$R^k = R^1 \times \cdots \times R^1$ (k factors).

نظام الأعداد الحقيقية الموسعة

The *extended real number system* is $\boxed{R^1}$ with two symbols, ∞ and $-\infty$, adjoined, and with the obvious ordering. If $-\infty \le a \le b \le \infty$, the *interval* $[a, b]$ and the *segment* (a, b) are defined to be

$-\infty \xrightarrow{\forall x\in\mathbb{R}} +\infty$

$-\infty < x < +\infty$

Text Book

$$[a, b] = \{x\colon a \le x \le b\}, \qquad (a, b) = \{x\colon a < x < b\}.$$

Real & Complex Analysis by W. Rudin

We also write

$$[a, b) = \{x\colon a \le x < b\}, \qquad (a, b] = \{x\colon a < x \le b\}.$$

If $E \subset [-\infty, \infty]$ and $E \ne \varnothing$, the least upper bound (supremum) and greatest lower bound (infimum) of E exist in $[-\infty, \infty]$ and are denoted by $\sup E$ and $\inf E$.

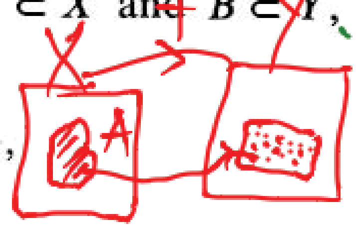
Sometimes (but only when $\sup E \in E$) we write $\max E$ for $\sup E$.

The symbol

$$f\colon X \to Y$$

$\boxed{\text{Always } D_f = X}$

means that f is a *function* (or *mapping* or *transformation*) of the set X into the set Y; i.e., f assigns to each $x \in X$ an element $f(x) \in Y$. If $A \subset X$ and $B \subset Y$, the *image* of A and the *inverse image* (or pre-image) of B are



$$f(A) = \{y\colon y = f(x) \text{ for some } x \in A\},$$
$$f^{-1}(B) = \{x\colon f(x) \in B\}.$$

Note that $f^{-1}(B)$ may be empty even when $B \ne \varnothing$.

The *domain* of f is X. The *range* of f is $f(X)$.

If $f(X) = Y$, f is said to map X *onto* Y.

We write $f^{-1}(y)$, instead of $f^{-1}(\{y\})$, for every $y \in Y$. If $f^{-1}(y)$ consists of at most one point, for each $y \in Y$, f is said to be *one-to-one*. If f is one-to-one, then f^{-1} is a function with domain $f(X)$ and range X.

If $f\colon X \to [-\infty, \infty]$ and $E \subset X$, it is customary to write $\sup_{x\in E} f(x)$ rather than $\sup f(E)$.

$\mathbb{R}^*$

If $f\colon X \to Y$ and $g\colon Y \to Z$, the *composite function* $g \circ f\colon X \to Z$ is defined by the formula

$-\infty = \sup \varnothing$ $\quad \forall \alpha \in \mathbb{R}^*; \ \forall x; (x \in \varnothing \Rightarrow x \le \alpha)$

$$(g \circ f)(x) = g(f(x)) \qquad (x \in X).$$

(*a*) A collection τ of subsets of a set X is said to be a *topology in X* if τ has the following three properties:

$X = \{a, b, c\}$

$\tau = \{\varnothing, \{a\}, \{a, b\}, X\}$

So α is an upper bound for $\varnothing$

(i) $\varnothing \in \tau$ and $X \in \tau$.

(ii) If $V_i \in \tau$ for $i = 1, \ldots, n$, then $V_1 \cap V_2 \cap \cdots \cap V_n \in \tau$.

(iii) If $\{V_\alpha\}$ is an arbitrary collection of members of τ (finite, countable, or uncountable), then $\bigcup_\alpha V_\alpha \in \tau$.

(*b*) If τ is a topology in X, then X is called a *topological space*, and the members of τ are called the *open sets* in X.

(*c*) If X and Y are topological spaces and if f is a mapping of X into Y, then f is said to be *continuous* provided that $f^{-1}(V)$ is an open set in X for every open set V in Y.

1.3 Definition

(*a*) A collection $\mathfrak{M}$ of subsets of a set X is said to be a *σ-algebra in X* if $\mathfrak{M}$ has the following properties:

(u)

(i) $X \in \mathfrak{M}$. $\Rightarrow \emptyset \in \mathfrak{M}$

(ii) If $A \in \mathfrak{M}$, then $A^c \in \mathfrak{M}$, where A^c is the complement of A relative to X.

(iii) If $A = \bigcup_{n=1}^{\infty} A_n$ and if $A_n \in \mathfrak{M}$ for $n = 1, 2, 3, \ldots$, then $A \in \mathfrak{M}$.

(*b*) If $\mathfrak{M}$ is a σ-algebra in X, then X is called a *measurable space*, and the members of $\mathfrak{M}$ are called the *measurable sets* in X. اندازه پذیر

(*c*) If X is a measurable space, Y is a topological space, and f is a mapping of X into Y, then f is said to be *measurable* provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y.

1.4 Comments on Definition 1.2 The most familiar topological spaces are the *metric spaces*. We shall assume some familiarity with metric spaces but shall give the basic definitions, for the sake of completeness.

A *metric space* is a set X in which a *distance function* (or *metric*) ρ is defined, with the following properties: $\rho : X \times X \to \mathbb{R}$

(*a*) $0 \le \rho(x, y) < \infty$ for all x and $y \in X$.

(*b*) $\rho(x, y) = 0$ if and only if $x = y$.

(*c*) $\rho(x, y) = \rho(y, x)$ for all x and $y \in X$.

(*d*) $\rho(x, y) \le \rho(x, z) + \rho(z, y)$ for all x, y, and $z \in X$.

Property (*d*) is called the *triangle inequality*.

If $x \in X$ and $r \ge 0$, the *open ball* with center at x and radius r is the set $\{y \in X : \rho(x, y) < r\}$.

If X is a metric space and if τ is the collection of all sets $E \subset X$ which are arbitrary unions of open balls, then τ is a topology in X. This is not hard to verify; the intersection property depends on the fact that if $x \in B_1 \cap B_2$, where B_1 and B_2 are open balls, then x is the center of an open ball $B \subset B_1 \cap B_2$. We leave this as an exercise.

For instance, in the real line R^1 a set is open if and only if it is a union of open segments (a, b). In the plane R^2, the open sets are those which are unions of open circular discs.

Another topological space, which we shall encounter frequently, is the extended real line $[-\infty, \infty]$; its topology is defined by declaring the following sets to be open: (a, b), $[-\infty, a)$, $(a, \infty]$, and any union of segments of this type.

1.6 Comments on Definition 1.3 Let $\mathfrak{M}$ be a σ-algebra in a set X. Referring to Properties (i) to (iii) of Definition 1.3(a), we immediately derive the following facts.

(a) Since $\varnothing = X^c$, (i) and (ii) imply that $\varnothing \in \mathfrak{M}$.

(b) Taking $A_{n+1} = A_{n+2} = \cdots = \varnothing$ in (iii), we see that $\boxed{A_1 \cup A_2 \cup \cdots \cup A_n}$ $\in \mathfrak{M}$ if $A_i \in \mathfrak{M}$ for $i = 1, \ldots, n$.

(c) Since

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c \right)^c,$$

قانون دمورگان

$\mathfrak{M}$ is closed under the formation of countable (and also finite) intersections.

(d) Since $A - B = B^c \cap A$, we have $A - B \in \mathfrak{M}$ if $A \in \mathfrak{M}$ and $B \in \mathfrak{M}$.

The prefix σ refers to the fact that (iii) is required to hold for all *countable* unions of members of $\mathfrak{M}$. If (iii) is required for finite unions only, then $\mathfrak{M}$ is called an *algebra* of sets.

1.7 Theorem *Let Y and Z be topological spaces, and let $g: Y \to Z$ be continuous.*

$f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$

(a) *If X is a topological space, if $f: X \to Y$ is continuous, and if $h = g \circ f$, then $h: X \to Z$ is continuous.*

(b) *If X is a measurable space, if $f: X \to Y$ is measurable, and if $h = g \circ f$, then $h: X \to Z$ is measurable.*

Stated informally, continuous functions of continuous functions are continuous; continuous functions of measurable functions are measurable.

PROOF If V is open in Z, then $g^{-1}(V)$ is open in Y, and

$$h^{-1}(V) = f^{-1}(g^{-1}(V)).$$

If f is continuous, it follows that $h^{-1}(V)$ is open, proving (a).

If f is continuous, it follows that $h^{-1}(V)$ is open, proving (a).
If f is measurable, it follows that $h^{-1}(V)$ is measurable, proving (b). ////

$E \subseteq \mathbb{R}$, then $+\infty$ is an upper bound
$-\infty$ " " lower "

E has ~~the~~ sup, since $\begin{cases} E \text{ is bd above, so } \sup E \in \mathbb{R} \\ E \text{ is not bd above, so } \sup E = +\infty \end{cases}$

since $+\infty$ is the only upper bound for E

E is called countable (شمارا) if there exists a 1-1 correspondence between E & $\mathbb{N}$.

دانشجویان باید دسترسی به کتاب آنالیز حقیقی و مختلط (رودین) ترجمه مجید رسولیان داشته باشند.

If $f: X \to Y$ is a function, then

$$\boxed{\begin{array}{l} f^{-1}(C) = \{x \in X : f(x) \in C\} \\ \text{So } x \in f^{-1}(C) \Leftrightarrow f(x) \in C \end{array}}$$

$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

$x \in f^{-1}(A \cup B) \Leftrightarrow f(x) \in A \cup B \Leftrightarrow f(x) \in A \vee f(x) \in B \Leftrightarrow x \in f^{-1}(A) \vee x \in f^{-1}(B) \Leftrightarrow x \in f^{-1}(A) \cup f^{-1}(B)$

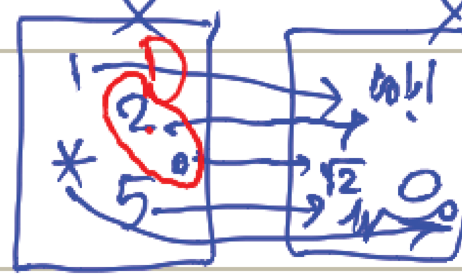
$$\boxed{\begin{array}{l} f(D) = \{f(x) : x \in D\} \\ \text{So } \begin{array}{l} x \in D \Rightarrow f(x) \in f(D) \\ y \in f(D) \Rightarrow \exists x \in D;\ y = f(x) \end{array} \end{array}}$$

$f(A \cap B) \subseteq f(A) \cap f(B)$

$y \in f(A \cap B) \Rightarrow \exists x \in A \cap B;\ y = f(x) \Rightarrow$

$x\in A, x\in B, y=f(x) \Rightarrow y\in f(A), y\in f(B)$

$\Rightarrow y\in f(A)\cap f(B)$.



