Chapter 5
(8 and 9 of 2nd ed.)
Numerical Methods for Unconstrained Optimum Design

Why Do We Need to Study the Numerical Methods for Non-linear Optimization

Graphical and analytical methods are inappropriate for many complicated engineering design problems. Because:

1. The number of design variables and constraints can be large.
2. The functions for the design problem can be highly nonlinear.
3. In many engineering applications, objective and/or constraint functions can be implicit in terms of design variables.

General Concepts Related to Numerical Algorithms

Unconstrained optimization problem classification

- **One-dimensional or line search problems**
  To find a scalar α^* to minimize a function $f(\alpha)$
- **Multidimensional problems**
  To find points x^* to minimize a function $f(x)=f(x_1,x_2,\ldots,x_n)$

A General Algorithm

Vector form:

$$x^{(k+1)}=x^{(k)}+\Delta x^{(k)},\ k=0,1,2,\ldots$$

Component form:

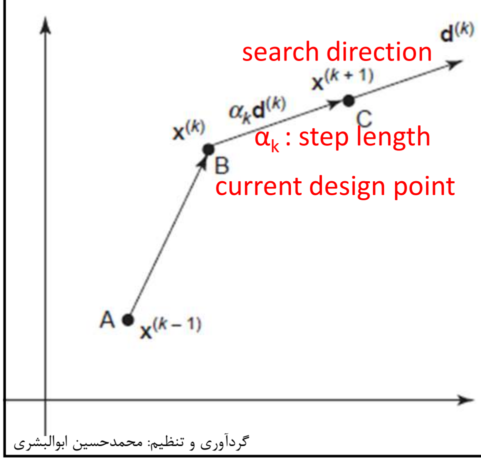
$$x_i^{(k+1)}=x_i^{(k)}+\Delta x_i^{(k)},\ i=1 \text{ to } n; \qquad k=0,1,2,\ldots$$

$$\Delta x^{(k)}=\alpha_k\, d^{(k)}$$

$d^{(k)}$: a “desirable” search direction in the design space

α_k: a positive scalar called the step size in that direction

If the direction $d^{(k)}$ is any “good,” then the step size must be greater than 0.



Thus, the process of computing $\Delta x^{(k)}$ involves solving two separate subproblems:

- The direction finding (Methods are:)
  1) Steepest descent, تندترین کاهش
  2) Conjugate gradients, گرادیان مزدوج
  3) Newton, نیوتن
  4) Quasi-Newton شبه نیوتن
- The step length determination (scaling along the direction)(Methods are:)
  1) Equal Interval, جستجوی بازه یکسان
  2) Golden Section, جستجوی بخش طلایی
  3) Polynomial interpolation درون‌یابی چند جمله‌ای

The process of moving from one design point to the next is illustrated in Fig.

In summary:

The basic idea of numerical methods for nonlinear optimization problems is:

- To start with a reasonable estimate for the optimum design.
- Cost and constraint functions and their derivatives are evaluated at that point.
- Based on them, the design is moved to a new point.
- The process is continued until either optimality conditions or some other stopping criteria are met.

This iterative process represents an organized search through the design space for points that represent local minima. Thus, the procedures are often called the search techniques or direct methods of optimization.

A general algorithm for iterative process that is applicable to both constrained and unconstrained problems

Step 1. Estimate a reasonable starting design $x^{(0)}$. Set the iteration counter $k=0$.

Step 2. Compute a search direction $d^{(k)}$ in the design space. This calculation generally requires a cost function value and its gradient for **unconstrained** problems and, in addition, constraint functions and their gradients for **constrained** problems.

Step 3. Check for convergence of the algorithm. If it has converged, stop; otherwise, continue.

Step 4. Calculate a positive step size α_k in the direction $d^{(k)}$.

Step 5. Update the design as follows, set $k=k+1$ and go to Step 2:

$$x^{(k+1)}=x^{(k)}+\alpha_k d^{(k)} \qquad (8.4)$$

Descent Direction and Descent Step

If $x^{(k)}$ is not a minimum point, then we should be able to find another point $x^{(k+1)}$ with a smaller cost function value than the one at $x^{(k)}$.

$$f(x^{(k+1)})<f(x^{(k)}) \qquad (8.5)$$

Substitute $x^{(k+1)}$ from Eq. (8.4) into the preceding inequality to obtain

$$f(x^{(k)}+\alpha_k d^{(k)})<f(x^{(k)}) \qquad (8.6)$$

Approximating the left side of Eq. (8.6) by the linear Taylor's expansion at the point $x^{(k)}$, we get

$$f(x^{(k)})+\alpha_k (c^{(k)}.d^{(k)})<f(x^{(k)}) \qquad (8.7)$$

where

$$c^{(k)}=\nabla f(x^{(k)})$$

Since $\alpha_k>0$, it may be dropped without affecting the inequality. Therefore, we get the condition

$$c^{(k)}.d^{(k)}<0 \quad \text{descent condition} \qquad (8.8)$$

Geometrically: The angle between the vectors $c^{(k)}$ and $d^{(k)}$ must be between 90° and 270° ($cd \cos \beta<0$)

EXAMPLE 8.1 Check for the Descent Condition

For the function

$$f(x)=x_1^2-x_1x_2+2x_2^2-2x_1+e^{(x_1+x_2)} \qquad \text{(a)}$$

check if the direction d=(1,2) at the point (0,0) is a descent direction for the function f.

