Chapter 2
Design optimization Formulation

What is the Design Optimization?

- Optimization is a component of design process
- The design of systems can be formulated as problems of optimization where a measure of performance is to be optimized while satisfying all the constraints.

Three Components of Design Optimization

1. Design variables – A set of parameters that describes the system (dimensions, material, load, …)
2. Design constraints – All systems are designed to perform within a given set of constraints. The constraints must be influenced by the design variables (max. or min. values of design variables).
3. Objective function – A criterion is needed to judge whether or not a given design is better than another (cost, profit, weight, deflection, stress, ….).

Optimum Design – Problem Formulation

The formulation of an optimization problem is extremely important, care should always be exercised in defining and developing expressions for the constraints.

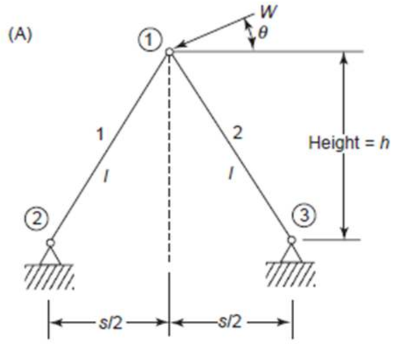
The optimum solution will only be as good as the formulation.

The following three steps shall be followed to transcribe a verbal statement of the design problem to mathematical formulation

1. Identify and define design variables.
2. Identify the cost function and develop an expression for it in terms of design variables.
3. Identify constraints and develop expressions for them in terms of design variables.

Problem Formulation
Design of a two-bar structure

The problem is to design a two-member bracket to support a force W without structural failure. Since the bracket will be produced in large quantities, the design objective is to minimize its mass while also satisfying certain fabrication and space limitation.



In formulating the design problem, we need to define structural failure more precisely. Member forces F_1 and F_2 can be used to define failure condition.

$$-F_1 \sin\alpha + F_2 \sin\alpha = W \cos\theta$$

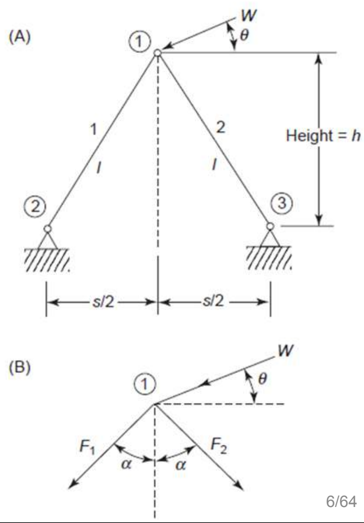
$$-F_1 \cos\alpha - F_2 \cos\alpha = W \sin\theta$$

$$\sin\alpha = \frac{s}{2l}, \quad \cos\alpha = \frac{h}{l}$$

$$F_1 = -0.5Wl\left[\frac{\sin\theta}{h} + \frac{2\cos\theta}{s}\right]$$

$$F_2 = -0.5Wl\left[\frac{\sin\theta}{h} - \frac{2\cos\theta}{s}\right]$$

$$l = \sqrt{h^2 + (0.5s)^2}$$



Problem Formulation

Design Variables

An important first step in the proper formulation of the problem is to identify design variables for the system.

Note on identifying design variables

1. Design variables should be independent of each other as far as possible. If they are not, then there must be some equality constraints between them (explained later). Conversely, if there are equality constraints in the problem, then the design variables are dependent.
2. A minimum number of design variables required to formulate a design optimization problem properly exists.

Problem Formulation

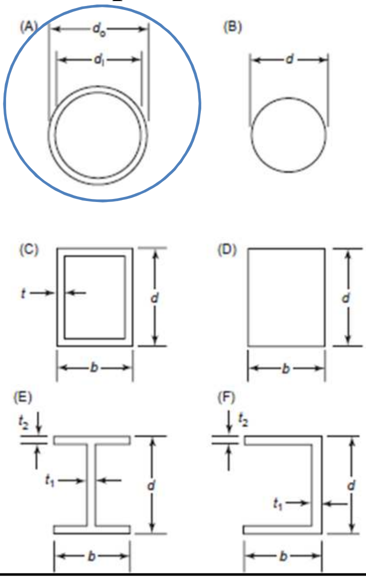
Note on identifying design variables (Cont'd)

3. As many independent parameters as possible should be designated as design variables at the problem formulation phase. Later on, some of the variables can be assigned fixed values.

4. A numerical value should be given to each variable once design variables have been defined to determine if a trial design of the system is specified.

Problem Formulation

Represent all the design variables for a problem in the vector x. Two-Bar Structure:



x_1 = height h of the truss

x_2 = span s of the truss

x_3 = outer diameter of member 1

x_4 = inner diameter of member 1

x_5 = outer diameter of member 2

x_6 = inner diameter of member 2

Problem Formulation

Objective Function

A criterion must be selected to compare various designs

1. It must be a scalar function whose numerical values could be obtained once a design is specified.
2. It must be a function of design variables, $f(x)$.
3. The objective function is minimized or maximized (minimize cost, maximize profit, minimize weight, maximize ride quality of a vehicle, minimize the cost of manufacturing, ….)
4. Multi-objective functions; minimize the weight of a structure and at the same time minimize the deflection or stress at a certain point.

Multi objective problem: مسائل چند هدفه

در این مسایل چند هدف با هم باید ماکزیمم و یا مینیمم بشوند.

راه حل‌های این گونه مسائل:

۱- تابع هدف جدید را به صورت زیر تعریف می‌کنیم:

$$F(X) = w_1 f_1 + w_2 f_2 + \dots$$

نقش wها

- ایجاد تعادل بین هدفها (هم مرتبه کردن هدف‌ها) و یا بر عکس
- بر هم زدن توازن به نفع یک هدف

۲- یکی از هدف‌ها به عنوان هدف (مهمترین تابع هزینه) و بقیه هدف‌ها را به عنوان قید در نظر می‌گیریم.

Objective Function

Two-Bar Structure: Mass is selected as the objective function

$$\text{Mass} = \frac{\pi\rho}{8}(4x_1^2 + x_2^2)^{1/2}(x_3^2 + x_5^2 - x_4^2 - x_6^2)$$

x_1 = height h of the truss

x_2 = span s of the truss

x_3 = outer diameter of member 1

x_4 = inner diameter of member 1

x_5 = outer diameter of member 2

x_6 = inner diameter of member 2

Design Constraints (section 2.5) قیود طراحی:

Feasible Design

A design meeting all the requirements is called a feasible (acceptable) design. An infeasible design does not meet one or more requirements.