$f(D)=\{$اباما$, \sqrt{2}\}$

$f(1)=$اباما$\in f(D)$ but $1\notin D$

Recall that a function is ...

تنبیه

$f\subseteq X\times Y$ تابع است هرگاه هیچ دو زوج متمایزی دارای مولفه‌های اول مساوی نباشند

تابع با ۳ چیز مشخص می‌شود. مجموعه آغاز، مجموعه پایان و مجموعه زوج‌های مرتب f

تابع با ۳ چیز دیگر به طور معادل مشخص می‌شود:

ضابطه، D_f، مجموعه پایان، مجموعه آغاز

اما وقتی مثلاً در ریاضی عمومی با تابع‌هایی از $\mathbb{R}$ به $\mathbb{R}$ سروکار داریم

تابع فقط با ضابطه‌اش مشخص می‌شود زیرا از روی ضابطه دامنه به عنوان

بزرگترین زیرمجموعه $\mathbb{R}$ که برای هر x در آن $f(x)$ معنی دارد

قابل مشخص‌سازی است. مانند

$f(x)=\frac{1}{x}$

$\Leftarrow \quad f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$

$\cdot\ x \in f^{-1}(g^{-1}(V)) \Leftrightarrow f(x) \in g^{-1}(V) \Leftrightarrow \underbrace{g(f(x))}_{(g \circ f)(x)} \in V \Leftrightarrow x \in (g \circ f)^{-1}(V)$

1.8 Theorem *Let u and v be real measurable functions on a measurable space X, let Φ be a continuous mapping of the plane into a topological space Y, and define*

$$h(x) = \Phi(u(x), v(x))$$

for $x \in X$. Then $h: X \to Y$ is measurable.

$h(x) = \Phi(u(x), v(x))$

PROOF Put $f(x) = (u(x), v(x))$. Then f maps X into the plane. Since $h = \Phi \circ f$, Theorem 1.7 shows that it is enough to prove the measurability of f.

If R is any open rectangle in the plane, with sides parallel to the axes, then R is the cartesian product of two segments I_1 and I_2, and $\Phi \circ f$

$$\underline{f^{-1}(R) = u^{-1}(I_1) \cap v^{-1}(I_2),} \quad 1$$

which is measurable, by our assumption on u and v. Every open set V in the plane is a countable union of such rectangles R_i, and since

$$f^{-1}(V) = f^{-1}\left(\bigcup_{i=1}^{\infty} R_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(R_i),$$

$f^{-1}(V)$ is measurable. ////

1.9 Let X be a measurable space. The following propositions are corollaries of Theorems 1.7 and 1.8:

$u(x) + iv(x)$

(*a*) *If $f = u + iv$, where u and v are real measurable functions on X, then f is a complex measurable function on X.*

$f(x) = \Phi((u(x), v(x)))$

This follows from Theorem 1.8, with $\Phi(z) = z$.

(*b*) *If $f = u + iv$ is a complex measurable function on X, then u, v, and $|f|$ are real measurable functions on X.*

$u(x) = \mathrm{Re}(f(x))$

This follows from Theorem 1.7, with $g(z) = \mathrm{Re}(z)$, $\mathrm{Im}(z)$, and $|z|$.

(*c*) *If f and g are complex measurable functions on X, then so are $f + g$ and fg.*

For real f and g this follows from Theorem 1.8, with

2

$$\Phi(s, t) = s + t$$

and $\Phi(s, t) = st$. The complex case then follows from (*a*) and (*b*).

(*d*) *If E is a measurable set in X and if*

3

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

then χ_E is a measurable function then χ_E is a measurable function.

This is obvious. We call χ_E the *characteristic function* of the set E. The letter χ will be reserved for characteristic functions throughout this book.

(*e*) *If f is a complex measurable function on X, there is a complex measurable function α on X such that $|\alpha| = 1$ and $f = \alpha|f|$.*

PROOF Let $E = \{x : f(x) = 0\}$, let Y be the complex plane with the origin removed, define $\varphi(z) = z/|z|$ for $z \in Y$, and put

$$\alpha(x) = \varphi(f(x) + \chi_E(x)) \qquad (x \in X).$$

If $x \in E$, $\alpha(x) = 1$; if $x \notin E$, $\alpha(x) = f(x)/|f(x)|$. Since φ is continuous on Y and since E is measurable (why?), the measurability of α follows from (*c*), (*d*), and Theorem 1.7. ////

We now show that σ-algebras exist in great profusion.

1.10 Theorem *If $\mathscr{F}$ is any collection of subsets of X, there exists a smallest σ-algebra $\mathfrak{M}^*$ in X such that $\mathscr{F} \subset \mathfrak{M}^*$.*

This $\mathfrak{M}^*$ is sometimes called the σ-algebra *generated* by $\mathscr{F}$.

PROOF Let Ω be the family of all σ-algebras $\mathfrak{M}$ in X which contain $\mathscr{F}$. Since the collection of all subsets of X is such a σ-algebra, Ω is not empty. Let $\mathfrak{M}^*$ be the intersection of all $\mathfrak{M} \in \Omega$. It is clear that $\mathscr{F} \subset \mathfrak{M}^*$ and that $\mathfrak{M}^*$ lies in every σ-algebra in X which contains $\mathscr{F}$. To complete the proof, we have to show that $\mathfrak{M}^*$ is itself a σ-algebra.

If $A_n \in \mathfrak{M}^*$ for $n = 1, 2, 3, \ldots$, and if $\mathfrak{M} \in \Omega$, then $A_n \in \mathfrak{M}$, so $\bigcup A_n \in \mathfrak{M}$, since $\mathfrak{M}$ is a σ-algebra. Since $\bigcup A_n \in \mathfrak{M}$ for *every* $\mathfrak{M} \in \Omega$, we conclude that $\bigcup A_n \in \mathfrak{M}^*$. The other two defining properties of a σ-algebra are verified in the same manner. ////

1.11 Borel Sets Let X be a topological space. By Theorem 1.10, there exists a smallest σ-algebra $\mathscr{B}$ in X such that every open set in X belongs to $\mathscr{B}$. The members of $\mathscr{B}$ are called the *Borel sets* of X.

In particular, closed sets are Borel sets (being, by definition, the complements of open sets), and so are all countable unions of closed sets and all countable intersections of open sets. These last two are called F_σ's and G_δ's, respectively, and play a considerable role. The notation is due to Hausdorff. The letters F and

G were used for closed and open sets, respectively, and σ refers to union (*Summe*), δ to intersection (*Durchschnitt*). For example, every half-open interval $[a, b)$ is a G_δ and an F_σ in R^1.

Since $\mathscr{B}$ is a σ-algebra, we may now regard X as a measurable space, with the Borel sets playing the role of the measurable sets; more concisely, we consider the measurable space $(X, \mathscr{B})$. If $f: X \to Y$ is a continuous mapping of X, where Y is any topological space, then it is evident from the definitions that $f^{-1}(V) \in \mathscr{B}$ for every open set V in Y. In other words, *every continuous mapping of X is Borel measurable.*

Borel measurable mappings are often called *Borel mappings*, or *Borel functions.*

1.12 Theorem *Suppose $\mathfrak{M}$ is a σ-algebra in X, and Y is a topological space. Let f map X into Y.*

(*a*) *If Ω is the collection of all sets $E \subset Y$ such that $f^{-1}(E) \in \mathfrak{M}$, then Ω is a σ-algebra in Y.*
(*b*) *If f is measurable and E is a Borel set in Y, then $f^{-1}(E) \in \mathfrak{M}$.*
(*c*) *If $Y = [-\infty, \infty]$ and $f^{-1}((\alpha, \infty]) \in \mathfrak{M}$ for every real α, then f is measurable.*
(*d*) *If f is measurable, if Z is a topological space, if $g: Y \to Z$ is a Borel mapping, and if $h = g \circ f$, then $h: X \to Z$ is measurable.*

Part (*c*) is a frequently used criterion for the measurability of real-valued functions. (See also Exercise 3.) Note that (*d*) generalizes Theorem 1.7(*b*).

PROOF (*a*) follows from the relations

$$f^{-1}(Y) = X,$$

$$f^{-1}(Y - A) = X - f^{-1}(A),$$

and
$$f^{-1}(A_1 \cup A_2 \cup \cdots) = f^{-1}(A_1) \cup f^{-1}(A_2) \cup \cdots.$$

To prove (*b*), let Ω be as in (*a*); the measurability of f implies that Ω contains all open sets in Y, and since Ω is a σ-algebra, Ω contains all Borel sets in Y.

To prove (*c*), let Ω be the collection of all $E \subset [-\infty, \infty]$ such that $f^{-1}(E) \in \mathfrak{M}$. Choose a real number α, and choose $\alpha_n < \alpha$ so that $\alpha_n \to \alpha$ as $n \to \infty$. Since $(\alpha_n, \infty] \in \Omega$ for each n, since

$$[-\infty, \alpha) = \bigcup_{n=1}^{\infty} [-\infty, \alpha_n] = \bigcup_{n=1}^{\infty} (\alpha_n, \infty]^c,$$

and since (*a*) shows that Ω is a σ-algebra, we see that $[-\infty, \alpha) \in \Omega$. The same and since (a) shows that Ω is a σ-algebra, we see that $[-\infty, \alpha) \in \Omega$. The same is then true of

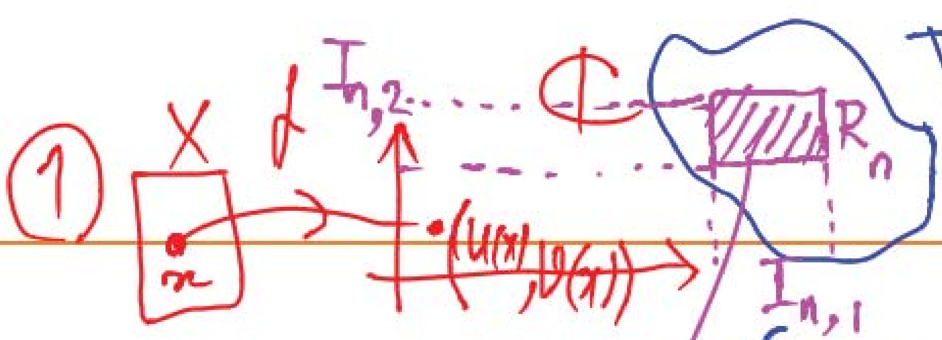
$$(\alpha, \beta) = [-\infty, \beta) \cap (\alpha, \infty].$$

Since every open set in $[-\infty, \infty]$ is a countable union of segments of the above types, Ω contains every open set. Thus f is measurable.

