= تعریف حد =

$|x-a| \leftarrow x$ نزدیک

$a-\delta \quad a \quad a+\delta$ $\qquad 0 \quad M \quad +\infty \notin \mathbb{R}$ $\qquad \mathbb{R} \not\ni -\infty \quad -M \quad 0$

تعریف همسایگی

$N_\delta(a)=\{x: |x-a|<\delta\}$ $\quad a-\delta<x<a+\delta$

$N_M(+\infty)=(M,+\infty)=\{x: x>M\}$

$N_M(-\infty)=(-\infty,-M)=\{x: x<-M\}$

تعریف $\mathbb{R}^*=\mathbb{R}\cup\{-\infty,+\infty\}$, $-\infty<x<+\infty$, $+\infty+(+\infty)=+\infty$, ...

اعداد حقیقی توسیع یافته

هرقدر که بخواهیم $f(x)$ را به l نزدیک کنیم مشروط بر اینکه x به قدر کافی به a نزدیک و برابر نشود

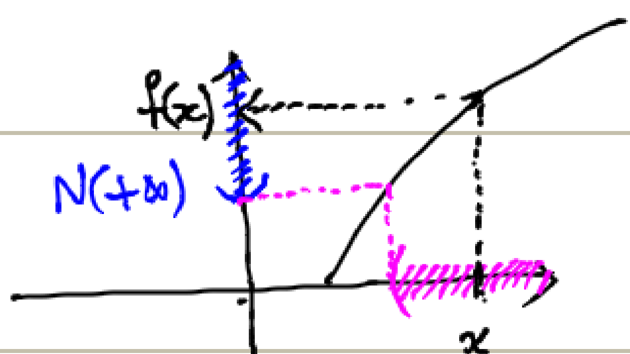
تعریف Let $a, l \in \mathbb{R}^*$.

$$\boxed{\forall N_\varepsilon(l)\ \exists N_\delta(a)\ \forall x\in D_f,\ x\neq a;\quad x\in N_\delta(a)\Rightarrow f(x)\in N_\varepsilon(l)}$$

$$a,l\in\mathbb{R}\Rightarrow \forall\varepsilon\ \exists\delta\ \forall x\neq a;\quad |x-a|<\delta\Rightarrow |f(x)-l|<\varepsilon$$

$$a=+\infty,\ l\in\mathbb{R}\Rightarrow \forall\varepsilon\ \exists\delta\ \forall x;\quad x>\delta\Rightarrow |f(x)-l|<\varepsilon$$

$$a=-\infty,\ l=+\infty\Rightarrow\ 〃\ 〃\ 〃;\quad x<-\delta\Rightarrow f(x)>\varepsilon$$



$\lim f(x)=+\infty$

$$\lim_{x\to a} f(x) = +\infty$$

$\forall \varepsilon\ \exists \delta\ \forall x;\ |x-a|<\delta \Rightarrow f(x) > \varepsilon$

$\forall N(+\infty)\ \exists N(a)\ \forall x;\ x \in N(a) \Rightarrow f(x) \in N(+\infty)$

$$\lim_{x\to +\infty} f(x) = +\infty$$

$\forall N(+\infty)\ \exists N(+\infty)\ \forall x;\ x \in N(+\infty) \Rightarrow f(x) \in N(+\infty)$

$$\lim_{x\to +\infty} f(x) = l$$

$$\lim_{x\to -\infty} f(x) = -\infty$$

1.1. Banach Algebras

نام درس: آنالیز تابعی   مرجع: نرماتیک جارودی + کتاب مورفی

فصل اول
از کتاب مورفی

We begin by setting up the basic vocabulary needed to discuss Banach algebras and by giving some examples.

An *algebra* is a vector space A together with a bilinear map

$$A \times A \to A, \quad (a,b) \mapsto ab,$$

such that

$$a(bc) = (ab)c \quad (a,b,c \in A).$$

B is a subalg of A if B is closed under: {addition, multiplication, scalar mult.

A *subalgebra* of A is a vector subspace B such that $b, b' \in B \Rightarrow bb' \in B$. Endowed with the multiplication got by restriction, B is itself an algebra.

A norm $\|.\|$ on A is said to be *submultiplicative* if

زیر ضربی

$$\|ab\| \le \|a\|\|b\| \quad (a,b \in A).$$

$\|x\|=0 \iff x=0$
$\|\lambda x\|=|\lambda|\|x\|$
$\|x+y\| \le \|x\|+\|y\|$

In this case the pair $(A, \|.\|)$ is called a *normed algebra*. If A admits a unit 1 ($a1 = 1a = a$, for all $a \in A$) and $\|1\| = 1$, we say that A is a *unital normed algebra*.

جبر نرمدار یکدار

جبر مجموعه‌ای است همراه با عمل $+$، $\cdot$، $\cdot_s$ به طوری که $(A,+,\cdot)$ حلقه و $(A,+,\cdot_s)$ فضای برداری است و
$\lambda \cdot_s (ab) = (\lambda \cdot_s a)b = a\cdot(\lambda \cdot_s b)$

If A is a normed algebra, then it is evident from the inequality

$$\|ab - a'b'\| \le \|a\|\|b - b'\| + \|a - a'\|\|b'\| \quad 1$$

that the multiplication operation $(a,b) \mapsto ab$ is jointly continuous.

A complete normed algebra is called a *Banach algebra*. A complete unital normed algebra is called a *unital Banach algebra*.

A subalgebra of a normed algebra is obviously itself a normed algebra with the norm got by restriction. The closure of a 2 subalgebra is a subalgebra. A closed subalgebra of a Banach algebra is a Banach algebra.

So $\bar{B}$ is a Ban subalg of a Banach alg A when B is a subalg.

1.1.1. Example. If S is a set, $\ell^\infty(S)$, the set of all bounded complex-valued functions on S, is a unital Banach algebra where the operations are defined pointwise:

$$\sup_{x\in S}|f(x)| < +\infty$$

$$(f+g)(x) = f(x) + g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$(\lambda f)(x) = \lambda f(x),$$

هر جلسه باید درس جلسه قبل را یاد گرفته باشید حتی اگر غایب بوده‌اید.

جاهای مطلبی را نفهمیده رد نشوید.

and the norm is the sup-norm

$$\|f\|_\infty = \sup_{x \in S} |f(x)|.$$

Exercise. Prove that $\ell^\infty(S)$ is complete (3)

1.1.2. *Example.* If Ω is a topological space, the set $C_b(\Omega)$ of all bounded continuous complex-valued functions on Ω is a closed subalgebra of $\ell^\infty(\Omega)$. (4) Thus, $C_b(\Omega)$ is a unital Banach algebra.

If Ω is compact, $C(\Omega)$, the set of continuous functions from Ω to $\mathbf{C}$, is of course equal to $C_b(\Omega)$. → the constant function $f \equiv 1$. $x-1$ $x+1$ $\mathbb{R}$ is l.c.H.S.

1.1.3. *Example.* If Ω is a locally compact Hausdorff space, we say that a continuous function f from Ω to $\mathbf{C}$ *vanishes at infinity*, if for each positive number ε the set $\{\omega \in \Omega \mid |f(\omega)| \geq \varepsilon\}$ is compact. We denote the set of such functions by $C_0(\Omega)$. It is a closed subalgebra of $C_b(\Omega)$, and therefore, a Banach algebra. It is unital if and only if Ω is compact, (5) and in this case $C_0(\Omega) = C(\Omega)$. The algebra $C_0(\Omega)$ is one of the most important examples of a Banach algebra, and we shall see it used constantly in C*-algebra theory (the functional calculus). (6)

قانون: در مرحله بعدی خوانده شود، دقت اگر غایب بوده باشید

1.1.4. *Example.* If (Ω, μ) is a measure space, the set $L^\infty(\Omega, \mu)$ of (classes of) essentially bounded complex-valued measurable functions on Ω is a unital Banach algebra with the usual (pointwise-defined) operations and the essential supremum norm $f \mapsto \|f\|_\infty$. clearly:

→ $\|f\|_\infty = \inf\{\lambda : |f(x)| \leq \lambda \text{ a.e.}\} < \infty$ every bd function is ess. bd.

$\ell^\infty(\Omega) \subset L^\infty(\Omega, \mu)$ $\boxed{\Omega}$

1.1.5. *Example.* If Ω is a measurable space, let $B_\infty(\Omega)$ denote the set of all bounded complex-valued measurable functions on Ω. Then $B_\infty(\Omega)$ is a closed subalgebra of $\ell^\infty(\Omega)$, so it is a unital Banach algebra. This example will be used in connection with the spectral theorem in Chapter 2.

1.1.6. *Example.* The set A of all کلیه continuous functions on the closed unit disc $\mathbf{D}$ in the plane which are analytic on the interior of $\mathbf{D}$ is a closed subalgebra of $C(\mathbf{D})$, so A is a unital Banach algebra, called the $\boxed{\textit{disc algebra.}}$ This is the motivating example in the theory of function algebras, where many aspects of the theory of analytic functions are extended to a Banach algebraic setting. open unit ball $\{z : |z| < 1\}$

$f(z) = \frac{1}{1-z}$

All of the above examples are of course *abelian*—that is, $ab = ba$ for all elements a and b—but the following examples are not, in general.

1.1.7. *Example.* If X is a normed vector space, denote by $B(X)$ the set of all bounded linear maps from X to itself (the *operators* on X). It is routine to show that $B(X)$ is a normed algebra with the pointwise-defined operations for addition and scalar multiplication, multiplication given by $(u, v) \mapsto u \circ v$, and norm the *operator norm*:

$$\|u\| = \sup_{x \neq 0} \frac{\|u(x)\|}{\|x\|} = \sup_{\|x\| \leq 1} \|u(x)\|.$$

If X is a Banach space, $B(X)$ is complete and is therefore a Banach algebra.

7

1.1.8. *Example.* The algebra $M_n(\mathbf{C})$ of $n \times n$-matrices with entries in $\mathbf{C}$ is identified with $B(\mathbf{C}^n)$. It is therefore a unital Banach algebra. Recall that an *upper triangular* matrix is one of the form

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & \dots & \dots & \lambda_{1n} \\ 0 & \lambda_{22} & \dots & \dots & \lambda_{2n} \\ 0 & 0 & \lambda_{33} & \dots & \lambda_{3n} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \lambda_{nn} \end{pmatrix}$$

(all entries below the main diagonal are zero). These matrices form a subalgebra of $M_n(\mathbf{C})$.

We shall be seeing many more examples of Banach algebras as we proceed. Most often these will be non-abelian, but in the first three sections of this chapter we shall be principally concerned with the abelian case.

If $(B_\lambda)_{\lambda \in \Lambda}$ is a family of subalgebras of an algebra A, then $\cap_{\lambda \in \Lambda} B_\lambda$ is a subalgebra, also. Hence, for any subset S of A, there is a smallest subalgebra B of A containing S (namely, the intersection of all the subalgebras

8

containing S). This algebra is called the subalgebra of A *generated* by S. If S is the singleton set $\{a\}$, then B is the linear span of all powers a^n $(n = 1, 2, \dots)$ of a. If A is a normed algebra, the closed algebra C *generated* by a set S is the smallest closed subalgebra containing S. It is plain that $C = \bar{B}$, where B is the subalgebra generated by S.

① If X & Y are Ban spaces, then $X \times Y$ together with $\|(x,y)\|_1 = \|x\| + \|y\|$ or $\|(x,y)\|_\infty = \max\{\|x\|, \|y\|\}$ is a Ban sp. (Conversely, if $X \times Y$ is complete, then X & Y are complete)

It is not hard to see that the norm top on $X \times Y$ is exactly the <u>product top</u> on $X \times Y$.

توپولوژی تیخونوف

If $(x_n, y_n) \to (x, y)$ in $\|\cdot\|_1$ on $X \times Y$, then $\boxed{x_n \to x}$ & $y_n \to y$:

Let $\varepsilon > 0$. Since $(x_n, y_n) \to (x, y)$, $\exists N \; \forall n \geq N$; $\underbrace{\|(x_n, y_n) - (x, y)\|}_{(x_n - x,\, y_n - y)} < \varepsilon$.

$\|x_n - x\| \leq \|x_n - x\| + \|y_n - y\| =$

Hence $\forall n \geq N$; $\|x_n - x\| < \varepsilon$. Thus $x_n \to x$.

Conversely if $x_n \to x$ & $y_n \to y$, then $(x_n, y_n) \to (x, y)$. (Exercise)

We intend to show that $\begin{cases} A \times A \to A \\ (a, b) \mapsto ab \end{cases}$ is jointly <u>cts</u>:

Let $(a_n, b_n) \to (a, b)$. So $a_n \to a$ & $b_n \to b$.

$$\underset{\downarrow\, 0}{\|a_n b_n - ab\|} = \|a_n b_n - ab_n + ab_n - ab\| \leq \underset{\downarrow\, \|b\|}{\|b_n\|}\underset{\downarrow\, 0}{\|a_n - a\|} + \underset{\text{scalar}}{\|a\|}\underset{\downarrow\, 0}{\|b_n - b\|}$$

the right hand side tends to 0, so $\|a_n b_n - ab\| \to 0$.

Hence $a_n b_n \to ab$.

Exercise: The multip. is separately cts, i.e., let b be fixed and $a_n \to a$. Then $a_n b \to ab$

(2) Let B be a subalg of A. Then $\overline{B}$ is also a subalg of A.

<u>Proof</u>: Let $b, b' \in \overline{B}$, $\lambda \in \mathbb{C}$. $\exists \{b_n\}$ & $\{b'_n\}$ in B s.t. $b_n \to b$ & $b'_n \to b'$.

Since the multip. in the field is jointly cts, $\underbrace{b_n b_n'}_{\in B\ (\text{subalg})} \longrightarrow bb'$. So $bb' \in \overline{B}$. Similarly, $b + b' \in \overline{B}$ & $\lambda b \in \overline{B}$. $\square$

Exercise. In a metric space (X,d), $x \in \overline{A} \iff \exists \{a_n\}$ in A s.t. $a_n \longrightarrow x$

Exercise. Let (X,d) be a complete metric space & $F \subseteq X$. Then (F,d) is complete <u>iff</u> F is closed in X.

(3) Let $\{f_n\}$ be a Cauchy seq. in $\ell^\infty(S)$. $\forall x \in S$, $\{f_n(x)\}$ is Cauchy in $\mathbb{C}$: Let $\varepsilon > 0$. So $\exists N\ \forall m,n \geq N$; $\underbrace{\|f_n - f_m\|_\infty}_{\sup_{t\in S} |(f_n - f_m)(t)|} < \varepsilon$.

$$|f_n(x) - f_m(x)| \leq \sup_{t \in S} |f_n(t) - f_m(t)|$$

Since $\mathbb{C}$ is complete $\exists f(x) \in \mathbb{C}$; $f_n(x) \longrightarrow f(x)$. Thus we have a function $f: S \longrightarrow \mathbb{C}$. We show that (i) f is bd (ii) $f_n \longrightarrow f$ in $\ell^\infty(S)$:

<u>Note</u>. $\|\cdot\|: X \longrightarrow \mathbb{R}$, $x \longmapsto \|x\|$ (X normed space) is cts, since $\big|\|x\| - \|y\|\big| \leq \|x - y\|$

مثلثی

(i)

Let $\varepsilon = 1$. Since $\{f_n\}$ is Cauchy $\exists N\ \forall m, n \geq N$; $\|f_n - f_m\| < 1$ N

So $\forall x$: $|f_n(x) - f_N(x)| < 1$ whence $|f_n(x)| \leq 1 + |f_N(x)| \leq 1 + \|f_N\|$

$|f_n(x)| - |f_N(x)| \leq$ ... $|f_n(x)| \leq 1 + |f_N(x)| \leq 1 + \|f_N\|_\infty$

لذا؛

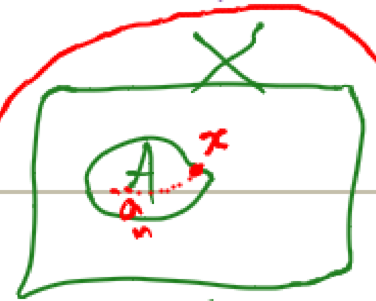
By Taking limit as $n \to \infty$ we have $|f(x)| \leq 1 + \|f_N\|_\infty \quad (x \in S)$

Thus $\|f\|_\infty \leq 1 + \|f_N\|_\infty < \infty$. Therefore $f \in \ell^\infty(S)$.

(ii) Let $\varepsilon > 0$. $\exists N \, \forall m, n \geq N \, \forall x;\ \underbrace{|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}}_{\|f_n - f_m\|_\infty < \varepsilon/2}$. Tend m to infinity to get بی‌نهایت $\forall n \geq N \, \forall x;\ \underbrace{|f_n(x) - f(x)| < \varepsilon/2}_{\|f_n - f\| \leq \frac{\varepsilon}{2} < \varepsilon}$.

Therefore $f_n \to f$ in $\ell^\infty(S)$ or $f_n \xrightarrow{\|\cdot\|_\infty} f$. $\square$

(4) $C_b(\Omega)$ is closed in $\ell^\infty(\Omega)$



Let $\{f_n\}$ be a seq. in $C_b(\Omega)$ such that $f_n \xrightarrow{\|\cdot\|_\infty} f \in \ell^\infty(\Omega)$. So

$\forall \varepsilon \exists N \forall n \geq N;\ \|f_n - f\|_\infty < \varepsilon$ or

$\forall \varepsilon \exists N \forall n \geq N \forall x;\ |f_n(x) - f(x)| < \varepsilon$

Hence $f_n \xrightarrow{u} f$ (uniformly).

Since f_n's are cts, so is f.

Thus $f \in C_b(\Omega)$. $\square$

A is closed in $X \Leftrightarrow$ $\forall \{a_n\}$ in A, if $a_n \to x$, then $x \in A$ ($x \in X$)

Proof. ($\Longrightarrow$) Let $\{a_n\}$ be a seq in A & $a_n \to x \in X$. So $x \in \bar{A}$. Since A is closed, $x \in \bar{A} = A$.

($\Leftarrow$) Always $A \subseteq \bar{A}$. We shall show that $\bar{A} \subseteq A$ (then A is closed).

Let $x \in \bar{A}$. $\exists \{a_n\}$ in A s.t. $a_n \to x$.

By our assumption, $x \in A$. ✓

(5) $C_0(\Omega)$ is unital iff Ω is compact.

$\underbrace{C_0(\Omega) \text{ is unital}}_{1\in C_0(\Omega)}$

$(\Rightarrow)$ Let $1\in C_0(\Omega)$. By the definition, $\underbrace{\{x\in\Omega : \underbrace{|1(x)|}_{1} \geq \tfrac{1}{2}\}}_{\Omega}$ is compact.

$(\Leftarrow)$ Suppose that Ω is compact. $\forall \varepsilon$, $\underbrace{\{x\in\Omega : |1(x)|\geq\varepsilon\}}\subseteq \underbrace{\Omega}_{\text{compact}}$

$[\varepsilon,\infty)$ تعریف معکوس بسته، بسته است. 1 پیوسته است.

$\underbrace{1^{-1}(\underbrace{[\varepsilon,\infty)}_{\text{closed}})}_{\text{closed}}$

so $\{x\in\Omega : |1(x)|\geq\varepsilon\}$ is compact. □

(6) If Ω is compact, then $C_0(\Omega) \overset{\subseteq}{=} C_b(\Omega)$. Then we denote them by $C(\Omega)$.

A

Since if $f\in C_b(\Omega)$, then $\forall\varepsilon$ $\underbrace{\{x : |f(x)|\geq\varepsilon\}}_{\substack{\text{closed}\\ \text{compact}}} \subseteq \underbrace{\Omega}_{\text{compact}}$.

دقت کنید موقع خواندن روی b بالا
هر جمله را خوب بفهمید و بعد بنویسید
بعد بروید

(7) $\forall A\in M_n(\mathbb{C})\ \exists\, \underset{A}{T}\in B(\underbrace{\mathbb{C}^n}_{\text{Hilbert sp}})$; $T_A : \mathbb{C}^n \to \mathbb{C}^n$, $X=\begin{bmatrix} x_1\\ \vdots \\ x_n\end{bmatrix} \mapsto AX$

خیلی مهم (مثال)

وقتی اسم شرح کنید میفهمید
را خوب یاد دارید، تلاش کنید آن

$\langle (z_1,\dots,z_n),(w_1,\dots,w_n)\rangle = \sum_{i=1}^{n} z_i\bar{w}_i$

$\mathbb{C}^n \underset{\text{onto}}{\overset{\text{isomor}}{\longleftrightarrow}} M_{n\times 1}(\mathbb{C})$

$(z_1,\dots,z_n) \longleftrightarrow \begin{bmatrix} x_1 \\ \vdots \end{bmatrix}$

$T(\alpha X+Y)=$

را با دیگر بکشید.