Solution. If d=(1,2) is a descent direction, then it must satisfy Inequality (8.8). To verify this, we calculate the gradient c of the function $f(x)$ at (0,0) and evaluate ($c\cdot d$), as

$$c=(2x_1-x_2-2+e^{(x_1+x_2)},-x_1+4x_2+e^{(x_1+x_2)})=(-1,1) \qquad \text{(b)}$$

$$(c\cdot d)=(-1,1)\begin{bmatrix}1\\2\end{bmatrix}=-1+2=1>0 \qquad \text{(c)}$$

Inequality (8.8) is violated, and thus the given d is not a descent direction for the function $f(x)$.

Convergence of Algorithms

The central idea behind numerical methods of optimization is to search for the optimum point in an iterative manner, generating a sequence of designs. It is important to note that the success of a method depends on the guarantee of convergence of the sequence to the optimum point.

The property of convergence to a local optimum point irrespective of the starting point is called global convergence of the numerical method.
For **unconstrained** problems, a convergent algorithm must reduce the cost function at each iteration until a minimum point is reached.

Note: the algorithms converge to a local minimum point only, as opposed to a global minimum, since they only use the local information about the cost function and its derivatives in the search process.

Rate of Convergence

In practice, a numerical method may take a large number of iterations to reach the optimum point. Therefore, it is important to employ methods having a faster rate of convergence.

Rate of convergence of an algorithm is usually measured by the numbers of iterations and function evaluations needed to obtain an acceptable solution.

Rate of convergence is a measure of how fast the difference between the solution point and its estimates goes to zero.

Faster algorithms usually use second-order information about the problem functions when calculating the search direction. They are known as Newton methods.

Many algorithms also approximate second-order information using only the first-order information.

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Basic Ideas and Algorithms for Step Size Determination

This is often called the one-dimensional search (or, line search) problem.

Definition of One-Dimensional Minimization Subproblem

For an optimization problem with several variables, the direction finding problem must be solved first. Then, a step size must be determined by searching for the minimum of the cost function along the search direction. This is always a one-dimensional minimization problem.

$$x^{(k+1)}=x^{(k)}+\alpha_k d^{(k)} \qquad (8.9a)$$

$$f(x^{(k+1)})=f(x^{(k)}+\alpha_k d^{(k)})=\bar{f}(\alpha) \qquad (8.9b)$$

where $\bar{f}(\alpha)$ is the new function with α as the only independent variable (in the sequel, we shall drop the overbar for functions of single variable).

Note that at $\alpha=0$, $f(0)=f(x^{(k)})$ from Eq. (8.9b), which is the current value of the cost function.
It is important:

- To understand this reduction of a function of n variables to a function of only one variable since this fundamental step is used in almost all optimization methods.

- To understand the geometric significance of Eq. (8.9b).

We shall elaborate on these ideas later.

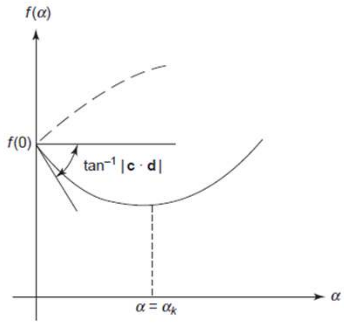
If $x^{(k)}$ is not a minimum point, then it is possible to find a descent direction $d^{(k)}$ at the point and reduce the cost function further. Recall that a small move along $d^{(k)}$ reduces the cost function.

Therefore, using Eqs. (8.5) and (8.9b), the descent condition for the cost function can be expressed as the inequality:

$$f(\alpha)<f(0) \qquad (8.10)$$

Since $f(\alpha)$ is a function of single variable, we can plot $f(\alpha)$ versus α. To satisfy Inequality (8.10) ($f(\alpha)<f(0)$), the curve $f(\alpha)$ versus α must have a negative slope at the point α=0. Such a curve is shown by the solid line in Fig.
It must be understood that if the search direction is that of descent, the graph of $f(\alpha)$ versus α cannot be the one shown by the dashed curve because any positive α would cause the function $f(\alpha)$ to increase, violating Inequality (8.10) ($f(\alpha)<f(0)$).
This would also be a contradiction as $d^{(k)}$ is a direction of descent for the cost function. Therefore, the graph of $f(\alpha)$ versus α must be the solid curve in Fig. for all problems.

In fact, the slope of the curve $f(\alpha)$ at α=0 is calculated as $f'(0)=c^{(k)}\cdot d^{(k)}$ (see eq. (8.11) next slide), which is negative as seen in inequality (8.8) ($c^{(k)}.d^{(k)}<0$). This discussion shows that if $d^{(k)}$ is a descent direction, then α must always be a positive scalar in Eq. (8.8). Thus, the one-dimensional minimization problem is to find $\alpha_k=\alpha$ such that $f(\alpha)$ is minimized.



