$$

$$dI_{xy} = dI_{x'y'} + \left(\sqrt{50}\, y^{\frac{1}{2}} dy\right)(0)(y)$$

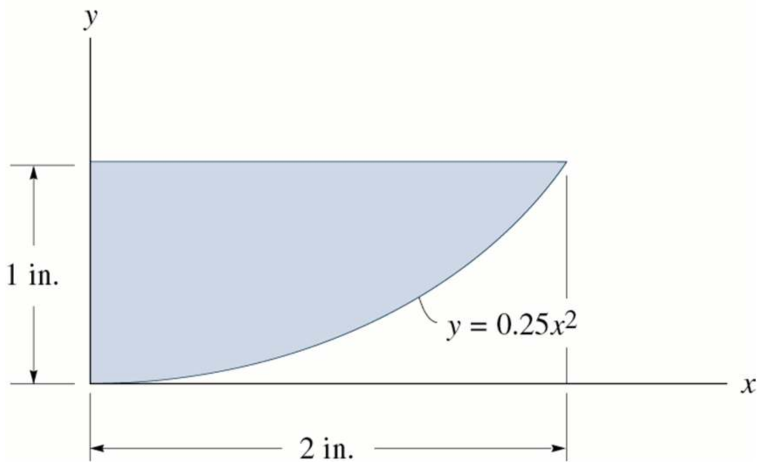
$$dI_{xy} = 0 \quad Theref\ \ ore$$

$$I_{xy} = \int dI_{xy} = 0$$

Moments of Inertia

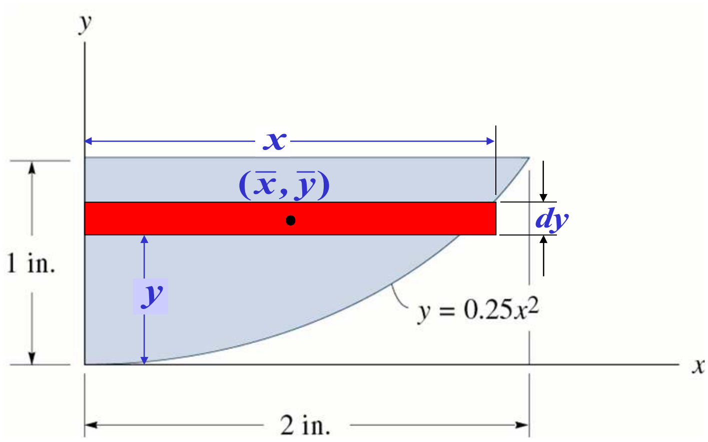
Sample Problem 10.6

Determine the product of inertia of the shaded area with respect to the x and y axes



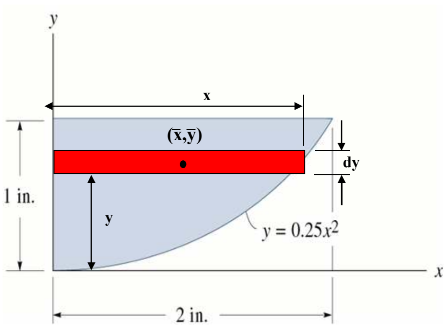
Moments of Inertia

Sample Problem 10.6



Moments of Inertia

Sample Problem 10.6



$$I_{xy} = \int_A dI_{xy} = \int_A xy\,dA$$

$x = 2y^{\frac{1}{2}}$

$dA = x\,dy = 2y^{\frac{1}{2}}dy$ and

$\bar{x} = \frac{x}{2}; \quad \bar{y} = y$

$$dI_{xy} = d\bar{I}_{x'y'} + dA\,\bar{x}\,\bar{y}$$

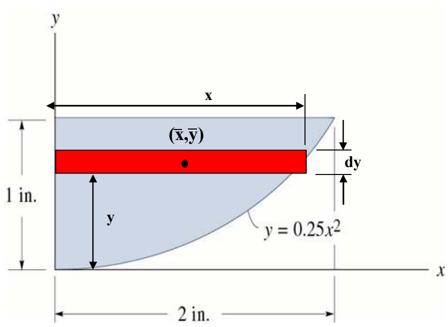
$$dI_{xy} = 0 + \left(2y^{\frac{1}{2}}dy\right)\left(\frac{x}{2}\right)(y)$$

$$dI_{xy} = \left(2y^{\frac{1}{2}}dy\right)\left(y^{\frac{1}{2}}\right)(y)$$

$$dI_{xy} = 2y^2 dy$$

Moments of Inertia

Sample Problem 10.6



$$dI_{xy} = 2y^2\,dy$$

$$I_{xy} = \int_A dI_{xy} = \int_0^1 2y^2\,dy$$

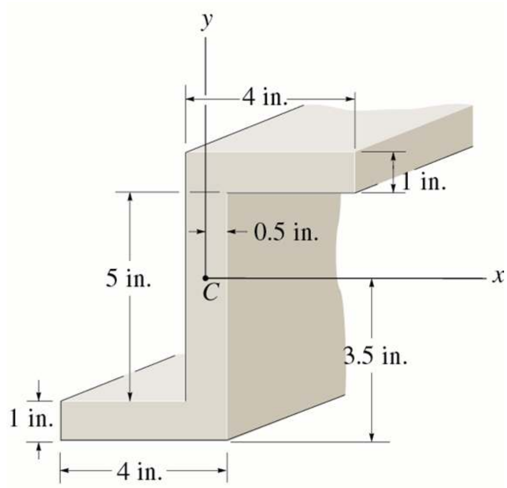
$$I_{xy} = \frac{2}{3}y^3\Big|_0^1$$

$$I_{xy} = .667 \quad in^4$$

Moments of Inertia

Sample Problem 10.7

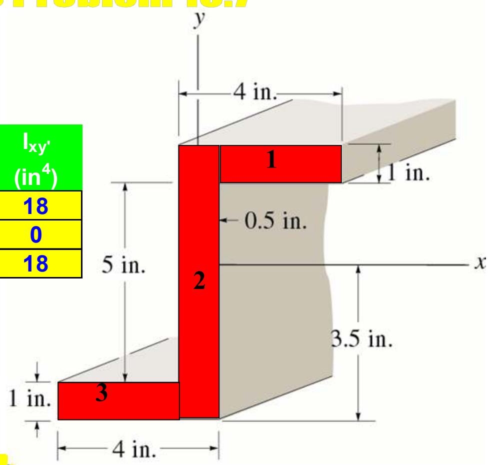
Determine the product of inertia of the shaded area with respect to the x and y axes that have their origin located at the centroid C.