$A(\alpha X+Y)=\alpha AX+AY$
$=\alpha T_A(X)+T_A(Y)$

$$\forall T\in B(\mathbb{C}^n)\ \exists A_T\in M_n(\mathbb{C});\quad A_T=\Big[\ \underbrace{Te_j}_{\text{the } j\text{-th column}}\ \Big]\qquad e_j:=(0,\cdots,0,\underbrace{1}_{j\text{-th}},0,\cdots,0)$$

We have a 1-1 correspondence.

$$A\longrightarrow T_A\longrightarrow A_{T_A},\quad A=A_{T_A}\qquad\qquad T\longrightarrow A_T\longrightarrow T_{A_T},\quad T=T_{A_T}$$

<u>Example</u>. (1) Let $A=\begin{bmatrix}2&1\\0&1\end{bmatrix}$. Then

$$\begin{cases}T_A:\mathbb{C}^2\longrightarrow\mathbb{C}^2\\ \begin{bmatrix}x\\y\end{bmatrix}\mapsto\begin{bmatrix}2&1\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2x+y\\y\end{bmatrix}\end{cases}\quad \text{So } T_A:\mathbb{C}^2\to\mathbb{C}^2,\ T_A(x,y)=(2x+y,y).$$

(2) Let $\begin{cases}T:\mathbb{C}^2\longrightarrow\mathbb{C}^2\\ T(x,y)=(2x+y,y)\end{cases}$. Then $A_T=\big[\,T\underbrace{e_1}_{(1,0)}\ \ T\underbrace{e_2}_{(0,1)}\,\big]=\begin{bmatrix}2&1\\0&1\end{bmatrix}$

Thus there is a linear isomorphism between $B(\mathbb{C}^n)$ & $M_n(\mathbb{C})$, so we can put a norm, called the operator norm, on $M_n(\mathbb{C})$ via $\|A\|:=\|T_A\|$. This norm is a <u>very important</u> object. There are other important norms on $M_n(\mathbb{C})$, VTO norms on $M_n(\mathbb{C})$:

$$\|[a_{ij}]\|_m = \max_{1\le i,j\le n} |a_{ij}| \;,\quad \|[a_{ij}]\|_\sigma = \sum_{1\le i,j\le n} |a_{ij}| \;,\quad \|[a_{ij}]\|_c = \max_{1\le j\le n} \Big(\sum_{i=1}^{n} |a_{ij}|^2\Big)^{\frac{1}{2}}$$

$$\|[a_{ij}]\|_r = \max_{1\le i\le n} \Big(\sum_{j=1}^{n} |a_{ij}|^2\Big)^{\frac{1}{2}}$$

__Def.__ $\|\cdot\|_1$ & $\|\cdot\|_2$ are equivalent if $\exists \alpha,\beta>0$ s.th. $\|x\|_1 \le \alpha \|x\|_2$, $\|x\|_2 \le \beta \|x\|_1$.

__Exercise.__ Show that $\|\cdot\|_m, \|\cdot\|_\sigma, \|\cdot\|_c, \|\cdot\|_r$ and $\|\cdot\|$ are equivalent.

Hint: $\|[a_{ij}]\|_\sigma = \sum_{1\le i,j\le n} |a_{ij}| \le \sum_{1\le i,j\le n} \underbrace{\max_{1\le i,j\le n} |a_{ij}|}_{\|[a_{ij}]\|_m} = \underbrace{n^2}_{\alpha} \|[a_{ij}]\|_m$

(8) If $\{B_i\}_{i\in I}$ is a family of subalgebras of A, then $B = \bigcap_{i\in I} B_i$ is a subalg. If S is a subset of A, then the subalg $\langle S\rangle$ generated by S is $\bigcap B$. This is the "smallest" subalg of A containing S:
$\underbrace{}_{\text{which contains } S}$

$S \subseteq B$
B is a subalg of A

$S \subseteq D \overset{\text{subalg}}{\leq} A \Rightarrow \langle S \rangle \subseteq D$ (زیرا D یکی از Bهای بالایی است)

$S \subseteq \langle S \rangle \overset{\text{subalg}}{\leq} A$

In particular, $\langle a \rangle = \bigcap\limits_{\substack{a \in B \leq A \\ \text{subalg}}} B$ $\quad \left\{ P(a) : \begin{array}{l} P \text{ is a polynomial} \\ P(0) = 0 \end{array} \right\} \rightarrow a \in B_0$ is a subalgebra

$a, a^2, a^3, \cdots$
λa^n $\quad \in \langle a \rangle$
$P(a) = \lambda_1 a + \cdots + \lambda_n a^n$

If $a \in D \leq A \Longrightarrow B_0 \subseteq D$

$\therefore B_0 = \langle a \rangle$

closed subalg generated by a is $\overline{B_0}$.

شنبه ۱۳۹۱/۱۱/۲۴

A *left* (respectively, *right*) *ideal* in an algebra A is a vector subspace I of A such that

max ideal

$$\forall I, \quad a \in A \text{ and } b \in I \Rightarrow ab \in I \quad (\text{respectively, } ba \in I).$$

$M \subseteq I \subseteq A \Rightarrow M = I$ or $I = A$

An *ideal* in A is a vector subspace that is simultaneously a left and a right ideal in A. Obviously, 0 and A are ideals in A, called the *trivial* ideals. A *maximal* ideal in A is a proper ideal (that is, it is not A) that is not contained in any other proper ideal in A. Maximal left ideals are defined similarly. 1

$\neq A$

3

An ideal I is *modular* if there is an element u in A such that $a - au$ and $a - ua$ are in I for all $a \in A$. It follows easily from Zorn's lemma that every proper modular ideal is contained in a maximal ideal. 2

If ω is an element of a locally compact Hausdorff space Ω, and $M_\omega = \{f \in C_0(\Omega) \mid f(\omega) = 0\}$, then M_ω is a modular ideal in the algebra $C_0(\Omega)$. This is so because there is an element $u \in C_0(\Omega)$ such that $u(\omega) = 1$, and hence, $f - uf \in M_\omega$ for all $f \in C_0(\Omega)$. Since M_ω is of codimension one in $C_0(\Omega)$ (as $M \oplus \mathbf{C}u = C_0(\Omega)$), it is a maximal ideal. 4

If I is an ideal of A, then A/I is an algebra with the multiplication given by

$= \{a + I : a \in A\}$

$$(a + I)(b + I) = ab + I.$$

$\lambda(a+I) + (b+I) = (\lambda a + b) + I$

If I is modular, then A/I is unital (if $a - au, a - ua \in I$ for all $a \in A$, then $u + I$ is the unit). Conversely, if A/I is unital then I is modular.

If A is unital, then obviously all its ideals are modular, and therefore, A posesses maximal ideals.

If $(I_\lambda)_{\lambda \in \Lambda}$ is a family of ideals of an algebra A, then $\cap_{\lambda \in \Lambda} I_\lambda$ is an ideal of A. Hence, if $S \subseteq A$, there is a smallest ideal I of A containing S. We call I the ideal *generated* by S. If A is a normed algebra, then the closure of an ideal is an ideal. The closed ideal J *generated* by a set S is the smallest closed ideal containing S. It is clear that J is the closure of the ideal generated by S.

5

1.1.1. Theorem. *If I is a closed ideal in a normed algebra A, then A/I is a normed algebra when endowed with the quotient norm*

$$\|a + I\| = \inf_{b \in I} \|a + b\|.$$

۱۹ آبان ۹۸

Proof. Let $\varepsilon > 0$ and suppose that a, b belong to A. Then $\varepsilon + \|a + I\| > \|a + a'\|$ and $\varepsilon + \|b + I\| > \|b + b'\|$ for some $a', b' \in I$. Hence,

$$(\varepsilon + \|a + I\|)(\varepsilon + \|b + I\|) > \|a + a'\|\|b + b'\| \geq \overbrace{\|ab + c\|}^{\|(a+a')(b+b')\|} \geq \inf_{x \in I} \|ab + x\|,$$

where $c = a'b + ab' + a'b' \in I$. Thus, $(\varepsilon + \|a + I\|)(\varepsilon + \|b + I\|) \geq \|ab + I\|$. Letting $\varepsilon \to 0$, we get $\|a + I\|\|b + I\| \geq \|ab + I\|$; that is, the quotient norm is submultiplicative. □

A *homomorphism* from an algebra A to an algebra B is a linear map $\varphi: A \to B$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$. Its kernel $\ker(\varphi)$ is an ideal in A and its image $\varphi(A)$ is a subalgebra of B. We say φ is *unital* if A and B are unital and $\varphi(1) = 1$.

$x \mapsto x + I$

If I is an ideal in A, the quotient map $\pi: A \to A/I$ is a homomorphism.

If φ, ψ are continuous homomorphisms from a normed algebra A to a normed algebra B, then $\varphi = \psi$ if φ and ψ are equal on a set S that generates A as a normed algebra (that is, A is the closed algebra generated by S). This follows from the observation that the set $\{a \in A \mid \varphi(a) = \psi(a)\}$ is a closed subalgebra of A.

1.2. The Spectrum and the Spectral Radius

Let $\mathbf{C}[z]$ denote the algebra of all polynomials in an indeterminate z with complex coefficients. If a is an element of a unital algebra A and $p \in \mathbf{C}[z]$ is the polynomial

$$p = \lambda_0 + \lambda_1 z^1 + \cdots + \lambda_n z^n,$$

we set

$$p(a) = \lambda_0 1 + \lambda_1 a^1 + \cdots + \lambda_n a^n.$$

The map

$$\mathbf{C}[z] \to A, \quad p \mapsto p(a),$$

is a unital homomorphism.

We say that $a \in A$ is *invertible* if there is an element b in A such that $ab = ba = 1$. In this case b is unique and written a^{-1}. The set

$$\mathrm{Inv}(A) = \{a \in A \mid a \text{ is invertible}\}$$

$\left.\begin{matrix} ab=1 \\ ca=1 \end{matrix}\right\} \Rightarrow b=c$

is a group under multiplication.

$b = 1.b = (ca)b = c(ab) = c1 = c$

We define the *spectrum* of an element a to be the set

$$\sigma(a) = \sigma_A(a) = \{\lambda \in \mathbf{C} \mid \lambda 1 - a \notin \mathrm{Inv}(A)\}.$$

We shall henceforth find it convenient to write $\lambda 1$ simply as λ.

1.2.1. *Example.* Let $A = C(\Omega)$, where Ω is a compact Hausdorff space. Then $\sigma(f) = f(\Omega)$ for all $f \in A$.

6

1.2.2. *Example.* Let $A = \ell^\infty(S)$, where S is a non-empty set. Then $\sigma(f) = (f(S))^-$ (the closure in $\mathbf{C}$) for all $f \in A$.

1.2.3. *Example.* Let A be the algebra of upper triangular $n \times n$-matrices. If $a \in A$, say

$$a = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1n} \\ 0 & \lambda_{22} & \dots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_{nn} \end{pmatrix}$$

it is elementary that

$$\sigma(a) = \{\lambda_{11}, \lambda_{22}, \dots, \lambda_{nn}\}.$$

7

Similarly, if $A = M_n(\mathbf{C})$ and $a \in A$, then $\sigma(a)$ is the set of eigenvalues of a.

Thus, one thinks of the spectrum as simultaneously a generalisation of the range of a function and the set of eigenvalues of a finite square matrix.

1.2.1. *Remark.* If a, b are elements of a unital algebra A, then $1 - ab$ is invertible if and only if $1 - ba$ is invertible. This follows from the observation that if $1 - ab$ has inverse c, then $1 - ba$ has inverse $1 + bca$.

A consequence of this equivalence is that $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$ for all $a, b \in A$.

$\lambda \notin \sigma(ab) - \{0\} \Leftrightarrow ab - \lambda 1$ is inv. $\Leftrightarrow \lambda^{-1} ab - 1$ is inv $\Leftrightarrow \lambda^{-1} ba - 1$ is inv $\Leftrightarrow ba - \lambda 1$ is inv $\Leftrightarrow \lambda \notin \sigma(ba) - \{0\}$

① $(\Sigma := \{I : I \underset{\neq}{\triangleleft} A\}, \subseteq)$ POSet — left ideal

Partially ordered set

[illegible]

Let A has a unit 1. $1a = a1 = a$

Poset + $\forall a, b;\ aRb \vee bRa$

If $\{I_\alpha\}$ is a chain (totally ordered set)

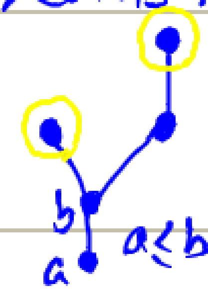
$x, y \in \bigcup I_\alpha \Rightarrow \exists \alpha, \beta;\ x \in I_\alpha,\ y \in I_\beta \xrightarrow[\text{w.l.o.g.}]{I_\alpha \subseteq I_\beta} x, y \in I_\beta \Rightarrow x + y \in I_\beta \Rightarrow x + y \in \bigcup I$

Then $\forall \alpha;\ I_\alpha \subseteq \bigcup_\alpha I_\alpha \underset{\neq}{\triangleleft} A$ (An ideal I is proper iff $1 \notin I$)

$\bigcup_\alpha I_\alpha \in \Sigma$

So the chain $\{I_\alpha\}$ has an upper bound. By Zorn's lemma Σ has a maximal element M.

maximal ideal



(2) Let I_0 be a proper modular ideal of A. Then $\exists$ a maximal ideal M s.t. $I_0 \subseteq M$.

Hint: $(\Sigma = \{J : I_0 \subseteq J \underset{\neq}{\triangleleft} A\}, \subseteq)$ is poset.

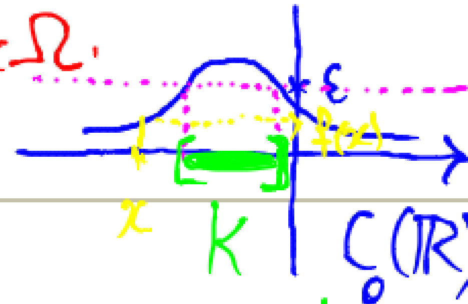
Every chain $\{J_\alpha\}$ has an upper bound $(\bigcup_\alpha J_\alpha)$. So Σ has a maximal element M.

(3) $u \in A$ is a modular element of I iff $\forall a \in A;\ \begin{matrix} a - au \in I \\ a - ua \in I \end{matrix}$

iff $a + I = au + I$ ... iff $a + I = (a+I)(u+I)$ ... iff $u + I$ is the identity of $\frac{A}{I}$

If A has the identity 1, then 1 is modular for all ideals.

Example: Let Ω be a l.c.H.s. & $\omega\in\Omega$.



$M=\{f\in C_0(\Omega): f(\omega)=0\} \trianglelefteq C_0(\Omega)$

$\exists u\in C_0(\Omega);\ 0\le u\le 1,\ u(\omega)=1$

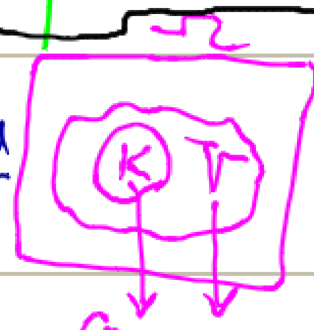
$V=\Omega,\ K=\{\omega\}$

$\forall f\in C_0(\Omega);\ (f-uf)(\omega)=f(\omega)-\underbrace{u(\omega)}_{1}f(\omega)=0$

$\underbrace{\qquad\qquad\qquad}_{f-uf\in M}$

$\therefore M$ is modular.

Urysohn's Lemma



$\forall\ \underset{\text{compact}}{K}\subseteq\underset{\text{open}}{V}\ \exists f\in C_c(\Omega);$

$0\le f\le 1,\ f|_K=1,\ f|_{V^c}=0$

$C_c(\Omega)=\{f:\Omega\to\mathbb{C} \mid f \text{ is cts \& } \overline{\{x\in\Omega: f(x)\neq 0\}} \text{ is compact}\} \trianglelefteq$

$C_c(\Omega)$ is dense in $C_0(\Omega)$ w.r.t. $\|\cdot\|_\infty$.

In order to show that M is a max ideal, we prove that

$M\oplus\mathbb{C}u=C_0(\Omega)$

First $M\cap\mathbb{C}u=\{0\}$, second $\forall f\in C_0(\Omega);\ f=\underbrace{f-f(\omega)u}_{\in M}+\underbrace{f(\omega)u}_{\in\mathbb{C}u}$

Now we see that $\dfrac{C_0(\Omega)}{M}\simeq\mathbb{C}u\simeq\underset{\text{field}}{\mathbb{C}}$. Thus if $M\subseteq I\trianglelefteq A$,

then $\dfrac{I}{M}\trianglelefteq\underset{\text{field}}{\dfrac{A}{M}}$. Hence $\dfrac{I}{M}=\{0_{\frac{A}{M}}\}$ or $\dfrac{I}{M}=\dfrac{A}{M}$. So $I=M$ or $I=A$.

Thus M is a maximal ideal.

Direct sum of normed spaces

Let X, Y be linear spaces. $X\times Y$ is a linear space:

$(x,y)+\lambda(x',y')=(x+\lambda x', y+\lambda y')$

We denote this space by $X\oplus Y$ and call it the external direct sum of X & Y.

Next, let X and Y be subspaces of a linear space Z such that $X\cap Y=\{0\}$. Then subspace $X+Y=\{x+y : x\in X, y\in Y\}$ of Z is called the internal direct sum of X, Y and denote it by $X\oplus Y$. $X\cap Y=\{0\}$ implies that if $x+y=x'+y'$ then $\underbrace{x-x'}_{\in X}=\underbrace{y'-y}_{\in Y}\overset{\downarrow}{=}0$. Hence $x=x'$, $y=y'$. Thus each vector of $X\oplus Y$ is represented uniquely as $\underset{\substack{\cap\\ X}}{x}+\underset{\substack{\cap\\ Y}}{y}$.

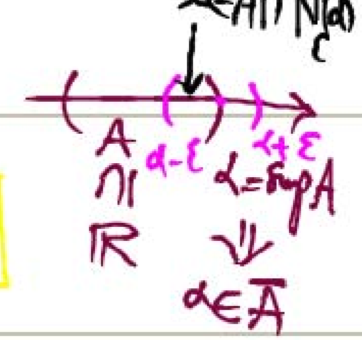
There is no difference between int & Ext direct sums:

Let X & Y be arbitrary linear spaces. Consider the ext. dir. sum $\underset{\substack{\| \\ X\times Y}}{X\oplus Y}$. Put $\begin{cases} X_1=\{(x,0) \mid x\in X\}\leq X\oplus Y \\ Y_1=\{(0,y)\mid y\in Y\}\leq X\oplus Y \\ X_1\cap Y_1=\{(0,0)\}\end{cases}$ Then $f: \underset{ext}{X\oplus Y}\to \underset{int}{X\oplus Y}$, $(x,y)\mapsto(x,0)+(0,y)$ is an isomorphism between linear spaces.

Now let $X \leq Z$, $Y \leq Z$, $X \cap Y = \{0\}$. Then $\begin{cases} \varphi : \underset{\text{int}}{\overset{X+Y}{\overset{\shortparallel}{X \oplus Y}}} \to \underset{\text{ext}}{\overset{X \times Y}{\overset{\shortparallel}{X \oplus Y}}} \\ x+y \mapsto (x,y) \end{cases}$

is a linear space isomorphism. $\square$

(5) Let I be a closed ideal of A. Then $\frac{A}{I}$ is an algebra. Put $\|a+I\| = \inf\{\|a+x\| : x \in I\}$.



$\|a+I\| = 0 \Leftrightarrow \inf\limits_{x \in I} \|a+x\| = 0 \Leftrightarrow 0 \in \overline{\{\|a+x\| : x \in I\}}$

$\Leftrightarrow \exists \{x_n\}$ in I ; $\underbrace{\|a+x_n\|}_{-x_n \to a} \to 0 \Leftrightarrow \underbrace{-x_n}_{\in I} \to a \Leftrightarrow a \in \overline{I} = \boxed{I}$

$\Leftrightarrow a + I = 0$

$\inf A = -\sup(-A)$

If I is a closed subspace of A, then $\frac{A}{I}$ is a normed space.

