روش سیمپلکس دو گامی

برای حل مسائلی که قیدهای مساوی یا "≥0" دارند، از روش سیمپلکس دو گامی استفاده می‌شود.

- ابتدا با کم کردن متغیر زیادتی، قیدهای "≥0" را به مساوی تبدیل می‌کنیم.
- سپس به آن‌ها (قیدهای مساوی یا "≥0") متغیرهای مصنوعی (artificial) اضافه می‌نماییم.
- آن‌گاه گام اول را به روشی که توضیح داده می‌شود انجام می‌دهیم.
- در گام دوم یک مسئلهٔ LP معمولی در پیش داریم.

گام اول:

۱. اضافه کردن متغیرهای مصنوعی که باعث وسیع‌تر شدن چند ضلعی محدب مسئله می‌شود.

۲. متغیرهای مصنوعی را اصلی به حساب می‌آوریم تا به یک جواب قابل‌قبول اولیه دست پیدا کنیم که یک نقطه رأس از ناحیه جدید توسعه یافته است.

۳. متغیرهای مصنوعی را از مسئله حذف می‌کنیم. برای این کار یک تابع هزینهٔ مصنوعی W که مجموع متغیرهای مصنوعی است تعریف می‌کنیم.

$$\min W = x_{n+1} + x_{n+2} + \ldots + x_{n+m} = \sum_{i=1}^{m} x_{n+i}$$

6.4.4 Phase I Algorithm

- The artificial cost function is used to determine the pivot element.
- The original cost function is treated as a constraint and the elimination step is also executed for it. This way, the real cost function is in terms of the nonbasic variables only at the end of Phase I, and the Simplex method can be continued during Phase II.
- All artificial variables become nonbasic at the end of Phase I.
- Since w is the sum of all the artificial variables, its minimum value is clearly zero. When $w=0$, an extreme point of the original feasible set is reached. w is then discarded in favor of f and iterations continue in Phase II until the minimum of f is obtained.

6.4.4 Phase I Algorithm (cont'd)

- Suppose, however, that w cannot be driven to zero. This will be apparent when

None of the reduced cost coefficients for the artificial cost function is negative and yet w is greater than zero.

Clearly, this means that we cannot reach the original feasible set and, therefore,

No feasible solution exists for the original design problem, i.e., it is an infeasible problem.

At this point the designer should re-examine the formulation of the problem, which may be:

Over-constrained

or

Improperly formulated

6.4.5 Phase II Algorithm

In the final tableau from Phase I, the artificial cost row is replaced by the actual cost function equation and the Simplex iterations continue.

- مسئله بهینه سازی برای گام ۱:

تابع هزینهٔ مصنوعی

$$\min W = x_{n+1} + x_{n+2} + \ldots + x_{n+m} = \sum_{i=1}^{m} x_{n+i}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + x_{n+1} = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + x_{n+2} = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_m + x_{n+m} = b_m \\ x_i \geq 0 \quad i = 1 \text{ to } n+m \end{cases}$$

$$W = \sum_{i=1}^{m} b_i - \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} x_j$$

- ضرایب هزینهٔ کاهش یافته:

$$c'_j = -\sum_{i=1}^{m} a_{ij} \;;\; j = 1 \text{ to } n$$

مثال ۴.۱۱ (6.11): استفاده از متغیرهای مصنوعی برای قیود نوع "≥"

- برای مسالهٔ LP زیر با استفاده از روش سیمپلکس جواب بهین را به دست می‌آوریم.

$$\max \quad z = y_1 + 2y_2$$
$$3y_1 + 2y_2 \le 12$$
$$2y_1 + 3y_2 \ge 6$$
$$y_1 \ge 0$$

y_2 از نظر علامت آزاد است

$y_1 \to x_1$

$y_2 \to x_2 - x_3$

- ابتدا مسئله را به شکل استاندارد تبدیل می‌کنیم:

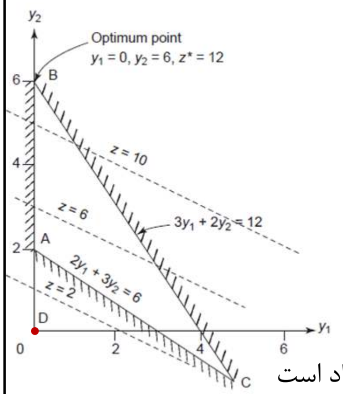
$$\min \ f = -x_1 - 2x_2 + 2x_3$$
$$3x_1 + 2x_2 - 2x_3 + x_4 = 12$$
$$2x_1 + 3x_2 - 3x_3 - x_5 + x_6 = 6$$
$$x_i \ge 0 \quad i = 1 \text{ to } 6$$

متغیر کمبود (x_4)
متغیر زیادتی (x_5)
متغیر مصنوعی (x_6)

- تابع هزینهٔ مصنوعی :

$$W = x_6 = 6 - 2x_1 - 3x_2 + 3x_3 + x_5$$

اصلی	x_1	x_2	x_3	x_4	x_5	x_6	b	b/a
x_4	3	2	-2	1	0	0	12	6
x_6	2	3	-3	0	-1	1	6	2
تابع هزینه	-1	-2	2	0	0	0	f-0	
هزینهٔ مصنوعی	-2	-3	3	0	1	0	W-6	



$$x_4 = 12,\ x_6 = 6$$

$$x_1 = x_2 = x_3 = x_5 = 0$$

$$y_1 = x_1 = 0$$

$$y_2 = x_2 - x_3 = 0 - 0 = 0$$

- $y_1=y_2=0$ Point D (infeasible)