Graph of f(α) versus α.
FIGURE 8-3

8.2.2 Analytical Method to Compute Step Size

If $f(\alpha)$ is a simple function, then we can use the analytical procedure to determine α_k (necessary and sufficient conditions of Section 4.3).
The necessary condition is $df(\alpha_k)/d\alpha=0$, and
The sufficient condition is $d^2f(\alpha_k)/d\alpha^2>0$.
We shall illustrate the analytical line search procedure with Example 8.2. Note that differentiation of $f(x^{(k+1)})$ in Eq. (8.9b) with respect to α, using the chain rule of differentiation and setting it to zero, gives

$$\frac{df(x^{(k+1)})}{d\alpha} = \frac{\partial f^T(x^{(k+1)})}{\partial \alpha}\frac{d(x^{(k+1)})}{d\alpha} = \nabla f(x^{(k+1)}).d^{(k)} = c^{(k+1)}.d^{(k)} = 0 \quad (8.11)$$

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Since the dot product of two vectors is zero in Eq. (8.11), $(c^{(k+1)}.d^{(k)}=0)$, the gradient of the cost function at the new point is orthogonal to the search direction at the kth iteration, i.e., $c^{(k+1)}$ is normal to $d^{(k)}$. The condition in Eq. (8.11) is important for two reasons:

(1) It can be used directly to obtain an equation in terms of step size α whose smallest root gives the exact step size, and
(2) It can be used to check the accuracy of the step size in a numerical procedure to calculate α and thus it is called the line search termination criterion.

Many times numerical line search methods will give an approximate or inexact value of the step size along the search direction. The line search termination criterion is useful for determining the accuracy of the step size; i.e., for checking $c^{(k+1)}\cdot d^{(k)} = 0$.

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EXAMPLE 8.2 Analytical Step Size Determination

Let a direction of change for the function

$$f(x) = 3x_1^2 + 2x_1x_2 + 2x_2^2 + 7 \quad \text{(a)}$$

at the point (1,2) be given as (-1,-1). Compute the step size α_k to minimize *f(x)* in the given direction.

Solution. For the given point $x^{(k)}=(1,2)$, $f(x^{(k)})=22$, and $d^{(k)}=(-1,-1)$. We first check to see if $d^{(k)}$ is a direction of descent using Inequality (8.8) ($c^{(k)}.d^{(k)}<0$). The gradient of the function at (1,2) is given as $c^{(k)}=(10,10)$ and $c^{(k)} \cdot d^{(k)}=10(-1)+10(-1)=-20<0$.
Therefore, (-1,-1) is a direction of descent. The new point $x^{(k+1)}$ using Eq. (8.9a) ($x^{(k+1)}=x^{(k)}+\alpha_k d^{(k)}$) is given as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{(k+1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \text{or} \quad x_1^{(k+1)} = 1-\alpha;\; x_2^{(k+1)} = 2-\alpha \quad \text{(b)}$$

Substituting these equations into the cost function of Eq. (a), we get

$$f(x^{(k+1)}) = 3(1-\alpha)^2 + 2(1-\alpha)(2-\alpha) + 2(2-\alpha)^2 + 7$$
$$= 7\alpha^2 - 20\alpha + 22 = f(\alpha) \quad \text{(c)}$$

Therefore, along the given direction (-1,-1), *f(x)* becomes a function of the single variable α.
Note from Eq. (c) that $f(0)=22$, which is the cost function value at the current point, and that $f'(0)=-20<0$, which is the slope of $f(\alpha)$ at $\alpha=0$ (also recall that $f'(0)=c^{(k)} \cdot d^{(k)}$).
Now using the necessary and sufficient conditions of optimality for $f(\alpha)$, we obtain

$$\frac{df}{d\alpha} = 14\alpha_k - 20 = 0, \quad \alpha_k = \frac{10}{7}; \; \frac{d^2f}{d\alpha^2} = 14 > 0 \quad \text{(d)}$$

Therefore, $\alpha_k=10/7$ minimizes *f(x)* in the direction (-1,-1). The new point is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{(k+1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{10}{7}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} \\ \frac{4}{7} \end{bmatrix} \quad \text{(e)}$$

$f(x)^{(k+1)}=f(-3/7, 4/7) = 54/7$

This is a substantial reduction from the cost function value of 22 at the previous point.

Note that Eq. (d) for calculation of step size α can also be obtained by directly using the condition given in Eq. (8.11)($\nabla f(x^{(k+1)}).d^{(k)} = 0$).

Using Eq. (b), $x_1^{(k+1)} = 1-\alpha;\ x_2^{(k+1)} = 2-\alpha$ the gradient of f at the new design point in terms of α is given as

$$c^{(k+1)} = (6x_1 + 2x_2,\ 2x_1 + 4x_2) = (10-8\alpha,\ 10-6\alpha) \quad \text{(f)}$$

Using the condition of Eq. (8.11), we get 14α-20=0 which is same as Eq. (d).