If I " " " ideal " " " " " " " algebra

since $(a+I)(b+I) = ab + I$ should be well-defined:

If $a+I = a'+I$ $(\Leftrightarrow a-a' \in I)$, then
$b+I = b'+I$

از حفظ کردن مطالب بدون فهمیدن آن جداً اجتناب کنید

$\underbrace{a-a'}_{x} \in I$, $\underbrace{b-b'}_{y} \in I$. So $ab - a'b' = ab - (a-x)(b-y) = \underbrace{ay}_{\in I} - \underbrace{xb}_{\in I} + \underbrace{xy}_{\in I}$

$\in I$. Thus $ab + I = a'b' + I$. ////

(6) $f \in C(\overset{\text{abelian alg}}{\Omega}) \Rightarrow \sigma(f) = f(\Omega)$

$\lambda \notin \sigma(f) \Leftrightarrow f - \lambda 1$ is invertible $\Leftrightarrow \exists g \in C(\Omega);\ \underbrace{g(f-\lambda 1) = 1}_{g = \frac{1}{f-\lambda 1} \in C(\Omega)}$

$\overset{\forall x, p}{\Leftrightarrow} (f - \lambda 1)(x) \neq 0 \Leftrightarrow \forall x; f(x) \neq \lambda \Leftrightarrow \lambda \notin f(\Omega)$

$f \neq 0$

$\equiv$

$\exists x; f(x) \neq 0$

$\forall x; f(x) \neq 0$

(7) If $A \in \underset{\substack{\sim \\ B(\mathbb{C}^n)}}{M_n}(\mathbb{C})$, then $\sigma(A)$ = the set of all eigenvalues of A

$\lambda \notin \sigma(A) \Leftrightarrow A - \lambda I$ is invertible

$A : \mathbb{C}^n \xrightarrow{\text{linear}} \mathbb{C}^n$ is 1-1 iff it is onto.

$\Leftrightarrow A - \lambda I$ is 1-1 $\Leftrightarrow \det(A - \lambda I) \neq 0$

$\Leftrightarrow \lambda$ is not an eigenvalue of A

$\boxed{(\sim P \Leftrightarrow \sim q) \equiv P \Leftrightarrow q}$

λ is an eigenvalue of a matrix A if $\exists x \neq 0;\ Ax = \lambda x$

If $T : X \to Y$ is linear, then

(1) T is invertible iff T is 1-1 & onto

(2) In the category of normed spaces, a bd linear map T is invertible iff T is 1-1, onto & T^{-1} cts, since

in this category morphisms are bd linear maps.

Remark. In general, T^{-1} is not even if T is bd. The open mapping theorem says that if X & Y are Ban, then T^{-1} is bd.

Every linear map on a finite dimensional normed space is always $\underset{\text{bd}}{\underline{\text{cts}}}$.

So when we are dealing with matrices, there is no continuity discussion.

$f(A)=\{f(x): x\in A\}$

$T: X\to Y$ is bd iff $\sup_{x\neq 0}\frac{\|Tx\|}{\|x\|}<\infty$ iff $\exists M\ \forall x, \|Tx\|\le M\|x\|$

When T is bd we can take $M=\|T\|$.

1.2.1. Theorem. *Let a be an element of a unital algebra A. If $\sigma(a)$ is non-empty and $p \in \mathbf{C}[z]$, then*

For example, If $P(z)=z^2+1$, then $\sigma(a^2+1)=P(\sigma(a))=\{P(\lambda): \lambda\in Sp(a)\}$ λ^2+1

$$\sigma(p(a)) = p(\sigma(a)).$$

Proof. We may suppose that p is not constant. If $\mu \in \mathbf{C}$, there are elements $\lambda_0, \ldots, \lambda_n$ in $\mathbf{C}$, where $\lambda_0 \neq 0$, such that

$z=\lambda_i$, $P(\lambda_i)=\mu$ $\Leftarrow$ $P(z)$

$$p - \mu = \lambda_0(z - \lambda_1)\ldots(z - \lambda_n),$$

and therefore,

$$p(a) - \mu = \lambda_0(a - \lambda_1)\ldots(a - \lambda_n).$$

It is clear that $p(a) - \mu$ is invertible if and only if $a - \lambda_1, \ldots, a - \lambda_n$ are. It follows that $\mu \in \sigma(p(a))$ if and only if $\mu = p(\lambda)$ for some $\lambda \in \sigma(a)$, and therefore, $\sigma(p(a)) = p(\sigma(a))$. $\square$

$P(a)-\mu$ is not inv $\Leftrightarrow \exists i$; $a-\lambda_i$ is not inv $\Leftrightarrow \exists i$ $\lambda_i\in\sigma(a)$ $\Leftrightarrow$ $\mu=P(\lambda_i)$ for some $\lambda_i\in\sigma(a)$

The spectral mapping property for polynomials is generalised to continuous functions in Chapter 2, but only for certain elements in certain algebras. There is a version of Theorem 1.2.1 for analytic functions and Banach algebras (see [Tak, Proposition 2.8], for example). We shall not need this, however.

1.2.2. Theorem. *Let A be a unital Banach algebra and a an element of A such that $\|a\| < 1$. Then $1 - a \in \mathrm{Inv}(A)$ and*

$$(1 - a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

$\|a^n\|\le\|a\|^n$, so by the comparison test, $\sum_{n=0}^{\infty}\|a^n\|$ converges and

Proof. Since $\sum_{n=0}^{\infty} \|a^n\| \leq \sum_{n=0}^{\infty} \|a\|^n = (1 - \|a\|)^{-1} < +\infty$, the series $\sum_{n=0}^{\infty} a^n$ is convergent, to b say, in A, and since $(1 - a)(1 + \cdots + a^n) = 1 - a^{n+1}$ converges to $(1 - a)b = b(1 - a)$ and to 1 as $n \to \infty$, the element b is the inverse of $1 - a$. $\square$

Since $\sum a^n<\infty$, $\lim_{n\to\infty} a^n=0$

The series in Theorem 1.2.2 is called the *Neumann* series for $(1 - a)^{-1}$.

1.2.3. Theorem. *If A is a unital Banach algebra, then $\mathrm{Inv}(A)$ is open in A, and the map*

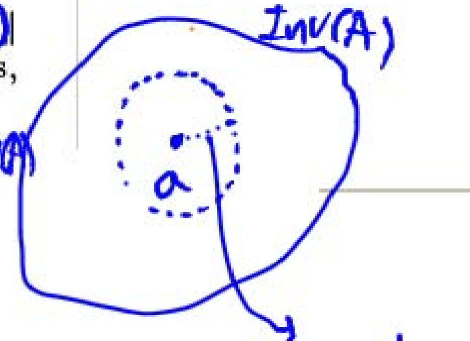
$$\mathrm{Inv}(A) \to A, \quad a \mapsto a^{-1},$$

is differentiable.

aa^{-1}

Proof. Suppose that $a \in \mathrm{Inv}(A)$ and $\|b - a\| < \|a^{-1}\|^{-1}$. Then $\|ba^{-1} - 1\| \leq \|b - a\|\|a^{-1}\| < 1$, so $ba^{-1} \in \mathrm{Inv}(A)$, and therefore, $b \in \mathrm{Inv}(A)$. Thus, $\mathrm{Inv}(A)$ is open in A.

$b=(ba^{-1})a$ So $N(a)\subseteq \mathrm{Inv}(A)$ $\|a^{-1}\|^{-1}$



If $b \in A$ and $\|b\| < 1$, then $1 + b \in \mathrm{Inv}(A)$ and

$$\|(1 + b)^{-1} - 1 + b\| = \|\sum_{n=0}^{\infty}(-1)^n b^n - 1 + b\| = \|\sum_{n=2}^{\infty}(-1)^n b^n\|$$

$\|1-b\|$

$$\le \sum_{n=2}^{\infty} \|b\|^n = \|b\|^2/(1 - \|b\|).$$

$\|a^{-1}\|^{-1}$

Let $a \in \mathrm{Inv}(A)$ and suppose that $\|c\| < \frac{1}{2}\|a^{-1}\|^{-1}$. Then $\|a^{-1}c\| < 1/2 < 1$, so (with $b = a^{-1}c$),

$$\|(1 + a^{-1}c)^{-1} - 1 + a^{-1}c\| \le \|a^{-1}c\|^2/(1 - \|a^{-1}c\|)^{-1} \le 2\|a^{-1}c\|^2,$$

since $1 - \|a^{-1}c\| > 1/2$. Now define u to be the linear operator on A given by $u(b) = -a^{-1}ba^{-1}$. Then,

$$\|(a+c)^{-1} - a^{-1} - u(c)\| = \|(1 + a^{-1}c)^{-1}a^{-1} - a^{-1} + a^{-1}ca^{-1}\|$$

$$\le \|(1 + a^{-1}c)^{-1} - 1 + a^{-1}c\|\|a^{-1}\| \le 2(\|a^{-1}\|^3\|c\|^2).$$

Consequently,

$$\lim_{c \to 0} \frac{\|(a+c)^{-1} - a^{-1} - u(c)\|}{\|c\|} = 0,$$

and therefore, the map $\sigma : b \mapsto b^{-1}$ is differentiable at $b = a$ with derivative $\sigma'(a) = u$. □

$A = \mathbb{C}$

$\mathrm{Inv}(A) = \mathbb{C} - \{0\}$

$\sigma : \mathrm{Inv}(A) \to \mathrm{Inv}(A)$

$\sigma(z) = \frac{1}{z}$

$\sigma'(z) = \frac{-1}{z^2}$

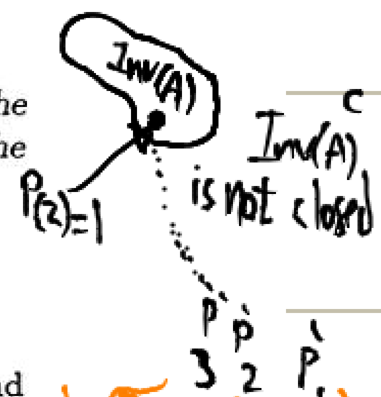
$\sigma'(a) : A \to A$

$b \mapsto \underbrace{\frac{-1}{a^2} b}_{-a^{-1}ba^{-1}}$

The algebra $\mathbf{C}[z]$ is a normed algebra where the norm is defined by setting

$$\|p\| = \sup_{|\lambda| \le 1} |p(\lambda)|.$$

Observe that $\mathrm{Inv}(\mathbf{C}[z]) = \mathbf{C} \setminus \{0\}$, so the polynomials $p_n = 1 + z/n$ are not invertible. But $\lim_{n\to\infty} p_n = 1$, which shows that $\mathrm{Inv}(\mathbf{C}[z])$ is not open in $\mathbf{C}[z]$. Thus, the norm on $\mathbf{C}[z]$ is not complete.



1.2.4. Lemma. *Let A be a unital Banach algebra and let $a \in A$. The spectrum $\sigma(a)$ of a is a closed subset of the disc in the plane of centre the origin and radius $\|a\|$, and the map*

$$\mathbf{C} \setminus \sigma(a) \to A, \ \lambda \mapsto (a - \lambda)^{-1},$$

is differentiable.

$\lambda \mapsto a - \lambda \mapsto (a-\lambda)^{-1}$

Proof. If $|\lambda| > \|a\|$, then $\|\lambda^{-1}a\| < 1$, so $1 - \lambda^{-1}a$ is invertible, and therefore, so is $\lambda - a$. Hence, $\lambda \notin \sigma(a)$. Thus, $\lambda \in \sigma(a) \Rightarrow |\lambda| \le \|a\|$. The set $\sigma(a)$ is closed, that is, $\mathbf{C} \setminus \sigma(a)$ is open, because $\mathrm{Inv}(A)$ is open in A. Differentiability of the map $\lambda \mapsto (a - \lambda)^{-1}$ follows from Theorem 1.2.3. □

$\lambda_n \in \sigma(a)$, $\lambda_n \to \lambda$

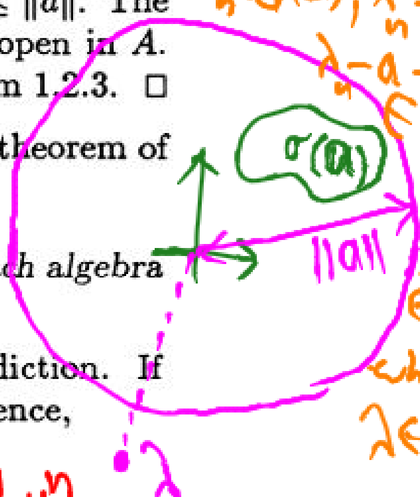
$\lambda_n - a \to \lambda - a$

$\in \mathrm{Inv}(A)^c$

So $\lambda - a \in \mathrm{Inv}(A)^c$

whence $\lambda \in \sigma(a)$

The following result can be thought of as the fundamental theorem of Banach algebras.



1.2.5. Theorem (Gelfand). *If a is an element of a unital Banach algebra A, then the spectrum $\sigma(a)$ of a is non-empty.*

Proof. Suppose that $\sigma(a) = \emptyset$ and we shall obtain a contradiction. If $|\lambda| > 2\|a\|$, then $\|\lambda^{-1}a\| < \frac{1}{2}$, and therefore, $1 - \|\lambda^{-1}a\| > \frac{1}{2}$. Hence,

$$\|(1-\lambda^{-1}a)^{-1}\| - 1 \le \|(1 - \lambda^{-1}a)^{-1} - 1\| = \|\sum_{n=1}^{\infty}(\lambda^{-1}a)^n\| \le \sum_{n=1}^{\infty}\|\lambda^{-1}a\|^n$$

$\therefore \sigma(a)$ is closed

$$\le \frac{\|\lambda^{-1}a\|}{\ldots} \le 2\|\lambda^{-1}a\| < 1$$

$$\leq \frac{\quad}{1-\|\lambda^{-1}a\|} \leq 2\|\lambda^{-1}a\| < 1.$$

Consequently, $\|(1-\lambda^{-1}a)^{-1}\| < 2$, and therefore,

$$\|(a-\lambda)^{-1}\| = \|\lambda^{-1}(1-\lambda^{-1}a)^{-1}\| < 2/|\lambda| < \|a\|^{-1}$$

($a \neq 0$ since $\sigma(a) = \emptyset$). Moreover, since the map $\lambda \mapsto (a-\lambda)^{-1}$ is continuous, it is bounded on the (compact) disc $2\|a\|\mathbf{D}$. Thus, we have shown that this map is bounded on all of $\mathbf{C}$; that is, there is a positive number M such that $\|(a-\lambda)^{-1}\| \leq M$ $(\lambda \in \mathbf{C})$.

If $\tau \in A^*$, the function $\lambda \mapsto \tau((a-\lambda)^{-1})$ is entire, and bounded by $M\|\tau\|$, so by Liouville's theorem in complex analysis, it is constant. In particular, $\tau(a^{-1}) = \tau((a-1)^{-1})$. Because this is true for all $\tau \in A^*$, we have $a^{-1} = (a-1)^{-1}$, so $a = a-1$, which is a contradiction. □

$\lambda=0$ 6 $\lambda=1$

$\lambda \mapsto (a-\lambda)^{-1}$ is differentiable with the derivative u. Then $\lambda \mapsto \tau((a-\lambda)^{-1})$ is also diff with der $\tau o u$.

It is easy to see that there are algebras in which not all elements have non-empty spectrum. For example, if $\mathbf{C}(z)$ denotes the field of quotients of $\mathbf{C}[z]$, then $\mathbf{C}(z)$ is an algebra, and the spectrum of z in this algebra is empty.

1.2.6. Theorem (Gelfand–Mazur). *If A is a unital Banach algebra in which every non-zero element is invertible, then $A = \mathbf{C}1$.* 7

Proof. This is immediate from Theorem 1.2.5. □

If a is an element of a unital Banach algebra A, its *spectral radius* is defined to be

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda| \leq \|a\|$$

$|\lambda| > \|a\| \Rightarrow a-\lambda$ is inv.

By Remark 1.2.1, $r(ab) = r(ba)$ for all $a, b \in A$.

1.2.4. *Example.* If $A = C(\Omega)$, where Ω is a compact Hausdorff space, then $r(f) = \|f\|_\infty$ $(f \in A)$.

$(sp(f) = f(\Omega))$

$r(f) = \sup_{\lambda \in sp(f)} |\lambda| = \sup_{\lambda \in f(\Omega)} |\lambda| = \sup_{t \in \Omega} |f(t)| = \|f\|_\infty$

1.2.5. *Example.* Let $A = M_2(\mathbf{C})$ and 8

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $\|a\| = 1$, but $r(a) = 0$, since $a^2 = 0$. 9

9' $r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = 0$, $n \geq 2$

1 If $S, T \in B(X)$, then $ST \in B(X)$ since

$\|STx\| = \|S(Tx)\| \leq \|S\|\,\|Tx\| \leq \underbrace{\|S\|\,\|T\|}_{M}\,\|x\|$

$\therefore ST \in B(X)$. Moreover, $\|ST\| = \sup_{x \neq 0} \dfrac{\|(ST)(x)\|}{\|x\|} \leq \|S\|\,\|T\|$.

Note:
If $f,g\in \ell^\infty(S)$, then $\|f\cdot g\|_\infty \leq \|f\|_\infty \|g\|_\infty$ since

$|(f\cdot g)(x)| = |f(x)g(x)| = |f(x)||g(x)| \leq \|f\|_\infty \|g\|_\infty$

$\|f\cdot g\| = \sup\limits_{x\neq 0} |(f\cdot g)(x)| \leq \|f\|_\infty \|g\|_\infty$.

(2) $P(z)=\lambda_0$, then $\underbrace{\sigma(\underbrace{P(a)}_{\lambda_0 1})}_{\{\lambda_0\}} = P(\sigma(a)) = \{\lambda_0\}$

$a-\lambda 1$ is the same as $a-\lambda$

$sp(\lambda_0 1) = \{\lambda\in\mathbb{C} : \underbrace{\lambda_0 1 - \lambda 1}_{(\lambda_0-\lambda)1} \text{ is not inv.}\} = \{\lambda_0\}$

(3) For example $P(z)=2z^2-3$, $\mu=1$

$P(z)-\mu = 2z^2-4 = 2(z^2-2) = 2(z-\underbrace{\sqrt{2}}_{\lambda_1})(z-\underbrace{(-\sqrt{2})}_{\lambda_2})$

$\{b_ib_j=b_jb_i$

(4) If $a=b_1b_2b_3$ is invertible, then $\exists c;\ ac=ca=1$. Hence

$b_1(b_2b_3c)=ac=1$, $(cb_2b_3)b_1=c(b_1b_2b_3)=ca=1$

So b_1 is inv. Similarly b_2, b_3 are inv.

(5) Let X & Y be normed spaces. A map $f: X \to Y$ is called differentiable at $x_0 \in X$, if $\exists\, T \in B(X,Y)$ such that
$$\lim_{h \to 0} \frac{\|f(x_0+h) - f(x_0) - Th\|}{\|h\|} = 0.$$
Then T is called the derivative of f at x_0.

N.B. If $X = Y = \mathbb{R}$, $T \in B(\mathbb{R}, \mathbb{R}) \simeq \mathbb{R}$

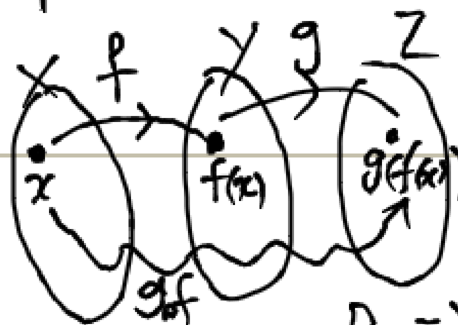
$\mathbb{C}$ $\quad$ $\mathbb{C}$ $\mathbb{C}$ $\quad$ $\mathbb{C}$

$\begin{cases} \varphi: B(\mathbb{R},\mathbb{R}) \xrightarrow{\text{linear}} \mathbb{R} \\ \quad \varphi \mapsto \varphi(1) \end{cases}$

$\begin{cases} \psi: \mathbb{R} \xrightarrow{\text{linear}} B(\mathbb{R},\mathbb{R}) \\ \quad r \mapsto \varphi_r: \mathbb{R} \to \mathbb{R} \\ \qquad\qquad x \mapsto rx \end{cases}$

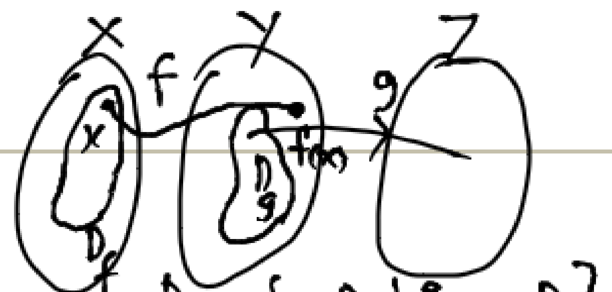
$\varphi \circ \psi = Id$, $\psi \circ \varphi = Id$

Composition of two functions (High School)



$D_{g\circ f} = X$

"$f: X \to Y$ is a function" means that $D_f = X$



$D_{g\circ f} = \{x \in D_f \mid f(x) \in D_g\}$

"$f: X \to Y$ is a function" means that $D_f \subseteq X$.

(6) $[\forall \tau \in A^*: \tau(x) = 0] \Rightarrow x = 0$

If $x \neq 0$, by the Hahn-Banach theorem, $\exists \tau_0: \tau_0(x) = \|x\|$

So $\|x\| = 0$. Hence $x = 0$.