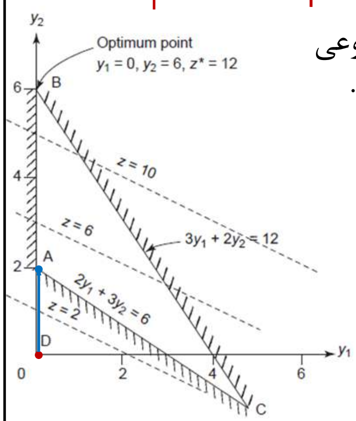
y_2 از نظر علامت آزاد است

9/45

اصلی	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	5/3	0	0	1	2/3	-2/3	8
x_2	2/3	1	-1	0	-1/3	1/3	2
تابع هزینه	1/3	0	0	0	-2/3	2/3	f+4
هزینهٔ مصنوعی	0	0	0	0	0	1	W-0

$$x_4 = 8,\ x_2 = 2$$

$$x_1 = x_3 = x_5 = x_6 = 0$$



پایان گام اول زیرا کلیه ضرایب سطر هزینه مصنوعی مربوط به متغیرهای غیراصلی نامنفی‌اند و $W=0$ است.

$$y_1 = x_1 = 0$$

$$y_2 = x_2 - x_3 = 2 - 0 = 2$$

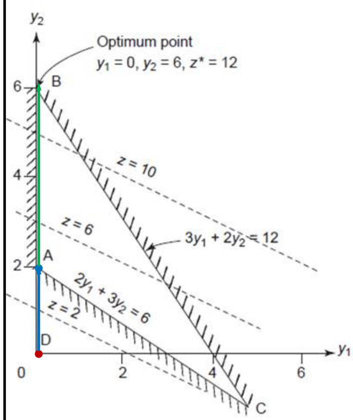
- $y_1=0$, $y_2=2$ Point A

10/45

اصلی	x_1	x_2	x_3	x_4	x_5	x_6	b
x_5	5/2	0	0	3/2	1	-1	12
x_2	3/2	1	-1	1/2	0	0	6
تابع هزینه	2	0	0	1	0	0	f+12

$$x_5 = 12,\ x_2 = 6$$
$$x_1 = x_3 = x_4 = x_6 = 0$$

پایان گام دوم. زیرا کلیه ضرایب سطر هزینه مربوط به متغیرهای غیراصلی نامنفی‌اند (ستون مربوط به متغیر مصنوعی x_6 را در نظر نمی‌گیریم).



$$y_1 = x_1 = 0$$
$$y_2 = x_2 - x_3 = 6 - 0 = 6$$

$y_1=0$, $y_2=6$ Point *B* (Optimum)

f+12=0 , f=-12

Use of Artificial Variables for Equality Constraints (Infeasible Problem)

مثال ۴.۱۲ (6.12)

maximize $Z=x_1+4x_2$

subject to $x_1+2x_2\le 5$

$2x_1+x_2=4$

$x_1-x_2\ge 3$

$x_1,x_2 \ge 0$

$$\min\ f = -x_1 - 4x_2$$

$$\begin{cases} x_1 + 2x_2 + x_3 = 5 \\ 2x_1 + x_2 + x_5 = 4 \\ x_1 - x_2 - x_4 + x_6 = 3 \\ x_i \ge 0 \quad i = 1 \text{ to } 6 \end{cases}$$

$$W = x_5 + x_6 = 7 - 3x_1 + x_4$$

$$W - 7 = -3x_1 + x_4$$

$x_1=0$, $x_2=0$

Point A

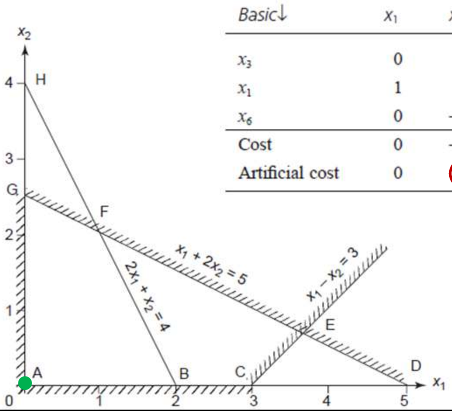
TABLE 6-15 Solution for Example 6.12 (Infeasible Problem)

Initial tableau: x_5 is identified to be replaced with x_1 in the basic set.

Basic↓	x_1	x_2	x_3	x_4	x_5	x_6	b	Ratio
x_3	1	2	1	0	0	0	5	$\frac{5}{1}=5$
x_5	2	1	0	0	1	0	4	$\frac{4}{2}=2$
x_6	1	−1	0	−1	0	1	3	$\frac{3}{1}=3$
Cost	−1	−4	0	0	0	0	$f-0$	
Artificial cost	−3	0	0	1	0	0	$w-7$	

Second tableau: End of Phase I.

Basic↓	x_1	x_2	x_3	x_4	x_5	x_6	b
x_3	0	$\frac{3}{2}$	1	0	$-\frac{1}{2}$	0	3
x_1	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	2
x_6	0	$-\frac{3}{2}$	0	−1	$-\frac{1}{2}$	1	1
Cost	0	$-\frac{7}{2}$	0	0	$\frac{1}{2}$	0	$f+2$
Artificial cost	0	$\frac{3}{2}$	0	1	$\frac{3}{2}$	0	$w-1$



- مسئله‌ای که جواب ندارد:

اگر هزینهٔ مصنوعی صفر نشده باشد و تمامی ضرایب سطر هزینهٔ مصنوعی مربوط به متغیرهای غیراصلی نامنفی باشند (که نتوان ستون لولا را تعیین کرد)، مسئله غیرقابل‌قبول می‌باشد.