8.2.3 Concepts Related to Numerical Methods to Compute Step Size

In Example 8.2:

• It was possible to simplify expressions and obtain an explicit form for the function $f(\alpha)$.

• Also, the functional form of $f(\alpha)$ was quite simple.

Therefore, it was possible to use the necessary and sufficient conditions of optimality to find the minimum of $f(\alpha)$ and analytically calculate the step size α_k.

For many problems:

• It is not possible to obtain an explicit expression for $f(\alpha)$.

• Moreover, even if the functional form of $f(\alpha)$ is known, it may be too complicated to lend itself to analytical solution.

Therefore, a numerical method must be used to find α_k to minimize $f(x)$ in the known direction $d^{(k)}$.

Usually, we must make some assumptions on the form of the line search function to compute step size by numerical methods.
For example, it must be assumed that a minimum exists and that it is unique in some interval of interest. A function with this property is called the *unimodal function.*

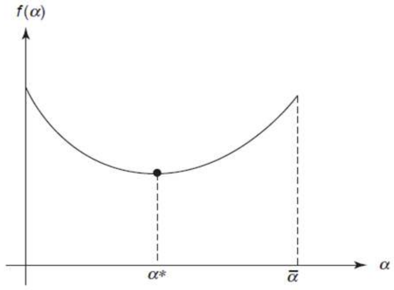


FIGURE 8-4 Unimodal function $f(\alpha)$.

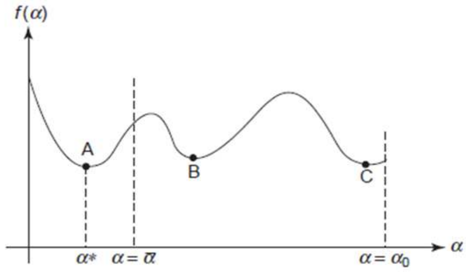


FIGURE 8-5 Nonunimodal function $f(\alpha)$ for $0 \le \alpha \le \alpha_0$ (unimodal for $0 \le \alpha \le \bar{\alpha}$).

Figure 8-4 shows the graph of such a function that decreases continuously until the minimum point is reached. Comparing Figs. 8-3 and 8-4, we observe that *f(α)* is a unimodal function in some interval.
Therefore, it has a unique minimum.

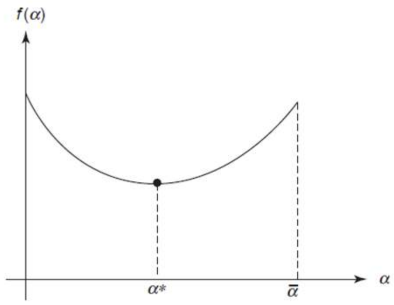


FIGURE 8-4 Unimodal function $f(\alpha)$.

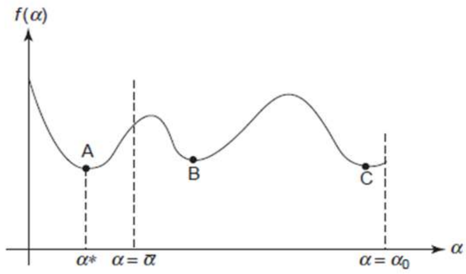


FIGURE 8-5 Nonunimodal function $f(\alpha)$ for $0 \le \alpha \le \alpha_0$ (unimodal for $0 \le \alpha \le \bar{\alpha}$).

Most one-dimensional search methods assume the line search function to be a unimodal function. This may appear to be a severe restriction on the methods; however, it is not.
For functions that are not unimodal, we can think of locating only a local minimum point that is closest to the starting point, i.e., closest to $\alpha=0$.
This is illustrated in Fig. 8-5, where the function $f(\alpha)$ is not unimodal for $0 \le \alpha \le \alpha_0$. Points *A*, *B*, and *C* are all local minima. If we restrict α to lie between 0 and $\bar{\alpha}$, however, there is only one local minimum point *A* because the function $f(\alpha)$ is unimodal for for $0 \le \alpha \le \bar{\alpha}$. Thus, the assumption of unimodality is not as restrictive as it appears.

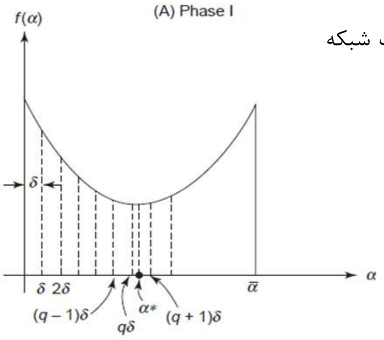
The line search problem then is to find α in an interval $0 \leq \alpha \leq \bar{\alpha}$ at which the function *f(α)* has a global minimum.
This statement of the problem, however, requires some modification.
Since we are dealing with numerical methods, it is not possible to locate the exact minimum point α^*.
In fact, what we determine is the interval in which the minimum lies, i.e., some lower and upper limits α_l and α_u for α^*.
The interval (α_l, α_u) is called the interval of uncertainty and is designated as $I=\alpha_u-\alpha_l$. (بازه عدم اطمینان)
Most numerical methods iteratively reduce the interval of uncertainty until it satisfies a specified tolerance ε, i.e., $I<\varepsilon$.
Once this stopping criterion is satisfied, α^* is taken as $0.5(\alpha_l+\alpha_u)$.
Methods based on the preceding philosophy are called interval reducing methods.