Moments of Inertia

Sample Problem 10.7

Segment	A (in^2)	d_x (in)	d_y (in)	$I_{xy'}$ (in^4)
1	3(1)	2	3	18
2	7(1)	0	0	0
3	3(1)	-2	-3	18



$$I_{xy} = \sum_i \left(I_{xy}\right)_i = 36.0 \quad in^4$$

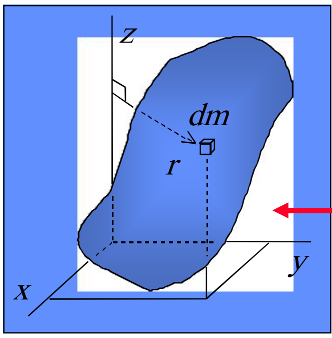
§ 9.11-9.15 Mass Moments of Inertia

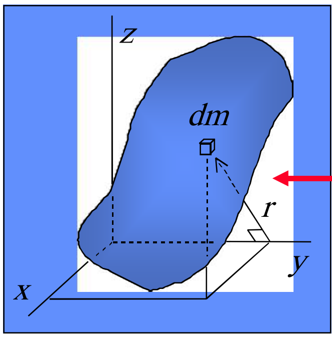
The mass moment of inertia is a measure of a body's "resistance" to angular acceleration

$$M = I\alpha$$

$$I = \int_m r^2 dm = \int_m r^2 \rho dV = \underbrace{\rho \int_m r^2 dV}_{\text{if } \rho = \text{constant}}$$

units: $(mass)(length)^2$





Applications:

$$I_z = \int r^2\, dm = \int (x^2+y^2)\, dm$$

$$I_y = \int r^2\, dm = \int (x^2+z^2)\, dm$$

similarly ...

$$I_x = \int r^2\, dm = \int (y^2+z^2)\, dm$$

Parallel axis theorem:

$$I = I_G + md^2$$

I_G = moment of inertia about mass center G

m = mass of body

d = perpendicular distance between parallel axes.

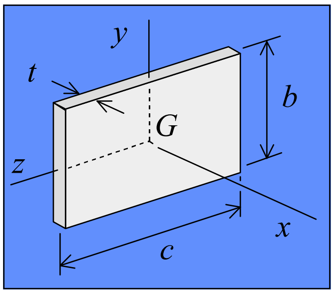
Radius of gyration, k:

$$I = mk^2 \text{ or } k = \sqrt{I/m}$$

Composite bodies:

$$I = \Sigma(I_G + md^2)$$

For homogeneous thin flat plate bodies, the mass moment of inertia is directly related to the area moment of inertia!



$(t << b \;\&\; c)$

area moments

$$I_z^{area} = \frac{1}{12} cb^3 \quad \Leftarrow I_z^{area} = \int y^2 dA$$

$$I_y^{area} = \frac{1}{12} bc^3 \quad J_c^{area} = \frac{1}{12} bc\left(b^2 + c^2\right)$$

mass moments

$$I_z^{mass} = \int y^2 \underbrace{dm}_{\rho t dA}$$

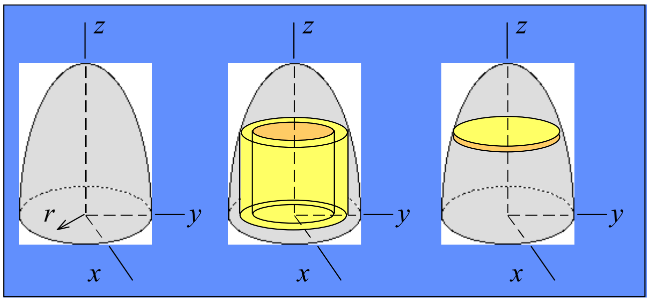
$$= \rho t \int y^2 dA \quad (\rho t \text{ constant})$$

$$= \rho t\, I_z^{area} = \frac{\rho t}{12} cb^3 \qquad = \frac{m}{12} b^2$$

also $I_y^{mass} = \rho t\, I_y^{area} = \frac{\rho t}{12} bc^3$

$$I_G^{mass} = \rho t\, J_c^{area} = \frac{\rho t}{12} bc\left(b^2 + c^2\right)$$

Mass Moments of Inertia for Solids of Revolution



Usually, we will take $r=x$ or $r=y$ (choice depends on how the eqn. for the generating curve is given).

shell element

$$dV = 2\pi rz\, dr$$

$$dI_z = r^2\, dm$$

$$= r^2 \rho\, dV$$

disk element

$$dV = \pi r^2\, dz$$

$$dI_z = (r^2/2)\, dm$$

$$= (r^2/2)\, \rho\, dV$$

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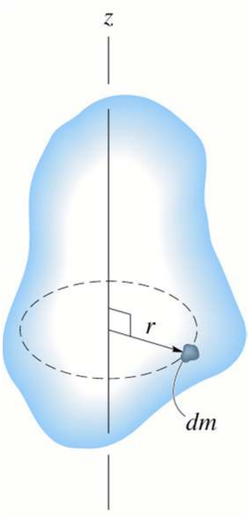
Moments of Inertia

Mass Moment of Inertia



$$I = \int_m r^2 dm$$

$$I_G = \int_m r_G^{\,2} dm$$

always positive
usually defined about the center of mass G

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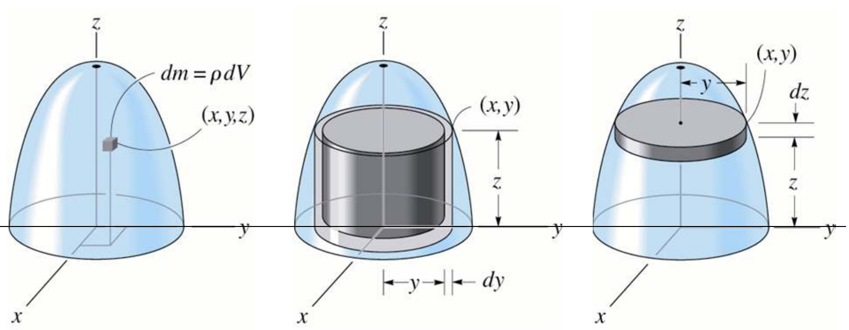
Moments of Inertia