- جواب قابل‌قبول اصلی تباهیده (صفر):

امکان دارد در یک چرخهٔ سیمپلکس، یک متغیر مقدارش صفر شود، یعنی جواب قابل‌قبول اصلی تباهیده شود. این موقعیت از نظر طراحی مطلوب نیست زیرا ما معمولاً نمی‌خواهیم یک متغیر طراحی صفر شود.

مثال ۴.۱۴ (6.14)

$$\begin{cases} \max \ \ z = x_1 + 4x_2 \\ x_1 + 2x_2 \le 5 \\ 2x_1 + x_2 \le 4 \\ 2x_1 + x_2 \ge 4 \\ x_1 - x_2 \ge 1 \\ x_1, x_2 \ge 0 \end{cases} \Rightarrow \begin{cases} \min \ \ f = -x_1 - 4x_2 \\ x_1 + 2x_2 + x_3 = 5 \\ 2x_1 + x_2 + x_4 = 4 \\ 2x_1 + x_2 - x_5 + x_7 = 4 \\ x_1 - x_2 - x_6 + x_8 = 1 \\ x_i \ge 0, \quad i = 1 \text{ to } 8 \end{cases}$$

TABLE 6-17 Solution for Example 6.14 (Degenerate Basic Feasible Solution)

Initial tableau: x_8 is identified to be replaced with x_1 in the basic set.

Basic↓	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b	Ratio
x_3	1	2	1	0	0	0	0	0	5	$\frac{5}{1}=5$
x_4	2	1	0	1	0	0	0	0	4	$\frac{4}{2}=2$
x_7	2	1	0	0	−1	0	1	0	4	$\frac{4}{2}=2$
x_8	1	−1	0	0	0	−1	0	1	1	$\frac{1}{1}=1$
Cost	−1	−4	0	0	0	0	0	0	$f-0$	
Artificial	−3	0	0	0	1	1	0	0	$w-5$	

Second tableau: x_7 is identified to be replaced with x_2 in the basic set.

مصنوعی

Basic↓	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b	Ratio
x_3	0	3	1	0	0	1	0	−1	4	$\frac{4}{3}$
x_4	0	3	0	1	0	2	0	−2	2	$\frac{2}{3}$ ←
x_7	0	3	0	0	−1	2	1	−2	2	$\frac{2}{3}$ ←
x_1	1	−1	0	0	0	−1	0	1	1	Negative
Cost	0	−5	0	0	0	−1	0	1	$f+1$	
Artificial	0	−3	0	0	1	−2	0	3	$w-2$	

هر کدام از سطرهای با نسبت یکسان انتخاب کنیم فرقی نمی‌کند.

15/45

TABLE 6-17 *Continued*

Third tableau: x_4 is identified to be replaced with x_5 in the basic set. End of Phase I.

مصنوعی

Basic↓	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b	Ratio
x_3	0	0	1	0	1	−1	−1	1	2	$\frac{2}{1}=2$
x_4	0	0	0	1	1	0	−1	0	0	$\frac{0}{1}=0$
x_2	0	1	0	0	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$	Negative
x_1	1	0	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{3}$	Negative
Cost	0	0	0	0	$-\frac{5}{3}$	$\frac{7}{3}$	$\frac{5}{3}$	$-\frac{7}{3}$	$f+\frac{13}{3}$	
Artificial	0	0	0	0	0	0	1	1	$w-0$	

Final tableau: End of Phase II.

Basic↓	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
x_3	0	0	1	−1	0	−1	0	1	2
x_5	0	0	0	1	1	0	−1	0	0
x_2	0	1	0	$\frac{1}{3}$	0	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{2}{3}$
x_1	1	0	0	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{5}{3}$
Cost	0	0	0	$\frac{5}{3}$	0	$\frac{7}{3}$	0	$-\frac{7}{3}$	$f+\frac{13}{3}$

It is theoretically possible for the Simplex method to fail by cycling between two degenerate basic feasible solutions. However, in practice this usually does not happen.

16/45

روش سیمپلکس جایگزین: Big-M method

- به جای استفاده از سطر هزینه مصنوعی، متغیرهای مصنوعی با یک ضریب بزرگ به تابع هزینهٔ اصلی اضافه می‌شود. جملات اضافی به عنوان جریمهٔ داشتن متغیرهای مصنوعی در مسئله عمل می‌کنند. برای حذف متغیرهای مصنوعی که اصلی هم هستند، از روش سیمپلکس استفاده می‌شود.

• Two important considerations for using the Big-M method.

The use of the penalty *M* may not always force the artificial variable to zero level by the final iteration. This can occur in the case where the given LP has no feasible solution. If any artificial variable is positive in the final iteration ($w \neq 0$) then the LP has no feasible solution space.

Theoretically, the application of the Big-M technique requires that *M* approaches infinity but to computerize the solution algorithm, *M* must be finite while being “sufficiently large.” The pitfall in this case is, however, if *M* is too large it can lead to substantial round-off error yielding an incorrect optimal solution.