Constraints Characteristics:

All restrictions placed on a design are collectively called constraints.

Each constraints must be influenced by one or more design variables. Only then it is meaningful and does it have influence on optimum design.

Implicit /Explicit Constraints تقسیم بندی قیود

Explicit: Some constraints are simple such as min. and max. values of design variables

صریح: روابط قید بر حسب متغیرهای طراحی باشد.

مثال: اگر A به عنوان متغیر طراحی سطح مقطع باشد: $A \le 3\,\text{mm}^2$

Implicit: Some are more complex and indirectly influenced by design variables

Example: Deflection at a point in large structure which is impossible to be expressed as an explicit function of the design variables.

- ضمنی: روابط قید بر حسب متغیرهای طراحی موجود نباشد. مانند تغییر مکان یک نقطه از مسئله خرپای دو بعدی

Linear and Nonlinear Constraints

Constraint functions having only first-order terms in design variables are called *linear constraints.*

More general problems have *nonlinear constraint* functions as well.

- خطی: برحسب متغیرهای طراحی مرتبه اول باشد $A \le 3\,\text{mm}^2$
- غیرخطی: بر حسب متغیرهای طراحی مرتبه اول نباشد.

Equality and Inequality Constraints

قیود مساوی و نامساوی

Design problems may have *equality* as well as *inequality* constraints. A feasible design must satisfy precisely all the equality constraints.

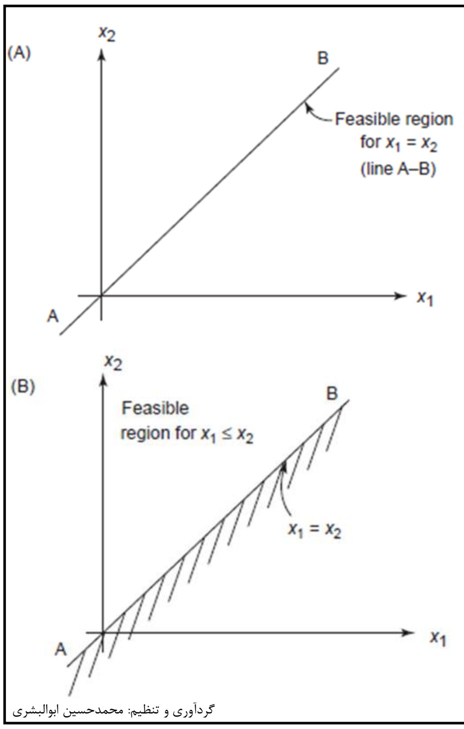
A machine must move precisely by delta (equality).

$$s = \Delta$$

Stress must not exceed the allowable stress of the material (inequality)

$$\sigma \le \sigma_a$$

It is easier to find feasible designs for a system having only inequality/equality constraints. ?



فضای قابل قبول برای یک قید نامساوی بیشتر از زمانی است که همان قید به صورت مساوی باشد.

قیدهای مثال خرپای دو عضوی:

۱- تنش اعضا از حد مجاز بیشتر نشود. $|\sigma| \le \sigma_a \qquad \sigma_a > 0$

$$F_1 = -0.5Wl\left[\frac{\sin\theta}{h} + \frac{2\cos\theta}{s}\right]$$

$$F_2 = -0.5Wl\left[\frac{\sin\theta}{h} - \frac{2\cos\theta}{s}\right]$$

$$A_1 = \frac{\pi}{4}(x_3^2 - x_4^2), \quad A_2 = \frac{\pi}{4}(x_5^2 - x_6^2)$$

$$\sigma_1 = \frac{F_1}{A_1} = \frac{Wl}{2}\cdot\frac{1}{\frac{\pi}{4}\left(x_3^2 - x_4^2\right)}\left[\frac{\sin\theta}{h} + \frac{2\cos\theta}{s}\right] \le \sigma_a$$

$$\sigma_2 = \frac{2Wl}{\pi\left(x_5^2 - x_6^2\right)}\left[\frac{\sin\theta}{x_1} - \frac{2\cos\theta}{x_2}\right] \le \sigma_a$$

Stress constraints need more attention. Please consider them again:

$$\frac{2Wl}{\pi\left(x_3^2 - x_4^2\right)}\left[\frac{\sin\theta}{x_1} + \frac{2\cos\theta}{x_2}\right] \le \sigma_a$$

$$\frac{2Wl}{\pi\left(x_5^2 - x_6^2\right)}\left[\frac{\sin\theta}{x_1} - \frac{2\cos\theta}{x_2}\right] \le \sigma_a$$

Could you suggest anything?

If $\frac{\sin\theta}{x_1} < \frac{2\cos\theta}{x_2}$ in the second eq., then F_2 is a compression force and the stress constraint for member 2 becomes:

$$\frac{-2Wl}{\pi\left(x_5^2 - x_6^2\right)}\left[\frac{\sin\theta}{x_1} - \frac{2\cos\theta}{x_2}\right] \le \sigma_a$$

۲- حدود متغیرهای طراحی رعایت شود.

$$x_{il} \le x_i \le x_{iu} \qquad \text{i =1 to 6}$$

Where x_{il} and x_{iu} are the minimum and maximum values for the *ith* design variable. These constraints are necessary to impose fabrication and physical space limitations.