Mass Moment of Inertia



$$I = \int_m r^2 dm = \int_V r^2 \rho \, dV = \rho \int_V r^2 dV$$

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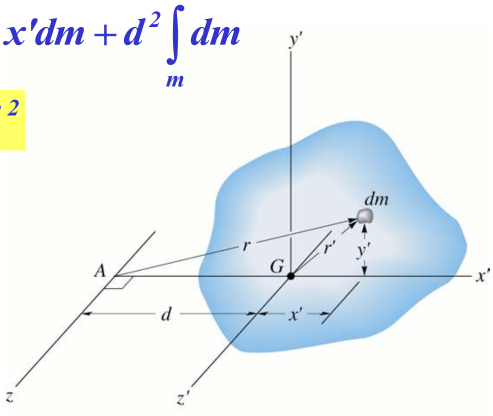
Moments of Inertia

Paralell Axis Theorem

$$I = \int_m r^2 dm = \int_m \left[(d + x')^2 + y'^2 \right] dm$$

$$I = \int_m \left[x'^2 + y'^2 \right] dm + 2d \int_m x' dm + d^2 \int_m dm$$

$$I = I_G + md^2$$



Moments of Inertia

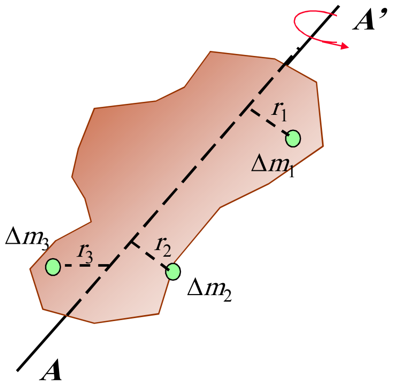
Radius of Gyration

$$I = mk^2$$

$$k = \sqrt{\frac{I}{m}}$$

Composite Bodies

$$I_G = \sum_i \left[(I_G)_i + m_i d_i^{\,2} \right]$$



Moments of inertia of mass are encountered in dynamics. They involve the rotation of a rigid body about an axis. The mass moment of inertia of a body with respect to an axis *AA'* is defined as

$$I = \int r^2 dm$$

where r is the distance from AA' to the element of mass.

The ***radius of gyration*** of the body is defined as

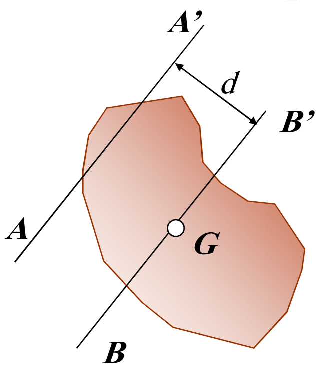
$$k = \sqrt{\frac{I}{m}}$$

The moments of inertia of mass with respect to the coordinate axes are

$$I_x = \int (y^2 + z^2)\ dm$$

$$I_y = \int (z^2 + x^2)\ dm$$

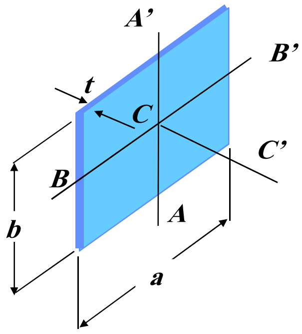
$$I_z = \int (x^2 + y^2)\ dm$$



The ***parallel-axis theorem*** also applies to mass moments of inertia.

$$I = \bar{I} + d^2 m$$

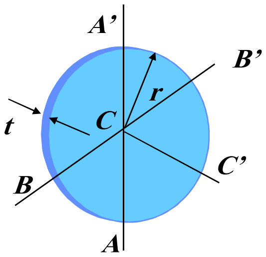
$\bar{I}$ is the mass moment of inertia with respect to the centroidal BB' axis, which is parallel to the AA' axis. The mass of the body is m.



The moments of inertia of ***thin plates*** can be readily obtained from the moments of inertia of their areas. For a ***rectangular plate***, the moments of inertia are

$$I_{AA'} = \frac{1}{12}ma^2 \qquad I_{BB'} = \frac{1}{12}mb^2$$

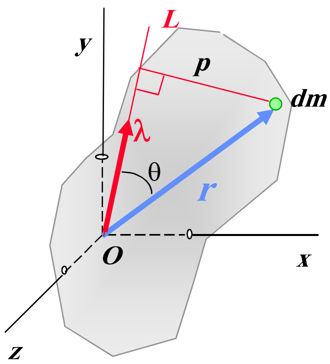
$$I_{CC'} = I_{AA'} + I_{BB'} = \frac{1}{12}m(a^2 + b^2)$$



For a ***circular plate*** they are

$$I_{AA'} = I_{BB'} = \frac{1}{4}mr^2$$

$$I_{CC'} = I_{AA'} + I_{BB'} = \frac{1}{2}mr^2$$



The moment of inertia of a body ***with respect to an arbitrary axis OL*** can be determined. The components of the unit vector λ along line *OL* are λ_x, λ_y, and λ_z.

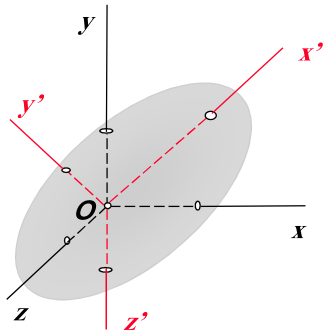
The ***products of inertia*** are

$$I_{xy} = \int xy\,dm \qquad I_{yz} = \int yz\,dm$$

$$I_{zx} = \int zx\,dm$$

The moment of inertia of the body with respect to *OL* is

$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$



By plotting a point Q along each axis OL at a distance $OQ = 1/\sqrt{I_{OL}}$ from O, we obtain the ***ellipsoid of inertia*** of a body. The principal axes x', y', and z' of this ellipsoid are the principal axes of inertia of the body, that is each product of inertia is zero, and we express I_{OL} as

$$I_{OL} = I_{x'}\lambda^2_{x'} + I_{y'}\lambda^2_{y'} + I_{z'}\lambda^2_{z'}$$

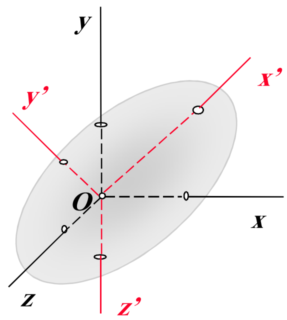
where $I_{x'}$, $I_{y'}$, $I_{z'}$ are the ***principal moments of inertia*** of the body at O.