The basic procedure for these methods can be divided into two phases.

In phase one, the location of the minimum point is bracketed and the initial interval of uncertainty is established.

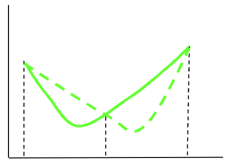
In the second phase, the interval of uncertainty is refined by eliminating regions that cannot contain the minimum.
This is done by computing and comparing function values in the interval of uncertainty.

8.2.4 Equal Interval Search (جستجوی بازه یکسان)



(A) Phase I: Initial bracketing of minimum.

ابتدا تابع در بازه $0 \le \alpha \le \bar{\alpha}$ در نقاط متعددی با استفاده از یک شبکه یکنواخت محاسبه می‌شود.



If the function has started to increase, i.e.,

$$f(q\delta)<f((q+1)\delta) \qquad (8.12)$$

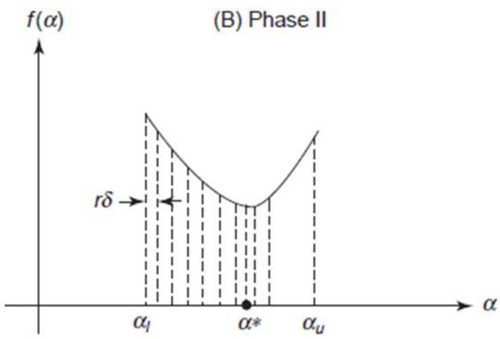
then the minimum has been surpassed.

Note that once Eq. (8.12) is satisfied for points q and $(q+1)$, the minimum can be between either the points $(q-1)$ and q or the points q and $(q+1)$. To account for both possibilities, we take the minimum to lie between the points $(q-1)$ and $(q+1)$.

$\alpha_l=(q-1)\delta,$ $\alpha_u=(q+1)\delta,$ $I=\alpha_u-\alpha_l=2\delta,$

Reducing the Interval of Uncertainty—Phase II

Establishment of the lower and upper limits on the minimum value of a indicates the end of Phase I. In Phase II, we restart the search process from the lower end of the interval of uncertainty $\alpha=\alpha_l$ with some reduced value for the increment δ, say $r\delta$, where $r \ll 1$.



(B) Phase II: Reducing the interval of uncertainty.

Then the preceding process of Phase I is repeated from $\alpha=\alpha_l$ with the reduced δ, and the minimum is again bracketed. Now the interval of uncertainty I is reduced to $2r\delta$.

The value of the increment is further reduced to, say, $r^2\delta$, and the process is repeated until the interval of uncertainty is reduced to an acceptable value ε.

Note that the method is convergent for unimodal functions and can be easily coded into a computer program.

The efficiency of a method such as the equal interval search depends on the number of function evaluations needed to achieve the desired accuracy.
Clearly, this depends on the initial choice for the value of δ.

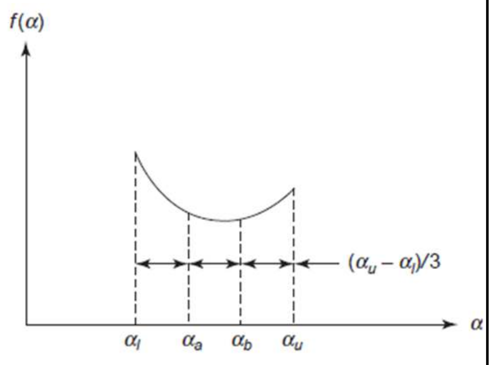
If δ is very small	Disadvantage: The process may take many function evaluations to initially bracket the minimum. Advantage: the interval of uncertainty at the end of the Phase I is fairly small. Subsequent improvements for the interval of uncertainty require fewer function evaluations.

It is usually advantageous to start with a larger value of δ and quickly bracket the minimum point. Then, the process is continued until the accuracy requirement is satisfied.

8.2.5 Alternate Equal Interval Search (جایگزین جستجوی بازه یکسان)

A slightly different computational procedure can be followed to reduce the interval of uncertainty in Phase II once the minimum has been bracketed in Phase I.

The procedure is to evaluate the function at two new points, say α_a and α_b in the interval of uncertainty. The points α_a and α_b are located at a distance of $I/3$ and $2I/3$ from the lower limit α_l, respectively, where $I=\alpha_u-\alpha_l$. That is,



An alternate equal interval search process.

$$\alpha_a = \alpha_l + \frac{1}{3}I \quad ; \quad \alpha_b = \alpha_l + \frac{2}{3}I = \alpha_u - \frac{1}{3}I$$

8.2.6 Golden Section Search (جستجوی بخش طلایی)

Golden section search is an improvement over the alternate equal interval search and is one of the better methods in the class of interval reducing methods.