The problem can be summarized as follows:

Find design variables x_1, x_2, x_3, x_4, x_5, and x_6 to minimize the objective function

$$\text{Mass} = \frac{\pi\rho}{8}(4x_1^2 + x_2^2)^{1/2}(x_3^2 + x_5^2 - x_4^2 - x_6^2)$$

subject to the following constraints:

$$\frac{2Wl}{\pi\left(x_3^2 - x_4^2\right)}\left[\frac{\sin\theta}{x_1} + \frac{2\cos\theta}{x_2}\right] \le \sigma_a$$

$$\frac{2Wl}{\pi\left(x_5^2 - x_6^2\right)}\left[\frac{\sin\theta}{x_1} - \frac{2\cos\theta}{x_2}\right] \le \sigma_a$$

$$\frac{-2Wl}{\pi\left(x_5^2 - x_6^2\right)}\left[\frac{\sin\theta}{x_1} - \frac{2\cos\theta}{x_2}\right] \le \sigma_a$$

$$x_{il} \le x_i \le x_{iu} \qquad \text{i=1 to 6}$$

این مسئله چند قید دارد؟ ۱۵

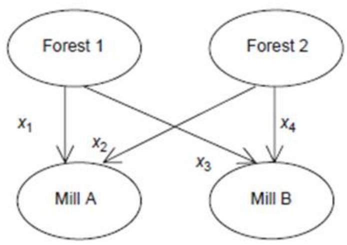
Saw Mill Operation :عملکرد کارخانه چوب بری

Step 1: Project/Problem Statement A company owns two saw mills and two forests. Table 2-1 shows the capacity of each mill (logs/day) and the distances between forests and mills (km). Each forest can yield up to 200 logs/day for the duration of the project, and the cost to transport the logs is estimated at \$0.15/km/log. At least 300 logs are needed each day. The goal is to minimize the total cost of transportation of logs each day.

Step 2: Data and Information Collection Data are given in Table 2-1 and in the problem statement.

TABLE 2-1 Data for Saw Mill Operation

	Distance, km		
Mill	*Forest 1*	*Forest 2*	*Mill capacity/day*
A	24.0	20.5	240 logs
B	17.2	18.0	300 logs



Step 3: Identification/Definition of Design Variables

The design problem is to determine how many logs to ship from Forest *i* to Mill *j*. Therefore, the design variables for the problem are identified and defined as follows:

x_1 = number of logs shipped from Forest 1 to Mill A
x_2 = number of logs shipped from Forest 2 to Mill A
x_3 = number of logs shipped from Forest 1 to Mill B
x_4 = number of logs shipped from Forest 2 to Mill B

Note that if we assign numerical values to these variables, an operational plan for the project is specified and the cost of transportation of logs per day can be calculated. The selected design may or may not satisfy all other requirements.

Step 4: Identification of a Criterion to Be Optimized

The design objective is to minimize the daily cost of transporting the logs to the mills. The cost of transportation, which depends on the distance between the forests and the mills, is:

$$\text{cost} = 24(0.15)x_1 + 20.5(0.15)x_2 + 17.2(0.15)x_3 + 18(0.15)x_4$$

or
$$= 3.6x_1 + 3.075x_2 + 2.58x_3 + 2.7x_4$$

Step 5: Identification of Constraints The constraints for the problem are based on the capacity of the mills and the yield of the forests:

$$x_1 + x_2 \le 240 \quad \text{(Mill A)} \qquad x_1 + x_3 \le 200 \quad \text{(Forest 1)}$$

$$x_3 + x_4 \le 300 \quad \text{(Mill B)} \qquad x_2 + x_4 \le 200 \quad \text{(Forest 2)}$$

The constraint on the number of logs needed for each day is expressed as

$$x_1 + x_2 + x_3 + x_4 \ge 300$$

For a realistic problem formulation, all design variables must be nonnegative, i.e., $x_i \ge 0; \quad i=1 \text{ to } 4$ Usually Missed!

The problem has four design variables, five inequality constraints, and four nonnegativity constraints on the variables. Note that all functions of the problem are linear in design variables, so it is a linear programming problem. Note also that for a meaningful solution, all design variables must have integer values. Such problems are called integer programming problems, which require special solution methods. 6.83: x_1*=0, x_2*=0, x_3 *=200, x_4*=100; f*=786.

طراحی کابینت:

یک کابینت از قطعات c_1 و c_2 و c_3 مونتاژ می‌شود:

تعداد برای هر کابینت	$8c_1$	$5c_2$	$15c_3$
	c_1	c_2	c_3
تعداد پیچ یا پرچ هر قطعه	5	6	3
هزینه پیچ کردن	0.7	1.0	0.6
هزینه پرچ کردن	0.6	0.8	1.0

۱۰۰ کابینت مورد نیاز است و ظرفیت پیچ کردن در روز ۶۰۰۰ و ظرفیت پرچ کردن در روز ۸۰۰۰ عدد می‌باشد.

تعداد قطعاتی که باید پیچ یا پرچ شوند را چنان تعیین کنید که هزینه ساخت مینیمم شود.

رابطه‌سازی اول :

در این رابطه‌سازی فرض می‌شود هر C_1,C_2,C_3 یا پیچ می‌شود یا پرچ می‌شود.

برای هر ۱۰۰ کابینت:

x_1 = تعداد C_1 هایی که باید پیچ شوند

x_2 = تعداد C_1 هایی که باید پرچ شوند

x_3 = تعداد C_2 هایی که باید پیچ شوند

x_4 = تعداد C_2 هایی که باید پرچ شوند

x_5 = تعداد C_3 هایی که باید پیچ شوند

x_6 = تعداد C_3 هایی که باید پرچ شوند

هزینه = $0.70(5)x_1 + 0.6(5)x_2 + 1.0(6)x_3 + 0.8(6)x_4 + 0.6(3)x_5 + 1.0(3)x_6$

قیود مسئله:

تعداد قطعات مورد نیاز c_2,c_1 و c_3 برای ۱۰۰ کابینت در روز عبارتند از:

$x_1+x_2=800$ (تعداد c_1 ها)

$x_3+x_4=500$ (تعداد c_2 ها)

$x_5+x_6=1500$ (تعداد c_3 ها)

ظرفیت پیچ کردن $5x_1+6x_3+3x_5 \leq 6000$

ظرفیت پرچ کردن $5x_2+6x_4+3x_6 \leq 8000$

قید نامنفی بودن متغیرهای طراحی $x_i \geq 0, \quad i=1 \text{ to } 6$

Usually missed!

رابطه‌سازی دوم:

اگر این قید که هر قطعه‌ای باید پیچ یا پرچ شود را رها سازیم، داریم:

در این رابطه‌سازی هر C_1,C_2,C_3 می‌تواند مخلوطی از پیچ یا پرچ داشته باشد.

x_1= تعداد پیچ‌هایی که برای همه C_1 ها نیاز است،
x_2= تعداد پیچ‌هایی که برای همه C_2 ها نیاز است،
x_3= تعداد پیچ‌هایی که برای همه C_3 ها نیاز است،
x_4= تعداد پرچ‌هایی که برای همه C_1 ها نیاز است،
x_5= تعداد پرچ‌هایی که برای همه C_2 ها نیاز است،
x_6= تعداد پرچ‌هایی که برای همه C_3 ها نیاز است،

هدف همچنان مینیمم کردن هزینه کل ساخت ۱۰۰ کابینت است.