To prove (d), let $V \subset Z$ be open. Then $g^{-1}(V)$ is a Borel set of Y, and since

$$h^{-1}(V) = f^{-1}(g^{-1}(V)),$$

(b) shows that $h^{-1}(V) \in \mathfrak{M}$. ////



$$V^{\text{open}} = \dot{\bigcup_{n=1}^{\infty}} R_n$$

هر مجموعه باز در صفحه اجتماع شمارایی از مستطیل‌های باز جدا از هم می‌باشد

$$R_n = I_{n,1} \times I_{n,2}$$

$$f^{-1}(V) = f^{-1}\Big(\dot{\bigcup_{n=1}^{\infty}} R_n\Big) = \dot{\bigcup_{n=1}^{\infty}} f^{-1}(R_n) = \dot{\bigcup_{n=1}^{\infty}} \Big(\underbrace{u^{-1}(I_{n,1})}_{\text{meas}} \cap \underbrace{v^{-1}(I_{n,2})}_{\text{meas}}\Big) \in \mathfrak{M}$$

open

$f^{-1}(R) = u^{-1}(I_1) \cap v^{-1}(I_2)$ where $R = I_1 \times I_2$

$u, v : X \longrightarrow \mathbb{R}$ are meas. func.

$$x \in f^{-1}(R) \Leftrightarrow \underset{(u(x), v(x))}{f(x)} \in R = I_1 \times I_2 \Leftrightarrow \underbrace{u(x) \in I_1}_{x \in u^{-1}(I_1)}, \underbrace{v(x) \in I_2}_{x \in v^{-1}(I_2)} \Leftrightarrow x \in u^{-1}(I_1) \cap v^{-1}(I_2)$$

② $\Phi(t,s)=t+s$

~~$\Phi(f,g)=f+g$, f,g are complex valued~~ :

$\Phi(\mathrm{Re}f, \mathrm{Re}g) = \underbrace{\mathrm{Re}f + \mathrm{Re}g}_{\mathrm{Re}(f+g)}$ is meas, by Th 1.8

$\Phi(\mathrm{Im}f, \mathrm{Im}g) = \underbrace{\mathrm{Im}f + \mathrm{Im}g}_{\mathrm{Im}(f+g)}$ is " " " .

$\Rightarrow f+g = \mathrm{Re}(f+g) + i\,\mathrm{Im}(f+g)$ is meas, by Cor. 1.9(a).

③ E is a meas set iff $\chi_E(x)=\begin{cases}1 & x\in E\\ 0 & x\notin E\end{cases}$ is a meas func.

($\Leftarrow$) Let χ_E be meas. Since $E=\chi_E^{-1}(\{1\})$ (open), E is meas.

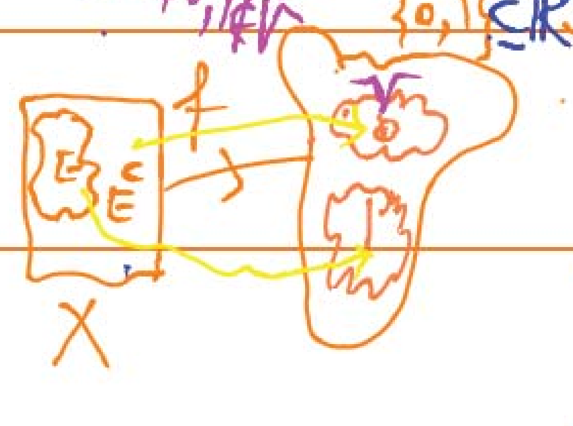
($\Rightarrow$) Let E be meas. Since $\chi^{-1}(V)=\begin{cases}E^c & 0\in V, 1\notin V\\ E & 0\notin V, 1\in V\\ X & 0\in V, 1\in V\\ \emptyset & 0\notin V, 1\notin V\end{cases}$ $\tau=\{\emptyset,\{0\},\{1\},\{0,1\}\}$

E open

is meas; we conclude that χ_E is meas.

(X,d)

$\tau_d=\{V\subseteq X : V \text{ is open}\}$ the top induced by d



(X,τ)

$\tau_Y=\{V\cap Y : V\in\tau\}$ is called the subspace top on Y.

1.8 Theorem *Let u and v be real measurable functions on a measurable space X, let Φ be a continuous mapping of the plane into a topological space Y, and define*

$$h(x) = \Phi(u(x),\ v(x))$$

for $x \in X$. Then $h: X \to Y$ is measurable.

$h(x) = \Phi(u(x), v(x))$

PROOF Put $f(x) = (u(x),\ v(x))$. Then f maps X into the plane. Since $h = \Phi \circ f$, Theorem 1.7 shows that it is enough to prove the measurability of f.

$\Phi \circ f$

If R is any open rectangle in the plane, with sides parallel to the axes, then R is the cartesian product of two segments I_1 and I_2, and

$$\underline{f^{-1}(R) = u^{-1}(I_1) \cap v^{-1}(I_2),} \quad 1$$

which is measurable, by our assumption on u and v. Every open set V in the plane is a countable union of such rectangles R_i, and since

$$f^{-1}(V) = f^{-1}\left(\bigcup_{i=1}^{\infty} R_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(R_i),$$

$f^{-1}(V)$ is measurable. ////

1.9 Let X be a measurable space. The following propositions are corollaries of Theorems 1.7 and 1.8:

$u(x) + i v(x)$

(a) *If $f = u + iv$, where u and v are real measurable functions on X, then f is a complex measurable function on X.*

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This follows from Theorem 1.8, with $\Phi(z) = z$.

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This follows from Theorem 1.7, with $g(z) = \mathrm{Re}\,(z)$, $\mathrm{Im}\,(z)$, and $|z|$.

(c) *If f and g are complex measurable functions on X, then so are $f + g$ and fg.*

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2

$$\Phi(s,\ t) = s + t$$

and $\Phi(s, t) = st$. The complex case then follows from (a) and (b).

(d) *If E is a measurable set in X and if*

3

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

then χ_E is a measurable function then χ_E is a measurable function.

This is obvious. We call χ_E the *characteristic function* of the set E. The letter χ will be reserved for characteristic functions throughout this book.

(e) *If f is a complex measurable function on X, there is a complex measurable function α on X such that $|\alpha| = 1$ and $f = \alpha |f|$.*

$X - f^{-1}(\underbrace{X-\{0\}}_{\text{open}}) = f^{-1}(\{0\})$

$\alpha(x) = \begin{cases} \frac{f(x)}{|f(x)|} & f(x) \neq 0 \\ 1 & f(x) = 0 \end{cases}$

PROOF Let $E = \{x : f(x) = 0\}$, let Y be the complex plane with the origin removed, define $\varphi(z) = z/|z|$ for $z \in Y$, and put

meas

$$\alpha(x) = \varphi(f(x) + \chi_E(x)) \qquad (x \in X).$$

cts meas

$\begin{cases} \varphi(1) = 1 & x \in E \\ \frac{f(x)}{|f(x)|} & x \notin E \end{cases}$

If $x \in E$, $\alpha(x) = 1$; if $x \notin E$, $\alpha(x) = f(x)/|f(x)|$. Since φ is continuous on Y and since E is measurable (why?), the measurability of α follows from (c), (d), and Theorem 1.7. ////

$z = e^{i\theta}|z|$

We now show that σ-algebras exist in great profusion.

$f(x) = \alpha(x)|f(x)|$

1.10 Theorem *If $\mathscr{F}$ is any collection of subsets of X, there exists a smallest σ-algebra $\mathfrak{M}^*$ in X such that $\mathscr{F} \subset \mathfrak{M}^*$.*

4

This $\mathfrak{M}^*$ is sometimes called the σ-algebra *generated* by $\mathscr{F}$.

PROOF Let Ω be the family of all σ-algebras $\mathfrak{M}$ in X which contain $\mathscr{F}$. Since the collection of all subsets of X is such a σ-algebra, Ω is not empty. Let $\mathfrak{M}^*$ be the intersection of all $\mathfrak{M} \in \Omega$. It is clear that $\mathscr{F} \subset \mathfrak{M}^*$ and that $\mathfrak{M}^*$ lies in every σ-algebra in X which contains $\mathscr{F}$. To complete the proof, we have to show that $\mathfrak{M}^*$ is itself a σ-algebra.

If $A_n \in \mathfrak{M}^*$ for $n = 1, 2, 3, \ldots$, and if $\mathfrak{M} \in \Omega$, then $A_n \in \mathfrak{M}$, so $\bigcup A_n \in \mathfrak{M}$, since $\mathfrak{M}$ is a σ-algebra. Since $\bigcup A_n \in \mathfrak{M}$ for *every* $\mathfrak{M} \in \Omega$, we conclude that $\bigcup A_n \in \mathfrak{M}^*$. The other two defining properties of a σ-algebra are verified in the same manner. ////

1.11 Borel Sets Let X be a topological space. By Theorem 1.10, there exists a smallest σ-algebra $\mathscr{B}$ in X such that every open set in X belongs to $\mathscr{B}$. The members of $\mathscr{B}$ are called the *Borel sets* of X.

In particular, closed sets are Borel sets (being, by definition, the complements of open sets), and so are all countable unions of closed sets and all countable intersections of open sets. These last two are called F_σ's and G_δ's, respectively, and play a considerable role. The notation is due to Hausdorff. The letters F and

G were used for closed and open sets, respectively, and σ refers to union (*Summe*), δ to intersection (*Durchschnitt*). For example, every half-open interval $[a, b)$ is a G_δ and an F_σ in R^1.

Since $\mathscr{B}$ is a σ-algebra, we may now regard X as a measurable space, with the Borel sets playing the role of the measurable sets; more concisely, we consider the measurable space $(X, \mathscr{B})$. If $f: X \to Y$ is a continuous mapping of X, where Y is any topological space, then it is evident from the definitions that $f^{-1}(V) \in \mathscr{B}$ for every open set V in Y. In other words, *every continuous mapping of X is Borel measurable.*

Borel measurable mappings are often called *Borel mappings*, or *Borel functions.*

1.12 Theorem *Suppose $\mathfrak{M}$ is a σ-algebra in X, and Y is a topological space. Let f map X into Y.*

$X \xrightarrow{f} Y$

(*a*) *If Ω is the collection of all sets $E \subset Y$ such that $f^{-1}(E) \in \mathfrak{M}$, then Ω is a σ-algebra in Y.*
(*b*) *If f is measurable and E is a Borel set in Y, then $f^{-1}(E) \in \mathfrak{M}$.*
(*c*) *If $Y = [-\infty, \infty]$ and $f^{-1}((\alpha, \infty]) \in \mathfrak{M}$ for every real α, then f is measurable.*
(*d*) *If f is measurable, if Z is a topological space, if $g: Y \to Z$ is a Borel mapping, and if $h = g \circ f$, then $h: X \to Z$ is measurable.*

Part (*c*) is a frequently used criterion for the measurability of real-valued functions. (See also Exercise 3.) Note that (*d*) generalizes Theorem 1.7(*b*).