(http://businessmanagementcourses.org/Lesson09TheBigMMethod.pdf,2013/11/30)

این روش را با ذکر مثالی تشریح می‌کنیم:
مثال: روش بیگ ام برای قیود نوع ”≥0“

مثال ۴.۱۵ (7.1)

- تابع مقابل را با قیود داده شده ماکزیمم کنید.

$$\begin{cases} \max\ z = y_1 + 2y_2 \\ 3y_1 + 2y_2 \le 12 \\ 2y_1 + 3y_2 \ge 6 \\ y_1 \ge 0\ ,\ y_2 \text{ :sign-free} \end{cases} \Rightarrow \begin{cases} \min\ f = -x_1 - 2x_2 + 2x_3 \\ 3x_1 + 2x_2 - 2x_3 + x_4 = 12 \\ 2x_1 + 3x_2 - 3x_3 - x_5 + x_6 = 6 \\ x_i \ge 0 \quad i = 1 \text{ to } 6 \end{cases}$$

$$y_2 = x_2 - x_3$$

- حال با توجه به ضرایب متغیرها در معادلات، ضریب بزرگ $M=10$ را در نظر می‌گیریم.
- توجه: با $M=20$ یا $M=2$ نیز جواب فرقی نمی‌کند. بنابراین روش بستگی به مقدار M (در این محدوده تغییرات) ندارد.

$$\min\ f = -x_1 - 2x_2 + 2x_3 + 10x_6$$

$$x_6 = 6 - 2x_1 - 3x_2 + 3x_3 + x_5$$

$$\min\ f = -x_1 - 2x_2 + 2x_3 + 10(6 - 2x_1 - 3x_2 + 3x_3 + x_5)$$

$$\min\ f = -21x_1 - 32x_2 + 32x_3 + 10x_5 + 60$$

اصلی	x_1	x_2	x_3	x_4	x_5	x_6	b	b/a
x_4	3	2	-2	1	0	0	12	6
x_6	2	3	-3	0	-1	1	6	2
هزینه	-21	-32	32	0	10	0	f-60	

اصلی	x_1	x_2	x_3	x_4	x_5	x_6	b	b/a
x_4	5/3	0	0	1	2/3	-2/3	8	
x_2	2/3	1	-1	0	-1/3	1/3	2	
هزینه	1/3	0	0	0	-2/3	32/3	f+4	

اصلی	x_1	x_2	x_3	x_4	x_5	x_6	b	
x_5	5/2	0	0	3/2	1	-1	12	
x_2	3/2	1	-1	1/2	0	0	6	
هزینه	2	0	0	1	0	10	f+12	

این روش بهتر از روش دوگامی است زیرا سطر مربوط به هزینه مصنوعی وجود ندارد. جواب مانند قبل است.

$x_5 = 12,\ x_2 = 6$

$x_1 = x_3 = x_4 = x_6 = 0$

f= -12

مسائل زیر را حل کرده و تا دو هفته دیگر تحویل فرمایید:

4) 38, 39, 41, 59

تمرین‌های قبلی:

4) 1,2,3, 9, 16, 20, 21, 26, 34,37

تحلیل پس بهینگی: Post Optimality Analysis

جواب یک مسئله LP به پارامترهای بردارهای b، c و ماتریس A دارد. این پارامترها در مسائل عملی در معرض خطا هستند. بنابراین ما افزون به جواب بهین علاقمند به فهم چگونگی تغییرات این جواب نسبت به این تغییرات هستیم. این پارامترها ممکن است پیوسته یا گسسته باشند.

- پارامترهای گسسته: تحلیل حساسیت(sensitivity analysis)
- پارامترهای پیوسته: برنامه‌ریزی پارامتریک

تغییرات پارامتری اصلی:

1) تغییرات حدود منابع : b_i
2) تغییرات ضرایب تابع هزینه: c_j

مورد بحث قرار می‌گیرد.

3) تغییرات ضرایب قیود: a_{ij}
4) اثر افزودن بر تعداد قیود
5) اثر افزایش تعداد متغیرها

Following Theorem gives a way of recovering the multipliers for the constraints of an LP problem from the **final tableau.**

Theorem 6.5 Lagrange Multiplier Values

Let the standard LP problem be solved using the Simplex method.

(1) For "≤ type" constraints, Lagrange Multiplier equals the reduced cost coefficient in the slack variable column associated with the constraint.

(2) For "=" and "≥ type" constraints, the Lagrange Multiplier equals the reduced cost coefficient in the artificial variable column associated with the constraint.

$$\Delta f = -v_i^* b_i - u_j^* e_j$$

(3) The Lagrange Multiplier is always:

(4) ≥0 for the "≤ type" constraint,
≤0 for the "≥ type" constraint,
free in sign for the "= type" constraint.

$$\frac{\partial f}{\partial e_j} = -u_j^*$$

$$\frac{\partial f}{\partial b_i} = -v_i^*$$

Using Theorem 4.7, we obtain the derivative of the cost function with respect to the right side parameters, and change in the optimum cost function:

$$\frac{\partial f}{\partial e_i} = -y_i; \quad \Delta f = -y_i \Delta e_i = -y_i (e_{i\,\text{new}} - e_{i\,\text{old}}) \qquad (6.23)$$

e_i: right side parameter of any constraint

y_i: Lagrange multiplier of the ith constraint

• Theorem 6.5 and Eq. (6.23) are applicable only if changes in the right side parameters are within certain limits.

• The calculation for Δf remains valid for simultaneous changes to multiple constraints.