The principal axes of inertia are determined by solving the cubic equation

$$K^3 - (I_x + I_y + I_z)K^2 + (I_x I_y + I_y I_z + I_z I_x - I_{xy}^2 - I_{yz}^2 - I_{xz}^2)K - (I_x I_y I_z - I_x I_{yz}^2 - I_y I_{zx}^2 - I_z I_{xy}^2 - 2 I_{xy} I_{yz} I_{zx}) = 0$$

The roots K_1, K_2, and K_3 of this equation are the principal moments of inertia. The direction cosines of the principal axis corresponding to each root are determined by using Eq. (9.54) and the identity

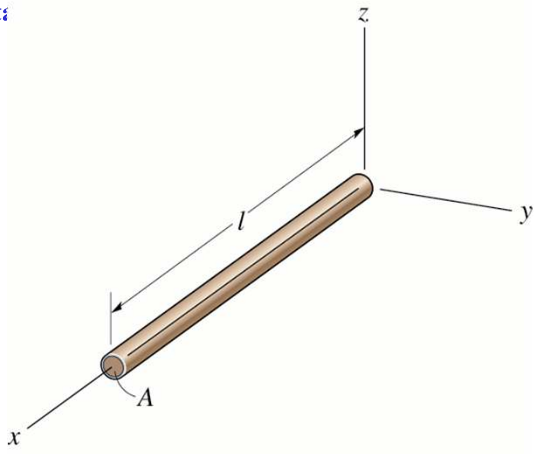
$$\lambda^2_x + \lambda^2_y + \lambda^2_z = 1$$



Moments of Inertia

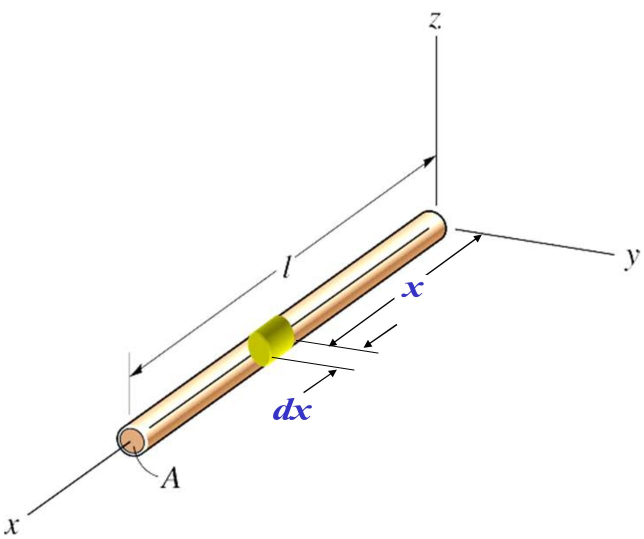
Sample Problem 10.8

Determine the mass moment of inertia I_y for the slender rod. The rod's density ρ and cross-sectional area A are constant. Express the results in terms of the rod's tot



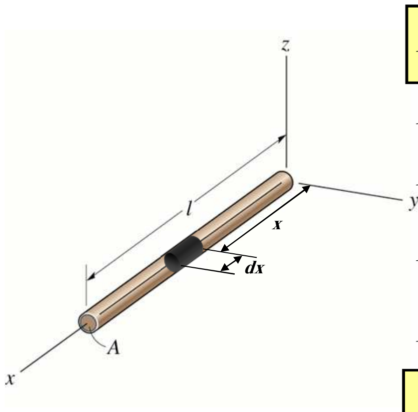
Moments of Inertia

Sample Problem 10.8



Moments of Inertia

Sample Problem 10.8



$$I_y = \int_M x^2 dm$$

$$I_y = \int_0^l x^2 \rho A dx$$

$$I_y = \rho A \int_0^l x^2 dx$$

$$I_y = \rho A \left. \frac{1}{3} x^3 \right|_0^l$$

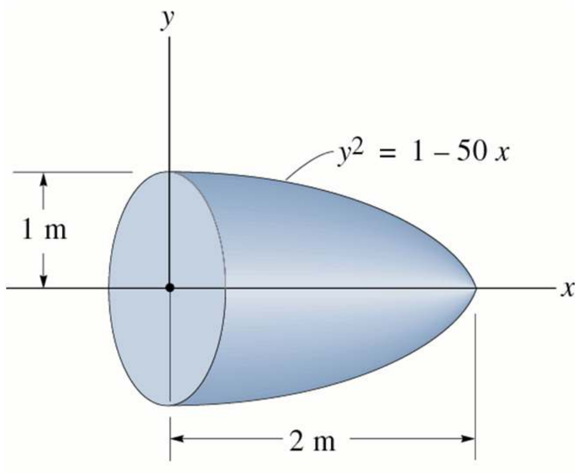
$$I_y = \frac{1}{3} \rho A l^3 = \frac{1}{3} (\rho A l) l^2$$

$$I_y = \frac{1}{3} m l^2$$

Moments of Inertia

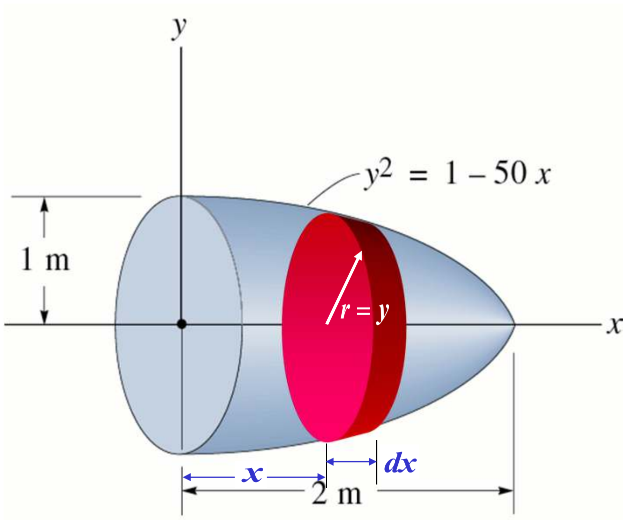
Sample Problem 10.9

The solid is formed by revolving the shaded area around the x axis. Determine the radius of gyration k_x. The density of the material is ρ=5 Mg/m^3.