هزینه $= 0.70x_1 + 1.0x_2 + 0.6x_3 + 0.6x_4 + 0.8x_5 + 1.0x_6$

قیود مسئله:

کل پیچ‌ها و پرچ‌هایی که برای c_2,c_1 و c_3 مورد نیاز است با قیود مساوی زیر بیان می‌شود.

$x_1+x_4=4000$ (برای c_1)
$x_2+x_5=3000$ (برای c_2)
$x_3+x_6=4500$ (برای c_3)
$x_1+x_2+x_3\le 6000$ ظرفیت پیچ کردن
$x_4+x_5+x_6\le 8000$ ظرفیت پرچ کردن
$x_i\ge 0,\quad i=1 \text{ to } 6$ قید نامنفی بودن متغیرهای طراحی

رابطه‌سازی سوم:

اگر بخواهیم همه کابینت‌ها مثل هم باشند:

x_1= تعداد C_1ها که در یک کابینت باید پیچ شوند،
x_2= تعداد C_1ها که در یک کابینت باید پرچ شوند،
x_3= تعداد C_2ها که در یک کابینت باید پیچ شوند،
x_4=تعداد C_2ها که در یک کابینت باید پرچ شوند،
x_5= تعداد C_3ها که در یک کابینت باید پیچ شوند،
x_6= تعداد C_3ها که در یک کابینت باید پرچ شوند،

هزینه ۱۰۰ کابینت در روز:

$$\text{هزینه} = 100[5(0.70)x_1 + 5(0.6)x_2 + 6(1.0)x_3 + 6(0.8)x_4 + 3(0.6)x_5 + 3(1.0)x_6]$$
$$= 350x_1 + 300x_2 + 600x_3 + 480x_4 + 180x_5 + 300x_6$$

قیود مسئله:

چون هر کابینت $5c_2, 8c_1$ و $15c_3$ نیاز دارد:

$x_1+x_2=8$ (برای c_1)

$x_3+x_4=5$ (برای c_2)

$x_5+x_6=15$ (برای c_3)

ظرفیت پیچ کردن $100(5x_1+6x_3+3x_5)\le 6000$

ظرفیت پرچ کردن $100(5x_2+6x_4+3x_6)\le 8000$

قید نامنفی بودن متغیرهای طراحی $x_i\ge 0, \quad i=1 \text{ to } 6$

توابع هزینه و قیود در هر سه رابطه‌سازی خطی هستند.
سه رابطه‌سازی سه جواب (مقادیر متغیرهای طراحی) بهینه مختلف خواهند داشت. ولی بر حسب اتفاق $f^* = 7500$ در همه یکسان است.
برای یک جواب معنی دار با این رابطه‌سازی‌ها، تمامی متغیرهای طراحی باید مقادیر اعداد صحیح داشته باشند. این‌ها را مسائل برنامه ریزی صحیح می‌نامند.

نکات:

۱. مسئله ممکن است رابطه‌سازی‌های مختلفی داشته باشد.
۲. علیرغم رابطه‌سازی‌های مختلف، جواب بهین می‌تواند یکسان یا متفاوت باشد.
۳. در فرایند رابطه‌سازی، کلیه فرضیات باید به طور دقیق بیان شود.
۴. فرایند رابطه‌سازی از حل مسئله جداست.
۵. در صورت رابطه‌سازی‌های مختلف، باید مسئله را برای هر رابطه‌سازی حل کرد و بهترین جواب را انتخاب کرد.

Example – Design of a Can of Coke

Design a can to hold at least the specified amount of coke and meet other design requirement. The cans will be produced in billions, so it is desirable to minimize the cost of manufacturing. Since the cost can be related directly to the surface area of the sheet metal used, it is reasonable to minimize the sheet metal required to fabricate the can.

Fabrication, handling, aesthetic (appreciating of beauty), and shipping considerations impose the following restrictions on the size of the can

1. The diameter of the can should be no more than 8 *cm*. Also, it should not be less than 3.5 *cm*.
2. The height of the can should be no more than 18 *cm* and no less than 8 *cm*.
3. The can is required to hold at least 400 *ml* of fluid.

Design variables

D = diameter of the can (cm)

H = height of the can (cm)

Objective function

The design objective is to minimize the surface area

$$f(D,H) = \pi DH + \frac{\pi}{2}D^2, \quad \text{cm}^2$$

The constraints must be formulated in terms of design variables.

The first constraint is that the can must hold at least 400 *ml* of fluid.

$$\frac{\pi}{4} D^2 H \geq 400, \quad \text{cm}^3$$

The other constraints on the size of the can are:

$$3.5 \leq D \leq 8; \quad 8 \leq H \leq 18, \quad \text{cm}$$

The problem has 2 independent design variable and 5 explicit constraints. The objective function and first constraint are nonlinear in design variable whereas the remaining constraints are linear.

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Minimum Weight Design of a Symmetric Three-Bar Truss

To support a force *P*, the truss must satisfy various performance and technological constraints, such as member crushing, member buckling, failure by excessive deflection of node 4, and failure by resonance when natural frequency of the structure is below a given threshold.

$$u = \frac{\sqrt{2} l P_u}{A_1 E}; v = \frac{\sqrt{2} l P_v}{(A_1 + \sqrt{2} A_2) E}$$

$$\sigma_1 = \frac{1}{\sqrt{2}} \left[\frac{P_u}{A_1} + \frac{P_v}{(A_1 + \sqrt{2} A_2)} \right]$$

$$\sigma_2 = \frac{\sqrt{2} P_v}{(A_1 + \sqrt{2} A_2)}$$

$$\sigma_3 = \frac{1}{\sqrt{2}} \left[-\frac{P_u}{A_1} + \frac{P_v}{(A_1 + \sqrt{2} A_2)} \right]$$