مثال ۴.۱۶ (6.15): به دست آوردن ضریب لاگرانژ از جدول نهایی
تابع مقابل را با قیود داده شده ماکزیمم کنید:

شکل استاندارد

$$\begin{aligned} \max\ z &= 5x_1 - 2x_2 \\ 2x_1 + x_2 &\le 9 \\ x_1 - 2x_2 &\le 2 \\ -3x_1 + 2x_2 &\le 3 \\ x_1, x_2 &\ge 0 \end{aligned} \qquad \begin{aligned} \min\ f &= -5x_1 + 2x_2 \\ 2x_1 + x_2 + x_3 &= 9 \\ x_1 - 2x_2 + x_4 &= 2 \\ -3x_1 + 2x_2 + x_5 &= 3 \\ x_i &\ge 0, \text{i=1 to 5} \end{aligned}$$

جدول آخر

متغیرهای کمبود:
x_3, x_4, x_5

اصلی	x_1	x_2	x_3	x_4	x_5	b
x_2	0	1	0.2	-0.4	0	1
x_1	1	0	0.4	0.2	0	4
x_5	0	0	0.8	1.4	1	13
هزینه	0	0	1.6	1.8	0	$f+18$
	c_1'	c_2'	c_3'	c_4'	c_5'	

ضرایب لاگرانژ (y) مربوط به قیود "$\le$" برابر است با ضرایب هزینهٔ کاهش یافته مربوط به متغیرهای کمبود.

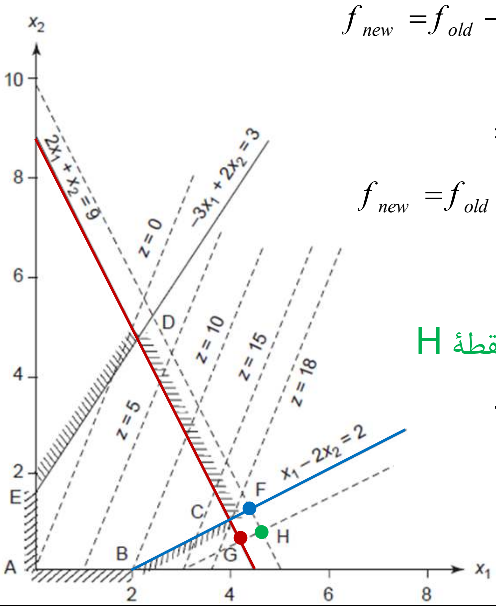
- قید اول $y_1 = c_3' = 1.6 \qquad \dfrac{\partial f}{\partial e_1} = -1.6$
- قید دوم $y_2 = c_4' = 1.8 \qquad \dfrac{\partial f}{\partial e_2} = -1.8$
- قید سوم $y_3 = c_5' = 0 \qquad \dfrac{\partial f}{\partial e_3} = 0$

- اگر قید اول را از 9 به 10 تبدیل کنیم:

$$\Delta f = -y_i \Delta e_i = -y_i (e_{i\,\text{new}} - e_{i\,\text{old}})$$

$$f_{new} = f_{old} - y_i \Delta e_i = -18 - 1.6(10-9) = -19.6$$

نقطهٔ F



- اگر قید دوم را از 2 به 3 تغییر دهیم:

$$f_{new} = f_{old} - y_i \Delta e_i = -18 - 1.8(3-2) = -19.8$$

نقطهٔ G

- اگر هر دو را همزمان تغییر دهیم به نقطهٔ H می‌رسیم:

$$f_{new} = f_{old} - \sum y_i \Delta e_i = -18 - 1.6(10-9) - 1.8(3-2) = -21.4$$

مثال ۴.۱۷ (6.16): به دست آوردن ضرایب لاگرانژ از جدول نهایی

$$\max \quad z = x_1 + 4x_2$$
$$x_1 + 2x_2 \le 5$$
$$2x_1 + x_2 = 4$$
$$x_1 - x_2 \ge 1$$
$$x_1 \text{ and } x_2 \ge 0$$

شکل استاندارد:

$$\min \quad f = -x_1 - 4x_2$$
$$x_1 + 2x_2 + x_3 = 5$$
$$2x_1 + x_2 + x_5 = 4$$
$$x_1 - x_2 - x_4 + x_6 = 1$$
$$x_i \ge 0 \quad i = 1 \text{ to } 6$$

جدول آخر دوگامی

Final tableau: Reduced cost coefficients in nonbasic columns are nonnegative; the tableau gives the optimum point. End of Phase I. End of Phase II.

Basic↓	x_1	x_2	x_3	x_4	x_5	x_6	b
x_3	0	0	1	−1	−1	1	2
x_2	0	1	0	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$
x_1	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{3}$
Cost	0	0	0	$\frac{7}{3}$	$\frac{5}{3}$	$-\frac{7}{3}$	$f+\frac{13}{3}$
	(c'_1)	(c'_2)	(c'_3)	(c'_4)	(c'_5)	(c'_6)	
Artificial	0	0	0	0	1	1	$w-0$

x_3, slack variable; x_4, surplus variable; x_5, x_6, artificial variables.

1. For $x_1+2x_2 \le 5$: $y_1=0$ (c'_3 in the slack variable column x_3)
2. For $2x_1+x_2=4$: $y_2=5/3$ (c'_5 in the artificial variable column x_5)
3. For $x_1-x_2 \ge 1$: $y_3=-7/3$ (c'_6 in the artificial variable column x_6)

When the right side of the third constraint is changed from 1 to 2 (i.e., x_1-x_2≥2), the cost function f=(-x_1-4x_2) changes by

$$\Delta f = -y_3 \Delta e_3 = -\left(-\frac{7}{3}\right)(2-1) = \frac{7}{3}$$

That is, the cost function will increase by 7/3, from -13/3 to -2 (z=2). (-13/3+7/3=-2)