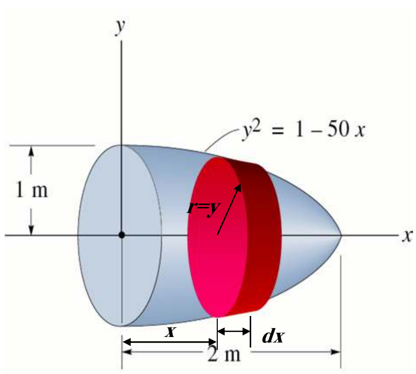
Moments of Inertia

Sample Problem 10.9



Moments of Inertia

Sample Problem 10.9

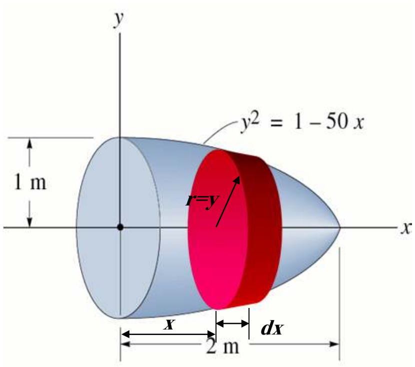


$$dm = \rho dV$$
$$dm = \rho \pi y^2 dx$$
$$dm = \rho \pi (1 - 0.5x) dx$$

$$dI_x = \frac{1}{2} dm y^2$$
$$dI_x = \frac{1}{2} [\rho \pi (1 - 0.5x) dx](1 - 0.5x)$$
$$dI_x = \frac{\rho \pi}{2} (.25x^2 - x + 1) dx$$

Moments of Inertia

Sample Problem 10.9



$$k_x = \sqrt{\frac{I_x}{m}}$$

$$m = \int_m dm$$

$$m = \int_0^2 \rho\pi\,(1 - 0.5x)\,dx$$

$$m = \rho\pi\left(x - \frac{0.5}{2}x^2\right)\Big|_0^2$$

$$m = \rho\pi$$

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Moments of Inertia

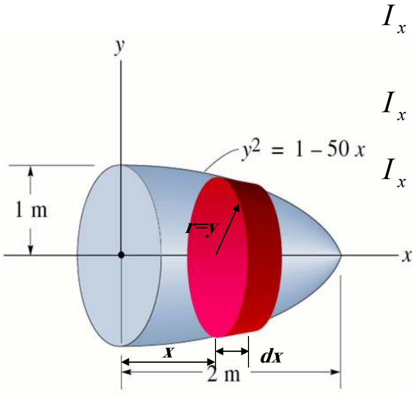
Sample Problem 10.9

$$I_x = \int dI_x$$

$$I_x = \int_0^2 \frac{\rho\pi}{2}\left(.25x^2 - x + 1\right)dx$$

$$I_x = \frac{\rho\pi}{2}\left(\frac{.25x^3}{3} - \frac{x^2}{2} + x\right)\Big|_0^2$$

$$I_x = .3333\,\rho\pi$$



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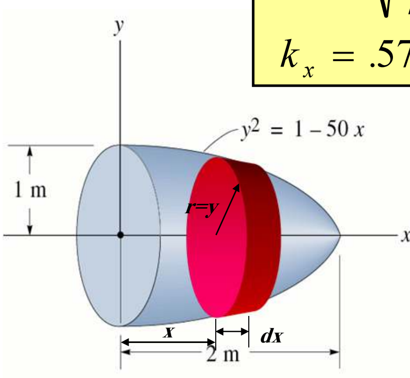
Moments of Inertia

Sample Problem 10.9

$$k_x = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{.3333\ \rho\pi}{\rho\pi}}$$

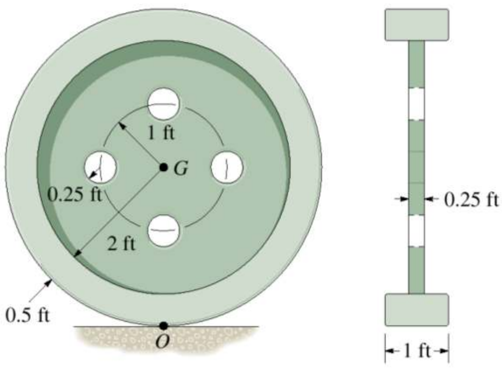
$$k_x = .577\ m$$



Moments of Inertia

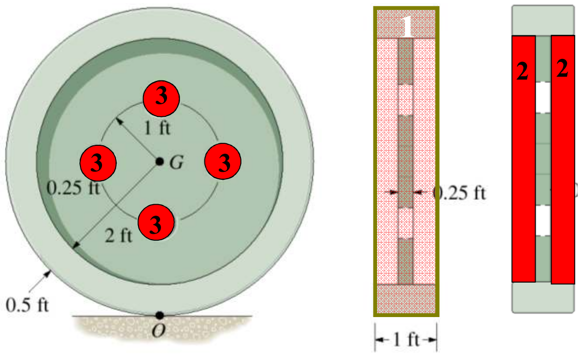
Sample Problem 10.10

Determine the moment of inertia of the wheel about an axis which is perpendicular to the page and passes through the center of mass G. The material has a specific weight $\gamma = 90\ lb/ft^3$.



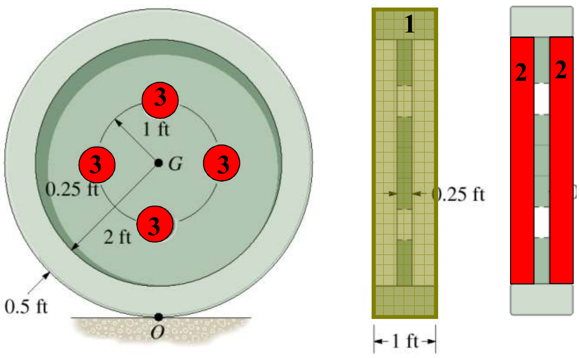
Moments of Inertia

Sample Problem 10.10



Moments of Inertia

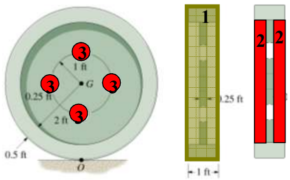
Sample Problem 10.10



$$I_G = \sum_i \left[(I_G)_i + m_i d_i^2 \right]$$

Moments of Inertia

Sample Problem 10.10



note that

$$I_G = \frac{1}{2} mr^2 \text{ then}$$

$$I_G = \sum_i \left[(I_G)_i + m_i d_i^2 \right]$$

$$I_G = \frac{1}{2}\left[\frac{\pi (2.5)^2 (1)(90)}{32.2}\right](2.5)^2 - \frac{1}{2}\left[\frac{\pi (2)^2 (.75)(90)}{32.2}\right](2)^2$$

$$4\left\{\frac{1}{2}\left[\frac{\pi (.25)^2 (.25)(90)}{32.2}\right](.25)^2 + \left[\frac{\pi (.25)^2 (.25)(90)}{32.2}\right](1)^2\right\}$$

$$I_G = 118 \ slug \cdot ft2$$

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§ 9.1-9.5 Area moments of inertia