$A^c \in \Omega \Leftarrow f^{-1}(A^c) = f^{-1}(A)^c \in \mathfrak{M} \Leftarrow f^{-1}(A) \in \mathfrak{M}$

PROOF (*a*) follows from the relations

$$f^{-1}(Y) = X,$$

$$f^{-1}(Y - A) = X - f^{-1}(A), \qquad \Uparrow\ A \in \Omega$$

and
$$f^{-1}(A_1 \cup A_2 \cup \cdots) = f^{-1}(A_1) \cup f^{-1}(A_2) \cup \cdots.$$

$f^{-1}(E) \in \mathfrak{M}$

To prove (*b*), let Ω be as in (*a*); the measurability of f implies that Ω contains all open sets in Y, and since Ω is a σ-algebra, Ω contains all Borel sets in Y. $\tau_Y \subseteq \Omega \Rightarrow \langle \tau_Y \rangle \subseteq \Omega \Leftrightarrow \forall E$ Borel, $E \in \Omega$

To prove (*c*), let Ω be the collection of all $E \subset [-\infty, \infty]$ such that $f^{-1}(E) \in \mathfrak{M}$. Choose a real number α, and choose $\alpha_n < \alpha$ so that $\alpha_n \to \alpha$ as $n \to \infty$. Since $(\alpha_n, \infty] \in \Omega$ for each n, since

$$[-\infty, \alpha) = \bigcup_{n=1}^{\infty} [-\infty, \alpha_n] = \bigcup_{n=1}^{\infty} (\alpha_n, \infty]^c,$$

and since (*a*) shows that Ω is a σ-algebra, we see that $[-\infty, \alpha) \in \Omega$. The same and since (a) shows that Ω is a σ-algebra, we see that $[-\infty, \alpha) \in \Omega$. The same is then true of

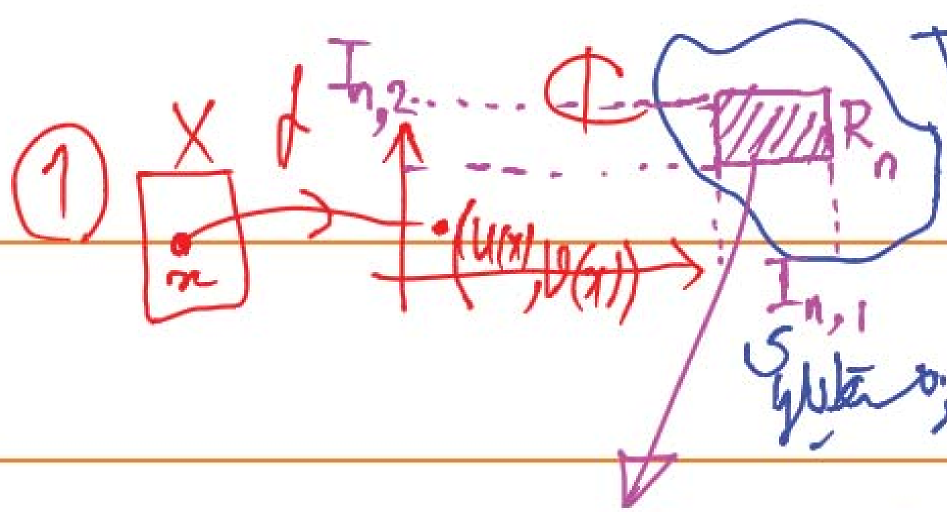
$$(\alpha, \beta) = [-\infty, \beta) \cap (\alpha, \infty].$$

Since every open set in $[-\infty, \infty]$ is a countable union of segments of the above types, Ω contains every open set. Thus f is measurable.

To prove (d), let $V \subset Z$ be open. Then $g^{-1}(V)$ is a Borel set of Y, and since

$$h^{-1}(V) = f^{-1}(g^{-1}(V)),$$

(b) shows that $h^{-1}(V) \in \mathfrak{M}$. ////



V^{open}

$V^{open} = \dot{\bigcup}_{n=1}^{\infty} R_n$

(هر باز در صفحه مختلط، اجتماع شمارایی از مستطیل‌های
باز جدا از هم می‌باشد)

$R_n = I_{n,1} \times I_{n,2}$

$$f^{-1}(V) = f^{-1}\Big(\dot{\bigcup_{n=1}^{\infty}} R_n\Big) = \dot{\bigcup_{n=1}^{\infty}} f^{-1}(R_n) = \dot{\bigcup_{n=1}^{\infty}} \Big(\underbrace{u^{-1}(\overset{open}{I_{n,1}})}_{meas} \cap \underbrace{v^{-1}(I_{n,2})}_{meas}\Big) \in \mathfrak{M}$$

$$f^{-1}(R) = u^{-1}(I_1) \cap v^{-1}(I_2) \quad \text{where } R = I_1 \times I_2$$

$u, v : X \longrightarrow \mathbb{R}$ are meas. func.

$$x \in f^{-1}(R) \Leftrightarrow \underset{\substack{\| \\ (u(x), v(x))}}{f(x)} \in R = I_1 \times I_2 \Leftrightarrow \underbrace{u(x) \in I_1}_{x \in u^{-1}(I_1)}, \underbrace{v(x) \in I_2}_{x \in v^{-1}(I_2)} \Leftrightarrow x \in u^{-1}(I_1) \cap v^{-1}(I_2)$$

(2) $\Phi(t,s)=t+s$

$\cancel{\Phi(f,g)=f+g}$, $\cancel{f,g \text{ are complex valued}}$

$\Phi(\mathrm{Re}f, \mathrm{Re}g) = \underbrace{\mathrm{Re}f+\mathrm{Re}g}_{\mathrm{Re}(f+g)}$ is meas, by Th 1.8

$\Phi(\mathrm{Im}f, \mathrm{Im}g) = \underbrace{\mathrm{Im}f+\mathrm{Im}g}_{\mathrm{Im}(f+g)}$ is " " ".

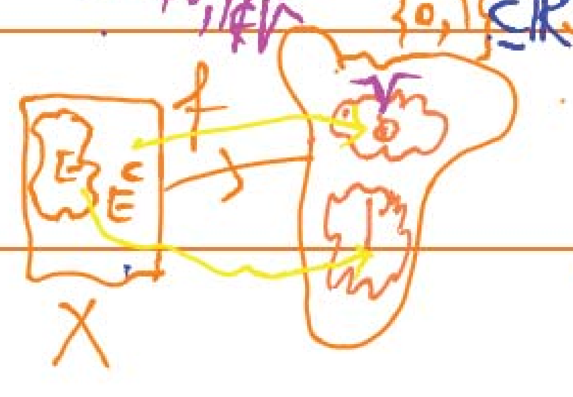
$\Rightarrow f+g = \mathrm{Re}(f+g)+i\,\mathrm{Im}(f+g)$ is meas, by Cor. 1.9(a).

(3) E is a meas set iff $\chi_E(x)=\begin{cases}1 & x\in E\\ 0 & x\notin E\end{cases}$ is a meas func.

($\Leftarrow$) Let χ_E be meas. Since $E=\chi_E^{-1}(\{1\})$ (open), E is meas.

($\Rightarrow$) Let E be meas. Since $\chi^{-1}(V)=\begin{cases}E^c & 0\in V, 1\notin V\\ E & 0\notin V, 1\in V\\ X & 0\in V, 1\in V\\ \emptyset & 0\notin V, 1\notin V\end{cases}$ (E open; $V \in \{\emptyset,\{0\},\{1\},\{0,1\}\}$, $V \subseteq \{0,1\}$) is meas; we conclude that χ_E is meas.

$\tau_d=\{V\subseteq X : V \text{ is open}\}$ the top induced by d



$\tau_Y=\{V\cap Y : V\in\tau\}$ is called the subspace top on Y.

ادله باید به صورت ... تعریف

فرض

یکی از مراحل قبلی بود

(4) $\mathcal{F} \subseteq 2^X = P(X) \Rightarrow \underbrace{\bigcap_{\mathcal{F} \subseteq \mathcal{M} \text{ is } \sigma\text{-alg}} \mathcal{M}}_{} \overset{\langle \mathcal{F} \rangle}{\overset{\shortparallel}{}}$ is a σ-alg

and is the smallest σ-alg containing $\mathcal{F}$.

کوچکترین

<u>proof</u>: Let $A \in \bigcap \mathcal{M}$. Then $A \in \mathcal{M}$ for all $\mathcal{M}$.

So $A^c \in \mathcal{M}$ for all $\mathcal{M}$. Hence $A^c \in \bigcap \mathcal{M}$

The rest is clear.

If $\mathcal{N}$ is σ-alg & $\mathcal{F} \subseteq \mathcal{N}$ then

$\mathcal{N} \in \{\mathcal{M} \mid \mathcal{M} \text{ is } \sigma\text{-alg} \ \& \ \mathcal{F} \subseteq \mathcal{M}\}$. So $\bigcap_{\mathcal{F} \subseteq \mathcal{M} \ \sigma\text{-alg}} \mathcal{M} \subseteq \mathcal{N}$

If $\mathcal{F} = \tau$ (a top on X), then $\langle \tau \rangle$ $\overset{\langle \mathcal{F} \rangle}{}$ $\supseteq$ the σ-alg of Borel sets. Example: (a,b), $[a,b]$, $(a,b] = \bigcap_{n=1}^{\infty} (a, b+\frac{1}{n})$ in $\mathbb{R}$

In particular $\bigcap\limits_{n=1}^{\infty} \underbrace{G_n}_{\text{open}}$, $\bigcup\limits_{n=1}^{\infty} \underbrace{F_n}_{\text{closed}} \in \langle \tau \rangle$

a G_δ-set   an F_σ-set

Recall that a basis for a to is a set $\mathcal{B}$ of subsets of X such that each open set can be written as a union of elements of $\mathcal{B}$.

Example: $\{N_r(x) : x \in \mathbb{R}, r \in \mathbb{R}^{>0}\}$ is a basis for Euclidean top on $\mathbb{R}$.

Also $\{I_1 \times I_2 : I_1 \& I_2 \text{ are open intervals}\}$ is a basis for the Euclidean top on $\mathbb{R}^2$.

In addition, $\{(a,b) : a,b \in \mathbb{R}\} \cup \{(a,\infty] : a \in \mathbb{R}\}$

$\cup \{[-\infty, b) : b \in \mathbb{R}\}$ is a basis for a top on $[-\infty, +\infty]$

1.13 Definition Let $\{a_n\}$ be a sequence in $[-\infty, \infty]$, and put

$$b_k = \sup \{a_k, a_{k+1}, a_{k+2}, \ldots\} \qquad (k = 1, 2, 3, \ldots) \tag{1}$$

and

$$\beta = \inf \{b_1, b_2, b_3, \ldots\}. \tag{2}$$

We call β the *upper limit* of $\{a_n\}$, and write

$$\beta = \limsup_{n \to \infty} a_n. \tag{3}$$

The following properties are easily verified: First, $b_1 \geq b_2 \geq b_3 \geq \cdots$, so that $b_k \to \beta$ as $k \to \infty$; secondly, there is a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that $a_{n_i} \to \beta$ as $i \to \infty$, and β is the largest number with this property.