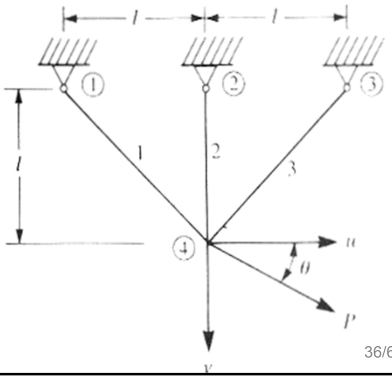
Lowest eigenvalue of the structure, ξ

$$\zeta = \frac{3EA_1}{\rho l^2 (4A_1 + \sqrt{2} A_2)}$$

$$\zeta \geq (2\pi\omega_0)^2$$

Member buckling constraints

$$-\sigma_1 \leq \frac{\pi^2 E \beta A_1}{2l^2}; \quad -\sigma_2 \leq \frac{\pi^2 E \beta A_2}{l^2}; \quad -\sigma_3 \leq \frac{\pi^2 E \beta A_1}{2l^2}$$



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Three–bar Truss

Design Variables: (Because of symmetry)

$$A_1, A_2 (A_1 = A_3)$$

Min volume

$$\text{volume} = l(2\sqrt{2}A_1 + A_2)$$

Subject to:

$$\sigma_i \le \sigma_a$$

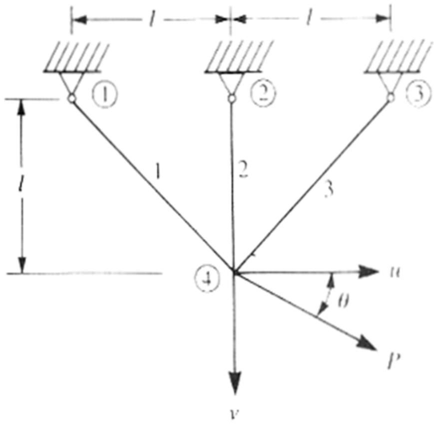
$$\zeta_1 \ge \alpha$$

$$u_4 \le \delta_1$$

$$v_4 \le \delta_2$$

$$\text{buckling of } ith \text{ member} \le \beta_i$$

$$A_1, A_2 \ge A_{\min}$$



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2.7– یک الگوی استاندارد برای مسائل طراحی بهین

یک بردار n بعدی $X = (x_1, x_2, \ldots x_n)$ چنان بیابید که:

تابع هزینه $f(X) = f(x_1, x_2, \ldots x_n)$

را نسبت به: قیود مساوی

$$h_j(X) \equiv h_j(x_1, x_2, \ldots x_n) = 0; \quad j = 1 \text{ to } p$$

و قیود نامساوی

$$g_i(X) \equiv g_i(x_1, x_2, \ldots x_n) \le 0; \quad \text{i=1 to m}$$

مینیمم نماید.

شرطهای $x_{il} \le x_i \le x_{iu}$ و $x_i \ge 0$ در قیود نامساوی گنجانده شده است.

در مقالات مختلف قیود صریح روی متغیرهای طراحی نامهای گوناگون دارند:

$x_{il} \le x_i \le x_{iu}$

side constraints, technological constraints,

simple bounds, sizing constraints

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Optimum solution does not change if

تابع هزینه را در یک عدد مثبت ضرب کرد و یا یک عدد ثابت به آن اضافه کرد

$$\alpha f(x), f(x) + \beta$$

$$\alpha > 0, \ \beta \text{ any constant}$$

بدیهی است که مقدار تابع هزینه تغییر می‌کند

قیدهای مساوی را در یک عدد ضرب کرد $\beta h_j(x) = 0$

قیدهای نامساوی را در یک عدد مثبت ضرب کرد $\alpha g_i(x) \le 0$

مجموعه قیود یا مجموعه طراحی‌های قابل‌قبول

$$S = \{x \mid h_j(x) = 0; j = 1 \text{ to } p, g_i(x) \le 0; i = 1 \text{ to } m\}$$

اگر تعداد قیدها زیاد شود ناحیه قابل‌قبول کوچکتر می‌شود و برعکس.
در اثر کوچک شدن ناحیه قابل‌قبول، ممکن است مینیمم تابع هزینه افزایش پیدا کند.

قیود فعال / غیرفعال / نقض شده

$$g_i(x) \le 0 \begin{cases} g_i(x^*) = 0 \\ g_i(x^*) < 0 \end{cases}$$

قید در نقطه‌ی طراحی x^* فعال است.
قید در نقطه‌ی طراحی x^* غیرفعال است.

قیدهای مساوی غیرفعال نمی‌توانند باشند.

$$\begin{cases} g_i(x^*) > 0 \\ h_j(x^*) \ne 0 \end{cases}$$

قید در نقطه‌ی طراحی x^* نقض شده است.

Standard Design Optimization Model

$$\min f(x)$$
$$\text{s.t.} \quad h_j(x) = 0, \quad j = 1 \; to \; p$$
$$g_i(x) \le 0, \quad i = 1 \; to \; m$$
$$x_{il} \le x_i \le x_{iu}, \quad i = 1 \; to \; n$$

If p>n : over-determined system of eqns. (redundant eqn)

If p<n : optimum soln. is possible

If p=n : no optimization is necessary

No restriction on inequality constraints

Optimization Types:
- Unconstrained optimization
- Constrained optimization

Linear Programming :

all $f(x), h_j(x), g_i(x)$ are linear in x

Observations on the standard model:

$$\min f(x) \equiv \max[-f(x)] \equiv \max[\frac{1}{f(x)}]$$

$$g_j(x) \ge 0 \equiv -g_j(x) \le 0$$

Observations on the Standard Model

- The functions $f(x)$, $h_j(x)$, and $g_i(x)$ must depend on some or all of the design variables.
- The number of independent equality constraints must be less than or at most equal to the number of design variables.
- There is no restriction on the number of inequality constraints.
- Some design problems may not have any constraints (unconstrained optimization problems).
- Linear programming is needed if all the functions $f(x)$, $h_j(x)$, and $g_i(x)$ are linear in design variables x, otherwise use nonlinear programming.

Discrete and Integer Design Variables

متغیرهای طراحی صحیح و گسسته

- صحیح: مانند تعداد ماشین، نفر...
- گسسته: مانند ضخامت ورق، تیر آهن‌ها، پیچ مهره‌ها...

- متغیرهای صحیح و یا گسسته، به مسئله طراحی قیود اضافی تحمیل می‌کنند.
- مقدار تابع هزینه بهین وقتی متغیرهای طراحی صحیح و یا گسسته باشند، در مقایسه با حالت پیوسته، به احتمال زیاد افزایش می‌یابد.