The *moments of inertia* of an area are defined to be:

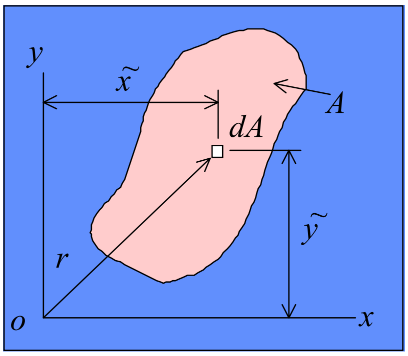
$$I_x = \int_A \tilde{y}^2 dA$$

$$I_y = \int_A \tilde{x}^2 dA$$

$$J_o = \int_A \tilde{r}^2 dA$$

$$I_{xy} = \int_A \tilde{x}\tilde{y} \ dA$$

skim



J_o is the *polar moment of inertia.* Since $r^2=x^2+y^2$,

$$J_0 = I_x + I_y$$

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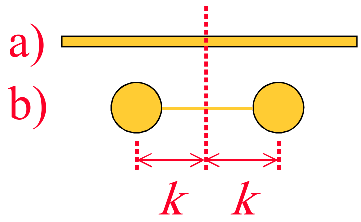
Radius of Gyration of an Area

The *radius of gyration* of an area is *defined* to be:

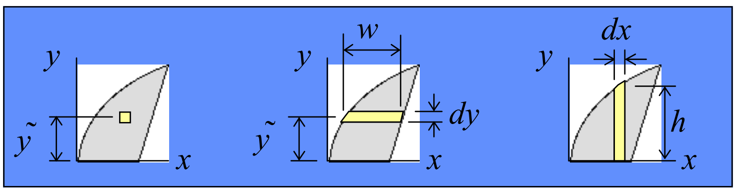
$$k_x = \sqrt{\frac{I_x}{A}} \qquad k_y = \sqrt{\frac{I_y}{A}} \qquad k_o = \sqrt{\frac{J_o}{A}}$$

The *radius of gyration* has units of length. For a complicated object (one that doesn't lend itself to easy integration), it may be easier to specify this radius. It is equivalent to the whole area existing at a single distance, k, from the axis.



Important Subtlety in Evaluating Moments of Inertia using single integrals

Consider evaluation of I_x*:* $\quad I_x = \int \tilde{y}^2 dA$

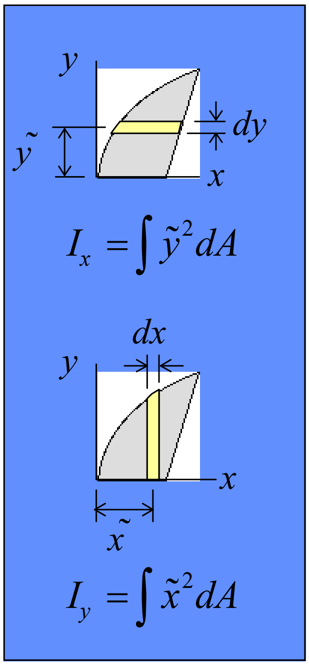


double integral	*single integral*	*single integral*
$I_x = \int \tilde{y}^2\, dxdy$	$I_x = \int \tilde{y}^2\, w(y)\, dy$	$I_x = \int \tilde{y}^2\, h(x)\, dx$
OK because entire area increment has the same *y*-distance.	*OK* because entire area increment has the same *y*-distance.	*NO GOOD* because *y*-distance is different throughout the area slice. $(\tilde{y}^2\, dxdy \neq \tilde{y}^2\, hdx)$

So, for *this* lecture (more next time):

- when computing I_x we must use an area slice parallel to the x-axis, and
- when computing I_y we must use an area slice parallel to the y-axis.

In §9.6 we will use the *parallel axis theorem* to remove this restriction.



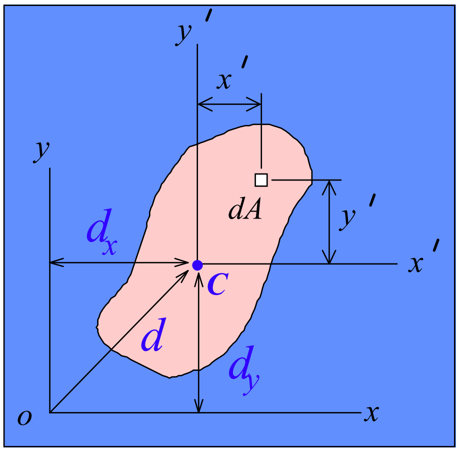
§ 9.6-9.7 Parallel axis theorem; Composite Areas

Relations between moments of inertia referred to parallel axes

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$I_y = \bar{I}_{y'} + Ad_x^2$$

$$J_o = \bar{J}_c + Ad^2$$



$\bar{I}_{x'}$, $\bar{I}_{y'}$, $\bar{J}_c$ = moments about *centroidal* axes *proof:*

$$I_x = \int_A y^2 dA = \int_A (y' + d_y)^2 \, dA$$

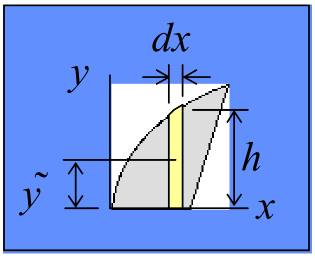
$$= \underbrace{\int_A y'^2 dA}_{\bar{I}_{x'}} + 2d_y \underbrace{\int_A y' dA}_{0} + d_y^2 \underbrace{\int_A dA}_{A}$$

$$= \bar{I}_{x'} + A d_y^2$$

What is the parallel axis theorem good for?

In practice, it will often be convenient for us to compute moments of inertia about centroidal axes. Using the parallel axis theorem, we can then express moments of inertia about other axes that are needed.