We shall demonstrate that same result is obtained when the third constraint is written in the "≤ form" (-x_1+x_2≤-1). The Lagrange multiplier for the constraint is 7/3, which is the negative of the preceding value. When the right side of the third constraint is changed to 2 (i.e., it becomes -x_1+x_2≤-2), the cost function f=(-x_1-4x_2) changes by

$$\Delta f = -y_3 \Delta e_3 = -\frac{7}{3}(-2-(-1)) = \frac{7}{3}$$

When the right side of the equality constraint is changed to 5 from 4, the cost function changes by

$$\Delta f = -y_2 \Delta e_2 = -\frac{5}{3}(5-4) = -\frac{5}{3}$$

Ranging Right Side Parameters

$$\max\{r_i < 0\} \le \Delta_k \le \min\{r_i > 0\}; \ r_i = -\frac{b'_i}{a'_{ij}}, \ i = 1 \text{ to } m \qquad (6.24)$$

Note: $r_i = 0$ is not included

b'_i=right side parameter for the ith constraint in the final tableau

a'_{ij}=parameters in the jth column of the final tableau; the jth column corresponds to x_j which is the slack variable for a "≤ type" constraint, or the artificial variable for an equality, or "≥ type" constraint

r_i=negative of the ratios of the right sides with the parameters in the jth column

Δ_k=possible change in the right side of the kth constraint; the slack or the artificial variable for the kth constraint determines the index j of the column whose elements are used in the Inequalities (6.24).

If there is no lower or upper bound on Δ_k, i.e., the limit is ∞.

The new right side parameters b''_i due to a change of Δ_k in b_k are given as

$$b''_i = b'_i + \Delta_k a'_{ij}; \ i = 1 \text{ to } m \qquad (6.25)$$

مثال ۴.۱۸

$$\max \ z = 5x_1 - 2x_2$$
$$2x_1 + x_2 \le 9$$
$$x_1 - 2x_2 \le 2$$
$$-3x_1 + 2x_2 \le 3$$
$$x_1 \text{ and } x_2 \ge 0$$

EXAMPLE 6.17 Ranges for Resource Limits "≤Type" Constraints

For the first constraint, x_3 is the slack variable, and so $j=3$ in Inequalities (6.24) for calculation of range for Δ_1, the change to the constraint's right side. The ratios of the right side parameters with the elements in column 3, r_i of Eq. (6.24) are calculated as

Final tableau: Reduced cost coefficients in nonbasic columns are nonnegative; the tableau gives optimum point.

Basic↓	x_1	x_2	x_3	x_4	x_5	b
x_2	0	1	0.2	−0.4	0	1
x_1	1	0	0.4	0.2	0	4
x_5	0	0	0.8	1.4	1	13
Cost	0	0	**1.6**	**1.8**	0	$f+18$
	(c'_1)	(c'_2)	(c'_3)	(c'_4)	(c'_5)	

x_3, x_4, and x_5 are slack variables.

$$r_i = -\frac{b'_i}{a'_{i3}} = \left\{-\frac{1}{0.2}, -\frac{4}{0.4}, -\frac{13}{0.8}\right\} = \{-5.0, -10.0, -16.25\}$$

$$\max\{-5.0, -10.0, -16.25\} \le \Delta_1, \text{or } -5 \le \Delta_1$$

Since there is no positive r_i, there is no upper limit on Δ_1.

31/45

Thus, limits for Δ_1 are $-5 \le \Delta_1 \le \infty$ and the range on b_1 is obtained by adding the current value of $b_1=9$ to both sides, as

$$-5+9 \le b_1 \le \infty+9, \text{ or } 4 \le b_1 \le \infty \qquad \text{(c)}$$

For the second constraint ($k=2$), x_4 is the slack variable. Therefore, we will use elements in column x_4 of the final tableau ($\acute{a}_{i4}$, $j=4$) in the inequalities of Eq. (6.24). The ratios of the right side parameters with the elements in column 4, r_i of Eq. (6.24), are calculated as

$$r_i = -\frac{b'_i}{a'_{i4}} = \left\{-\frac{1}{-0.4}, -\frac{4}{0.2}, -\frac{13}{1.4}\right\} = \{2.5, -20.0, -9.286\}$$

$$\max\{-20.0, -9.286\} \le \Delta_2 \le \min\{2.5\}, \text{or } -9.286 \le \Delta_2 \le 2.5$$

Therefore, the allowed decrease in b_2 is 9.286 and the allowed increase is 2.5.

Adding 2 to the above inequality (the current value of b_2), the range on b_2 is given as

$$-7.286 \le b_2 \le 4.5$$

Similarly, for the third constraint, the ranges for Δ_3 and b_3 are:

$$-13 \le \Delta_3 \le \infty, \quad -10 \le b_3 \le \infty$$

گردآوری و تنظیم: محمدحسین ابوالبشری 32/45

New values of design variables

Let us calculate new values for the design variables if the right side of the first constraint is changed from 9 to 10. Note that this change is within the limits determined in the foregoing. In Eq. (6.25), *k=1*, so Δ_I=10-9=1

Also, *j*=3, so we use the third column from Table 6-18 in Eq. (6.25) and obtain new values of the variables as

$$x_2 = b_1'' = b_1' + \Delta_1 a_{13}' = 1 + (1)(0.2) = 1.2$$

$$x_1 = b_2'' = b_2' + \Delta_1 a_{23}' = 4 + (1)(0.4) = 4.4$$

$$x_5 = b_3'' = b_3' + \Delta_1 a_{33}' = 13 + (1)(0.8) = 13.8$$

point *F*

Similarly, if the right side of the second constraint is changed from 2 to 3, the new values of the variables, using Eq. (6.25) and the x_4 column from Table 6-18, are calculated as:

$$x_2 = b_1'' = b_1' + \Delta_2 a_{14}' = 1 + (1)(-0.4) = 0.6$$

$$x_1 = b_2'' = b_2' + \Delta_2 a_{24}' = 4 + (1)(0.2) = 4.2$$

$$x_5 = b_3'' = b_3' + \Delta_2 a_{34}' = 13 + (1)(1.4) = 14.4$$

This solution corresponds to point *G* in Fig. 6-7.