$f(\lambda) = a - \lambda$ $\quad$ $g(x) = x^{-1}$

⑦ Define $\theta: A \xrightarrow[\text{Onto}]{1\text{-}1} \mathbb{C}$

$a \mapsto \lambda_a$

$\emptyset \neq \sigma(a) = \{\lambda : a - \lambda 1 \text{ is not invertible}\} = \{\lambda : \underbrace{a - \lambda 1 = 0}_{a = \lambda_a 1}\}$

θ is isometry

Let $\sigma(a) = \{\lambda_a\}$ singleton

$\|a\| = \|\lambda_a 1\| = |\lambda_a|$

$\lambda_1, \lambda_2 \in \sigma(a) \Rightarrow \begin{cases} a = \lambda_1 1 \\ a = \lambda_2 1 \end{cases} \Rightarrow \lambda_1 1 = \lambda_2 1 \Rightarrow \lambda_1 = \lambda_2$

θ is linear

$\theta(a + \mu b) = \lambda_{a+\mu b}$

$$\left.\begin{array}{l} a + \mu b = \lambda_{a+\mu b} 1 = \theta(a+\mu b) 1 \\ \quad \| \\ \lambda_a 1 + \mu \lambda_b 1 = (\lambda_a + \mu \lambda_b) 1 = (\theta(a) + \mu\, \theta(b)) 1 \end{array}\right\} \Rightarrow \theta(a + \mu b) = \theta(a) + \mu \theta(b)$$

⑧ $A = M_2(\mathbb{C})$, $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{cases} a : \mathbb{C}^2 \to \mathbb{C}^2 \\ \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix} \end{cases}$

$$\|a\| = \sup_{\begin{bmatrix} x \\ y \end{bmatrix} \neq 0} \frac{\|a \begin{bmatrix} x \\ y \end{bmatrix}\|}{\|\begin{bmatrix} x \\ y \end{bmatrix}\|} = \sup \frac{\sqrt{y^2 + 0^2}}{\sqrt{x^2 + y^2}} \cancel{\leq} 1$$

If $\alpha \in A$, α is an upper bound for A, then $\sup A = \alpha$

max

$$\frac{\sqrt{1^2 + 0^2}}{\sqrt{0^2 + 1^2}} = 1$$

$\sigma(a) = \{\lambda : \lambda \text{ is an eigenvalue of } a\} = \{0\} \Rightarrow r(a) = 0$

$\det(a - \lambda I) = 0 \Rightarrow \left| \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \Rightarrow \begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 = 0 \Rightarrow \lambda = 0$

⑨ $a^2 = 0 \Rightarrow sp(a^2) = sp(a)^2 \Rightarrow \{0\} = sp(a)^2 = \{\lambda^2 : \lambda \in sp(a)\} \Rightarrow sp(a) = \{0\} \Rightarrow r(a) = 0$

1.2.7. Theorem (Beurling). *If a is an element of a unital Banach algebra A, then*

$$r(a) = \inf_{n\geq 1} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^n\|^{1/n} \leq \|x\|$$

Proof. If $\lambda \in \sigma(a)$, then $\lambda^n \in \sigma(a^n)$, so $|\lambda^n| \leq \|a^n\|$, and therefore, $r(a) \leq \inf_{n\geq 1} \|a^n\|^{1/n} \leq \liminf_{n\to\infty} \|a^n\|^{1/n}$.

Let Δ be the open disc in $\mathbf{C}$ centered at 0 and of radius $1/r(a)$ (we use the usual convention that $1/0 = +\infty$). If $\lambda \in \Delta$, then $1 - \lambda a \in \mathrm{Inv}(A)$. If $\tau \in A^*$, then the map

$$f: \Delta \to \mathbf{C},\ \lambda \mapsto \tau((1-\lambda a)^{-1}),$$

is analytic, so there are unique complex numbers λ_n such that

$$f(\lambda) = \sum_{n=0}^{\infty} \lambda_n \lambda^n \quad (\lambda \in \Delta).$$

However, if $|\lambda| < 1/\|a\| (\leq 1/r(a))$, then $\|\lambda a\| < 1$, so

$$(1-\lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n,$$

and therefore,

$$f(\lambda) = \sum_{n=0}^{\infty} \lambda^n \tau(a^n).$$

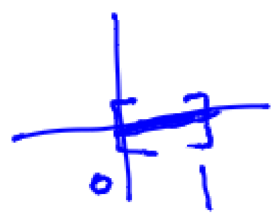
It follows that $\lambda_n = \tau(a^n)$ for all $n \geq 0$. Hence, the sequence $(\tau(a^n)\lambda^n)$ converges to 0 for each $\lambda \in \Delta$, and therefore *a fortiori*, it is bounded. Since this is true for each $\tau \in A^*$, it follows from the principle of uniform boundedness that $(\lambda^n a^n)$ is a bounded sequence. Hence, there is a positive number M (depending on λ, of course) such that $\|\lambda^n a^n\| \leq M$ for all $n \geq 0$, and therefore, $\|a^n\|^{1/n} \leq M^{1/n}/|\lambda|$ (if $\lambda \neq 0$). Consequently, $\limsup_{n\to\infty} \|a^n\|^{1/n} \leq 1/|\lambda|$. We have thus shown that if $r(a) < |\lambda^{-1}|$, then $\limsup_{n\to\infty} \|a^n\|^{1/n} \leq |\lambda^{-1}|$. It follows that $\limsup_{n\to\infty} \|a^n\|^{1/n} \leq r(a)$, and since $r(a) \leq \liminf_{n\to\infty} \|a^n\|^{1/n}$, therefore $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$. □

1.2.6. *Example.* Let A be the set of C^1-functions on the interval [0,1]. This is an algebra when endowed with the pointwise-defined operations, and a submultiplicative norm on A is given by

$$\|f\| = \|f\|_\infty + \|f'\|_\infty \qquad (f \in A).$$

(Exercise?)

It is elementary that A is complete under this norm, and therefore, A is a Banach algebra. Let $x: [0,1] \to \mathbf{C}$ be the inclusion, so $x \in A$. Clearly, $\|x^n\| = 1 + n$ for all n, so $r(x) = \lim(1+n)^{1/n} = 1 < 2 = \|x\|$.



$x(t) = t$

1.2.9. Theorem. *Let A be a unital Banach algebra.*

If $a \in A$, then e^a is invertible with inverse e^{-a}, and if a, b are commuting elements of A, then $e^{a+b} = e^a e^b$.

without proof

We shall see later that not every invertible element is of the form e^a.

$e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!}$ is converg

If an algebra is non-unital we can adjoin a unit to it. This is very helpful in many cases, and we shall frequently make use of it, but it does not reduce the theory to the unital case. There are situations where adjoining a unit is unnatural, such as when one is studying the group algebra $L^1(G)$ of a locally compact group G (see the addenda section of this chapter for the definition of this algebra).

$$\sum_{n=0}^{\infty} \left\|\frac{a^n}{n!}\right\| < \infty$$

$$\left\|\frac{a^n}{n!}\right\| \le \frac{\|a\|^n}{n!} \quad \text{Comp. test}$$

$$\sum_{n=0}^{\infty} \frac{\|a\|^n}{n!} = e^{\|a\|}$$

If A is an algebra, we set $\tilde{A} = A \oplus \mathbf{C}$ as a vector space. → External direct sum. We define a multiplication on $\tilde{A}$ making it a unital algebra by setting

$$(a,\lambda)+\delta(b,\mu)=(a+\delta b, \lambda+\delta\mu)$$

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu).$$

1

The unit is (0,1). The algebra $\tilde{A}$ is called the *unitization* of A. The map

$$A \to \tilde{A}, \quad a \mapsto (a, 0),$$

is an injective homomorphism, which we use to identify A as an ideal of $\tilde{A}$. We then write $a + \lambda$ for (a, λ). The map

(a,λ) is denoted by $a+\lambda$

$$\tilde{A} \to \mathbf{C}, \quad a + \lambda \mapsto \lambda,$$

$$(0,1) \mapsto 1$$

is a unital homomorphism with kernel A, called the *canonical* homomorphism.

If A is abelian, so is $\tilde{A}$.

If A is a normed algebra, we make $\tilde{A}$ into a normed algebra by setting

2

$$\|a + \lambda\| = \|a\| + |\lambda|.$$

Observe that A is a closed subalgebra of $\tilde{A}$, and that $\tilde{A}$ is a Banach algebra if A is one.

If A is a non-unital Banach algebra, then for $a \in A$ we set $\sigma_A(a) = \sigma_{\tilde{A}}(a)$, and $r(a) = \sup_{\lambda \in \sigma_A(a)} |\lambda|$. Note that 0 is an element of $\sigma_A(a)$ in this case.

1.3. The Gelfand Representation

The idea of this section is to represent an abelian Banach algebra as an algebra of continuous functions on a locally compact Hausdorff space. This is an extremely useful way of looking at these algebras, but in the case of the more "complicated" algebras, the picture it presents may be of limited accuracy.

We begin by proving some results on ideals and multiplicative linear functionals.

(3) $Av \subseteq A$. If $a \in A$, then $a = (av^{-1})v \in Av$. So $A \subseteq Av$.

1.3.1. Theorem. *Let I be a modular ideal of a Banach algebra A. If I is proper, so is its closure $\bar{I}$. If I is maximal, then it is closed.*

Proof. Let u be an element of A such that $a - au$ and $a - ua$ are in I for all $a \in A$. If $b \in I$ and $\|u - b\| < 1$, then the element $v = 1 - u + b$ is invertible in $\tilde{A}$. If $a \in A$, then $av = a - au + ab \in I$, so $A = Av \subseteq I$. This contradicts the assumption that I is proper, and shows that $\|u - b\| \geq 1$ for all $b \in I$. It follows that $u \notin \bar{I}$, so $\bar{I}$ is proper. If $u \in \bar{I}$, then $\exists \{b_n\}$ in I s.t. $b_n \to u$. So $\exists N; \|b_N - u\| < 1$ ✗.

If I is maximal, then $I = \bar{I}$, as $\bar{I}$ is a proper ideal containing I. □

$I \subseteq \bar{I} \subsetneq A$

1.3.1. *Remark.* If L is a left ideal of a Banach algebra A, it is *modular* if there is an element u in A such that $a - au \in L$ for all $a \in A$, and in this case its closure is a proper left ideal. Moreover, if L is a modular maximal left ideal, it is closed. The proofs are the same as for Theorem 1.3.1.

1.3.2. Lemma. *If I is a modular maximal ideal of a unital abelian algebra A, then A/I is a field.*

Proof. The algebra A/I is unital and abelian, with unit $u + I$ say. If J is an ideal of A/I and π is the quotient map from A to A/I, then $\pi^{-1}(J)$ is an ideal of A containing I. Hence, $\pi^{-1}(J) = A$ or I, by maximality of I. Therefore, $J = A/I$ or 0. Thus, A/I and 0 are the only ideals of A/I. Now suppose that $\pi(a)$ is a non-zero element of A/I. Then $J = \pi(a)(A/I)$ is a non-zero ideal of A/I, and therefore, $J = A/I$. Hence, there is an element b of A such that $(a + I)(b + I) = u + I$, so $a + I$ is invertible. This shows that A/I is a field. □

$\frac{A}{I} = \{a + I : a \in A\}$ with $(a+I) \overset{\delta}{+} (b+I) = (a \overset{\delta}{+} b) + I$ is an alg. $\pi: A \to \frac{A}{I}$, $\pi(a) = a + I$ is the quotient map. If $J \trianglelefteq \frac{A}{I}$, then $I \subset \pi^{-1}(J) \trianglelefteq A$

trum is motivated by the following result.

1.3.7. Theorem. *Let A be a unital Banach algebra generated by 1 and an element a. Then A is abelian and the map*

$$\hat{a}: \boxed{\Omega(A)} \to \sigma(a), \ \tau \mapsto \tau(a),$$

is a homeomorphism.

the closure of the set of all $p(a)$ s.t. p is a polynomial

the set of all nonzero hom $\tau: A \to \mathbb{C}$ (Character).

Proof. It is clear that A is abelian and that $\hat{a}$ is a continuous bijection, and because $\Omega(A)$ and $\sigma(a)$ are compact Hausdorff spaces, $\hat{a}$ is therefore a homeomorphism. □

① $(a,\lambda)(0,1) = (a\cdot 0 + \lambda\cdot 0 + 1\cdot a,\ \lambda\cdot 1) = (a,\lambda)$

② $\varphi : A \xrightarrow{\text{hom}} \tilde{A}$ is 1-1, $\|a\| = \|(a,0)\|$, $A \overset{\text{bi-ideal}}{\triangleleft} \tilde{A}$

$a \mapsto (a,0)$

$(a,0)(b,\mu) = (ab+\mu a, 0) \in A$

$\begin{cases} \varphi(ab) = (ab,0) = (a,0)(b,0) = \varphi(a)\varphi(b) \\ \varphi(a+\delta b) = \varphi(a) + \delta\varphi(b) \end{cases}$

Note: If A is already unital with the unit e,

then $(e,0)(a,\lambda) = (a+\lambda e, 0) \neq (a,\lambda)$

$(e,1)(a,\lambda) = (a+\lambda e + a, \lambda) \neq (a,\lambda)$

So e has no role in $\tilde{A}$.

② Let $(a_n, 0) \to (a,\lambda)$. Let $\varepsilon > 0$.

$\exists N\ \forall n \geq 0$; $\underbrace{\|(a_n, 0) - (a,\lambda)\|}_{|\lambda| \leq \|a_n - a\| + |\lambda|} < \varepsilon$.

Hence $|\lambda| < \varepsilon$. Since ε is arbitrary, $\lambda = 0$

Therefore $(a,\lambda) = (a,0) \in A$.

Let $\forall \varepsilon > 0$; $|\lambda| \leq \varepsilon$.

If $\lambda \neq 0$, then $|\lambda| \leq \frac{|\lambda|}{2} = \varepsilon$. So $1 \leq \frac{1}{2}$ ※

Note that if $\varphi: A \to B$ is a homomorphism between algebras A and B ~~and B is unital~~, then $\tilde{\varphi}: \tilde{A} \to \tilde{B}$, $a + \lambda \mapsto \varphi(a) + \lambda$, $(a \in A,\ \lambda \in \mathbf{C})$ is the unique unital homomorphism extending φ: $\tilde{\varphi}(a+0)=\varphi(a)+0=\varphi(a)$ or $\tilde{\varphi}|_A=\varphi$

If $\varphi: A \to B$ is a unital homomorphism between unital algebras, then $\varphi(\mathrm{Inv}(A)) \subseteq \mathrm{Inv}(B)$, so $\sigma(\varphi(a)) \subseteq \sigma(a)$ $(a \in A)$.

A *character* on an abelian algebra A is a non-zero homomorphism $\tau: A \to \mathbf{C}$. We denote by $\Omega(A)$ the set of characters on A.

1.3.3. Theorem. *Let A be a unital abelian Banach algebra.*

(1) *If $\tau \in \Omega(A)$, then $\|\tau\| = 1$.*

(2) *The set $\Omega(A)$ is non-empty, and the map*

~~$\tau(1)=\tau(1.1)=\tau(1).\tau(1)\Rightarrow\tau(1)=1$~~

If $a-\tau(a)$ is inv, then $\exists b;\ b(a-\tau(a)1)=1$. So $\tau(b)(\tau(a)-\tau(a)\tau(1))=\tau(1)$ X

$$\tau \mapsto \ker(\tau)$$

$\tau(1)=0\Rightarrow\tau(a)=\tau(a.1)=\tau(a)\tau(1)=0\ \forall a$ X $(\tau\neq 0)$ $\therefore \tau(1)\neq 0$

defines a bijection from $\Omega(A)$ onto the set of all maximal ideals of A.

Proof. If $\tau \in \Omega(A)$ and $a \in A$, then $\tau(a) \in \sigma(a)$, so $|\tau(a)| \le r(a) \le \|a\|$. Hence, $\|\tau\| \le 1$. Also, $\tau(1) = 1$, since $\tau(1) = \tau(1)^2$ and $\tau(1) \neq 0$. Hence, $\|\tau\| = 1$.

$\tau^{-1}(\{0\})$ $\quad$ $\exists a;\ \tau(a)\neq 0\Rightarrow a\notin\ker\tau\Rightarrow\ker\tau\neq A$

Let I denote the closed ideal $\ker(\tau)$. This is proper, since $\tau \neq 0$, and $I \oplus \mathbf{C}1 = A$, since $a - \tau(a) \in I$ for all $a \in A$. It follows that I is a maximal ideal of A. $\frac{A}{I}\simeq\mathbb{C}$ is a fi

If $\tau_1, \tau_2 \in \Omega(A)$ and $\ker(\tau_1) = \ker(\tau_2)$, then for each $a \in A$ we have $\tau_1(a - \tau_2(a)) = 0$, so $\tau_1(a) = \tau_2(a)$. Thus, $\tau_1 = \tau_2$. $\in\ker\tau_2$

If I is an arbitrary maximal ideal of A, then I is closed by Theorem 1.3.1 and A/I is a unital Banach algebra in which every non-zero element is invertible, by Lemma 1.3.2. Hence, by Theorem 1.2.6 $A/I = \mathbf{C}(1 + I)$. It follows that $A = I \oplus \mathbf{C}1$. Define $\tau: A \to \mathbf{C}$ by $\tau(b + \lambda) = \lambda$, $(b \in I,\ \lambda \in \mathbf{C})$. Then τ is a character and $\ker(\tau) = I$.

Thus, we have shown that the map $\tau \mapsto \ker(\tau)$ is a bijection from the characters onto the maximal ideals of A.

We have seen already that A admits maximal ideals (since it is unital). Therefore, $\Omega(A) \neq \emptyset$. $\square$

1.3.4. Theorem. *Let A be an abelian Banach algebra.*

(1) *If A is unital, then* $\quad a-\tau(a)1$ is not inv

$$\sigma(a) \supseteq \{\tau(a) \mid \tau \in \Omega(A)\} \quad (a \in A).$$

(2) *If A is non-unital, then*

$$\sigma(a) = \{\tau(a) \mid \tau \in \Omega(A)\} \cup \{0\} \quad (a \in A).$$

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Proof. If A is unital and a is an element of A whose spectrum contains λ, then the ideal $I = (a - \lambda)A$ is proper, so I is contained in a maximal ideal

is not inv. $\tau((a-\lambda)1) = 0$ So

$\ker(\tau)$, where $\tau \in \Omega(A)$. Hence, $\tau(a) = \lambda$. This shows that the inclusion $\sigma(a) \subseteq \{\tau(a) \mid \tau \in \Omega(A)\}$ holds, and the reverse inclusion is clear.

hom $\tau: A \to \mathbb{C}$

Now suppose that A is non-unital, and let $\tau_\infty : \tilde{A} \to \mathbf{C}$ be the canonical homomorphism. Then $\Omega(\tilde{A}) = \{\tilde{\tau} \mid \tau \in \Omega(A)\} \cup \{\tau_\infty\}$, where $\tilde{\tau}$ is the unique character on $\tilde{A}$ extending the character τ on A. Hence, by Condition (1), $\sigma(a) = \sigma_{\tilde{A}}(a) = \{\tau(a) \mid \tau \in \Omega(\tilde{A})\} = \{\tau(a) \mid \tau \in \Omega(A)\} \cup \{0\}$ for each $a \in A$. $\square$

Pa4(1)

$\tilde{\tau}: \tilde{A} \to \mathbb{C}$

$\tilde{\tau}(a+\lambda) = \tau(a) + \lambda$

$\tilde{\tau}_\infty : \tilde{A} \to \mathbb{C}$

$\tilde{\tau}_\infty(a+\lambda) = \lambda$

If A is an abelian Banach algebra, it follows from Theorem 1.3.4 that $\Omega(A)$ is contained in the closed unit ball of A^*. We endow $\Omega(A)$ with the relative weak* topology, and call the topological space $\Omega(A)$ the *character space*, or *spectrum*, of A.

$\tilde{\tau}_\infty\big|_A = 0$

1.3.5. Theorem. *If A is an abelian Banach algebra, then $\Omega(A)$ is a locally compact Hausdorff space. If A is unital, then $\Omega(A)$ is compact.*

Proof. It is easily checked that $\Omega(A) \cup \{0\}$ is weak* closed in the closed unit ball S of A^*. Since S is weak* compact (Banach–Alaoglu theorem), $\Omega(A) \cup \{0\}$ is weak* compact, and therefore, $\Omega(A)$ is locally compact.

If A is unital, then $\Omega(A)$ is weak* closed in S and thus compact. $\square$

If $\rho \in \Omega(\tilde{A})$, then $\rho\big|_A \in \Omega(A)$ or $\rho = 0$

Note that $\Omega(A)$ may be empty. This is the case for $A = 0$, for example.

Suppose that A is an abelian Banach algebra for which the space $\Omega(A)$ is non-empty. If $a \in A$, we define the function $\hat{a}$ by

$$\hat{a} : \Omega(A) \to \mathbf{C}, \ \tau \mapsto \tau(a).$$

Clearly the topology on $\Omega(A)$ is the smallest one making all of the functions $\hat{a}$ continuous. The set $\{\tau \in \Omega(A) \mid |\tau(a)| \geq \varepsilon\}$ is weak* closed in the closed unit ball of A^* for each $\varepsilon > 0$, and weak* compact by the Banach–Alaoglu theorem. Hence, $\hat{a} \in C_0(\Omega(A))$.