The *lower limit* is defined analogously: simply interchange sup and inf in (1) and (2). Note that

$$\liminf_{n \to \infty} a_n = -\limsup_{n \to \infty} (-a_n). \tag{4}$$

If $\{a_n\}$ converges, then evidently

$$\limsup_{n \to \infty} a_n = \liminf_{n \to \infty} a_n = \lim_{n \to \infty} a_n. \quad \text{(why?)} \tag{5}$$

Suppose $\{f_n\}$ is a sequence of extended-real functions on a set X. Then $\sup_n f_n$ and $\limsup_{n \to \infty} f_n$ are the functions defined on X by

$$\left(\sup_n f_n\right)(x) = \sup_n (f_n(x)), \tag{6}$$

$$\left(\limsup_{n \to \infty} f_n\right)(x) = \limsup_{n \to \infty} (f_n(x)). \tag{7}$$

If

$$f(x) = \lim_{n \to \infty} f_n(x), \tag{8}$$

the limit being assumed to exist at every $x \in X$, then we call f the *pointwise limit* of the sequence $\{f_n\}$.

1.14 Theorem *If $f_n : X \to [-\infty, \infty]$ is measurable, for $n = 1, 2, 3, \ldots$, and*

$$g = \sup_{n \geq 1} f_n, \qquad h = \limsup_{n \to \infty} f_n,$$

then g and h are measurable.

PROOF $g^{-1}((\alpha, \infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha, \infty])$. Hence Theorem 1.12(*c*) implies that g is measurable. The same result holds of course with inf in place of sup, and since

$$h = \inf_{k \geq 1} \left\{ \sup_{i \geq k} f_i \right\},$$

it follows that h is measurable. ////

Corollaries

(*a*) *The limit of every pointwise convergent sequence of complex measurable functions is measurable.*

(*b*) *If f and g are measurable* (*with range in* $[-\infty, \infty]$), *then so are* $\max\{f, g\}$ *and* $\min\{f, g\}$. *In particular, this is true of the functions*

$$f^+ = \max\{f, 0\} \quad \text{and} \quad f^- = -\min\{f, 0\}.$$

$= \sup\{f, 0, 0, \dots\}$

1.15 The above functions f^+ and f^- are called the *positive and negative parts* of f. We have $|f| = f^+ + f^-$ and $f = f^+ - f^-$, a standard representation of f as a difference of two nonnegative functions, with a certain minimality property:

$f(x) = f^+(x) - f^-(x)\ \forall x$

Proposition *If $f = g - h$, $g \geq 0$, and $h \geq 0$, then $f^+ \leq g$ and $f^- \leq h$.*

PROOF $f \leq g$ and $0 \leq g$ clearly implies $\boxed{\max\{f, 0\} \leq g.}$ ////

Let $x \in D_f = D_g = D_h$. (i) If $f(x) \geq 0$, then $\max\{f(x), 0\} = f(x) \leq g(x)$. (ii) If $f(x) < 0$, then $\max\{f(x), 0\} = 0 \leq g(x)$

Simple Functions

1.16 Definition A complex function s on a measurable space X whose range consists of only finitely many points will be called a *simple function*. Among these are the nonnegative simple functions, whose range is a finite subset of $[0, \infty)$. Note that we explicitly exclude ∞ from the values of a simple function.

If $\alpha_1, \dots, \alpha_n$ are the distinct values of a simple function s, and if we set $A_i = \{x : s(x) = \alpha_i\}$, then clearly

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i},$$

where χ_{A_i} is the characteristic function of A_i, as defined in Sec. 1.9(*d*).

It is also clear that s is measurable if and only if each of the sets A_i is measurable.

non-negative

1.17 Theorem *Let $f\colon X \to [0, \infty]$ be measurable. There exist simple measurable functions s_n on X such that*

Pointwise Free to read

(a) $0 \le s_1 \le s_2 \le \cdots \le f$.
(b) $s_n(x) \to f(x)$ *as* $n \to \infty$, *for every* $x \in X$.

$s_n \xrightarrow{P} f$

Proof Put $\delta_n = 2^{-n}$. To each positive integer n and each real number t corresponds a unique integer $k = k_n(t)$ that satisfies $k\delta_n \le t < (k+1)\delta_n$. Define

$$\varphi_n(t) = \begin{cases} k_n(t)\delta_n & \text{if } 0 \le t < n \\ n & \text{if } n \le t \le \infty. \end{cases} \tag{1}$$

Each φ_n is then a Borel function on $[0, \infty]$,

$$t - \delta_n < \varphi_n(t) \le t \qquad \text{if } 0 \le t \le n, \tag{2}$$

$0 \le \varphi_1 \le \varphi_2 \le \cdots \le t$, and $\varphi_n(t) \to t$ as $n \to \infty$, for every $t \in [0, \infty]$. It follows that the functions

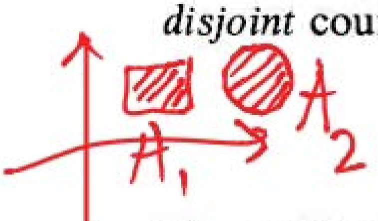
$$s_n = \varphi_n \circ f \tag{3}$$

satisfy (*a*) and (*b*); they are measurable, by Theorem 1.12(*d*). ////

Elementary Properties of Measures

1.18 Definition

(*a*) A *positive measure* is a function μ, defined on a σ-algebra $\mathfrak{M}$, whose range is in $[0, \infty]$ and which is *countably additive*. This means that if $\{A_i\}$ is a *disjoint* countable collection of members of $\mathfrak{M}$, then



$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i). \tag{1}$$

disjoint

To avoid trivialities, we shall also assume that $\mu(A) < \infty$ for at least one $A \in \mathfrak{M}$.

equivalently: $\mu(\emptyset) = 0$

(*b*) A *measure space* is a measurable space which has a positive measure defined on the σ-algebra of its measurable sets.

(*c*) A *complex measure* is a complex-valued countably additive function defined on a σ-algebra.

Note: What we have called a *positive measure* is frequently just called a *measure*; we add the word "positive" for emphasis. If $\mu(E) = 0$ for every $E \in \mathfrak{M}$, then μ is a positive measure, by our definition. The value ∞ is admissible for a positive measure; but when we talk of a *complex* measure μ, it is understood that $\mu(E)$ is a complex number, for every $E \in \mathfrak{M}$. The *real measures* form a subclass of the complex ones, of course.

1.19 Theorem *Let μ be a positive measure on a σ-algebra $\mathfrak{M}$. Then*

$\dot\cup\,\emptyset\,\dot\cup\,\emptyset$ $\qquad$ $+\mu(\emptyset)+\mu(\emptyset)+\cdots$

(*a*) $\mu(\emptyset) = 0$.

(*b*) $\mu(A_1 \dot\cup \cdots \dot\cup A_n) = \mu(A_1) + \cdots + \mu(A_n)$ *if* $A_1, \ldots, A_n$ *are pairwise disjoint members of* $\mathfrak{M}$.

$\subset =$

(*c*) $A \subset B$ *implies* $\mu(A) \leq \mu(B)$ *if* $A \in \mathfrak{M}$, $B \in \mathfrak{M}$.

Example

(*d*) $\mu(A_n) \to \mu(A)$ *as* $n \to \infty$ *if* $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathfrak{M}$, *and*

$A_n = [0, 1-\frac{1}{n}] \subseteq A_{n+1}$ $\qquad$ $B = A \dot\cup (B-A)$

$$A_1 \subset A_2 \subset A_3 \subset \cdots.$$

$0 \quad 1-\frac{1}{n}$ $\qquad$ $\mu(B) = \mu(A) + \underbrace{\mu(B-A)}_{\geq 0}$

(*e*) $\mu(A_n) \to \mu(A)$ *as* $n \to \infty$ *if* $A = \bigcap_{n=1}^{\infty} A_n$, $A_n \in \mathfrak{M}$,

$A = \bigcup_{n=1}^{\infty} A_n = [0,1)$ $\qquad$ $\Downarrow$

$$A_1 \supset A_2 \supset A_3 \supset \cdots,$$

$\mu(A_n) = 1-\frac{1}{n} \to 1 = \mu(A)$ $\qquad$ $\underbrace{\mu(B)}_{+\infty} - \underbrace{\mu(A)}_{+\infty} = \mu(B-A)$

and $\mu(A_1)$ *is finite.*

As the proof will show, these properties, with the exception of (*c*), also hold for complex measures; (*b*) is called *finite additivity*; (*c*) is called *monotonicity*.

$\sum_{i=2}^{\infty} \mu(A_i) = 0 \Rightarrow \mu(A_i) = 0 \ \forall i \geq 2$

PROOF

$\geq 0 \qquad \mu(\emptyset) = 0$

(*a*) Take $A \in \mathfrak{M}$ so that $\mu(A) < \infty$, and take $A_1 = A$ and $A_2 = A_3 = \cdots = \emptyset$ in 1.18(1).

$\mu(A) = \mu(\dot\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i) = \mu(A) + \sum_{i=2}^{\infty} \mu(A_i)$

(*b*) Take $A_{n+1} = A_{n+2} = \cdots = \emptyset$ in 1.18(1).

(*c*) Since $B = A \cup (B - A)$ and $A \cap (B - A) = \emptyset$, we see that (*b*) implies $\mu(B) = \mu(A) + \mu(B - A) \geq \mu(A)$.

(*d*) Put $B_1 = A_1$, and put $B_n = A_n - A_{n-1}$ for $n = 2, 3, 4, \ldots$. Then $B_n \in \mathfrak{M}$, $B_i \cap B_j = \emptyset$ if $i \neq j$, $A_n = B_1 \dot\cup \cdots \dot\cup B_n$, and $A = \dot\bigcup_{i=1}^{\infty} B_i$. Hence

(4)

$$\mu(A_n) = \sum_{i=1}^{n} \mu(B_i) \quad \text{and} \quad \mu(A) = \sum_{i=1}^{\infty} \mu(B_i).$$

Now (*d*) follows, by the definition of the sum of an infinite series.

(*e*) Put $C_n = A_1 - A_n$. Then $C_1 \subset C_2 \subset C_3 \subset \cdots$,

$$\mu(C_n) = \mu(A_1) - \mu(A_n),$$

X $\qquad$ $A_1 - A = \bigcup C_n$, and so (*d*) shows that

$$\mu(A_1) - \mu(A) = \mu(A_1 - A) = \lim_{n\to\infty} \mu(C_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n).$$

This implies (e). ////

1.20 Examples The construction of interesting measure spaces requires some labor, as we shall see. However, a few simple-minded examples can be given immediately:

(a) For any $E \subset X$, where X is any set, define $\mu(E) = \infty$ if E is an infinite set, and let $\mu(E)$ be the number of points in E if E is finite. This μ is called the *counting measure* on X.