To evaluate I_x use parallel axis theorem applied to the area slice:



$$I_x = \int dI_x$$

From || axis theorem: $dI_x = d\bar{I}_{x'} + \tilde{y}^2 dA$

where $\tilde{y}$ is the distance to the centroid of dA.

$$I_x = \int d\bar{I}_{x'} + \int \tilde{y}^2 dA$$

$$= \int \frac{1}{12} h^3 dx + \int \left(\frac{h}{2}\right)^2 h dx$$

Moments of Inertia for composite areas

$$I_x = \sum_{i=1}^{n} \left(\bar{I}_{x'} + A d_y^2 \right)_i \qquad I_y = \sum_{i=1}^{n} \left(\bar{I}_{y'} + A d_x^2 \right)_i$$

$$J_o = \sum_{i=1}^{n} \left(\bar{J}_c + A d^2 \right)_i$$

n = # of composite areas

$\bar{I}_{x'}, \ \bar{I}_{y'}, \ \bar{J}_c$ = moments of area i about *centroidal* axes of area i (easy to get for simple shapes)

گردآوری و تنظیم: محمدحسین ابوالبشری

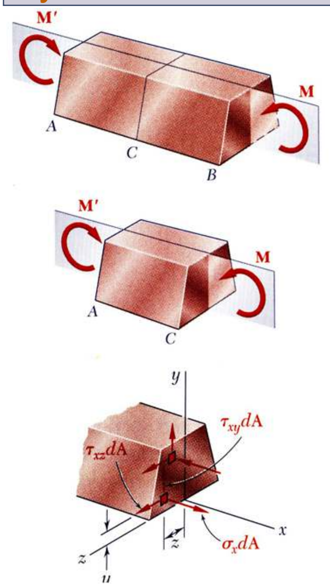
9 - ۷۱

MECHANICS OF MATERIALS

Third Edition

Beer • Johnston • DeWolf

Symmetric Member in Pure Bending

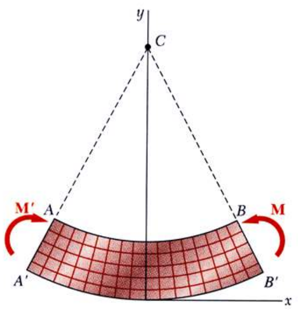


- Internal forces in any cross section are equivalent to a couple. The moment of the couple is the section *bending moment.*
- From statics, a couple M consists of two equal and opposite forces.
- The sum of the components of the forces in any direction is zero.
- The moment is the same about any axis perpendicular to the plane of the couple and zero about any axis contained in the plane.
- These requirements may be applied to the sums of the components and moments of the statically indeterminate elementary internal forces.

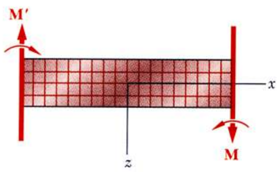
$$F_x = \int \sigma_x \, dA = 0$$
$$M_y = \int z\sigma_x \, dA = 0$$
$$M_z = \int -y\sigma_x \, dA = M$$

4 - ۷۲

Bending Deformations



(a) Longitudinal, vertical section (plane of symmetry)

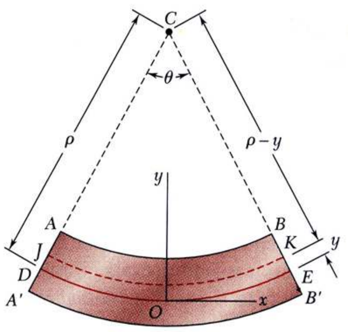


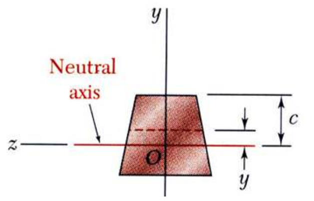
(b) Longitudinal, horizontal section

Beam with a plane of symmetry in pure bending:

- member remains symmetric
- bends uniformly to form a circular arc
- cross-sectional plane passes through arc center and remains planar
- length of top decreases and length of bottom increases
- a *neutral surface* must exist that is parallel to the upper and lower surfaces and for which the length does not change
- stresses and strains are negative (compressive) above the neutral plane and positive (tension) below it

Strain Due to Bending





Consider a beam segment of length L.

After deformation, the length of the neutral surface remains L. At other sections,

$$L' = (\rho - y)\theta$$

$$\delta = L - L' = (\rho - y)\theta - \rho\theta = -y\theta$$

$$\varepsilon_x = \frac{\delta}{L} = -\frac{y\theta}{\rho\theta} = -\frac{y}{\rho} \quad \text{(strain varies linearly)}$$

$$\varepsilon_m = \frac{c}{\rho} \quad \text{or} \quad \rho = \frac{c}{\varepsilon_m}$$

$$\varepsilon_x = -\frac{y}{c}\varepsilon_m$$

Stress Due to Bending

- For a linearly elastic material,

$$\sigma_x = E\varepsilon_x = -\frac{y}{c}E\varepsilon_m$$

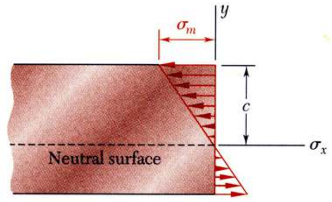
$$= -\frac{y}{c}\sigma_m \quad \text{(stress varies linearly)}$$

- For static equilibrium,

$$F_x = 0 = \int \sigma_x \, dA = \int -\frac{y}{c}\sigma_m \, dA$$

$$0 = -\frac{\sigma_m}{c}\int y \, dA$$

First moment with respect to neutral plane is zero. Therefore, the neutral surface must pass through the section centroid.