When the right sides of two or more constraints are changed simultaneously, we can use Eq. (6.25) to determine new values of the design variables. However, we have to make sure that the new right sides do not change the basic and nonbasic sets of variables, i.e., the vertex that gives the optimum solution is not changed. Or, in other words, no new constraint becomes active. As an example, let us calculate the new values of design variables using Eq. (6.25) when the right sides of the first and the second constraints are changed to 10 and 3 from 9 and 2, respectively:

$$x_2 = b_1'' = b_1' + \Delta_1 a_{13}' + \Delta_2 a_{14}' = 1 + (1)(0.2) + (1)(-0.4) = 0.8$$

$$x_1 = b_2'' = b_2' + \Delta_1 a_{23}' + \Delta_2 a_{24}' = 4 + (1)(0.4) + (1)(0.2) = 4.6$$

$$x_5 = b_3'' = b_3' + \Delta_1 a_{33}' + \Delta_2 a_{34}' = 13 + (1)(0.8) + (1)(1.4) = 15.2$$

It can be verified that the new solution corresponds to point *H* in Fig. 6-7.

Ranging Cost Coefficients

If a cost coefficient c_k is changed to $c_k+\Delta c_k$, we like to find an admissible range on Δc_k such that the optimum design variables are not changed.
Note that when the cost coefficients are changed, the feasible region for the problem does not change.
However, orientation of the cost function hyperplane and value of the cost function change.
Limits on the change Δc_k for the coefficient c_k depend on whether x_k is a basic variable at the optimum.
Thus, we must consider the two cases separately. Theorems 6.7 and 6.8 give ranges for the cost coefficients for the two cases.

Theorem 6.7 Range for Cost Coefficient of Nonbasic Variables

Let c_k be such that x^*_k is not a basic variable. If this c_k is replaced by any $c_k+\Delta c_k$, where $-c'_k \le \Delta c_k \le \infty$, then the optimum solution (design variables and the cost function) does not change. Here, $-c'_k$ is the reduced cost coefficient corresponding to x^*_k in the final tableau.

Theorem 6.8 Range for Cost Coefficient of Basic Variables

Let c_k be such that x^*_k is a basic variable, and let $x^*_k = b'_r$ (a superscript * is used to indicate optimum value). Then, the range for the change Δc_k in c_k for which the optimum design variables do not change is given as

$$\max\{d_j < 0\} \le \Delta c_k \le \min\{d_j > 0\};\ d_j = \frac{c'_j}{a'_{rj}} \qquad (6.26)$$

$$\max\{d_j < 0\} \le \Delta c_k \le \min\{d_j > 0\};\ d_j = \frac{c'_j}{a'_{rj}} \quad (6.26)$$

Note: $d_j = 0$ is not included

$\acute{a}_{rj}$=element in the rth row and the jth column of the final tableau. The index r is determined by the row that determines x^*_k. Index j corresponds to each of the nonbasic columns excluding artificial columns. (Note: if no $\acute{a}_{ij}$>0, then there is no upper limit; if no $\acute{a}_{ij}$<0, then there is no lower limit.)

$\acute{c}_j$=reduced cost coefficient in the jth nonbasic column excluding artificial variable columns

d_j=ratios of the reduced cost coefficients with the elements in the rth row corresponding to nonbasic columns excluding artificial columns

When Δc_k satisfies Inequality (6.26), the optimum value of the cost function is $f^*+\Delta c_k x^*_k$.

Ranges for Cost Coefficients "≤Type" Constraints

مثال ۴.۲۰ (6.19):

$$\max\ z = 5x_1 - 2x_2$$
$$2x_1 + x_2 \le 9$$
$$x_1 - 2x_2 \le 2$$
$$-3x_1 + 2x_2 \le 3$$
$$x_1 \text{ and } x_2 \ge 0$$

For range of x_2 For range of x_1

Final tableau: Reduced cost coefficients in nonbasic columns are nonnegative; the tableau gives optimum point.

Basic↓	x_1	x_2	x_3	x_4	x_5	b
x_2	0	1	0.2	−0.4	0	1
x_1	1	0	0.4	0.2	0	4
x_5	0	0	0.8	1.4	1	13
Cost	0	0	**1.6**	**1.8**	0	$f+18$
	(c'_1)	(c'_2)	(c'_3)	(c'_4)	(c'_5)	

x_3, x_4, and x_5 are slack variables.

The problem is solved as a minimization of the cost function $f=-5x_1+2x_2$. Therefore, we will find ranges for the cost coefficients c_1=-5 and c_2=2.

Note that since both x_1 and x_2 are basic variables, Theorem 6.8 will be used.