(b) Fix $x_0 \in X$, define $\mu(E) = 1$ if $x_0 \in E$ and $\mu(E) = 0$ if $x_0 \notin E$, for any $E \subset X$. This μ may be called the *unit mass* concentrated at x_0.

(c) Let μ be the counting measure on the set $\{1, 2, 3, \ldots\}$, let $A_n = \{n, n+1, n+2, \ldots\}$. Then $\bigcap A_n = \varnothing$ but $\mu(A_n) = \infty$ for $n = 1, 2, 3, \ldots$. This shows that the hypothesis

$$\mu(A_1) < \infty$$

is not superfluous in Theorem 1.19(e).

$$① \ \Big(\overset{g}{\sup_{n\ge 1} f_n}\Big)^{-1}((\alpha,\infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha,\infty])$$

$$x \in g^{-1}((\alpha,\infty]) \Leftrightarrow \underbrace{g(x)}_{\sup f_n} \in (\alpha,\infty] \Leftrightarrow \alpha < \sup_{n\ge 1} f_n(x) \Leftrightarrow \exists m;\ \alpha < f_m(x)$$

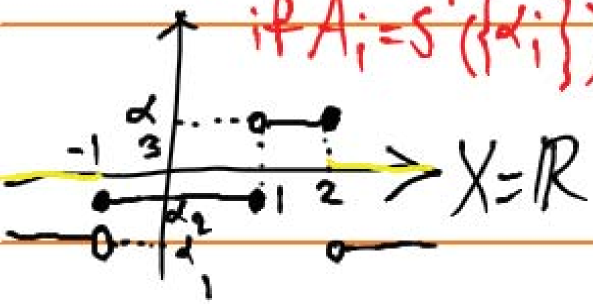
$$\Leftrightarrow \underset{\exists m}{f_m(x)} \in (\alpha,\infty] \Leftrightarrow \underset{\exists m}{x \in f_m^{-1}((\alpha,\infty])} \Leftrightarrow x \in \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha,\infty])$$

$$② \ \forall a \in \mathbb{R}\ \exists \overset{u}{a_+}, \overset{v}{a_-} \in \mathbb{R}^{\ge 0};\ \begin{cases} a = \overset{u}{a_+} - \overset{v}{a_-} \\ |a| = a_+ + a_- \end{cases}$$

$$a_+ = \frac{a+|a|}{2}, \quad a_- = \frac{|a|-a}{2}$$

$$\overline{a+|a| = 2a_+}^{\,u}$$

(3) A function $s: X \to [0,\infty]$ is called simple if its range, $s(X)$, is finite, say $s(X)=\{\alpha_1,\dots,\alpha_n\}$. Then if $A_i = s^{-1}(\{\alpha_i\})$, then $\boxed{s=\sum_{i=1}^{n} \alpha_i \chi_{A_i}}$. Note that A_is are disjoint.



$$s(x)=\begin{cases}\alpha_1 & x<-1 \;\&\; x>2 = s^{-1}(\{\alpha_1\})=A_1\\ \alpha_2 & -1\le x\le 1 = s^{-1}(\{\alpha_2\})=A_2\\ \alpha_3 & 1<x<2 = s^{-1}(\{\alpha_3\})=A_3\end{cases}$$

$X=\bigcup_{i=1}^{3} s^{-1}\{\alpha_i\}$, $s(x)=\sum_{i=1}^{3}\alpha_i \chi_{A_i}(x)$, where $A_i = s^{-1}(\{\alpha_i\})$ $\left(A_i\cap A_j = s^{-1}(\{\alpha_i\})\cap s^{-1}(\{\alpha_j\}) = s^{-1}(\{\alpha_i\}\cap\{\alpha_j\}) = s^{-1}(\emptyset) = \emptyset\right)$

Use induction to prove

(4) $A_n = B_1\cup\dots\cup B_n$.

Proof: Let $A_{n-1}=B_1\cup\dots\cup B_{n-1}$

$$x\in A_n = A_{n-1}\cup B_n \Rightarrow \begin{cases} x\in B_n \subseteq \bigcup_{i=1}^{n} B_i \\ x\in A_{n-1} = \bigcup_{i=1}^{n-1} B_i \subseteq \bigcup_{i=1}^{n} B_i\end{cases} \Rightarrow x\in \bigcup_{i=1}^{n} B_i$$

$$x\in \bigcup_{i=1}^{n} B_i \Rightarrow \exists\, 1\le j\le n;\ x\in B_j = A_j - A_{j-1} \Rightarrow x\in A_j \subseteq A_n \quad \checkmark$$
$1\le j\le n$

Integration of Positive Functions

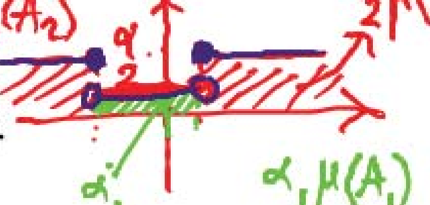
In this section, $\mathfrak{M}$ will be a σ-algebra in a set X and μ will be a positive measure on $\mathfrak{M}$.

1.23 Definition If $s\colon X \to [0, \infty)$ is a measurable simple function, of the form

$$\boxed{s = \sum_{i=1}^{n} \alpha_i \chi_{A_i},} \longrightarrow A_i = s^{-1}(\{\alpha_i\}) \tag{1}$$

where $\alpha_1, \ldots, \alpha_n$ are the distinct values of s (compare Definition 1.16), and if $E \in \mathfrak{M}$, we define

$$\int_{\mathbb{R}} s\, d\mu = \alpha_1 \mu(A_1) + \alpha_2 \mu(A_2)$$



$$\int_E s\, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E). \tag{2}$$

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If $f\colon X \to [0, \infty]$ is measurable, and $E \in \mathfrak{M}$, we define

$$\int_E f\, d\mu = \sup_{0 \le s \le f} \int_E s\, d\mu, \tag{3}$$

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The left member of (3) is called the *Lebesgue integral* of f over E, with respect to the measure μ. It is a number in $[0, \infty]$.

Observe that we apparently have two definitions for $\int_E f\, d\mu$ if f is simple, namely, (2) and (3). However, these assign the same value to the integral, since f is, in this case, the largest of the functions s which occur on the right of (3).

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(*a*) *If* $0 \le f \le g$, *then* $\int_E f\, d\mu \le \int_E g\, d\mu$.

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X

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(*f*) *If* $f \geq 0$, *then* $\int_E f \, d\mu = \int_X \chi_E f \, d\mu$.

This last result shows that we could have restricted our definition of integration to integrals over all of X, without losing any generality. If we wanted to integrate over subsets, we could then use (*f*) as the definition. It is purely a matter of taste which definition is preferred.

One may also remark here that every measurable subset E of a measure space X is again a measure space, in a perfectly natural way: The new measurable sets are simply those measurable subsets of X which lie in E, and the measure is unchanged, except that its domain is restricted. This shows again that as soon as we have integration defined over every measure space, we automatically have it defined over every measurable subset of every measure space.

1.25 Proposition *Let s and t be nonnegative measurable simple functions on X. For $E \in \mathfrak{M}$, define*

$$\varphi(E) = \int_E s \, d\mu. \tag{1}$$

Then φ is a measure on $\mathfrak{M}$. Also

$$\int_X (s + t) \, d\mu = \int_X s \, d\mu + \int_X t \, d\mu. \tag{2}$$

(This proposition contains provisional forms of Theorems 1.27 and 1.29.)

PROOF If s is as in Definition 1.23, and if $E_1, E_2, \ldots$ are disjoint members of $\mathfrak{M}$ whose union is E, the countable additivity of μ shows that

$$\begin{aligned} \varphi(E) &= \sum_{i=1}^{n} \alpha_i \, \mu(A_i \cap E) = \sum_{i=1}^{n} \alpha_i \sum_{r=1}^{\infty} \mu(A_i \cap E_r) \\ &= \sum_{r=1}^{\infty} \sum_{i=1}^{n} \alpha_i \, \mu(A_i \cap E_r) = \sum_{r=1}^{\infty} \varphi(E_r). \end{aligned}$$

Also, $\varphi(\varnothing) = 0$, so that φ is not identically ∞.

Next, let s be as before, let $\beta_1, \ldots, \beta_m$ be the distinct values of t, and let

Next, let s be as before, let $\beta_1, \ldots, \beta_m$ be the distinct values of t, and let $B_j = \{x: t(x) = \beta_j\}$. If $E_{ij} = A_i \cap B_j$, then

$$\int_{E_{ij}} (s + t)\, d\mu = (\alpha_i + \beta_j)\mu(E_{ij})$$

and

$$\int_{E_{ij}} s\, d\mu + \int_{E_{ij}} t\, d\mu = \alpha_i\, \mu(E_{ij}) + \beta_j\, \mu(E_{ij}).$$

Thus (2) holds with E_{ij} in place of X. Since X is the disjoint union of the sets E_{ij} $(1 \le i \le n,\ 1 \le j \le m)$, the first half of our proposition implies that (2) holds. ////

We now come to the interesting part of the theory. One of its most remarkable features is the ease with which it handles limit operations.

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(a) $0 \le f_1(x) \le f_2(x) \le \cdots \le \infty$ *for every* $x \in X$,
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Then f is measurable, and

$$\int_X f_n\, d\mu \to \int_X f\, d\mu \qquad \text{as } n \to \infty.$$

PROOF Since $\int f_n \le \int f_{n+1}$, there exists an $\alpha \in [0, \infty]$ such that

$$\int_X f_n\, d\mu \to \alpha \qquad \text{as } n \to \infty. \tag{1}$$

By Theorem 1.14, f is measurable. Since $f_n \le f$, we have $\int f_n \le \int f$ for every n, so (1) implies

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Let s be any simple measurable function such that $0 \le s \le f$, let c be a constant, $0 < c < 1$, and define

$$E_n = \{x: f_n(x) \ge cs(x)\} \qquad (n = 1, 2, 3, \ldots). \tag{3}$$

Each E_n is measurable, $E_1 \subset E_2 \subset E_3 \subset \cdots$, and $X = \bigcup E_n$. To see this equality, consider some $x \in X$. If $f(x) = 0$, then $x \in E_1$; if $f(x) > 0$, then $cs(x) < f(x)$, since $c < 1$; hence $x \in E_n$ for some n. Also

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Let $n \to \infty$, applying Proposition 1.25 and Theorem 1.19(*d*) to the last integral in (4). The result is

$$\alpha \geq c \int_X s \, d\mu. \tag{5}$$

Since (5) holds for every $c < 1$, we have

$$\alpha \geq \int_X s \, d\mu \tag{6}$$

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The theorem follows from (1), (2), and (7). ////

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X

$$\int g_N \, d\mu = \sum^{N} \int f_n \, d\mu. \tag{4}$$

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$\forall x \in C;\ x \le \beta$

If we let μ be the counting measure on a countable set, Theorem 1.27 is a statement about double series of nonnegative real numbers (which can of course be proved by more elementary means):