We call $\hat{a}$ the *Gelfand transform* of a.

Although the following result is very important, its proof is easy, because we have already done most of the work needed to demonstrate it.

1.3.6. Theorem (Gelfand Representation). *Suppose that A is an abelian Banach algebra and that $\Omega(A)$ is non-empty. Then the map*

$$A \to C_0(\Omega(A)), \ a \mapsto \hat{a},$$

is a norm-decreasing homomorphism, and

$$r(a) = \|\hat{a}\|_\infty \qquad (a \in A).$$

If A is unital, $\sigma(a) = \hat{a}(\Omega(A))$, and if A is non-unital, $\sigma(a) = \hat{a}(\Omega(A)) \cup \{0\}$, for each $a \in A$.

Proof. By Theorem 1.3.4 the spectrum $\sigma(a)$ is the range of $\hat{a}$, together with $\{0\}$ if A is non-unital. Hence, $r(a) = \|\hat{a}\|_\infty$, which implies that the map $a \mapsto \hat{a}$ is norm-decreasing. That this map is a homomorphism is easily checked. $\square$

The kernel of the Gelfand representation is called the *radical* of the algebra A. It consists of the elements a such that $r(a) = 0$. It therefore contains the nilpotent elements. If the radical is zero, A is said to be *semisimple*.

In a general algebra an element whose spectrum consists of the set $\{0\}$ is said to be *quasinilpotent*.

Let a, b be commuting elements of an arbitrary Banach algebra A. Then $r(a+b) \leq r(a) + r(b)$, and $r(ab) \leq r(a)r(b)$. To see this, wé may suppose that A is unital and abelian (if necessary, adjoin a unit and restrict to the closed subalgebra generated by $1, a$, and b). Then $r(a+b) = \|(a+b)\hat{}\|_\infty \leq \|\hat{a}\|_\infty + \|\hat{b}\|_\infty = r(a) + r(b)$ by Theorem 1.3.6. Similarly, $r(ab) = \|(ab)\hat{}\|_\infty \leq \|\hat{a}\|_\infty \|\hat{b}\|_\infty = r(a)r(b)$. Direct proofs of the first of these inequalities (that is, where the Gelfand representation is not invoked) tend to be messy.

The spectral radius is neither subadditive nor submultiplicative in general: Let $A = M_2(\mathbf{C})$ and suppose

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $r(a) = r(b) = 0$, since a and b have square zero, but $r(a+b) = r(ab) = 1$.

The interpretation of the character space as a sort of generalised spectrum is motivated by the following result.

1.3.7. Theorem. *Let A be a unital Banach algebra generated by 1 and an element a. Then A is abelian and the map*

$$\hat{a}\colon \Omega(A) \to \sigma(a), \; \tau \mapsto \tau(a),$$

is a homeomorphism.

[illegible]
۹۲,۱,۲۵

Proof. It is clear that A is abelian and that $\hat{a}$ is a continuous bijection, and because $\Omega(A)$ and $\sigma(a)$ are compact Hausdorff spaces, $\hat{a}$ is therefore a homeomorphism. $\square$

① If $\psi : \tilde{A} \to \tilde{B}$, $\psi|_A = \varphi$, then $\psi = \tilde{\varphi}$ since
unital homm

$$\psi(a+\lambda) = \psi(a) + \psi(\lambda 1) = \varphi(a) + \lambda \underbrace{\psi(1)}_{1} = \tilde{\varphi}(a+\lambda)$$

② $\varphi(\mathrm{Inv}(A)) \subseteq \mathrm{Inv}(B)$

$f : X \to Y$
$f(A) = \{f(a) : a \in A\}$

Let $a \in \mathrm{Inv}(A)$ and consider $\varphi(a)$.
So $\exists b \in A$; $ab = ba = 1_A$. Hence $\varphi(a)\varphi(b) = \varphi(b)\varphi(a) = \varphi(1_A) = 1_B$
So $\varphi(a)$ is inv.

③ $\lambda \notin \sigma(a) \Rightarrow \exists b\ (a - \lambda 1_A)b = 1_A \Rightarrow (\varphi(a) - \lambda 1_B)\varphi(b) = 1_B \Rightarrow$
$\varphi(a) - \lambda 1_B$ is inv $\Rightarrow \lambda \notin \sigma(\varphi(a))$

<u>Exercise</u> If $f : X \to \mathbb{C}$ is cts iff $\ker f$ is closed.

④ $\ker \tau \triangleleft A$
$\forall a \in A\ \forall b \in \ker \tau$; $\tau(ab) = \tau(a)\underbrace{\tau(b)}_{=0} = 0 \Rightarrow ab \in \ker \tau$

⑤ $a = (\underset{\substack{\cap \\ I = \ker \tau}}{a - \tau(a)1}) + \underset{\substack{\cap \\ \mathbb{C}}}{\tau(a)}1 \Rightarrow A = I + \mathbb{C}1$

(b) $\frac{A}{I} = \mathbb{C}(1+I) \simeq \mathbb{C} \Rightarrow A = I \oplus \mathbb{C}1$

$\lambda(1+I) \mapsto \lambda$

$a = b_a + \lambda_a 1 \qquad I \cap \mathbb{C}1 = \{0\}$

$a + I = \lambda_a(1+I) \quad \exists! \lambda_a \in \mathbb{C}$

$= \lambda_a 1 + I$

$a - \lambda_a 1 \in I \Rightarrow \exists b_a \in I;\ a - \lambda_a 1 = b_a$

Topological Vector Space (Chapter One of Funct. Anal.) by W. Rudin
(tVS)

<u>Def</u>. Let X be a vector space (= linear space). Let τ be a topology on X such that
① Every singleton is closed.
② $+ : \underbrace{X \times X}_{\text{Product top}} \to X$ & $\cdot \; \underbrace{\underbrace{\mathbb{F}}_{\mathbb{C} \text{ or } \mathbb{R}} \times X}_{\text{Prod. top}} \longrightarrow X$ are cts.

Recall that the product topology on $X \times Y$ is the topology whose basis is $\{U \times V \mid U \text{ is open in } X \text{ \& } V \text{ is open in } Y\}$. Both

$T_a : X \to X$, $x \mapsto a + x$ & $M_\lambda : X \to X$, $x \mapsto \lambda x$, $\lambda \neq 0$ are $\underbrace{\text{homeomorphism}}_{1\text{-}1,\ \text{ont},\ f \text{ cts},\ f^{-1} \text{ cts}}$.

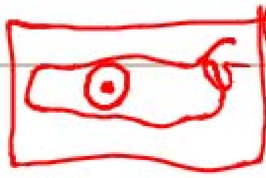
Recall that the notions of local basis, basis and subbasis in top. (See Wikipedia)

$\boxed{x, y \in E,\ 0 \le \lambda \le 1 \Downarrow \lambda x + (1-\lambda) y \in E}$

Types of t.v.s.'s

1. locally convex t.v.s (l.c.t.v.s): The origin has a local basis consisting of convex sets

2. metrizable t.v.s.: the topology is generated by a metric



(X,d)

G is open $\Leftrightarrow \forall x \in G\ \exists r_x > 0;\ N_{r_x}(x) \subseteq G$

Let (X,τ) be a t.v.s.
If $\exists d$ on X such that $\tau =$ the set of all G which are open in (X,d)

Def. $E \subseteq X$ (t.v.s.) is called bounded if $\forall$ nbh V of 0
$\exists s>0\ \forall t>s;\ E \subseteq tV$.

This is equivalent to the usual notion of boundedness in the setting of normed spaces:

E is bd $\Leftrightarrow \exists M\ \forall x \in E;\ \|x\| \leq M$

After Eid (بعد از عید)

A set $Y \subset X$ is called a *subspace* of X if Y is itself a vector space (with respect to the same operations, of course). One checks easily that this happens if and only if $0 \in Y$ and

$\{\alpha x : x \in Y\}$

$$\alpha Y + \beta Y \subset Y$$

for all scalars α and β.

$A+B = \{a+b : a \in A, b \in B\}$

A set $C \subset X$ is said to be *convex* if

$$\{tx+(1-t)y : x, y \in C\} = tC + (1-t)C \subset C \qquad (0 \le t \le 1).$$

In other words, it is required that C should contain $tx + (1-t)y$ if $x \in C$, $y \in C$, and $0 \le t \le 1$.

A set $B \subset X$ is said to be *balanced* if $\alpha B \subset B$ for every $\alpha \in \Phi$ with $|\alpha| \le 1$.

A vector space X has *dimension* n ($\dim X = n$) if X has a *basis* $\{u_1, \ldots, u_n\}$. This means that every $x \in X$ has a unique representation of the form

$$x = \alpha_1 u_1 + \cdots + \alpha_n u_n \qquad (\alpha_i \in \Phi).$$

If $\dim X = n$ for some n, X is said to have *finite dimension*. If $X = \{0\}$, then $\dim X = 0$.

Separation Properties

$A+V = \bigcup_{x \in A} x+V$ is open

1.10 Theorem *Suppose K and C are subsets of a topological vector space X, K is compact, C is closed, and $K \cap C = \varnothing$. Then 0 has a* symm *neighborhood V such that*

$$(K + V) \cap (C + V) = \varnothing.$$

Note that $K + V$ is a union of translates $x + V$ of V ($x \in K$). Thus $K + V$ is an open set that contains K. The theorem thus implies the existence of disjoint open sets that contain K and C, respectively.

PROOF. We begin with the following proposition, which will be useful in other contexts as well:

Lemma: *If W is a neighborhood of 0 in X, then there is a neighborhood U of 0 which is symmetric (in the sense that $U = -U$) and which satisfies $U + U \subset W$.*

To see this, note that $0 + 0 = 0$, that addition is continuous, and

To see this, note that $0 + 0 = 0$, that addition is continuous, and that 0 therefore has neighborhoods V_1, V_2 such that $V_1 + V_2 \subset W$. If 1

$$U = V_1 \cap V_2 \cap (-V_1) \cap (-V_2), \quad 2$$

then U has the required properties.

The proposition can now be applied to U in place of W and yields a new symmetric neighborhood U' of 0 such that

$$U_1 + U_1 + U_1 = 0 + U_1 + U_1 + U_1 \subseteq U' + U' + U' + U' \subset W. \qquad \subseteq U + U$$

It is clear how this can be continued.

If $K = \varnothing$, then $K + V = \varnothing$, and the conclusion of the theorem is obvious. We therefore assume that $K \neq \varnothing$, and consider a point $x \in K$. Since C is closed, since x is not in C, and since the topology of X is invariant under translations, the preceding proposition 3 shows that 0 has a symmetric neighborhood V_x such that $x + V_x + V_x + V_x$ does not intersect C; the symmetry of V_x shows then that

$$(x + V_x + V_x) \cap (C + V_x) = \varnothing. \qquad 4 \tag{1}$$

Since K is compact, there are finitely many points $x_1, \ldots, x_n$ in K such that $K \subseteq \bigcup_{x \in K} x + V_x$ open

$$K \subset (x_1 + V_{x_1}) \cup \cdots \cup (x_n + V_{x_n}).$$

Put $V = V_{x_1} \cap \cdots \cap V_{x_n}$. Then

$$K + V \subset \bigcup_{i=1}^{n} (x_i + V_{x_i} + \underbrace{V}_{\subseteq V_{x_i}}) \subset \bigcup_{i=1}^{n} (x_i + V_{x_i} + V_{x_i}),$$

and no term in this last union intersects $C + V$, by (1). This completes the proof. ////

Since $V \subseteq V_{x_i}$ & (4)

Since $C + V$ is open, it is even true that the *closure* of $K + V$ does not intersect $C + V$; in particular, the closure of $K + V$ does not intersect C. 5 The following special case of this, obtained by taking $K = \{0\}$, is of considerable interest.

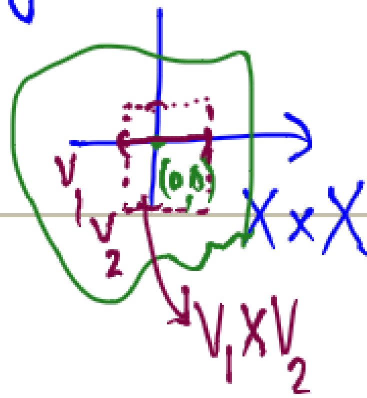
1.11 Theorem *If $\mathscr{B}$ is a local base for a topological vector space X, then every member of $\mathscr{B}$ contains the closure of some member of $\mathscr{B}$.* 6

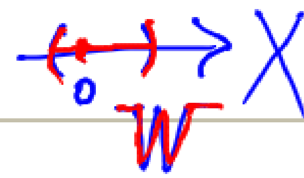
So far we have not used the assumption that every point of X is a closed set. We now use it and apply Theorem 1.10 to a pair of distinct points in place of K and C. The conclusion is that these points have disjoint neighborhoods. In other words, the Hausdorff separation axiom holds:

1.12 Theorem *Every topological vector space is a Hausdorff space.* 7

① $+ : X\times X \longrightarrow X$ is cts at $(0,0)$. So $\exists$ open sets V_1, V_2
$(x,y)\longmapsto x+y$

in X s.t. $V_1+V_2 \subseteq W$.





② Always $U = W\cap(-W)$ is symmetric:

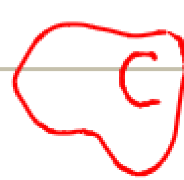
$x\in U \Rightarrow \begin{cases} x\in W \\ x\in -W \Rightarrow \exists\, y\in W;\ \boxed{x=-y} \Rightarrow y=-x\in -W \Rightarrow \boxed{y\in W\cap(-W)=U} \end{cases}$

$x\in -U$

$\therefore U\subseteq -U$.

Similarly $-U\subseteq U$. Hence $U=-U$ is symmetric.

③



$X-C$ open

$x\in V\subseteq X-C$

$\exists V^{\text{open}}$

$\exists V_x$ in X such that $0 \in V_x + V_x + V_x \subseteq V - x$
symm

$\therefore \; x + V_x + V_x + V_x \subseteq V \subseteq X - C$

So $(x + V_x + V_x + V_x) \cap C = \emptyset$

(4) $(x + V_x + V_x - V_x) \cap C = \emptyset$

So $(x + V_x + V_x) \cap (V_x + C) = \emptyset$ since

$y \in (x + V_x + V_x) \cap (V_x + C) \Rightarrow \begin{cases} y \in V_x + C \Rightarrow y = v_3 + c \\ y \in x + V_x + V_x \Rightarrow y = x + v_1 + v_2 \end{cases} \Rightarrow$

$\underbrace{x + v_1 + v_2 - v_3}_{\in x + V_x + V_x + V_x} = \underset{C}{\underset{\cap}{c}}$ $\times$.

(5) We know $(K+V) \cap (C+V) = \emptyset$. Hence $\overline{K+V} \cap (C+V) = \emptyset$ since if $x \in \overline{K+V} \cap (C+V)$, then $\begin{cases} x \in C+V \text{ open} \\ x \in \overline{K+V} \end{cases}$ Hence $(C+V) \cap (K+V) \neq \emptyset$ $\times$.

(6) First note that every topological space X has a local base $\mathcal{B}$ at each point $x_0 \in X$. It is sufficient to put $\mathcal{B} = \{V \subseteq X : x_0 \in V \;\&\; V \text{ is open}\}$.

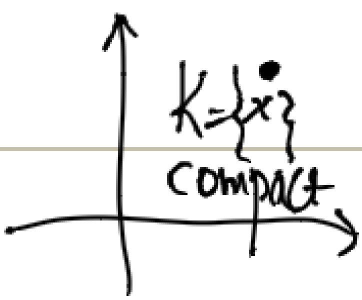
Let $W \in \mathcal{B}$. Put $K = \{0\}$ & $X - W = C$. We observe that $K \cap C = \emptyset$. By Theorem 1.10, $\exists \underset{\ni 0}{V}^{\text{open}}$ such that

$$\overline{(\underbrace{K}_{\{0\}}+V)} \cap (C+V) = \emptyset.$$

So $\overline{V} \cap C = \emptyset$.

Hence $\overline{V} \subseteq W$ (*). By the definition of local base, $\exists U \in \mathcal{B}$; $U \subseteq V$ (□). It follows from (*) & (□) that $\overline{U} \subseteq W$. □

(7)



$C = \{y\}$ closed. By Theorem 1.10, $\exists \underset{\ni 0}{V}^{\text{open}}$;

$$(K+V) \cap (C+V) = \emptyset$$

So $\underbrace{(x+V)}_{\text{is a nbh of } x} \cap \underbrace{(y+V)}_{\text{is a nbh of } y} = \emptyset$. □

1.13 Theorem *Let X be a topological vector space.*

(a) *If $A \subset X$ then $\bar{A} = \bigcap (A + V)$, where V runs through all neighborhoods of 0.*

(b) *If $A \subset X$ and $B \subset X$, then $\bar{A} + \bar{B} \subset \overline{A + B}$.*

(c) *If Y is a subspace of X, so is $\bar{Y}$.* 1

(d) *If C is a convex subset of X, so are $\bar{C}$ and C°.*

(e) *If B is a balanced subset of X, so is $\bar{B}$; if also $0 \in B^\circ$ then B° is balanced.*

(f) *If E is a bounded subset of X, so is $\bar{E}$.* ← ($\mathrm{diam}(E) = \mathrm{diam}\,\bar{E}$ in metric spaces)

PROOF. (a) $x \in \bar{A}$ if and only if $(x + V) \cap A \neq \varnothing$ for every neighborhood V of 0, and this happens if and only if $x \in A - V$ for every such V. Since $-V$ is a neighborhood of 0 if and only if V is one, the proof is complete. $y \in (x+V) \cap A \Rightarrow \exists v \in V;\ y = x + v \Rightarrow x = y - v \in A - V$

(b) Take $a \in \bar{A}$, $b \in \bar{B}$; let W be a neighborhood of $a + b$. There are neighborhoods W_1 and W_2 of a and b such that $W_1 + W_2 \subset W$. There exist $x \in A \cap W_1$ and $y \in B \cap W_2$, since $a \in \bar{A}$ and $b \in \bar{B}$. Then $x + y$ lies in $(A + B) \cap W$, so that this intersection is not empty. Consequently, $a + b \in \overline{A + B}$. $(a,b) \mapsto a+b$

(c) Suppose α and β are scalars. By the proposition in Section 1.7, $\alpha\bar{Y} = \overline{\alpha Y}$ if $\alpha \neq 0$; if $\alpha = 0$, these two sets are obviously equal. Hence it follows from (b) that 2

$$\alpha\bar{Y} + \beta\bar{Y} = \overline{\alpha Y} + \overline{\beta Y} \subset \overline{\alpha Y + \beta Y} \subset \bar{Y};$$

$\alpha \neq 0$ 2 $M_\alpha : X \to X$, $x \mapsto \alpha x$ is homeomorphism $\forall A;\ M_\alpha(\bar{A}) = \overline{M_\alpha(A)}$

the assumption that Y is a subspace was used in the last inclusion.

The proofs that convex sets have convex closures and that balanced sets have balanced closures are so similar to this proof of (c) that we shall omit them from (d) and (e).

(d) Since $C^\circ \subset C$ and C is convex, we have

$$\underbrace{tC^\circ + (1 - t)C^\circ}_{\text{open}} \subset C$$

if $0 < t < 1$. The two sets on the left are open; hence so is their sum. Since every open subset of C is a subset of C°, it follows that C° is convex.

(e) If $0 < |\alpha| \leq 1$, then $\alpha B^\circ = (\alpha B)^\circ$, since $x \to \alpha x$ is a homeomorphism. Hence $\alpha B^\circ \subset \alpha B \subset B$, since B is balanced. But αB° is open. So $\alpha B^\circ \subset B^\circ$. If B° contains the origin, then $\alpha B^\circ \subset B^\circ$ even for $\alpha = 0$.

(f) Let V be a neighborhood of 0. By Theorem 1.11, $\bar{W} \subset V$ for some neighborhood W of 0. Since E is bounded, $E \subset tW$ for all sufficiently large t. For these t, we have $\bar{E} \subset t\bar{W} \subset tV$. ///

1.14 Theorem *In a topological vector space X,*

(*a*) *every neighborhood of 0 contains a balanced neighborhood of 0, and*

(*b*) *every convex neighborhood of 0 contains a balanced convex neighborhood of 0.*

$\left\{ \begin{array}{l} \mathbb{F} \times X \to X \\ 0 \cdot 0 = 0 \end{array} \right.$ $\forall U \ni 0 \; \exists V \ni 0 \; \exists N(0);\; N(0) \cdot V \subseteq U$

PROOF. (*a*) Suppose U is a neighborhood of 0 in X. Since scalar multiplication is continuous, there is a $\delta > 0$ and there is a neighborhood V of 0 in X such that $\alpha V \subset U$ whenever $|\alpha| < \delta$. Let W be the union of all these sets αV. Then W is a neighborhood of 0, W is balanced, and $W \subset U$.