Solutions of Integer/Discrete variable Problems

1)

- Solve the problem assuming continuous design variables if that is possible.
- The nearest discrete/integer values are assigned to the variables.
- The design is checked for feasibility.
- With a few trials, the best feasible design close to the continuous optimum can be obtained. Note that there can be numerous combinations of discrete variables that can give feasible designs.

Solutions of Integer/Discrete variable Problems (Cont'd)

2) Adaptive numerical optimization procedure

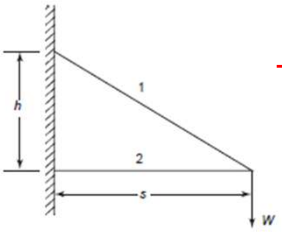
- Optimum solution with continuous variables is first obtained if that is possible.
- Only the variables that are close to their discrete or integer value are assigned that value.
- They are then held fixed and the problem is optimized again.
- The procedure is continued until all the variables have been assigned discrete or integer values.
- A few further trials may be made to improve the optimum cost function value.

Using the Graphical Solution method, you will be able to:

- Graphically solve any optimization problem having two design variables
- Plot constraints and identify their feasible/infeasible side
- Identify the feasible region/feasible set for the problem
- Plot objective function contours through the feasible region
- Graphically locate the optimum solution for a problem and identify active/inactive constraints
- Identify problems that may have multiple, unbounded, or infeasible solutions

Optimization using graphical method

3.8.1) A wall bracket is to be designed to support a load of *W*. The bracket should not fail under the load.



h=30 cm, *s*=40 cm, *W*=1.2 MN

Total volume of the bracket is to be minimized.

Bar 1: $\sigma_1 \le \sigma_a$

Bar 2: $\sigma_2 \le \sigma_a$

σ_a= allowable stress for the material16,000(N/cm^2)
σ_1 = stress in Bar 1 which is given as F_1/A_1, N/cm^2
σ_2 = stress in Bar 2 which is given as F_2/A_2, N/cm^2
A_1 = cross-sectional area of Bar 1 (cm^2)
A_2 = cross-sectional area of Bar 2 (cm^2)
F_1 = force due to load W in Bar 1 (N)
F_2 = force due to load W in Bar 2 (N)

Design variables: A_1 and A_2

Objective function: $f\,(A_1, A_2) = l_1 A_1 + l_2 A_2, \quad \text{cm}^3$

Forces on bar 1 and bar 2 are:

$$F_1 = (2.0E+06)\ \text{N},\ \ F_2 = (1.6E+06)\ \text{N}$$

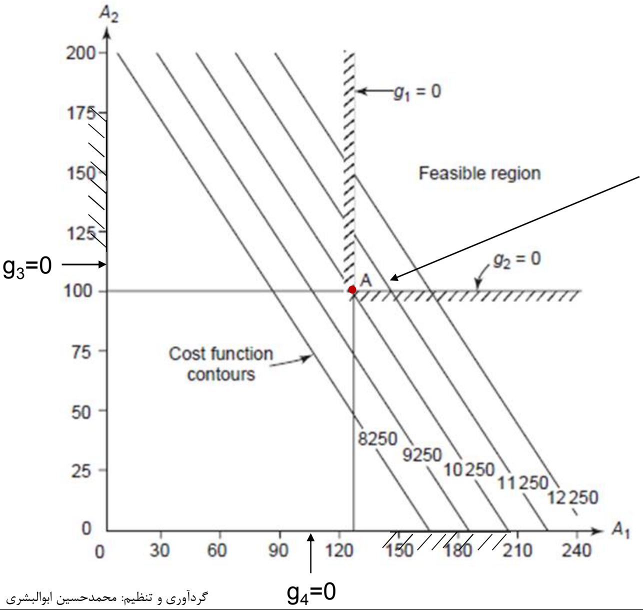
Stress constraints:

$$g_1 = \frac{(2.0E+06)}{A_1} - 16000 \le 0$$

$$g_2 = \frac{(1.6E+06)}{A_2} - 16000 \le 0$$

Usually missed! $g_3 \equiv -A_1 \le 0, \quad g_4 \equiv -A_2 \le 0$

Graphical Solution for the Wall Bracket problem



Optimum solution:

A_1=125,
A_2=100

3.1.1 Profit Maximization Problem

Step 1: Project/Problem Statement

A company manufactures two machines, A and B. Using available resources, either 28A or 14B machines can be manufactured daily. The sales department can sell up to 14A machines or 24B machines. The shipping facility can handle no more than 16 machines per day. The company makes a profit of $400 on each A machine and $600 on each B machine. How many A and B machines should the company manufacture every day to maximize its profit?

Step 2: Data and Information Collection

Defined in the project statement.

Step 3: Identification/Definition of Design Variables

The following two design variables are identified in the problem statement:

x_1 = number of A machines manufactured each day

x_2 = number of B machines manufactured each day

Step 4: Identification of a Criterion to Be Optimized

The objective is to maximize daily profit, which can be expressed in terms of design variables as

$$P = 400x_1 + 600x_2 \qquad \text{(a)}$$

Step 5: Identification of Constraints Design constraints are placed on manufacturing capacity, limitations on the sales personnel, and restrictions on the shipping and handling facility. The constraint on the shipping and handling facility is quite straightforward, expressed as

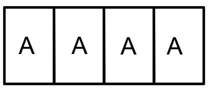
$$x_1 + x_2 \leq 16 \qquad \text{(b)}$$

Constraints on manufacturing:

It is assumed that if the company is manufacturing x_1 number A machines per day, then the remaining resources and equipment can be proportionately utilized to manufacture X_2 number B machines, and vice versa.

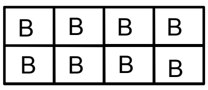
Therefore, noting that $x_1/28$ is the fraction of resources used to produce A machines and $x_2/14$ is the fraction used for B, the constraint is expressed as

Example for Capacity 4A or 8B



$$\frac{\text{No.of A}}{4} = \frac{4}{4} = 1$$

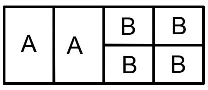
اگر تمام ظرفیت تولید به ماشین A اختصاص یابد.



$$\frac{\text{No. of B}}{8} = \frac{8}{8} = 1$$

اگر تمام ظرفیت تولید به ماشین B اختصاص یابد.