- For static equilibrium,

$$M = \int -y\sigma_x \, dA = \int -y\left(-\frac{y}{c}\sigma_m\right) dA$$

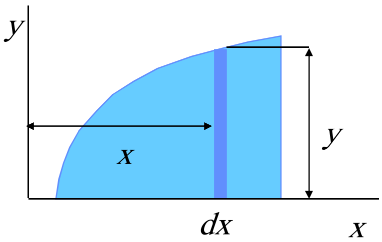
$$M = \frac{\sigma_m}{c}\int y^2 \, dA = \frac{\sigma_m I}{c}$$

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

Substituting $\sigma_x = -\frac{y}{c}\sigma_m$

$$\sigma_x = -\frac{My}{I}$$

Chapter 9 DISTRIBUTED FORCES: MOMENTS OF INERTIA

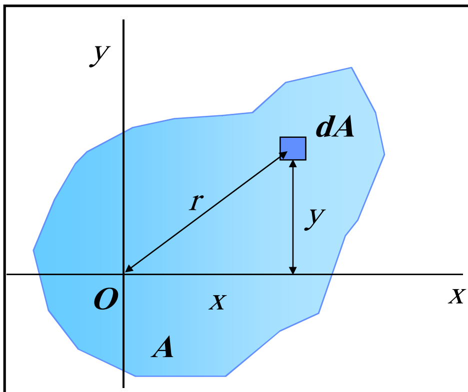


The ***rectangular moments of inertia*** I_x ***and*** I_y ***of an area*** are defined as

$$I_x = \int y^2 dA \qquad I_y = \int x^2 dA$$

These computations are reduced to single integrations by choosing dA to be a thin strip parallel to one of the coordinate axes. The result is

$$dI_x = \frac{1}{3}\, y^3 dx \qquad dI_y = \, x^2 y dx$$

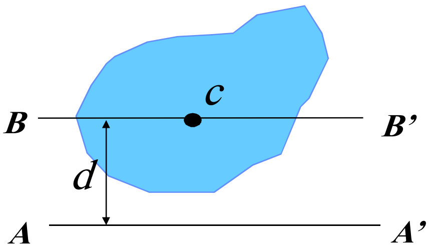


The ***polar moment of inertia of an area A*** with respect to the pole O is defined as

$$J_O = \int r^2 dA$$

The distance from O to the element of area dA is r. Observing that $r^2 = x^2 + y^2$, we established the relation

$$J_O = I_x + I_y$$

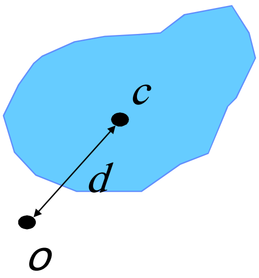


The ***parallel-axis theorem*** states that the moment of inertia I of an area with respect to any given axis AA' is equal to the moment of inertia $\bar{I}$ of the area with respect to the centroidal *axis BB' that is parallel to AA'* ***plus*** the product of the area A and the square of the distance d between the two axes:

$$I = \bar{I} + Ad^2$$

This expression can also be used to determine $\bar{I}$ when the moment of inertia with respect to AA' is known:

$$\bar{I} = I - Ad^2$$



A similar theorem can be used with the polar moment of inertia. The polar moment of inertia J_O of an area about O and the polar moment of inertia $\bar{J}_C$ of the area about its centroid are related to the distance d between points C and O by the relationship

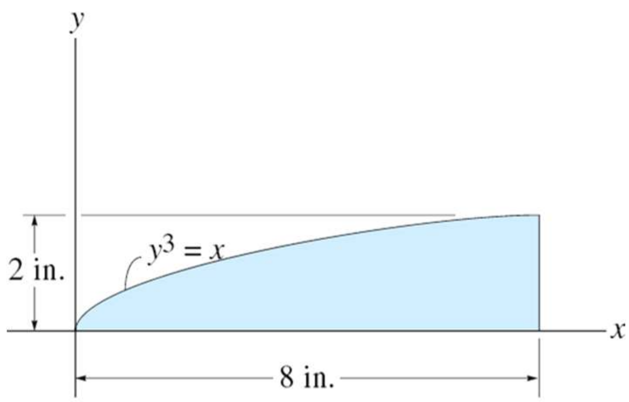
$$J_O = \bar{J}_C + Ad^2$$

The parallel-axis theorem is used very effectively to compute the ***moment of inertia of a composite area*** with respect to a given axis.

Moments of Inertia

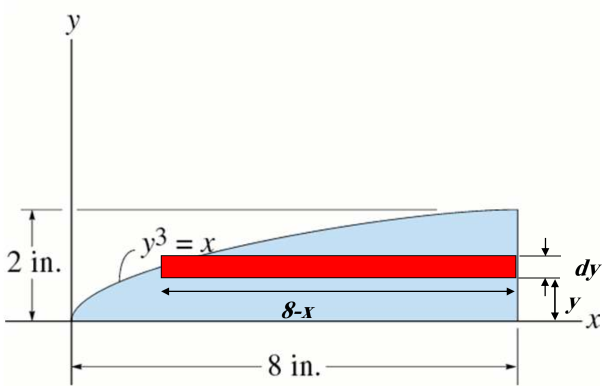
Sample Problem 10.2

Determine the moment of Inertia about the y-axis



Moments of Inertia

Sample Problem 10.2



Moments of Inertia

Sample Problem 10.2

$$dI_y = dI_{\bar{y}} + \overline{x^2}dA$$

$$dI_y = \frac{1}{12}hb^3 + \left[x + \frac{1}{2}(8-x)\right]^2 [(8-x)dy]$$

$$dI_y = \frac{1}{12}(dy)(8-x)^3 + \left[y^3 + \frac{1}{2}(8-y^3)\right]^2 [(8-y^3)dy]$$

$$dI_y = \frac{1}{12}(dy)(8-y^3)^3 + \left[\frac{1}{2}(y^3+8)\right]^2 [(8-y^3)dy]$$

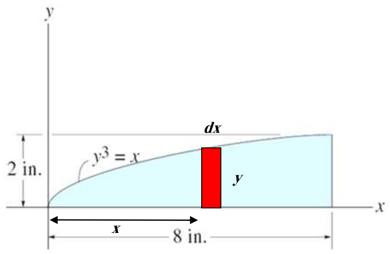
$$dI_y = \left\{\frac{1}{12}(8-y^3)^3 + \frac{1}{4}(y^3+8)^2(8-y^3)\right\}dy$$

$$I_y = \int_A dI_y = \int_0^2 \left\{\frac{1}{12}(8-y^3)^3 + \frac{1}{4}(y^3+8)^2(8-y^3)\right\}dy$$

$$I_y = 307 \ \ in^4$$

Moments of Inertia

Sample Problem 10.2



$$dA = y\,dx$$

$$I_y = \int_A x^2 dA = \int_0^8 x^2 y\,dx$$

$$I_y = \int_0^8 x^{\frac{7}{3}} dx = \frac{3}{10} x^{\frac{10}{3}} \Big|_0^8$$

$$I_y = 307 \quad in^4$$