$|\alpha| \leq 1, x \in W \Rightarrow \exists \beta;\ x \in \beta V \Rightarrow \alpha x = \alpha\beta v \in W$ $\quad |\alpha\beta| \leq |\beta| < \delta$

(*b*) Suppose U is a convex neighborhood of 0 in X. Let $A = \bigcap \alpha U$, where α ranges over the scalars of absolute value 1. Choose W as in part (*a*). Since W is balanced, $\alpha^{-1}W = W$ when $|\alpha| = 1$; hence $W \subset \alpha U$. Thus $W \subset A$, which implies that the interior A° of A is a neighborhood of 0. Clearly $A^\circ \subset U$. Being an intersection of convex sets, A is convex; hence so is A°. To prove that A° is a neighborhood with the desired properties, we have to show that A° is balanced; for this it suffices to prove that A is balanced. Choose r and β so that $0 \leq r \leq 1$, $|\beta| = 1$. Then

$$r\beta A = \bigcap_{|\alpha|=1} r\beta\alpha U = \bigcap_{|\alpha|=1} r\alpha U.$$

Since αU is a convex set that contains 0, we have $r\alpha U \subset \alpha U$. Thus $r\beta A \subset A$, which completes the proof. ////

Theorem 1.14 can be restated in terms of local bases. Let us say that a local base $\mathscr{B}$ is *balanced* if its members are balanced sets, and let us call $\mathscr{B}$ *convex* if its members are convex sets.

Corollary

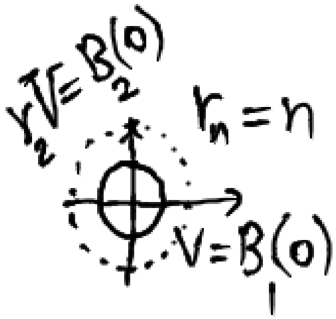
(*a*) *Every topological vector space has a balanced local base.*

(*b*) *Every locally convex space has a balanced convex local base.*

Recall also that Theorem 1.11 holds for each of these local bases.

1.15 Theorem *Suppose V is a neighborhood of 0 in a topological vector space X.*

(a) *If $0 < r_1 < r_2 < \cdots$ and $r_n \to \infty$ as $n \to \infty$, then*



$$X = \bigcup_{n=1}^{\infty} r_n V.$$

(b) *Every compact subset K of X is bounded.*

(c) *If $\delta_1 > \delta_2 > \cdots$ and $\delta_n \to 0$ as $n \to \infty$, and if V is bounded, then the collection*

$$\{\delta_n V : n = 1, 2, 3, \ldots\}$$

is a local base for X.

the inverse image of V under $\alpha \mapsto \alpha x$

$h: \mathbb{F} \to X$, $\alpha \mapsto \alpha x$

PROOF. (a) Fix $x \in X$. Since $\alpha \to \alpha x$ is a continuous mapping of the scalar field into X, the set of all α with $\alpha x \in V$ is open, contains 0, hence contains $1/r_n$ for all large n. Thus $(1/r_n)x \in V$, or $x \in r_n V$, for large n.

$\exists \delta;\ h(N_\delta(0)) \subseteq V$

(b) Let W be a balanced neighborhood of 0 such that $W \subset V$. By (a),

$r_n \to \infty \Rightarrow \frac{1}{r_n} \to 0 \Rightarrow \exists N\ \forall n \geq N;\ \frac{1}{r_n} \in N_\delta(0) \Rightarrow \frac{1}{r_n} x \in V$

$$K \subset \bigcup_{n=1}^{\infty} nW.$$

Since K is compact, there are integers $n_1 < \cdots < n_s$ such that

$$K \subset n_1 W \cup \cdots \cup n_s W = n_s W.$$

The equality holds because W is balanced. If $t > n_s$, it follows that $K \subset tW \subset tV$.

(c) Let U be a neighborhood of 0 in X. If V is bounded, there exists $s > 0$ such that $V \subset tU$ for all $t > s$. If n is so large that $s\delta_n < 1$, it follows that $V \subset (1/\delta_n)U$. Hence U actually contains all but finitely many of the sets $\delta_n V$. ////

تور

≡Net≡

Let $(D, \leq)$ be a Poset (Partially ordered set). If $\forall \alpha, \beta \in D\ \exists \gamma \in D$

such that $\alpha \le \gamma$ & $\beta \le \gamma$, then D is called a directed set.

Let (X, τ) be a top space. By a net we mean a function $\left\{\begin{array}{l} D \to X \\ \alpha \mapsto x_\alpha \end{array}\right.$ denoted by $(x_\alpha)_{\alpha \in D}$. A net (x_α) is said to converge to $x \in X$ if $\forall \underset{x}{\underset{\Psi}{G}} \in \tau\ \exists \alpha_0\ \forall \alpha \ge \alpha_0;\ x_\alpha \in G$. Then we write $x_\alpha \to x$.

Theorem (Bring its Proof by the next week) $x \in \bar{A}$ iff there exists a net (x_α) with $x_\alpha \in A$ such that $x_\alpha \to x$.

Theorem. $f: X \to Y$ is cts at $x_0 \in X$ iff $\forall \{x_\alpha\}$ in X s.t $x_\alpha \to x_0$ it holds $f(x_\alpha) \to f(x_0)$

(1) $x, y \in \bar{Y}$, $\lambda \in \mathbb{C}$ $\Rightarrow \exists (x_\alpha), (y_\alpha)$ in Y; $\begin{array}{l} x_\alpha \to x \\ y_\alpha \to y \end{array}$. $\underbrace{\lambda x_\alpha + y_\alpha}_{\in Y} \to \lambda x + y$.

So $\lambda x + y \in \bar{Y}$.

Note: $f: X \xrightarrow[\text{onto}]{1-1} Y$ is a $\overbrace{\text{homeomorphism}}^{f \,\&\, f^{-1} \text{ are cts}}$ iff one of the following (and so all) holds:

(1) $f(\bar{A}) = \overline{f(A)}$ (2) $f^{-1}(\bar{B}) = \overline{f^{-1}(B)}$ (3) $f(A^\circ) = f(A)^\circ$

(4) $f^{-1}(B^\circ) = f^{-1}(B)^\circ$ (5) f is cts & takes open sets to open sets (6) f is cts & takes closed sets to closed sets (7) ...

Note: A° is the "biggest" open subset of A, since $A^\circ \cup G$

$A = \bigcup G$
$\tau \ni G \subseteq A$

$\bar{A}$ is the "smallest" closed superset of A, since

$$\bar{A} = \bigcap_{A \subseteq F \text{ closed}} F .$$

③ If $f : \underbrace{X \times Y}_{\text{with Tychonov top}} \to Z$ is cts, then we say f is jointly continuous. Fix $x_0 \in X$, $y_0 \in Y$. Then

$$\begin{cases} f_{y_0} : X \to Z \\ \quad x \mapsto f(x, y_0) \\ f_{x_0} : Y \to Z \\ \quad f_{x_0}(y) = f(x_0, y) \end{cases}$$

are cts (we say f is separately cts)

The function $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is separately continuous but is not jointly continuous.

Seminorms and Local Convexity

1.33 Definitions A *seminorm* on a vector space X is a real-valued function p on X such that

$P(0)=P(2\cdot 0)=2P(0) \Rightarrow P(0)=0$

$0=P(0)=P(x+(-x))\le P(x)+P(-x)=2P(x) \Rightarrow P(x)\ge 0$

(a) $p(x+y) \le p(x) + p(y)$ and

(b) $p(\alpha x) = |\alpha|\, p(x)$

for all x and y in X and all scalars α.

Property (*a*) is called *subadditivity*. Theorem 1.34 will show that a seminorm p is a norm if it satisfies

or $P(x)=0 \Rightarrow x=0$

(c) $p(x) \neq 0$ if $x \neq 0$.

A family $\mathscr{P}$ of seminorms on X is said to be *separating* if to each $x \neq 0$ corresponds at least one $p \in \mathscr{P}$ with $p(x) \neq 0$.

Next, consider a convex set $A \subset X$ which is *absorbing*, in the sense that every ~~$x \in X$ lies in tA for some $t = t(x) > 0$.~~ equivalently $x\in A$ [For example, (*a*) of Theorem 1.15 implies that every neighborhood of 0 in a topological vector space is absorbing. Every absorbing set obviously contains 0.] The *Minkowski functional* μ_A of A is defined by

$$\mu_A(x) = \inf\{t > 0 : t^{-1}x \in A\} \qquad (x \in X).$$

Note that $\mu_A(x) < \infty$ for all $x \in X$, since A is absorbing. The seminorms on X will turn out to be precisely the Minkowski functionals of *balanced* convex absorbing sets.

1.34 Theorem *Suppose p is a seminorm on a vector space X. Then*

$\forall x\in X;\ P\left(\frac{1}{2P(x)}x\right)=\frac{1}{2P(x)}P(x)=\frac{1}{2}<1$

(a) $p(0) = 0$.

(b) $|p(x) - p(y)| \le p(x - y)$.

(c) $p(x) \geq 0$.

(d) $\{x : p(x) = 0\}$ *is a subspace of* X.

(e) *The set $B = \{x : p(x) < 1\}$ is convex, balanced, absorbing, and $p = \mu_B$.*

$\forall x\in B\ \forall\alpha;\ |\alpha|\le 1 \Rightarrow P(\alpha x)=|\alpha|P(x)<1 \Rightarrow \alpha x\in B$ (B is balanced)

PROOF. Statement (*a*) follows from $p(\alpha x) = |\alpha|\, p(x)$, with $\alpha = 0$. The subadditivity of p shows that

$$p(x) = p(x - y + y) \leq p(x - y) + p(y)$$

so that $p(x) - p(y) \leq p(x - y)$. This also holds with x and y interchanged. Since $p(x - y) = p(y - x)$, (b) follows. With $y = 0$, (b) implies (c). If $p(x) = p(y) = 0$ and α, β are scalars, (c) implies

$$0 \leq p(\alpha x + \beta y) \leq |\alpha| p(x) + |\beta| p(y) = 0.$$

This proves (d).

As to (e), it is clear that B is balanced. If $x \in B$, $y \in B$, and $0 < t < 1$, then

$$p(tx + (1 - t)y) \leq tp(x) + (1 - t)p(y) < 1.$$

Thus B is convex. If $x \in X$ and $s > p(x)$ then $p(s^{-1}x) = s^{-1}p(x) < 1$. This shows that B is absorbing and also that $\mu_B(x) \leq s$. Hence $\mu_B \leq p$. But if $0 < t \leq p(x)$ then $p(t^{-1}x) \geq 1$, and so $t^{-1}x$ is not in B. This implies $p(x) \leq \mu_B(x)$ and completes the proof. 1 so $t \leq \mu_B(x)$ ////

balanced

35 Theorem *Suppose A is a convex absorbing set in a vector space X. hen*

$\inf\{s>0 : s^{-1}x \in B\} = \mu_B(x) < t \Rightarrow \exists s \in \Gamma;\ s < t$

$\Rightarrow p(s^{-1}x) = s^{-1}p(x) \geq t^{-1}p(x) = p(t^{-1}x) \geq 1 \Rightarrow s^{-1}x \notin B$

) $\mu_A(x + y) \leq \mu_A(x) + \mu_A(y)$.

) $\mu_A(tx) = t\mu_A(x)$ *if* $t \geq 0$.

μ_A *is a seminorm if* [redacted]

If $B = \{x : \mu_A(x) < 1\}$ *and* $C = \{x : \mu_A(x) \leq 1\}$, *then* $B \subset A \subset C$ *and* $\mu_B = \mu_A = \mu_C$.

2

PROOF. If $t = \mu_A(x) + \varepsilon$ and $s = \mu_A(y) + \varepsilon$, for some $\varepsilon > 0$, then x/t and y/s are in A; hence so is their convex combination

$$\frac{x + y}{s + t} = \frac{t}{s + t} \cdot \frac{x}{t} + \frac{s}{s + t} \cdot \frac{y}{s}.$$

$= 1 - \dfrac{t}{s+t}$

This shows that $\mu_A(x + y) \leq s + t = \mu_A(x) + \mu_A(y) + 2\varepsilon$, and (a) is proved.

Property (b) is clear, and (c) follows from (a) and (b).

When we turn to (d), the inclusions $B \subset A \subset C$ show that $\mu_C \leq \mu_A \leq \mu_B$. To prove equality, fix $x \in X$, and choose s, t so that $\mu_C(x) <$

3

$s < t$. Then $x/s \in C$, $\mu_A(x/s) \le 1$, $\mu_A(x/t) \le s/t < 1$; hence $x/t \in B$, so that $\mu_B(x) \le t$. This holds for every $t > \mu_C(x)$. Hence $\mu_B(x) \le \mu_C(x)$. ////

.36 Theorem *Suppose $\mathscr{B}$ is a convex balanced local base in a topological vector space X. Associate to every $V \in \mathscr{B}$ its Minkowski functional μ_V. Then*

a) $V = \{x \in X : \mu_V(x) < 1\}$, for every $V \in \mathscr{B}$, and

b) $\{\mu_V : V \in \mathscr{B}\}$ is a separating family of continuous seminorms on X.

3

PROOF. If $x \in V$, then $x/t \in V$ for some $t < 1$, because V is open; hence $\mu_V(x) < 1$. If $x \notin V$, then $x/t \in V$ implies $t \ge 1$, because V is balanced; hence $\mu_V(x) \ge 1$. This proves (*a*).

Theorem 1.35 shows that each μ_V is a seminorm. If $r > 0$, it follows from (*a*) and Theorem 1.34 that

$r^{-1}(x-y) \in V \Rightarrow \mu_V\left(\frac{x-y}{r}\right) < 1$

$$|\mu_V(x) - \mu_V(y)| \le \mu_V(x - y) < r$$

if $x - y \in rV$. Hence μ_V is continuous. If $x \in X$ and $x \neq 0$, then $x \notin V$ for some $V \in \mathscr{B}$. For this V, $\mu_V(x) \ge 1$. Thus $\{\mu_V\}$ is separating. ////

A known fact: $(X, \|\cdot\|)$ already is a normed space

37 Theorem *Suppose $\mathscr{P}$ is a separating family of seminorms on a vector space X. Associate to each $p \in \mathscr{P}$ and to each positive integer n the set*

$p = \|\cdot\| : X \to \mathbb{R}$ is cts since

$$V(p, n) = \left\{x : p(x) < \frac{1}{n}\right\}.$$

$\big|\|x\| - \|y\|\big| \le \|x - y\|$

Let $\mathscr{B}$ be the collection of all finite intersections of the sets $V(p, n)$. Then $\mathscr{B}$ is a convex balanced local base for a topology τ on X, which turns X into a locally convex space such that

every $p \in \mathscr{P}$ is continuous, and

a set $E \subset X$ is bounded if and only if every $p \in \mathscr{P}$ is bounded on E.

4

PROOF. Declare a set $A \subset X$ to be open if and only if A is a (possibly empty) union of translates of members of $\mathscr{B}$. This clearly defines a translation-invariant topology τ on X; each member of $\mathscr{B}$ is convex and balanced, and $\mathscr{B}$ is a local base for τ.

5

Suppose $x \in X$, $x \neq 0$. Then $p(x) > 0$ for some $p \in \mathscr{P}$. Since x is not in $V(p, n)$ if $np(x) > 1$, we see that 0 is not in the neighborhood $x - V(p, n)$ of x, so that x is not in the closure of $\{0\}$. Thus $\{0\}$ is a closed set, and since τ is translation-invariant, every point of X is a closed set.

$\{0\}$ closed $\Rightarrow \{0\} + x = \{x\}$ closed

Next we show that addition and scalar multiplication are continuous. Let U be a neighborhood of 0 in X. Then

$X \times X \to X$

$(0,0) \mapsto \boxed{0+0=0}$

$$U \supset V(p_1, n_1) \cap \cdots \cap V(p_m, n_m) \ni 0 \tag{1}$$

for some $p_1, \ldots, p_m \in \mathscr{P}$ and some positive integers $n_1, \ldots, n_m$. Put U

$$V = V(p_1, 2n_1) \cap \cdots \cap V(p_m, 2n_m). \tag{2}$$

Since every $p \in \mathscr{P}$ is subadditive, $V + V \subset U$. (6) This proves that addition is continuous.

Suppose now that $x \in X$, α is a scalar, and U and V are as above. Then $x \in sV$ for some $s > 0$. Put $t = s/(1 + |\alpha| s)$. If $y \in x + tV$ and $|\beta - \alpha| < 1/s$, then similar to (3)

$\Phi \times X \to X$

$(\alpha, x) \mapsto \alpha x$

$$\beta y - \alpha x = \beta(y - x) + (\beta - \alpha)x$$

which lies in

$$|\beta| tV + \underbrace{|\beta - \alpha|}_{<s} sV \subset V + V \subset U$$

$\Rightarrow \beta y \in \alpha x + U$

$\alpha x + U$

since $|\beta| t \le 1$ and V is balanced. This proves that scalar multiplication is continuous.

Thus X is a locally convex space. The definition of $V(p, n)$ shows that every $p \in \mathscr{P}$ is continuous at 0. Hence p is continuous on X, by (*b*) of Theorem 1.34.

Finally, suppose $E \subset X$ is bounded. Fix $p \in \mathscr{P}$. Since $V(p, 1)$ is a neighborhood of 0, $E \subset kV(p, 1)$ for some $k < \infty$. Hence $p(x) < k$ for every $x \in E$. It follows that every $p \in \mathscr{P}$ is bounded on E.

Conversely, suppose E satisfies this condition, U is a neighborhood of 0, and (1) holds. There are numbers $M_i < \infty$ such that $p_i < M_i$ on E $(1 \le i \le m)$. If $n > M_i n_i$ for $1 \le i \le m$, it follows that $E \subset nU$, so that E is bounded. ////

(*)

① $\left(t < p(x) \Rightarrow t \le \mu_B(x) \right) \Longrightarrow p(x) \le \mu_B(x)$

فرض کنیم t , $\mu_B(x)$ کوچکتر از t هستند و سپس در $(*)$ t را از چپ به $p(x)$ میل دهیم.

② $t=\mu_A(x)+\varepsilon \Rightarrow \frac{x}{t}\in A$ ✓

$$\underset{\inf\underbrace{\{s:\, s^{-1}x\in A\}}_{\Gamma}}{\mu_A(x)}<t \Rightarrow \exists s\in\Gamma;\ s<t \Rightarrow t^{-1}s<1 \xRightarrow{A \text{ balanced}} t^{-1}s(s^{-1}x)\in A \Rightarrow t^{-1}x\in A$$

$s^{-1}x\in A$

③ $V^{\text{open}} \Rightarrow V$ is absorbing ✓

$$0\in V^{\text{open}} \xRightarrow{\boxed{\text{Th } 1.15}} X=\bigcup_{n=1}^{\infty} nV \Rightarrow \forall x\in X\ \exists n_0:\ x\in n_0V \Rightarrow n_0^{-1}x\in V$$

This shows that V is absorbing.

$$\begin{cases}\varphi:\mathbb{C}\,(\cancel{\mathbb{R}})\times X\to X\\ (r,x)\mapsto rx\end{cases}$$ is cts at, e.g., $(1,x)$.

for example

$\varphi\big((1-\delta,1+\delta)\times W\big)\subseteq V$

$\forall t \in (1, 1+\delta)$; $\underset{\frac{1}{s}x}{\underbrace{tx}} \in V$ ✓ x

$\frac{1}{1+\delta} < s := \frac{1}{t} < 1$

Let $E \subseteq X$ be a set.