Let $\beta < \alpha$. So β is not an upper bound for C. So $\exists x \in C;\ x > \beta$. So $x \in D$ & $x > \beta$. Hence $x \le \beta$ & $x > \beta$. X

Corollary *If $a_{ij} \ge 0$ for i and $j = 1, 2, 3, \ldots$, then*

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$C \subseteq D \Rightarrow \underbrace{\sup C}_{\alpha} \le \underbrace{\sup D}_{\beta}$

Let $x \in C$. Since $C \subseteq D$, $x \in D$. So $x \le \beta$. Hence β is an upper bound for C. Thus $\alpha \le \beta$. □

α = the least upper bound of C

(1)

(a) $0 \le f \le g \Rightarrow \{\int_E s\,d\mu : 0 \le s \le f\} \subseteq \{\int_E t\,d\mu : 0 \le t \le g\}$

$\Rightarrow \sup \qquad \le \sup$

$\Rightarrow \int_E f\,d\mu \le \int_E g\,d\mu.$

(b) $A \subseteq B \Rightarrow \forall\, 0 \le s \le f;\ \int_A s\,d\mu = \sum \alpha_i \mu(A_i \cap A) \le \sum_{i=1}^{n} \alpha_i \mu(A_i \cap B) = \int_B s\,d\mu \Rightarrow \sup_{0 \le s \le f} \int_A s\,d\mu \le \sup_{0 \le t \le f} \int_B t\,d\mu$

$\alpha_i \ge 0$

$A_i \cap A \subseteq A_i \cap B$

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$\int_A s\,d\mu \le \int_B s\,d\mu \le \sup_{0 \le t \le f} \int_B t\,d\mu$ an upper bound

$\therefore \sup \int_A s\,d\mu \le \sup \int_B t\,d\mu$

(c) $\int_E cf\,d\mu = c\int_E f\,d\mu \quad (c>0)$ $\quad ct=\sum_{i=1}^n c\alpha_i \chi_{A_i} \Leftarrow t=\sum_{i=1}^n \alpha_i \chi_{A_i}$

$$\int_E cf\,d\mu = \sup_{0\le s\le cf} \int_E s\,d\mu = \sup_{0\le \boxed{\frac{s}{c}}\le f} \int_E s\,d\mu = \sup_{0\le t\le f} \int_E ct\,d\mu = \sup_{0\le t\le f} \sum_{i=1}^n c\alpha_i \mu(A_i \cap E)$$

t

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$\int_E t\,d\mu$

$$\left.\begin{array}{l} cA=\{cx : x\in A\} \\ c>0 \end{array}\right\} \Rightarrow \underbrace{\sup(cA)}_{\alpha} = c\,\underbrace{\sup A}_{\beta}$$

$\forall x\in A,\ x\le\beta \Rightarrow \forall x\in A;\ cx\le c\beta \Rightarrow \forall y\in cA;$
$y\le c\beta \Rightarrow c\beta$ is an upper bound for cA $\Rightarrow \alpha\le c\beta$.

$\forall cx\in cA;\ cx\le\alpha \Rightarrow \forall x\in A;\ x\le\frac{\alpha}{c} \Rightarrow \beta\le\frac{\alpha}{c} \Rightarrow c\beta\le\alpha$

$\square$

(d) $f=0 \Rightarrow \int_E f\,d\mu = \sup\left\{\int_E s\,d\mu : 0 \le s \le \underset{\overset{\parallel}{0}}{f}\right\} = \sup\left\{\int_E 0\,d\mu\right\} = \sup\{0\} = 0$

$s = 0 = \underset{\overset{\parallel}{\alpha_1}}{0}\chi$ $X = A_1$

$\int_E 0\,d\mu = 0\mu(E\cap X) = 0$

(e) $\mu(E)=0 \Rightarrow \int_E f\,d\mu = \sup_{0\le s\le f} \int_E s\,d\mu = \sup_{\substack{0\le s\le f \\ \parallel \\ \sum_{i=1}^{n} \alpha_i \chi_{A_i}}} \sum_{i=1}^{n} \alpha_i \underbrace{\mu(E\cap A_i)}_{0 \text{ since } E\cap A_i \subseteq E} = \sup\{0\} = 0.$

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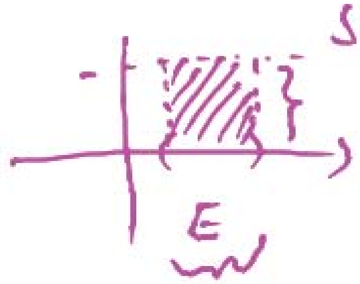

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$\sup_n x \in [-\infty, \infty]$

Then f is measurable, and

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Since (5) holds for every $c < 1$, we have

همه
با فرض این که c در رابطه (5) تو اینها (c
c→1 میل می دهیم

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To this end, let $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$. Then $\int_E s\,d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E)$

$s\chi_E = \sum_{i=1}^{n} \alpha_i \chi_{A_i}\chi_E$ ($\chi_{A_i \cap E}$) $\int_X s\chi_E\,d\mu = \sum_{i=1}^{n} \alpha_i \mu(\underbrace{X \cap A_i \cap E}_{A_i \cap E})$ ✓

(2) $\int_{E_{pq}} s+t\,d\mu = \sum_{\substack{1\le i\le n\\ 1\le j\le m}} (\alpha_i+\beta_j)\mu(\underbrace{E_{ij}\cap E_{pq}}) = (\alpha_p+\beta_q)\mu(E_{pq})$

$A_i \cap B_j \cap A_p \cap B_q$

A_p $(i=p)$ B_q $(j=q)$

(3) $\int_X (s+t)\,d\mu = \varphi_{s+t}(X) = \varphi_{s+t}(\dot{\bigcup}_{i,j} E_{ij}) = \sum_{i,j} \underbrace{\varphi_{s+t}(E_{ij})}_{\varphi_s(E_{ij})+\varphi_t(E_{ij})} = \sum_{i,j} \varphi_s(E_{ij}) + \sum_{i,j} \varphi_t(E_{ij}) = \varphi_s(\dot{\bigcup} E_{ij}) + \varphi_t(\dot{\bigcup} E_{ij}) = \varphi_s(X) + \varphi_t(X) = \int_X s\,d\mu + \int_X t\,d\mu.$ □

φ_{s+t} is meas

φ_s, φ_t are meas

Let $x \in X$.

(4) $0 \le S \le f$, $0 < c < 1$ $\Rightarrow$ $cS(x) < f(x) = \sup_n f_n(x)$ $\Rightarrow$ $cS(x)$ is not an upper bound for $\{f_1(x), f_2(x), \dots\}$ $\Rightarrow$ $cS(x) < f_m(x)$ for some m

$\Rightarrow x \in E_m \Rightarrow x \in \bigcup_{n=1}^{\infty} E_n \subseteq X$. $\therefore \bigcup_{n=1}^{\infty} E_n = X$

$\alpha = \sup A \iff \begin{cases} \boxed{\forall \varepsilon > 0 \ \exists x \in A;\ \alpha - \varepsilon < x} \\ \text{[illegible]} \ \alpha \end{cases}$

$\forall \varepsilon > 0$ $\alpha - \varepsilon < \alpha = \sup A$

$\alpha - \varepsilon$ is not an upper bound of A

1.28 Fatou's Lemma *If $f_n : X \to [0, \infty]$ is measurable, for each positive integer n, then*

$$\int_X \left(\liminf_{n\to\infty} f_n\right) d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu. \tag{1}$$

Strict inequality can occur in (1); see Exercise 8.

$\liminf_{n\to\infty} x_n = \sup_k \inf_{n \geq k} x_n$

$A \subseteq B \Rightarrow \inf A \geq \inf B$

PROOF Put

$\inf_{i \geq k} f_i$

$$g_k(x) = \inf_{i \geq k} f_i(x) \qquad (k = 1, 2, 3, \ldots; x \in X). \tag{2}$$

$= \lim_k \inf_{n \geq k} x_n$

$\{x_n : n \geq k\} \supseteq \{x_n : n \geq k+1\}$

$y_k = \inf$ 〃 $\leq \inf$ 〃 $= y_{k+1}$

increasing $\{\inf_{n \geq k} x_n\}_{k=1}^{\infty}$ y_k

Then $g_k \leq f_k$, so that

$$\int_X g_k \, d\mu \leq \int_X f_k \, d\mu \qquad (k = 1, 2, 3, \ldots). \tag{3}$$

Also, $0 \leq g_1 \leq g_2 \leq \cdots$, each g_k is measurable, by Theorem 1.14, and $g_k(x) \to \liminf f_n(x)$ as $k \to \infty$, by Definition 1.13. The monotone convergence theorem shows therefore that the left side of (3) tends to the left side of (1), as $k \to \infty$. Hence (1) follows from (3). ////

1.29 Theorem *Suppose $f : X \to [0, \infty]$ is measurable, and*

$(d\varphi = f \, d\mu) \equiv$

$$\varphi(E) = \int_E f \, d\mu \qquad (E \in \mathfrak{M}). \tag{1}$$

Then φ is a measure on $\mathfrak{M}$, and

$$\int_X g \, d\varphi = \int_X gf \, d\mu \tag{2}$$

for every measurable g on X with range in $[0, \infty]$.

PROOF Let $E_1, E_2, E_3, \ldots$ be disjoint members of $\mathfrak{M}$ whose union is E. Observe that

$= \chi_{\cup E_i} f \overset{(2)}{=} \left(\sum_{i=1}^{\infty} \chi_{E_i}\right) f$

$$\chi_E f = \sum_{j=1}^{\infty} \chi_{E_j} f \tag{3}$$

and that

$= \int_E f \, d\mu$

$$\varphi(E) = \int_X \chi_E f \, d\mu, \qquad \varphi(E_j) = \int_X \chi_{E_j} f \, d\mu. \tag{4}$$

X

$\int \chi_E f = \int \sum \chi_{E_i} f = \sum \int \chi f$

It now follows from Theorem 1.27 that

$$\varphi(E) = \sum_{j=1}^{\infty} \varphi(E_j). \tag{5}$$

Since $\varphi(\varnothing) = 0$, (5) proves that φ is a measure.

Next, (1) shows that (2) holds whenever $g = \chi_E$ for some $E \in \mathfrak{M}$. Hence (2) holds for every simple measurable function g, and the general case follows from the monotone convergence theorem. ////

Remark The second assertion of Theorem 1.29 is sometimes written in the form

$$\boxed{d\varphi = f\,d\mu.} \tag{6}$$

We assign no independent meaning to the symbols $d\varphi$ and $d\mu$; (6) merely means that (2) holds for every measurable $g \geq 0$.

Theorem 1.29 has a very important converse, the Radon-Nikodym theorem, which will be proved in Chap. 6.

Integration of Complex Functions

As before, μ will in this section be a positive measure on an arbitrary measurable space X.