اگر بخشی از ظرفیت تولید به ماشین A و بخشی به B اختصاص یابد.



$$\frac{\text{No.of A}}{4} + \frac{\text{No. of B}}{8} = \frac{2}{4} + \frac{4}{8} = 1$$

بنابراین:

$$\frac{x_1}{28} + \frac{x_2}{14} \leq 1 \qquad \text{(c)}$$

Similarly, the constraint on sales department resources is given as

$$\frac{x_1}{14}+\frac{x_2}{24}\le 1 \qquad \text{(d)}$$

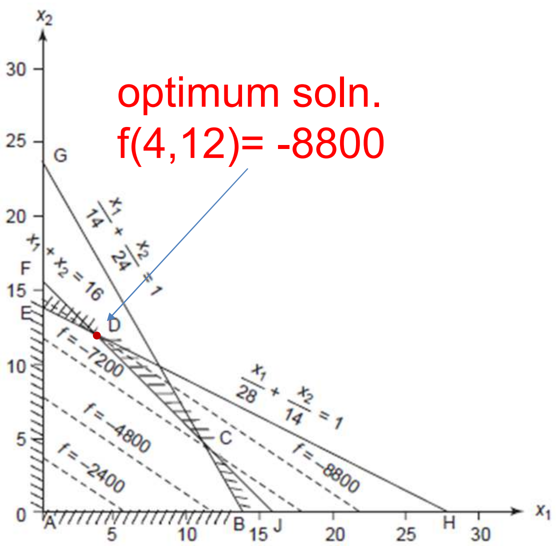
Finally, the design variables must be nonnegative as

$$x_1, x_2 \ge 0 \qquad \text{(e)}$$

Note:
- The formulation remains valid even when a design variable has zero value.
- No. of design variables: 2
- No. of inequality constraints: 5
- All functions of the problem are linear in variables x_1 and x_2. Therefore, it is a linear programming problem.

Graphical Solution of the Profit Maximization Problem

$$f = -400x_1 - 600x_2$$



(shipping and handling constraint)

$$x_1 + x_2 \le 16$$

(manufacturing constraint)

$$\frac{x_1}{28}+\frac{x_2}{14}\le 1$$

(limitation on sales department)

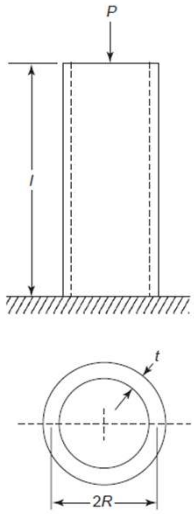
$$\frac{x_1}{14}+\frac{x_2}{24}\le 1$$

$$-x_1, -x_2 \le 0$$

FIGURE 6-1 Graphical solution for profit maximization LP problem. Optimum point = (4,12). Optimum cost = -8800.

Example of a Nonlinear Problem with Multiple Solution

The objective of this project is to design a minimum-mass tubular column of length *l* supporting a load *P* without buckling or overstressing.



R = mean radius of the column
t = wall thickness
Assuming: $(R >> t) \rightarrow A = 2\pi Rt; \quad I = \pi R^3 t$

$$\text{mass} = \rho(lA) = 2\rho l \pi Rt$$

$$\frac{P}{2\pi Rt} \le \sigma_a$$

$$P_{cr} = \frac{\pi^2 EI}{4l^2} \rightarrow P \le \frac{\pi^3 ER^3 t}{4l^2}$$

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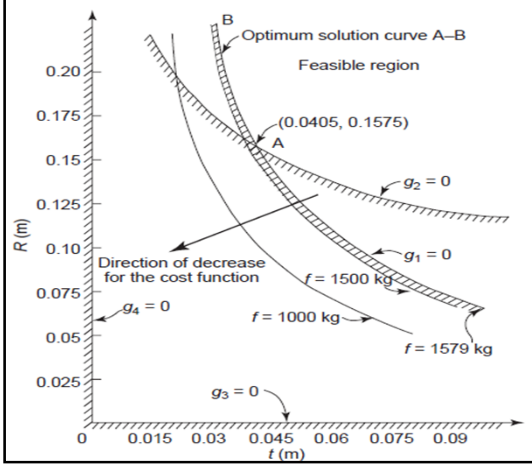
$$P = 10\text{MN}, E = 207\text{GPa}, \rho = 7833\,\text{kg/m}^3, l = 5.0\,\text{m}, \sigma_a = 248\text{MPa}$$

$$f(R,t) = 2\rho l \pi Rt = 2(7833)(5)\pi Rt = 2.4608 \times 10^5 Rt, \text{kg}$$

$$g_1(R,t) = \frac{P}{2\pi Rt} - \sigma_a = \frac{10\times 10^6}{2\pi Rt} - 248\times 10^6 \le 0 \text{ (stress constraint)}$$

$$g_2(R,t) = P - \frac{E\pi^3 R^3 t}{4l^2} = 10\times 10^6 - \frac{\pi^3(207\times 10^9)R^3 t}{4(5)(5)} \le 0 \text{ (bucling load constraint)}$$



$$g_3(R,t) = -R \le 0$$

$$g_4(R,t) = -t \le 0$$

شعاع حدود ۴ برابر ضخامت است و در استفاده از تقریب جدار نازک بودن می‌توان تجدید نظر کرد و روابط دقیق تری را استفاده کرد.