The basic idea of the method is still the same:

- Evaluate the function at predetermined points,
- Compare them to bracket the minimum in Phase I,
- And then converge on the minimum point in Phase II.

The method uses fewer function evaluations to reach the minimum point compared with other similar methods.

The number of function evaluations is reduced during both the phases, the initial bracketing phase as well as the interval reducing phase.

Initial Bracketing of Minimum—Phase I

In the equal interval methods, the initially selected increment δ is kept fixed to bracket the minimum initially. This can be an inefficient process if δ happens to be a small number.

An alternate procedure is to vary the increment at each step, i.e., multiply it by a constant $r>1$.

This way initial bracketing of the minimum is rapid; however, the length of the initial interval of uncertainty is increased. The golden section search procedure is such a variable interval search method.

In the method the value of r is not selected arbitrarily. It is selected as the golden ratio, which can be derived as 1.618 in several different ways.

$$q = 0; \quad \alpha_0 = \delta$$

$$q = 1; \quad \alpha_1 = \delta + 1.618\delta = 2.618\delta = \sum_{j=0}^{1} \delta(1.618)^j$$

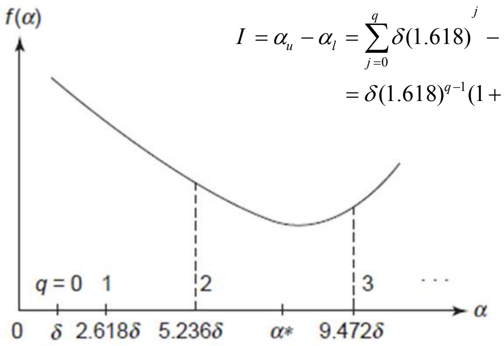
$$q = 2; \quad \alpha_2 = 2.618\delta + 1.618(1.618\delta) = 5.236\delta = \sum_{j=0}^{2} \delta(1.618)^j$$

$$q = 3; \quad \alpha_3 = 5.236\delta + (1.618^3\delta) = 9.472\delta = \sum_{j=0}^{3} \delta(1.618)^j$$

$$\alpha_q = \sum_{j=0}^{q} \delta(1.618)^j, \; q = 0,1,2,\cdots \qquad \alpha_u = \alpha_q = \sum_{j=0}^{q} \delta(1.618)^j, \; \alpha_l = \alpha_{q-2} = \sum_{j=0}^{q-2} \delta(1.618)^j$$

$$I = \alpha_u - \alpha_l = \sum_{j=0}^{q} \delta(1.618)^j - \sum_{j=0}^{q-2} \delta(1.618)^j = \delta(1.618)^{q-1} + \delta(1.618)^q$$

$$= \delta(1.618)^{q-1}(1 + 1.618) = 2.618(1.618)^{q-1}\delta$$



Initial bracketing of the minimum point in the golden section method.

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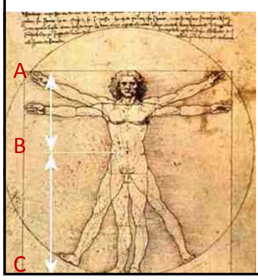
Reduction of Interval of Uncertainty—Phase II

The next task is to start reducing the interval of uncertainty by evaluating and comparing functions at some points in the established interval of uncertainty ***I***. The method uses two function values within the interval ***I***, just as in the alternate equal interval search of the Fig. However, the points α_a and α_b are not located at ***I***/3 from either end of the interval of uncertainty. Instead, they are located at a distance of 0.382***I*** (or 0.618***I***) from either end. The factor 0.382 is related to the golden ratio as we shall see in the following.



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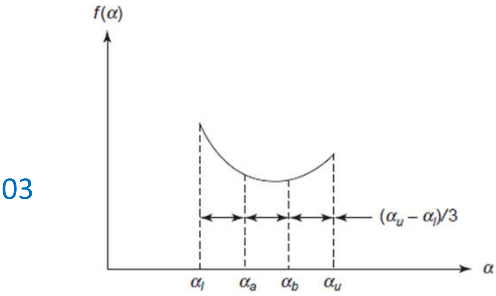

An alternate equal interval search process.

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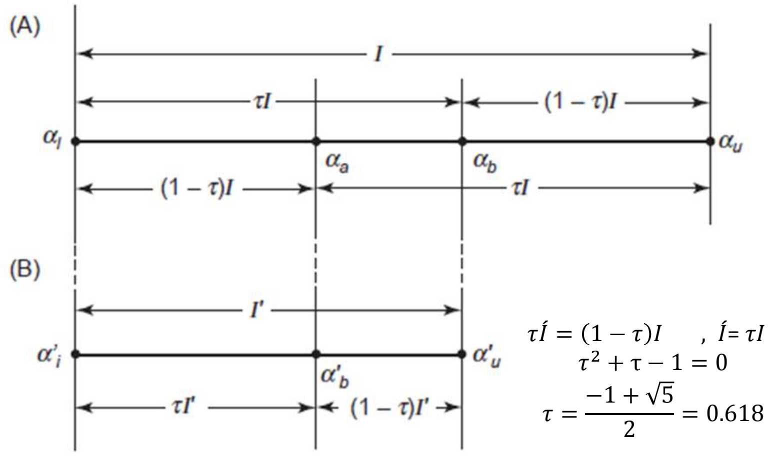


$$\tau \acute{I} = (1-\tau)I \quad , \acute{I} = \tau I$$
$$\tau^2 + \tau - 1 = 0$$
$$\tau = \frac{-1+\sqrt{5}}{2} = 0.618$$

Golden section partition.