Range of the first cost coefficient:

Since the second row determines the basic variable x_1, r=2 (the row number) for use in Inequalities (6.26). Columns 3 and 4 are nonbasic; therefore j=3,4 are the column indices for use in Eq. (6.26). After calculating the ratios d_j, the range for Δc_1 is calculated as

$$d_j = \frac{c'_j}{a'_{2j}} = \left\{\frac{1.6}{0.4}, \frac{1.8}{0.2}\right\} = \{4, 9\}; \quad -\infty \le \Delta c_1 \le \min\{4, 9\}, \text{or } -\infty \le \Delta c_1 \le 4$$

The range for c_1 is obtained by adding the current value of c_1=-5 to both sides of the above inequality,

$$-\infty \le c_1 \le -1 \qquad \text{(a)}$$

Thus, if c_1 changes from -5 to -4, the new cost function value is given as

$$f^*_{new} = f^* + \Delta c_1 x_1^* = -18 + [-4 - (-5)](4) = -14$$

That is, the cost function will increase by 4.

Range of the second cost coefficient:

For the second cost coefficient, r=1 (the row number) because the first row determines x_2 as a basic variable. After calculating the ratios d_j, the range for Δc_2 is calculated as

$$d_j = \frac{c'_j}{a'_{1j}} = \left\{\frac{1.6}{0.2}, \frac{1.8}{-0.4}\right\} = \{8, -4.5\}; \quad \max\{-4.5\} \le \Delta c_2 \le \min\{8\}, \text{or } -4.5 \le \Delta c_2 \le 8$$

The range for c_2 is obtained by adding the current value of c_2=2 to both sides of the above inequality,

$$-2.5 \le c_2 \le 10 \qquad \text{(b)}$$

Thus, if c_2 is changed from 2 to 3, the new cost function value is given as

$$f^*_{new} = f^* + \Delta c_2 x_1^* = -18 + (3-2)(1) = -17$$

Note that the range for the coefficients of the maximization function ($z=5x_1-2x_2$) can be obtained from Eqs. (a) and (b).
To determine these ranges, we multiply Eqs. (a) and (b) by -1. Therefore, the range for d_1=5 is given as $1 \le d_1 \le \infty$, and that for d_2=-2 is $-10 \le d_2 \le 2.5$.

EXAMPLE 6.20 (۴.۲۱)

Ranges for Cost Coefficients—Equality and "≥ Type" Constraints

Find ranges for the cost coefficients of the problem solved in Example 6.16.

Solution. The final tableau for the problem is given in Table 6-19 (next slide).

For range of x_2 For range of x_1 Nonbasic columns

Basic↓	x_1	x_2	x_3	x_4	x_5	x_6	**b**
x_3	0	0	1	-1	-1	1	2
x_2	0	1	0	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$
x_1	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{3}$
Cost	0	0	0	$\frac{7}{3}$	$\frac{5}{3}$	$-\frac{7}{3}$	$f+\frac{13}{3}$
	(c_1')	(c_2')	(c_3')	(c_4')	(c_5')	(c_6')	
Artificial	0	0	0	0	1	1	$w-0$

x_3, slack variable; x_4, surplus variable; x_5, x_6, artificial variables.

In the tableau, x_3 is a slack variable for the first constraint, x_4 is a surplus variable for the third constraint, and x_5 and x_6 are artificial variables for the second and third constraints, respectively. Since both x_1 and x_2 are basic variables, we will use Theorem 6.8 to find ranges for the cost coefficients c_1=-1 and c_2=-4. Note that the problem is solved as minimization of the cost function $f=-x_1-4x_2$. Columns 4, 5, and 6 are nonbasic. However, since artificial columns 5 and 6 must be excluded, only column 4 can be used in Eq. (6.26).

To find the range for Δc_1, r=3 is used because the third row determines x_1 as a basic variable. Using Inequalities (6.26) with r=3 and j=4, we have

$$\max\{\frac{7}{3}/(-\frac{1}{3})\} \le \Delta c_1 \le \infty; \quad \text{or} \quad -7 \le \Delta c_1 \le \infty \qquad \text{(a)}$$

The range for c_1 is obtained by adding the current value of c_1=-1 to both sides of the inequality,

$$-8 \le c_1 \le \infty \qquad \text{(b)}$$

Thus, if c_1 changes from -1 to -2, the new cost function value is given as

$$f^*_{new} = f^* + \Delta c_1 x_1^* = -\frac{13}{3} + [-2-(-1)](\frac{5}{3}) = -6 \qquad \text{(c)}$$

For the second cost coefficient, r =2 because the second row determines x_2 as a basic variable. Using Eq. (6.26) with r =2 and j =4, the range for Δc_2 is obtained as (see the tableau):

$$-\infty \le \Delta c_2 \le \min\{\frac{7}{3}/(\frac{2}{3})\}; \text{ or } -\infty \le \Delta c_2 \le 3.5$$

Thus the range for c_2 with current value c_2=-4 is given as $-\infty \le c_2 \le -0.5$. If c_2 changes from -4 to -3, the new value of the cost function is given as

$$f^*_{new} = f^* + \Delta c_2 x_1^* = -\frac{13}{3} + [-3-(-4)](\frac{2}{3}) = -\frac{11}{3} \qquad \text{(d)}$$

The ranges for coefficients of the maximization function ($z = x_1 + 4x_2$) are obtained by multiplying the above ranges by -1, as

$$-\infty \le d_1 \le 8 (-\infty \le \Delta d_1 \le 7) \text{ and } 0.5 \le d_2 \le \infty \ (-3.5 \le \Delta d_2 \le \infty) \qquad \text{(e)}$$

43/45

مسائل زیر را حل کرده و تا دو هفته دیگر تحویل فرمایید:

4) 68, 71,73

تمرین‌های قبلی:

4) 1,2,3, 9, 16, 20, 21, 26, 34,37, 38, 39, 41, 59

گردآوری و تنظیم: محمدحسین ابوالبشری 44/45