(4) the notions

(i) E is bd in a t.v.s when $\forall V \exists s \forall t > s$; $E \subseteq tV$

(ii) E is bd in a normed space when $\exists M \forall x \in E$; $\|x\| \le M$

are equivalent in normed spaces:

(i) $\Rightarrow$ (ii) Let $V = B(0)$. So $\exists s \forall t > s$; $E \subseteq tV$. Hence $E \subseteq (s+1)B(0) = \underbrace{B(0)}_{s+1}$

So $\forall x \in E$; $\|x\| \le \underbrace{s+1}_{M}$ ✓

(ii) $\Rightarrow$ (i) Let $0 \in V$ be open. $\exists \alpha$; $B_\alpha(0) \subseteq V$

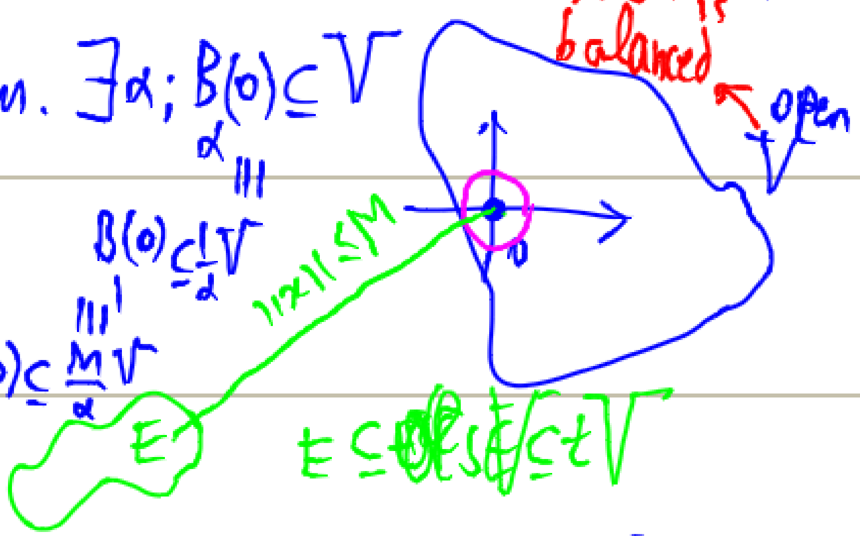
$\exists M \forall x \in E$; $\|x\| \le M$

$\underbrace{\qquad\qquad}$

$E \subseteq B_M(0)$

$B(0) \subseteq \frac{1}{\alpha} V$

$B_M(0) \subseteq \frac{M}{\alpha} V$



Put $s = \frac{M}{\alpha}$. Then $\forall t > s$; $E \subseteq B_M(0) \subseteq \frac{M}{\alpha} V = sV \subseteq tV$ ✓

(5) The set $\mathcal{B}$ of all sets of the form

$x + \bigcap_{i=1}^{K} V(P_i, n_i)$ is a base for a top. (this top

is the set of all arbitrary union of elements of $\mathcal{B}$)

is the set of all arbitrary union of elements of $\mathcal{B}$)

Note that always $\underbrace{\mathcal{S}}_{\text{subbase}} \subseteq \underbrace{\mathcal{B}}_{\text{base}} \subseteq \underbrace{\tau}_{\text{top}}$

Let $S \in \mathcal{S}$. $S = S \cap S \in \mathcal{B}$

Let $V \in \mathcal{B}$. $V = V \cup V \in \tau$

(b)

$x \in V, y \in V \Rightarrow \forall 1 \leq i \leq m$ $\begin{array}{l} P_i(x) < \frac{1}{2n_i} \\ P_i(y) < \frac{1}{2n_i} \end{array} \Rightarrow \overset{\forall 1 \leq i \leq m}{P_i(x+y)} \leq P_i(x) + P_i(y)$

$< \frac{1}{2n_i} + \frac{1}{2n_i} = \frac{1}{n_i} \Rightarrow x+y \in U$

Weak & Weak*-top

Let X be a vector space. Let $\mathcal{F}=\{f_\alpha\}_{\alpha\in I}$ be a separating family of real or complex-valued functionals on X. Put $P_\alpha(x)=|f_\alpha(x)|$, $\alpha\in I, x\in X$. Then $\{P_\alpha\}$ is a separating family of seminorms. Then l.c.t.v.s. generated by $\{P_\alpha\}_{\alpha\in I}$ is called the weak topology on X, denoted by $\sigma(X,\mathcal{F})$.

An element of a basis for $\sigma(X,\mathcal{F})$ is

$$\bigcap_{i=1}^{n}\left\{x : |f_{\alpha_i}(x)|<\frac{1}{n_i}\right\}=\boxed{\left\{x\in X : |f_{\alpha_i}(x)|<\varepsilon \right\}_{i=1,\dots,n}=V(f_1,\dots,f_n,\varepsilon)}$$

Then each f_α is cts at 0 (and so at each point):

$$\forall\varepsilon\ \exists \underset{\text{open}}{V(f,\varepsilon)}\ \forall x\in V(f,\varepsilon);\quad |f_\alpha(x)-f_\alpha(0)|=|f_\alpha(x)|<\varepsilon$$

this topology is the weakest top on X under which the f_α are cts:

Let τ be a top on X such that all f_α are cts.

Let $\underbrace{\{x : |f_\alpha(x)| < \varepsilon\}}_{= |f_\alpha|^{-1}((0,\varepsilon))}$ be an element of the subbase of $\sigma(X, \mathcal{F})$.

Since $\underset{: X \to \mathbb{C}}{f_\alpha}$ is cts $\underbrace{\text{with respect to}}_{\text{w.r.t.}}$ τ & $|\cdot| : \mathbb{C} \to \mathbb{R}$ is cts, so is $|f_\alpha| : X \to \mathbb{R}$.

Hence $|f_\alpha|^{-1}(\underbrace{(0,\varepsilon)}_{\text{open}}) \in \tau$. Thus $\sigma(X, \mathcal{F}) \subseteq \tau$. ✓

A net (x_i) in X is convergent to $x \in X$ w.r.t. $\sigma(X, \mathcal{F})$ iff $\forall f \in \mathcal{F}$; $f(x_i) \longrightarrow f(x)$ (♣).

<u>Lemma</u>. In a l.c.t.v.s. generated by $\{P_\alpha\}$, $x_i \to 0$ iff $P_\alpha(x_i) \to 0$ $\forall \alpha$

<u>Proof</u>. ($\Rightarrow$) We already observed that P_α's are cts, so if $x_i \to 0$, we have $P_\alpha(x_i) \to P_\alpha(0) = 0$ $\forall \alpha$ ✓

($\Leftarrow$) It is enough to consider elements of the basis of the top, when we want to prove that $x_i \to 0$:

Let $\{x \in X : P_{\alpha_k}(x) < \varepsilon,\ k = 1, \dots, n\}$ be an element of the basis $\mathcal{B}$. By our assumption $\forall\, 1 \le k \le n$, $P_{\alpha_k}(x_i) \xrightarrow{i} 0$.

$\forall_{1\le k\le n} \exists j_k \forall i \geq j_k;\ |P_{\alpha_k}(x_i) - 0| = P_{\alpha_k}(x_i) < \varepsilon$

Since $(x_i)_{i\in D}$ is a net, D is directed, so $\exists j\ \forall_{1\le k\le n},\ j_k \le j$.

Then $\forall_{1\le k\le n}\ \forall i \geq j;\ P_{\alpha_k}(x_i) < \varepsilon$

or equivalently $\forall i \geq j\ \underbrace{\forall_{1\le k\le n};\ P_{\alpha_k}(x_i) < \varepsilon}_{x_i \in \{x:\ P_{\alpha_k}(x) < \varepsilon,\ k=1,\dots,n\}}$.

Thus $x_i \to 0$. ✓

The proof of (♣):

$$x_i \to x \iff x_i - x \to 0 \iff \forall \alpha;\ P_\alpha(x_i - x) \to 0$$

$$\underset{P_\alpha = |f_\alpha|}{\iff} \forall \alpha;\ |\underbrace{f_\alpha(x_i - x)}_{f_\alpha(x_i) - f_\alpha(x)}| \to 0 \iff \forall \alpha;\ f_\alpha(x_i) \to f_\alpha(x). \ \square$$

Let $(X, \|\cdot\|)$ be a normed space & X^* be the dual of X, i.e. the space of all $\underset{\text{cts}}{\text{bd}}$ linear functionals $f: X \to \mathbb{C}$.

Then $\sigma(X, X^*)$ is called the weak top on X.

then $\sigma(X^*, X)$ is called ...

Let us recall that $\hat{}: X \hookrightarrow X^{**}$ is an isometric

$x \longmapsto \hat{x}$, $\begin{cases} \hat{x} : X^* \to \mathbb{C} \\ \hat{x}(f) = f(x) \end{cases}$

$|\hat{x}(f)| = |f(x)| \le \|f\|\,\|x\|$

$\|\hat{x}\| = \sup_{f \ne 0} \frac{|\hat{x}(f)|}{\|f\|} \le \|x\|$

By the Hahn-Ban theorem $\exists f$; $\|f\| = 1$, $f(x) = \|x\|$, $= \frac{|\hat{x}(f)|}{\|f\|}$

linear mapping (If this map is surjective, then X is called reflexive)

Let us identify $\hat{X} = \{\hat{x} : x \in X\}$ with X.

$\sigma(X^*, \underbrace{\hat{X}}_{= X})$ is called the weak*-top on X^*.

<u>Comments</u>: ① In $\sigma(X, X^*)$: $x_i \to x \Leftrightarrow \forall f \in X^*$; $f(x_i) \to f(x)$

② In $\sigma(X^*, \hat{X})$: $f_i \to f \Leftrightarrow \forall \hat{x} \in \hat{X}$; $\hat{x}(f_i) \to \hat{x}(f)$

$\Leftrightarrow \forall x \in X$; $f_i(x) \to f(x)$ (ie. f_i Pointwise converges to f)

قسمين - آلاغلو

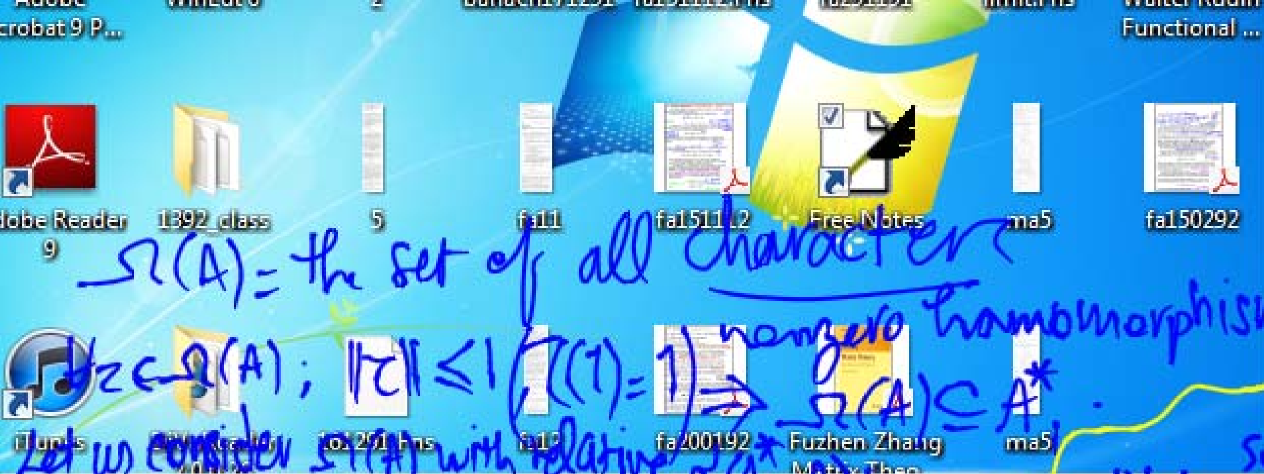
<u>Banach-Alaoglu Theorem</u>. The closed unit ball $\{f \in X^* : \|f\| \le 1\}$ of X^* is compact in $\sigma(X^*, X)$. $\boxed{\text{Project}}$

① Note that X & X^* already have normed structures:

X: $\|\cdot\|$; X^*: $\|f\| = \sup_{x \ne 0} \frac{|f(x)|}{\|x\|}$ $\to$ operator norm

② Note that we can consider a weak top on X^* by $\sigma(X^*, X^{**})$ $\leftarrow$ weak*-top

③ If $f_n \xrightarrow{\|\cdot\|} f$, $|f_n(x) - f(x)| \le \|f_n - f\|\,\|x\|$ $\forall x \Rightarrow f_n(x) \longrightarrow f(x)$ $\forall x \Rightarrow f_n \xrightarrow{w^*} f$



$\Omega(A)$ = the set of all character ~ nonzero homomorphism

$\forall \tau \in \Omega(A)$; $\|\tau\| \le 1$, $\tau(1)=1$ $\Rightarrow$ $\Omega(A) \subseteq A^*$.

Let us consider $\Omega(A)$ with relative $\sigma(A^*, A)$

(X, τ) Y subspace top $\tau|Y = \{G \cap Y : G \in \tau\}$

If A is an abelian Banach algebra, it follows from Theorem 1.3.4 that $\Omega(A)$ is contained in the closed unit ball of A^*. We endow $\Omega(A)$ with the relative weak* topology, and call the topological space $\Omega(A)$ the *character space*, or *spectrum*, of A.

① $\tau_\alpha \xrightarrow{w^*} \tau \in A^*$, $\tau \neq 0$ $\Rightarrow$ $\tau_\alpha(ab) = \tau_\alpha(a)\tau_\alpha(b)$ $\Rightarrow$ $\tau_\alpha \xrightarrow{\text{pointwise}} \tau$; $\tau(ab) = \tau(a)\tau(b)$ $\Downarrow$ $\tau \in \Omega(A)$

1.3.5. Theorem. *If A is an abelian Banach algebra, then $\Omega(A)$ is a locally compact Hausdorff space. If A is unital, then $\Omega(A)$ is compact.*

Proof. It is easily checked that $\Omega(A) \cup \{0\}$ is weak* closed in the closed unit ball S of A^*. Since S is weak* compact (Banach–Alaoglu theorem), $\Omega(A) \cup \{0\}$ is weak* compact, and therefore, $\Omega(A)$ is locally compact.

If A is unital, then $\Omega(A)$ is weak* closed in S and thus compact. □

$1 = \tau_\alpha(1) \to \tau(1) \Rightarrow \tau(1) = 1 \Rightarrow \tau \neq 0$

Note that $\Omega(A)$ may be empty. This is the case for $A = 0$, for example.

Suppose that A is an abelian Banach algebra for which the space $\Omega(A)$ is non-empty. If $a \in A$, we define the function $\hat{a}$ by

$X \hookrightarrow X^{**}$, $a \mapsto \hat{a}(f) = f(a)$

$$\hat{a}\colon \Omega(A) \to \mathbf{C},\ \tau \mapsto \tau(a).$$

Clearly the topology on $\Omega(A)$ is the smallest one making all of the functions $\hat{a}$ continuous. The set $\{\tau \in \Omega(A) \mid |\tau(a)| \ge \varepsilon\}$ is weak* closed in the closed unit ball of A^* for each $\varepsilon > 0$, and weak* compact by the Banach–Alaoglu theorem. Hence, $\hat{a} \in C_0(\Omega(A))$.

We call $\hat{a}$ the *Gelfand transform* of a.

$\tau_\alpha \xrightarrow{w^*} \tau \Rightarrow \tau_\alpha(a) \to \tau(a)\ \forall a$; $|\tau_\alpha(a)| \ge \varepsilon$ $\Rightarrow |\tau(a)| \ge \varepsilon\ \forall a$

Although the following result is very important, its proof is easy, because we have already done most of the work needed to demonstrate it.

1.3.6. Theorem (Gelfand Representation). *Suppose that A is an abelian Banach algebra and that $\Omega(A)$ is non-empty. Then the map*

$$A \to C_0(\Omega(A)),\ a \mapsto \hat{a},$$

is a norm-decreasing homomorphism, and

$\sup_{\lambda \in \sigma(a)} |\lambda| = \sup_{\lambda \in \sigma(\hat{a})} |\lambda| = \sup_{\tau} |\hat{a}(\tau)|$

$$r(a) = \|\hat{a}\|_\infty \qquad (a \in A).$$

$\|a\| \geq$ $\hat{a}(\tau)$

If A is unital, $\sigma(a) = \hat{a}(\Omega(A))$, and if A is non-unital, $\sigma(a) = \hat{a}(\Omega(A)) \cup \{0\}$, for each $a \in A$.

$\{\tau(a) : \tau \in \Omega(A)\}(\cup\{0\}) = \mathrm{ran}(\hat{a})$

Proof. By Theorem 1.3.4 the spectrum $\sigma(a)$ is the range of $\hat{a}$, together with $\{0\}$ if A is non-unital. Hence, $r(a) = \|\hat{a}\|_\infty$, which implies that the map $a \mapsto \hat{a}$ is norm-decreasing. That this map is a homomorphism is easily checked. □

The kernel of the Gelfand representation is called the *radical* of the algebra A. It consists of the elements a such that $r(a) = 0$. It therefore contains the nilpotent elements. If the radical is zero, A is said to be *semisimple*.

In a general algebra an element whose spectrum consists of the set $\{0\}$ is said to be *quasinilpotent*.

Let a, b be commuting elements of an arbitrary Banach algebra A. Then $r(a + b) \leq r(a) + r(b)$, and $r(ab) \leq r(a)r(b)$. To see this, we may suppose that A is unital and abelian (if necessary, adjoin a unit and restrict to the closed subalgebra generated by $1, a$, and b). Then $r(a + b) = \|(a + b)\hat{}\|_\infty \overset{(2)}{\leq} \|\hat{a}\|_\infty + \|\hat{b}\|_\infty = r(a) + r(b)$ by Theorem 1.3.6. Similarly, $r(ab) = \|(ab)\hat{}\|_\infty \leq \|\hat{a}\|_\infty \|\hat{b}\|_\infty = r(a)r(b)$. Direct proofs of the first of these inequalities (that is, where the Gelfand representation is not invoked) tend to be messy.

(2) $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

The spectral radius is neither subadditive nor submultiplicative in general: Let $A = M_2(\mathbf{C})$ and suppose

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $r(a) = r(b) = 0$, since a and b have square zero, but $r(a + b) = r(ab) = 1$.

The interpretation of the character space as a sort of generalised spectrum is motivated by the following result.

1.3.7. Theorem. *Let A be a unital Banach algebra generated by 1 and an element a. Then A is abelian and the map*

the closure of the set of polynomials in a

$$\hat{a} : \Omega(A) \to \sigma(a), \ \tau \mapsto \tau(a),$$

is a homeomorphism.

$f : X \xrightarrow[\text{onto}]{\text{cts, 1-1}} Y$, X compact, Y Compact Hausdorff; C closed, compact; f^{-1}; $(f^{-1})^{-1}(C) = f(C)$ closed

Proof. It is clear that A is abelian and that $\hat{a}$ is a continuous bijection, and because $\Omega(A)$ and $\sigma(a)$ are compact Hausdorff spaces, $\hat{a}$ is therefore a homeomorphism. □

To illustrate this, consider the disc algebra A. If u is its canonical

To illustrate this, consider the disc algebra A. If z is its canonical generator, then since $\sigma(z) = \mathbf{D}$, we have $\Omega(A) = \mathbf{D}$ by Theorem 1.3.7. In this case if $f \in A$, then $\hat{f}(\lambda) = f(\lambda)$, so the Gelfand transform is the identity map.

Compact

1.4. Compact and Fredholm Operators

This section is concerned with the elementary spectral theory of operators. We begin with the simplest non-trivial class of operators, the compact ones, a class that plays an important and fundamental role in operator theory. These operators behave much like operators on finite-dimensional vector spaces, and for this reason they are relatively easy to analyse.

A linear map $u: X \to Y$ between Banach spaces X and Y is *compact* if $u(S)$ is relatively compact in Y, where S is the closed unit ball of X. Equivalently, $u(S)$ is totally bounded. In this case $u(S)$ is bounded, and therefore, u is bounded.

$\equiv \overline{u(S)}$ compact

If X, Y are ~~Banach~~ normed spaces, we denote by $B(X, Y)$ the vector space of all bounded linear maps from X to Y. This is a Banach space when endowed with the operator norm. The set of all compact operators from X to Y is denoted by $K(X, Y)$.

The proof of the following is a routine exercise.

1.4.1. Theorem. (3) *Let X and Y be Banach spaces and $u \in B(X, Y)$. Then the following conditions are equivalent:*

(1) *u is compact;*
(2) *For each bounded set S in X, the set $u(S)$ is relatively compact in Y;*
(3) *For each bounded sequence (x_n) in X, the sequence $(u(x_n))$ admits a subsequence that converges in Y.*

It follows easily from Theorem 1.4.1 that $K(X, Y)$ is a vector subspace of $B(X, Y)$. Also, if $X' \xrightarrow{v} X \xrightarrow{u} Y \xrightarrow{w} Y'$ are bounded linear maps between Banach spaces and u is compact, then wu and uv are compact. Hence $K(X) = K(X, X)$ is an ideal in $B(X)$. $\{u \in K(X)\}$

$v \in B(X),\ u \in K(X) \Rightarrow \{uv \in K(X)\}$

$\{x_n\}$ bd $\Rightarrow \exists M\ \forall n\ \|x_n\| \leq M \Rightarrow \|vx_n\| \leq \|v\|\|x_n\| \leq M\|v\|\ \forall n \Rightarrow \{u(vx_n)\}$ has conv. subseq

1.4.2. Theorem. *If X is a Banach space, then $K(X) = B(X)$ if and only if X is finite-dimensional.*

$\{x_n\}$ bd $\Rightarrow \{ux_n\}$ has conv. subseq $\xRightarrow{\text{cts of } v} \{vux_n\}$ has conv. seq.