Two points are located at a distance of 0.618***I*** or 0.382***I*** from either end of the interval

EXAMPLE 8.3 Minimization of a Function by Golden Section Search

Consider the function $f(\alpha)=2-4\alpha+e^{\alpha}$. Use golden section search to find the minimum within an accuracy of ε=0.001. Use δ=0.5.

Solution. Analytically, the solution is α^*=1.3863, $f(\alpha^*)$=0.4548.
In the golden section search, we need to first bracket the minimum point (Phase I) and then iteratively reduce the interval of uncertainty (Phase II).
Table 8-1 shows various iterations of the method. In Phase I, the minimum point is bracketed in only four iterations as shown in the first part of the table. The initial interval of uncertainty is calculated as $I=\alpha_u-\alpha_l$=2.618034-0.5=2.118034 since f(2.618034)>f(1.309017) in Table 8-1. Note that this interval would be larger than the one obtained using equal interval searching.

Now, to reduce the interval of uncertainty in Phase II, let us calculate α_b as (α_l+0.618I) or $\alpha_b=\alpha_u-0.382I$ (calculations are shown in the second part of Table 8-1).

Note that α_a and $f(\alpha_a)$ are already known and need no further calculation.

This is the main advantage of the golden section search; only one additional function evaluation is needed in the interval of uncertainty in each iteration, compared with the two function evaluations needed for the alternate equal interval search.

We calculate α_b=1.809017 and $f(\alpha_b)$=0.868376. Note that the new calculation of the function is shown in boldface for each iteration. Since $f(\alpha_a)<f(\alpha_b)$, new limits for the reduced interval of uncertainty are α'_l=0.5 and α'_u=1.809017. Also, α'_b=1.309017 at which the function value is already known. We need to compute only $f(\alpha'_a)$ where $\alpha'_a=\alpha'_l+0.382(\alpha'_u-\alpha'_l)$=1.000.

Further refinement of the interval of uncertainty is repetitive and can be accomplished by writing a computer program.

TABLE 8-1 Golden Section Search for $f(\alpha) = 2 - 4\alpha + e^{\alpha}$ of Example 8.3

ε=0.001 , δ=0.5

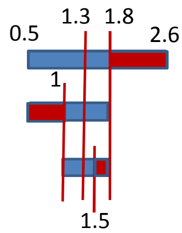
Phase 1: Initial bracketing of minimum

No., q	Trial step, α	Function value, $f(\alpha)$
1	0.000000	3.000000
2	$\alpha_l \to$ 0.500000	1.648721
3	0.5+1.618(0.5)= 1.309017	0.466464
4	1.309017+$(1.618)^2$(0.5)= $\alpha_u \to$ 2.618034	5.236610

Phase 2: Reducing interval of uncertainty

0.5+0.382(2.118034)   0.5+0.618(2.118034)

No.	α_l [$f(\alpha_l)$]	α_a [$f(\alpha_a)$]	α_b [$f(\alpha_b)$]	α_u [$f(\alpha_u)$]	I
1	0.500000 [1.648721]	1.309017 [0.466464]	**1.809017 [0.868376]**	2.618034 [5.236610]	2.118034
2	0.500000 [1.648721]	**1.000000 [0.718282]**	1.309017 [0.466464]	1.809017 [0.868376]	1.309017
3	1.000000 [0.718282]	1.309017 [0.466464]	**1.500000 [0.481689]**	1.809017 [0.868376]	0.809017
—	—	—	—	—	—
—	—	—	—	—	—
16	1.385438 [0.454824]	1.386031 [0.454823]	**1.386398 [0.454823]**	1.386991 [0.454824]	0.001553
17	1.386031 [0.454823] =	1.386398 [0.454823] =	**1.386624 [0.454823]** =	1.386991 [0.454823]	0.000960



α = 0.5(1.386398 + 1.386624) = 1.386511; $f(\alpha^*)$ = 0.454823.

Note: The new calculation for each iteration is shown as boldfaced and shaded; the arrows indicate direction of transfer of data.

9.1.1 Polynomial Interpolation

به منظور بالا بردن راندمان و دقت روش‌های عددی، معمولاً از روش بخش طلایی برای کاهش بازه عدم اطمینان و برای به دست آوردن مینیمم دقیق از درون‌یابی چند جمله‌ای استفاده می‌کنند. در هر دو فاز می‌توان از درون‌یابی چند جمله‌ای استفاده کرد.