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Graphical Solution of the Beam Design

Nonlinear problem with Multiple Solution

A beam of rectangular cross section is subjected to a bending moment of *M* (N·m) and a maximum shear force of *V* (N). The bending stress in the beam is calculated as $\sigma=6M/bd^2$ (Pa) and average shear stress is calculated as $\tau=3V/2bd$ (Pa), where *b* is the width and *d* is the depth of the beam. The allowable stresses in bending and shear are 10 MPa and 2 MPa, respectively.
It is also desirable that the depth of the beam not exceed twice its width and that the cross-sectional area of the beam is minimized.

d= depth of the beam, mm
b =width of the beam, mm

Min f(b,d)=bd M=40 kN·m V=150 kN

$$\sigma=\frac{6M}{bd^2}=\frac{6(40)(1000)(1000)}{bd^2}, \text{N/mm}^2 \qquad \sigma_a=10\,\text{MPa}=10\times10^6\,\text{N/m}^2=10\,\text{N/mm}^2$$

$$\tau=\frac{3v}{2bd}=\frac{3(150)(1000)}{2bd}, \text{N/mm}^2 \qquad \tau_a=2\,\text{MPa}=2\times10^6\,\text{N/m}^2=2\,\text{N/mm}^2$$

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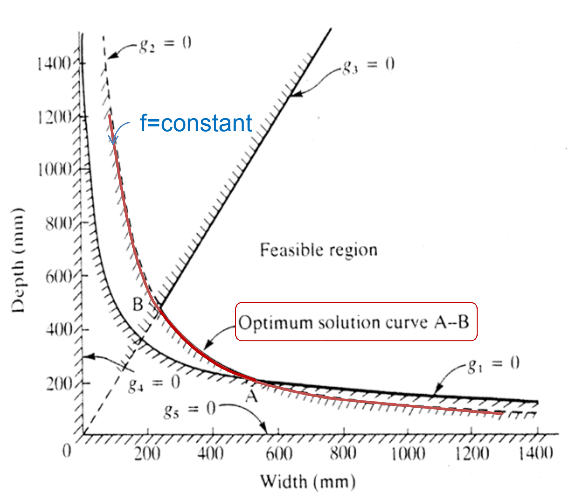
Graphical Solution of the Beam Design

$$g_1=\frac{6(40)(1000)(1000)}{bd^2}-10\le0 \text{ (bending stress)} \qquad g_3=d-2b\le0$$

$$g_2=\frac{3(150)(1000)}{2bd}-2\le0 \text{ (shear stress)} \qquad g_4=-b\le0;\quad g_5=-d\le0$$



In reality, *b* and *d* cannot both have zero value, so we should use some minimum value as lower bounds on them, i.e., $b\ge b_{min}$ and $d\ge d_{min}$.

Note that the cost function is parallel to the constraint g_2 (both functions have the same form: *bd* = constant).

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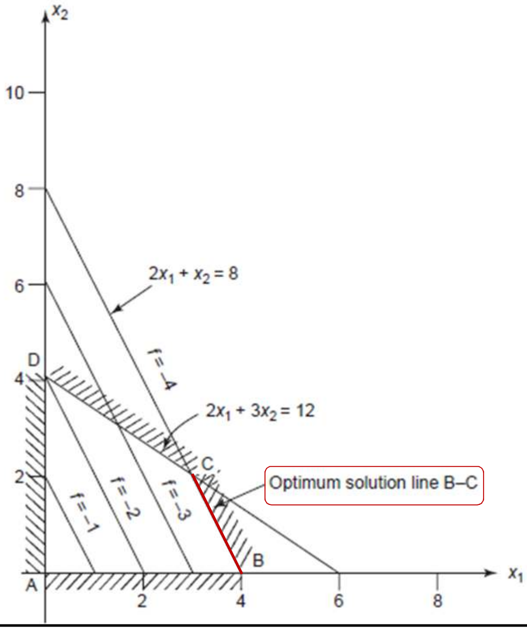
Example for a Multiple Solution



$$\min f = -x_1 - 0.5x_2$$
$$2x_1 + 3x_2 \le 12$$
$$2x_1 + x_2 \le 8$$
$$-x_1 \le 0, -x_2 \le 0$$

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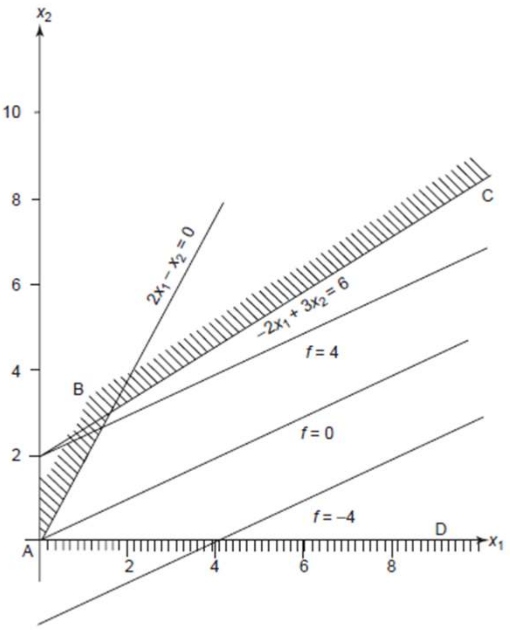
Example of an Unbounded Solution



مسئله با جواب بی کران

min $f(x) = -x_1 + 2x_2$

subject to

$-2x_1 + x_2 \le 0,$

$-2x_1 + 3x_2 \le 6,$

$-x_1 \le 0,\ -x_2 \le 0$

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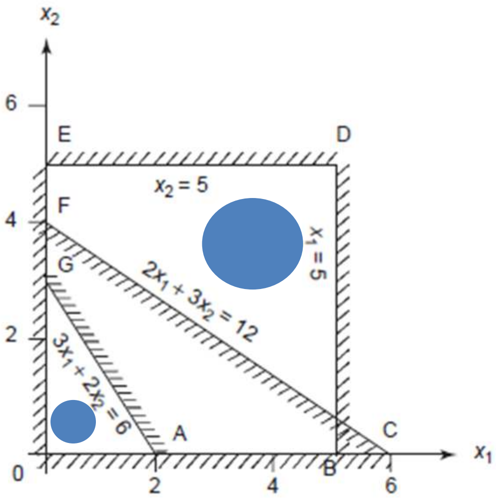
Infeasible Problem مسئله غیر قابل قبول

Conflicting requirements, inconsistent constraint equations or too many constraints on the system will result in no solution to the problem.



min $f(x)=x_1+2x_2$
subject to
$3x_1+2x_2 \le 6,$
$2x_1+3x_2 \ge 12,$
$x_1 \le 5,$
$x_2 \le 5,$
$x_1, x_2 \ge 0$

No region of design space that satisfies all constraints.

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مسائل زیر را حل کرده و تا دو هفته دیگر تحویل فرمایید:

2) 2,4,6,8,10,12,14,15,16,18,19,29,32,34,64,65

2) 2,8,12,13,17 با استفاده از Excel یا Matlab

There is the following section in the text book (2nd Ed.)
3.3 Use of MATLAB for Graphical Optimization

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