Proof. If S denotes the closed unit ball of X, then $K(X) = B(X) \Leftrightarrow \mathrm{id}_X$ is compact $\Leftrightarrow S$ is compact $\Leftrightarrow X$ is finite-dimensional. $\square$

4

1.4.3. Theorem. *If X, Y are Banach spaces, then $K(X, Y)$ is a closed vector space of $B(X, Y)$.*

Proof. We show that if a sequence (u_n) in $K(X, Y)$ converges to an

Proof. We show that if a sequence (u_n) in $K(X,Y)$ converges to an operator u in $B(X,Y)$, then u is compact. Let S denote the closed unit ball of X and let $\varepsilon > 0$. Choose an integer N such that $\|u_N - u\| < \varepsilon/3$. Since $u_N(S)$ is totally bounded, there are elements $x_1, \ldots, x_n \in S$, such that for each x in S, the inequality $\|u_N(x) - u_N(x_j)\| < \varepsilon/3$ holds for some index j. Hence, $= \overline{u_N(S)}$ compact

$$\begin{aligned}\|u(x) - u(x_j)\| &\leq \|u(x) - u_N(x)\| + \|u_N(x) - u_N(x_j)\| + \|u_N(x_j) - u(x_j)\| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

Thus, $u(S)$ is totally bounded, and therefore, $u \in K(X,Y)$. □

Recall that a linear map $u: X \to Y$ is of *finite rank* if $u(X)$ is finite-dimensional and that $\operatorname{rank}(u) = \dim(u(X))$.

If X and Y are Banach spaces and $u \in B(X,Y)$ is of finite rank, then $u \in K(X,Y)$. This is immediate from the fact that the closed unit ball of the finite-dimensional space $u(X)$ is compact.

It follows from this remark and Theorem 1.4.3 that norm-limits of finite-rank operators are compact, and it is natural to ask whether the converse is true. This is the case for Hilbert spaces, as we shall see in the next chapter, but it is not true for arbitrary Banach spaces. P. Enflo [Enf] has given an example of a Banach space for which there are compact operators that are not norm-limits of finite-rank operators.

If $u: X \to Y$ is a bounded linear map between Banach spaces, we define its *transpose* $u^* \in B(Y^*, X^*)$ by $u^*(\tau) = \tau \circ u$.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[rr, bend right, dashed, "\tau\circ u"'] & Y \arrow[r, "\tau"] & \mathbb{C} \end{tikzcd}$$

1.4.4. Theorem. *Let X, Y be Banach spaces and let $u \in K(X,Y)$. Then $u^* \in K(Y^*, X^*)$.*

Proof. Let S be the closed unit ball of X and let $\varepsilon > 0$. Since $u(S)$ is totally bounded, there exist elements $x_1, \ldots, x_n$ in S, such that if $x \in S$, then $\|u(x) - u(x_i)\| < \varepsilon/3$ for some index i. Define $v \in B(Y^*, \mathbf{C}^n)$ by setting $v(\tau) = (\tau u(x_1), \ldots, \tau u(x_n))$. Since the rank of v is finite, v is compact, and therefore $v(T)$ is totally bounded, where T is the closed unit ball of Y^*. Hence, there exist functionals $\tau_1, \ldots, \tau_m$ in T, such that if $\tau \in T$, then $\|v(\tau) - v(\tau_j)\| < \varepsilon/3$ for some index j. Observe that

$$\|v(\tau) - v(\tau_j)\| = \max_{1 \leq i \leq n} |u^*(\tau)(x_i) - u^*(\tau_j)(x_i)|.$$

Now suppose that $x \in S$. Then $\|u(x) - u(x_i)\| < \varepsilon/3$ for some index i, and $|u^*(\tau)(x_i) - u^*(\tau_j)(x_i)| < \varepsilon/3$. Hence,

$$|u^*(\tau)(x) - u^*(\tau_j)(x)| \leq |u^*(\tau)(x) - u^*(\tau)(x_i)| + |u^*(\tau)(x_i) - u^*(\tau_j)(x_i)|$$

$$+ |u^*(\tau_j)(x_i) - u^*(\tau_j)(x)|$$
$$\le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

It follows that $\|u^*(\tau) - u^*(\tau_j)\| \le \varepsilon$, so $u^*(T)$ is totally bounded and therefore u^* is compact. left to students □

≡ C^*_algebras ≡

A C^*-algebra is a Banach alg A endowed with an involution

$$\begin{cases} * : A \to A \\ a \mapsto a^* \\ (a+b)^* = a^* + b^* \\ (\lambda a)^* = \bar{\lambda} a^* \\ (ab)^* = b^* a^* \\ (a^*)^* = a \end{cases}$$

such that $\boxed{\|a^* a\| = \|a\|^2}$

C^*-condition

Examples: $\boxed{1}$ $\mathbb{C}$: $z^* = \bar{z}$ & $\|z^* z\| = |\bar{z} z| = \bar{z} z = |z|^2 = \|z\|^2$

$\|z\| = |z|$

$\boxed{2}$ $B(H)$: $\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$

$\langle Tx, y \rangle = \langle x, T^* y \rangle$

$\|T^* T\| = \|T\|^2$

$\boxed{3}$ $C(X)$: $\|f\|_\infty = \sup_{x \in X} |f(x)|$, $f^* = \bar{f}$ $(\bar{f}(x) = \overline{f(x)})$, $\|f^* f\|_\infty = \|f\|^2$

(3) u compact $\Rightarrow$ $\overline{u(D)}$ compact when D is bd

$S = B(0)$

D is bd $\Rightarrow \exists M;\ D \subseteq B_M(0) \Rightarrow u(D) \subseteq u(B_M(0))$

$= \{ ux : \|x\| \leq M \} = \{ u(My) : \|y\| \leq 1 \} = \{ Mu(y) : \|y\| \leq 1 \}$

$\frac{x}{M} = y$

$= Mu(S) \Rightarrow \overline{u(D)} \subseteq \overline{Mu(S)} = M\overline{u(S)}$ compact

λ_M is homeomorphism

Compact $\Rightarrow \overline{u(D)}$ is compact

$\left(\overline{u(D)} \text{ is compact when } D \text{ is bd} \right) \Rightarrow \forall \{x_n\}$ bd in X, $\{u(x_n)\}$ has covergent subseq:

Let $\{x_n\}$ be a bd sequence. Put $D = \{x_1, x_2, \dots\}$.

By our assumption, $\overline{u(D)} = \overline{\{ux_1, ux_2, \dots\}}$ is compact.

So $\{ux_n\} \subseteq \overline{u(D)}$ has a conv. subseq.

$K \subseteq X$ metric space is compact iff any $\{x_n\}$ in K has a conv. subseq

$\left(\forall \{x_n\} \text{ bd in } X, \{ux_n\} \text{ has conv. subseq in } Y \right) \Rightarrow u$ is compact

Let $S = B(0)$. We shall prove that $\overline{u(S)}$ is compact:

Let $\{y_n\}$ be a sequence in $\overline{u(S)}$.

$\forall n \ \exists x_n \in S; \ \|y_n - ux_n\| < \frac{1}{n}$

Since $\{x_n\} \subseteq S$, it is bd. So $\{ux_n\}$ has a convergent subseq,

say $\{ux_{n_k}\}_{k=1}^{\infty} \to y_0$ for some $y_0 \in Y$. We show that $\{y_{n_k}\}$ is convergent to y_0:

$0 \le \| y_{n_k} - y_0 \| \le \| y_{n_k} - u x_{n_k} \| + \| u x_{n_k} - y_0 \| \le \underbrace{\frac{1}{n_k}}_{\downarrow 0} + \underbrace{\| u x_{n_k} - y_0 \|}_{\downarrow 0}$ as $k \to \infty$

So $y_{n_k} \to y_0$ as $k \to \infty$. ✓

$\overline{I(S)} = \overline{S} = S$

(4) $\dim X < \infty \iff$ (we know) $S = B_1(0)$ is compact $\iff \begin{cases} I : X \to X \text{ is compact} \\ I(x) = x \end{cases}$

X is normed space

$K(X) \triangleleft B(X)$ ideal

$\iff I \in K(X) \iff K(X) = B(X)$

$\left.\begin{array}{l} I \triangleleft A \text{ ideal} \\ 1_A \in I \end{array}\right\} \Rightarrow \forall a \in A;\ a = a.1_A \in I$

$\Downarrow$

$I = A \overset{I \subseteq A}{\Longleftarrow} I \supseteq A$

Note: $\dim \mathbb{C}^n = n < \infty \Rightarrow K(\mathbb{C}^n) = B(\mathbb{C}^n) = M_n(\mathbb{C})$

Theorem. Every compact operator u is bd.

Let $S = B_1(0)$. The operator u is compact, so $\overline{u(S)}$ is compact. Hence $\overline{u(S)}$ is bd. So $u(S)$ is bd. Hence

$\|u\| = \sup_{x \in S} \|u(x)\| < \infty$. ✓

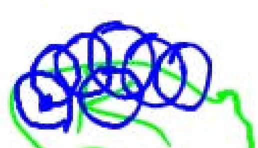
$\|u\| = \sup_{\|x\| \le 1} \|ux\| \le \infty$

$\boxed{\mathrm{diam}(\overline{K}) = \mathrm{diam}\, K}$

If $K \subseteq X$ is compact, then K is bd.

(X, d) metric space

Proof. X $\{B_1(x)\}_{x \in K}$



$\exists x_1, \dots, x_n;\ K \subseteq \bigcup_{i=1}^{n} B_1(x_i)$

Duality between X, X^*

Notation: $\boxed{1}$ $x, y, z, \ldots \in X$, $f, g, \ldots \in X^*$

$\boxed{2}$ $f(x) = \langle x, f \rangle$

$\boxed{3}$ $M^\perp = \{ f \in X^* : f(x) = 0 \ \forall x \in M \subseteq X \}$

$^\perp N = \{ x \in X : f(x) = 0 \ \forall f \in N \subseteq X^* \}$

Lemma 1. If $M \subseteq X$, then $M^\perp$ is a w^*-closed subspace of X^*.

Proof. $f, g \in M^\perp$, $\lambda \in \mathbb{C}$; $(\lambda f + g)(x) = \lambda \underbrace{f(x)}_{0} + \underbrace{g(x)}_{0} = 0 \ \forall x \in M$

$\Rightarrow \lambda f + g \in M^\perp$

Let $f_\alpha \in M^\perp$, $f_\alpha \xrightarrow{w^*} f$. Then $f(x) = \lim_\alpha \underbrace{f_\alpha(x)}_{0} = 0 \ \forall x \in M$. So $f \in M^\perp$.

Lemma 2. If $N \subseteq X^*$, then $^\perp N$ is norm closed subsp.

Proof. Let $\underset{\in\, {}^\perp N}{x_n} \xrightarrow{\|\cdot\|} x$. Let $f \in N$. $f(x) = \lim_n \underbrace{f(x_n)}_{0} = 0$. So $x \in {}^\perp N$.

continuity of f

Exercise $M \subseteq {}^\perp(M^\perp)$, $N \subseteq ({}^\perp N)^\perp$.

<u>Theorem</u>. Let M be a closed subspace of X.

Then $\left(\frac{X}{M}\right)^* \simeq M^\perp$, where $\simeq$ means isometrically isomorphism

$$\boxed{\|x+M\| = \inf_{z\in M} \|x+z\|}$$

<u>Proof</u>. $\begin{cases} \varphi : M^\perp \xrightarrow{\;\subseteq X^*\;} \left(\frac{X}{M}\right)^* \\ f \longmapsto \varphi(f), \begin{cases} \varphi(f) : \frac{X}{M} \to \mathbb{C} \\ \varphi(f)(x+M) = f(x) \end{cases} \end{cases}$

$$\begin{tikzcd}
x \arrow[d] \arrow[r, bend left, "f\in M^\perp"] & f(x) \\
x+M &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{C} \\
\frac{X}{M} \arrow[ur, dotted] &
\end{tikzcd}$$

We first show that $\varphi(f)$ is well-defined:

$x+M = x'+M \Rightarrow x - x' \in M \xrightarrow{f\in M^\perp} f(x-x') = 0 \Rightarrow f(x) = f(x') \Rightarrow$
$\varphi(f)(x+M) = \varphi(f)(x'+M)$

φ is well-defined, i.e., $\varphi(f) \in \left(\frac{X}{M}\right)^*$

Clearly $\varphi(f)$ is linear &

If T is <u>linear</u> & isometry, then T is 1-1

$Tx = Ty$

$T(x-y) = 0$

$\|x-y\| = \|T(x-y)\| = 0$

$x-y = 0$

$x = y$

$\|\varphi(f)(x+M)\| = \|\varphi(f)(x+z+M)\|$ $\quad z\in M$ arbitrary

$= \|f(x+z)\| \le \|f\|\,\|x+z\|$

گطو

So $\frac{\|\varphi(f)(x+M)\|}{\|f\|} \le \|x+z\|$. Hence $\frac{\|\varphi(f)(x+M)\|}{\|f\|} \le \inf_{z\in M} \|x+z\| = \|x+M\|$

Thus $\|\varphi(f)(x+M)\| \le \|f\|\,\|x+M\|$. So $\|\varphi(f)\| \le \|f\| < \infty$ (1) ✓

φ is linear, i.e., $\varphi(\lambda f+g)=\lambda\varphi(f)+\varphi(g)$

$\varphi(\lambda f+g)(x+M)=(\lambda f+g)(x)=\lambda f(x)+g(x)=\lambda\,\varphi(f)(x+M)+\varphi(g)(x+M)$

$=(\lambda\varphi(f)+\varphi(g))(x+M)$ ✓

φ is isometry, i.e, $\|\varphi(f)\|=\|f\|$:

We already proved $\|\varphi(f)\|\leq\|f\|$ (See ①). We shall show that $\|f\|\leq\|\varphi(f)\|$:

$\|f(x)\|=\|\varphi(f)(x+M)\|\overset{\text{def}}{\leq}\|\varphi(f)\|\underbrace{\|x+M\|}_{\inf_{m\in M}\|x+m\|}\overset{m=0}{\leq}\|\varphi(f)\|\,\|x\|$ ✓

Thus φ is also 1-1.

φ is surjective:

Let $g\in\left(\frac{X}{M}\right)^*$. Put $f=g\circ\pi\in X^*$

$$\begin{tikzcd} \frac{X}{M} \arrow[r, "g"] & \mathbb{C} \\ X \arrow[u, "\pi"] \arrow[ur, dotted, "?"'] & \end{tikzcd}$$

$\forall x\in M$; $f(x)=(g\circ\pi)(x)=g(\underbrace{\underbrace{\pi(x)}_{x+M}}_{0+M})=0$

So $f\in M^\perp$. We shall show that $\varphi(f)=g$

$\varphi(f)(x+M)=f(x)=g(\pi(x))=g(x+M)$. ✓ $\square$

<u>Exercise</u> $\frac{X^*}{M^\perp}\simeq M^*$ (M is closed subspace of X)

$\equiv B(H)$ as a C^*-algebra $\equiv$

Let $B(H)=\{T:H\to H \mid T$ is linear & bd$\}$, where H is a Hilbert space.

<u>Theorem</u> $\forall T\in B(H)\ \exists T^*\in B(H)\ \forall x,y\in H;\ \langle Tx,y\rangle=\langle x,T^*y\rangle$ and $(T+\lambda S)^*=T^*+\bar\lambda S^*$, $(TS)^*=S^*T^*$, $T^{**}=T$, $\|T^*T\| = \|T\|^2$
$(\|T\|=\|T^*\|)$

<u>Proof</u>. Let $T\in B(H)$. Let $y\in H$.

Exercise $\langle Tx,x\rangle=0\ \forall x \Rightarrow T=0$

Define $\begin{cases} f_y: H\to\mathbb{C} \\ f_y(x):=\langle Tx,y\rangle \end{cases}$.

$\forall f\in H^*=$ the dual of H $\exists z\in H\ \forall x\in H,\ f(x)=\langle x,z\rangle$ & $\|f\|=\|z\|$

Since $|f_y(x)|=|\langle Tx,y\rangle|\leq\|Tx\|\|y\|\leq\|T\|\|x\|\|y\|$,

we have $\|f_y\|\leq\|T\|\|y\|$. So $f_y\in H^*$. By the Riesz representation theorem $\exists z\in H;\ f_y(x)=\langle x,z\rangle\ \forall x\in H$

Put $T^*y=z$. Now we have $\begin{cases}T^*: H\to H \\ y\mapsto ?\ T^*y\end{cases}$. So $\langle Tx,y\rangle=\langle x,T^*y\rangle$. ✓

① Let y_1,y_2, $\alpha\in\mathbb{C}$ be given.
$\langle Tx,\alpha y_1\rangle+\langle Tx,y_2\rangle \overset{?}{=} \langle Tx,\alpha y_1\rangle+\langle Tx,y_2\rangle$

$T=S \Leftrightarrow \forall y: Tx=Sy$

$= \alpha\langle x, T y_1\rangle + \langle x, T y_2\rangle$

$\langle Tx, \alpha y_1 + y_2\rangle = \langle x, \alpha T^* y_1\rangle + \langle x, T^* y_2\rangle$

$\langle x, T^*(\alpha y_1 + y_2)\rangle = \langle x, \alpha T^* y_1 + T^* y_2\rangle$

$T^*(\alpha y_1 + y_2) = \alpha T^* y_1 + T^* y_2$

$\forall y \forall x; \langle x, Ty - Sy\rangle = 0 \Leftrightarrow \forall x \forall y; \langle x, Ty\rangle = \langle x, Sy\rangle$

$\forall x; \langle x, y\rangle = 0 \Rightarrow y = 0$ $\quad x = y$

$\forall x; \langle x, y\rangle = \langle x, z\rangle \Downarrow y = z$

$(\lozenge) \Rightarrow \|T^* y\| \leq \|T\| \|y\| \Rightarrow \|T^*\| = \sup_{y \neq 0} \frac{\|T^* y\|}{\|y\|} \leq \|T\| < \infty$

$\therefore T^* \in B(H)$ $\quad (\blacksquare)$

We shall prove that $T^{**} = T$:

$\forall x, y:$

$\langle x, (T^*)^* y\rangle = \langle T^* x, y\rangle = \overline{\langle y, T^* x\rangle} = \overline{\langle Ty, x\rangle} = \langle x, Ty\rangle$

We prove $(TS)^* = S^* T^*$

$\langle x, (TS)^* y\rangle = \langle \underbrace{(TS)x}_{T(Sx)}, y\rangle = \langle Sx, T^* y\rangle = \langle x, S^*(T^* y)\rangle = \langle x, (S^* T^*)(y)\rangle$

Finally we prove $\|T^* T\| = \|T\|^2$:

$\|T^* T\| \leq \|T^*\| \|T\| \underset{(\blacksquare)}{\leq} \|T\|^2$

$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^* T x, x\rangle = |\langle T^* T x, x\rangle| \leq \|T^* T x\| \|x\| \leq \|T^* T\| \|x\| \|x\|$ $\quad$ C-S inequality

$\forall x; \|Tx\|^2 \leq \|T^* T\| \|x\|^2$

$$\sup_{x\neq 0}\frac{\|Tx\|^2}{\|x\|^2}=\|T\|^2\leq\|T^*T\|$$

$\alpha=\sup A \Rightarrow \alpha^2=\sup\{x^2 : x\in A\}$
$A\subseteq\mathbb{R}^{\geq 0}$ ✓

We proved that $\|T^*\|\leq\|T\|$ $\forall T$ (See (VII)). So

$\|\underbrace{(T^*)^*}_{T}\|\leq\|T^*\|$. Hence $\|T\|\leq\|T^*\|$. ✓ $\square$

<u>Definition</u>.

- T self-adj if $T^*=T$ خودالحاق
- T normal if $T^*T=TT^*$ نرمال
- T unitary if $T^*T=TT^*=I$ یکانی
- T idempotent if $T^2=T$ خودتوان
- T projection if $T^2=T=T^*$ تصویری
- T partial isometry if $TT^*T=T$ طولپای جزئی
- T nilpotent of order n if $T^n=0$ پوچتوان

If (X,d), $\varphi: X\xrightarrow{1-1} Y$ onto, Y set, then we can put a metric on Y such that φ becomes isometry:

$\rho(y_1,y_2):=d(x_1,x_2)$, where $y_1=\varphi(x_1)$, $y_2=\varphi(x_2)$.

We know $\mathbb{C}^n$ is a Hilbert space $\langle(z_1,\dots,z_n),(w_1,\dots,w_n)\rangle=\sum_{j=1}^n z_j\overline{w_j}$

$\underbrace{B(\mathbb{C}^n)}_{H}$ and $M_n(\mathbb{C})$ are isomorphic as vector spaces.

$M_n(\mathbb{C}) \to B(\mathbb{C}^n)$

$A \rightsquigarrow T_A:\mathbb{C}^n\to\mathbb{C}^n$, $X\mapsto AX$

Put $\|A\|:=\|T_A\| = \sup_{X\neq 0}\frac{\|T_AX\|}{\|X\|}$ (operator norm)