Quadratic Curve Fitting

$$q(\alpha) = b_0 + b_1\alpha + b_2\alpha^2 \qquad (9.1)$$

$$b_0 + b_1\alpha_l + b_2\alpha_l^2 = f(\alpha_l)$$

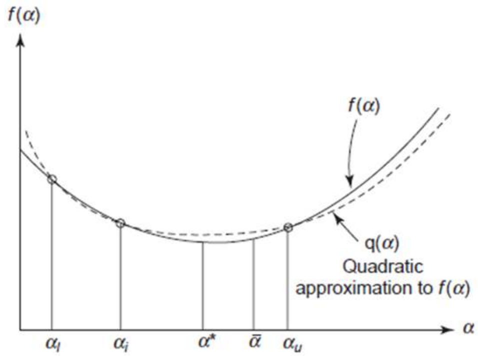
$$b_0 + b_1\alpha_i + b_2\alpha_i^2 = f(\alpha_i)$$

$$b_0 + b_1\alpha_u + b_2\alpha_u^2 = f(\alpha_u)$$

$$b_2 = \frac{1}{(\alpha_u - \alpha_i)}\left[\frac{f(\alpha_u) - f(\alpha_l)}{(\alpha_u - \alpha_l)} - \frac{f(\alpha_i) - f(\alpha_l)}{(\alpha_i - \alpha_l)}\right]$$

$$b_1 = \frac{f(\alpha_i) - f(\alpha_l)}{(\alpha_i - \alpha_l)} - a_2(\alpha_l + \alpha_i) \qquad (9.2)$$

$$b_0 = f(\alpha_l) - a_1\alpha_l - a_2\alpha_l^2$$



Quadratic approximation for a function f(α).

The minimum point of the quadratic curve $q(\alpha)$ in Eq. (9.1) is calculated by solving the necessary condition $dq/d\alpha=0$ and verifying the sufficiency condition $d^2q/d\alpha^2>0$, as

$$\bar{\alpha} = -\frac{1}{2b_2}b_1; \qquad \text{if } \frac{d^2q}{d\alpha^2} = 2b_2 > 0 \qquad (9.3)$$

Thus, if $b_2>0$, $\bar{\alpha}$ is minimum of $q(\alpha)$.

EXAMPLE 9.1 One-dimensional Minimization with Quadratic Interpolation

Find the minimum point of $f(\alpha)=2-4\alpha+e^{\alpha}$ of Example 8.3 by polynomial interpolation. Use the golden section search with δ=0.5 to bracket the minimum point initially.

Solution.

Iteration 1. From Example 8.3 the following information is known.

$$\alpha_l = 0.5, \qquad \alpha_i = 1.309017, \qquad \alpha_u = 2.618034$$
$$f(\alpha_l) = 1.648721, \quad f(\alpha_i) = 0.466464, \quad f(\alpha_u) = 5.236610$$

The coefficients b_0, b_1, and b_2 are calculated from Eq. (9.2) as

$$b_2 = \frac{1}{1.30902}\left(\frac{3.5879}{2.1180} - \frac{-1.1823}{0.80902}\right) = 2.410$$
$$b_1 = \frac{-1.1823}{0.80902} - (2.41)(1.80902) = -5.821$$
$$b_0 = 1.648271 - (-5.821)(0.50) - 2.41(0.25) = 3.957$$

Therefore, $\bar{\alpha} = 1.2077$ from Eq. (9.3), and $f(\bar{\alpha}) = 0.5149$.
Note that $\bar{\alpha} < \alpha_i$ and $f(\bar{\alpha}) < f(\alpha_i)$.
Thus, new limits of the reduced interval of uncertainty are

$$\alpha'_l = \bar{\alpha} = 1.2077 \qquad \alpha'_i = \alpha_i = 1.309017 \qquad \alpha'_u = \alpha_u = 2.618034$$

Iteration 2. We have the new limits for the interval of uncertainty, the intermediate point, and the respective values as

$$\alpha_l = 1.2077, \qquad \alpha_i = 1.309017, \qquad \alpha_u = 2.618034$$
$$f(\alpha_l) = 0.5149, \quad f(\alpha_i) = 0.466464, \quad f(\alpha_u) = 5.23661$$

The coefficients b_0, b_1, and b_2 are calculated as before,
b_0=5.7129, b_1=-7.8339, and b_2=2.9228.
Thus, $\bar{\alpha} = 1.34014$ and $f(\bar{\alpha}) = 0.4590$

Comparing these results with the optimum solution given in Table 8-1,

$$\bar{\alpha} = 1.316511 \qquad f(\bar{\alpha}) = 0.454823$$

We observe that $\bar{\alpha}$ and $f(\bar{\alpha})$ are quite close to the final solution. One more iteration can give a very good approximation to the optimum step size. Note that only five function evaluations are used to obtain a fairly accurate optimum step size for the function $f(\alpha)$.
Therefore, the polynomial interpolation approach can be quite efficient for one-dimensional minimization.

مسائل زیر را حل کرده و تا دو هفته دیگر تحویل فرمایید:

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تمرین‌های قبلی:

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