1.30 Definition We define $L^1(\mu)$ to be the collection of all complex measurable functions f on X for which

$$\int_X |f|\,d\mu < \infty.$$

$:X \to [0,\infty)$

$|a+ib| = \sqrt{a^2+b^2}$

$f = u+iv$

Note that the measurability of f implies that of $|f|$, as we saw in Proposition 1.9(b); hence the above integral is defined.

The members of $L^1(\mu)$ are called *Lebesgue integrable* functions (with respect to μ) or *summable functions*. The significance of the exponent 1 will become clear in Chap. 3.

$u^+ - u^- \quad v^+ - v^-$

1.31 Definition If $f = u + iv$, where u and v are real measurable functions on X, and if $f \in L^1(\mu)$, we define

$$\int_E f\,d\mu = \int_E u^+\,d\mu - \int_E u^-\,d\mu + i\int_E v^+\,d\mu - i\int_E v^-\,d\mu \in \mathbb{C} \tag{1}$$

for every measurable set E.

Here u^+ and u^- are the positive and negative parts of u, as defined in

Here u^+ and u^- are the positive and negative parts of u, as defined in Sec. 1.15; v^+ and v^- are similarly obtained from v. These four functions are measurable, real, and nonnegative; hence the four integrals on the right of (1) exist, by Definition 1.23. Furthermore, we have $u^+ \leq |u| \leq |f|$, etc., so that each of these four integrals is finite. Thus (1) defines the integral on the left as a complex number.

$|u| = \frac{u^+ + u^-}{2}$

Occasionally it is desirable to define the integral of a measurable function f with range in $[-\infty, \infty]$ to be

$f = f^+ - f^-$

$$\int_E f\,d\mu = \int_E f^+\,d\mu - \int_E f^-\,d\mu, \tag{2}$$

provided that at least one of the integrals on the right of (2) is finite. The left side of (2) is then a number in $[-\infty, \infty]$.

1.32 Theorem *Suppose f and $g \in L^1(\mu)$ and α and β are complex numbers. Then $\alpha f + \beta g \in L^1(\mu)$, and*

$$\int_X (\alpha f + \beta g)\,d\mu = \alpha \int_X f\,d\mu + \beta \int_X g\,d\mu. \tag{1}$$

PROOF The measurability of $\alpha f + \beta g$ follows from Proposition 1.9(*c*). By Sec. 1.24 and Theorem 1.27,

$|\alpha f + \beta g| \leq |\alpha|\,|f| + |\beta|\,|g|$

$$\begin{aligned}\int_X |\alpha f + \beta g|\,d\mu &\leq \int_X (|\alpha|\,|f| + |\beta|\,|g|)\,d\mu \\ &= \underbrace{|\alpha|}_{\in \mathbb{R}} \underbrace{\int_X |f|\,d\mu}_{\in \mathbb{R}} + \underbrace{|\beta|}_{\in \mathbb{R}} \underbrace{\int_X |g|\,d\mu}_{\in \mathbb{R}} < \infty.\end{aligned}$$

Thus $\alpha f + \beta g \in L^1(\mu)$.

To prove (1), it is clearly sufficient to prove

$$\boxed{\int_X \underbrace{(f + g)}_{h}\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu} \tag{2}$$

and

$$\int_X (\alpha f)\,d\mu = \alpha \int_X f\,d\mu, \tag{3}$$

and the general case of (2) will follow if we prove (2) for real f and g in $L^1(\mu)$.

Assuming this, and setting $h = f + g$, we have

$$h = h^+ - h^- \overset{= f+g}{=} f^+ - f^- + g^+ - g^-$$

or

or

$$h^+ + f^- + g^- = f^+ + g^+ + h^-. \tag{4}$$

By Theorem 1.27,

$$\int h^+ + \int f^- + \int g^- = \int f^+ + \int g^+ + \int h^-, \tag{5}$$

and since each of these integrals is finite, we may transpose and obtain (2).

That (3) holds if $\alpha \geq 0$ follows from Proposition 1.24(*c*). It is easy to verify that (3) holds if $\alpha = -1$, using relations like $(-u)^+ = u^-$. The case $\alpha = i$ is also easy: If $f = u + iv$, then

$$\int (if) = \int (iu - v) = \int (-v) + i \int u = - \int v + i \int u = i\left(\int u + i \int v \right)$$
$$= i \int f.$$

Left to the students

Combining these cases with (2), we obtain (3) for any complex α. ////

1.33 Theorem *If $f \in L^1(\mu)$, then*

very interesting

$$\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu.$$

Proof Put $z = \int_X f \, d\mu$. Since z is a complex number, there is a complex number α, with $|\alpha| = 1$, such that $\alpha z = |z|$. $\boxed{\text{Let } u \text{ be the real part of } \alpha f.}$ Then $u \leq |\alpha f| = |f|$. Hence

$\alpha = \frac{|z|}{z}$ $z \neq 0$

$$\underbrace{\left| \int_X f \, d\mu \right|}_{|z| = \alpha z} = \alpha \int_X f \, d\mu = \int_X \alpha f \, d\mu = \int_X u \, d\mu \leq \int_X |f| \, d\mu.$$

$u \leq |f|$

The third of the above equalities holds since the preceding ones show that $\int \alpha f \, d\mu$ is real. ////

$\int \mathrm{Re}(\alpha f) d\mu + \cancel{i \int \mathrm{Im}(\alpha f) d\mu}$

1.34 Lebesgue's Dominated Convergence Theorem *Suppose $\{f_n\}$ is a sequence of complex measurable functions on X such that*

below

$$f(x) = \lim_{n \to \infty} f_n(x) \tag{1}$$

x

exists for every $x \in X$. If there is a function $g \in L^1(\mu)$ such that

$$|f_n(x)| \le g(x) \qquad (n = 1, 2, 3, \ldots;\ x \in X), \tag{2}$$

then $f \in L^1(\mu)$,

$$\lim_{n\to\infty}\int_X |f_n - f|\,d\mu = 0, \tag{3}$$

$\forall \varepsilon\, \exists N\, \forall n \ge N;\ \left|\int f_n\,d\mu - \int f\,d\mu\right| < \varepsilon$

$\int (f_n - f)\,d\mu$ calculus

and

$$\lim_{n\to\infty}\int_X f_n\,d\mu = \int_X f\,d\mu. \tag{4}$$

$|f_n| + |f| \le g + g =$

PROOF Since $|f| \le g$ (4) and f is measurable, $f \in L^1(\mu)$ (5). Since $|f_n - f| \le 2g$, Fatou's lemma applies to the functions $2g - |f_n - f|$ and yields

$= \int \liminf (2g - |f_n - f|)\,d\mu \ge 0$

(4) $|f_n(x)| \le g(x)\ \forall n$
$\Downarrow$
$\lim_n |f_n(x)| \le \lim_n g(x)$
$\Downarrow$
$|f(x)| \le g(x)$
$\Downarrow$
(3) $\int |f| \le \int g < +\infty$

$f_n \to f$, $|f_n| \le g$, $\cap$ $L^1(\mu)$

$$\int_X 2g\,d\mu \underset{\text{F.L.}}{\le} \liminf_{n\to\infty}\int_X (2g - |f_n - f|)\,d\mu$$

$$\le \int_X 2g\,d\mu + \liminf_{n\to\infty}\left(-\int_X |f_n - f|\,d\mu\right)$$

$$= \int_X 2g\,d\mu - \limsup_{n\to\infty}\int_X |f_n - f|\,d\mu.$$

Since $\int 2g\,d\mu$ is finite, we may subtract it and obtain

$\liminf(x_n + y_n) \le \liminf x_n + \liminf y_n$

$$0 \le \liminf_n \int |f_n - f|\,d\mu \le \limsup_{n\to\infty}\int_X |f_n - f|\,d\mu \le 0. \tag{5}$$

If a sequence of nonnegative real numbers fails to converge to 0, then its upper limit is positive. Thus (5) implies (3). By Theorem 1.33, applied to $f_n - f$, (3) implies (4). ////

$g_k \xrightarrow{\text{increasing}} \liminf_{k} f_n = \liminf_{n\to\infty} f_n \overset{\text{M.C.T.}}{\Longrightarrow} \lim \int_X g_k\,d\mu = \int_X \liminf_{n\to\infty} f_n\,d\mu$

On the other hand, $\int g_n \le \int f_n$. So $\liminf_{n\to\infty}\int g_n\,d\mu \le \liminf \int f_n\,d\mu$. $\square$

$1 + 0 = \chi_A(x) + \chi_B(x)$ Either ... w.l.o.g

② $\chi_{A \dot\cup B} = \chi_A + \chi_B$:

$$\chi_{A \dot\cup B} = \begin{cases} 1 & x \in A \dot\cup B \quad x \in A \text{ or } x \in B \ (\ldots x \in A \ \& \ x \notin B) \\ 0 & x \notin A \dot\cup B \quad \text{neither } x \in A \text{ nor } x \in B \end{cases}$$

$0 + 0 = \chi_A(x) + \chi_B(x)$

X

1.35 Definition Let P be a property which a point x may or may not have. For instance, P might be the property "$f(x) > 0$" if f is a given function, or it might be "$\{f_n(x)\}$ converges" if $\{f_n\}$ is a given sequence of functions.

If μ is a measure on a σ-algebra $\mathfrak{M}$ and if $E \in \mathfrak{M}$, the statement "$\boxed{P \text{ holds almost everywhere on } E}$" (abbreviated to "$P$ holds a.e. on E") means that there exists an $N \in \mathfrak{M}$ such that $\mu(N) = 0$, $N \subset E$, and P holds at every point of $E - N$. This concept of a.e. depends of course very strongly on the given measure, and we shall write "a.e. $[\mu]$" whenever clarity requires that the measure be indicated.

For example, if f and g are measurable functions and if

$f = g$ a.e.

$$\underbrace{\mu(\{x : f(x) \neq g(x)\})}_{N} = 0, \tag{1}$$

we say that $f = g$ a.e. $[\mu]$ on X, and we may write $f \sim g$. This is easily seen to be an equivalence relation. The transitivity ($f \sim g$ and $g \sim h$ implies $f \sim h$) is a consequence of the fact that the union of two sets of measure 0 has measure 0.

$\int_N g\,d\mu \quad \int_{E-N} g\,d\mu$

$\varphi(A) = \int_A f\,d\mu$

Note that if $f \sim g$, then, for every $E \in \mathfrak{M}$,

$\int_E f\,d\mu = \int_N f\,d\mu + \int_{E-N} f\,d\mu$

$E = N \cup (E - N)$

$$\int_E f\,d\mu = \int_E g\,d\mu. \tag{2}$$

$\varphi(E) = \varphi(N) + \varphi(E-N)$

To see this, let N be the set which appears in (1); then E is the union of the disjoint sets $E - N$ and $E \cap N$; on $E - N$, $f = g$, and $\mu(E \cap N) = 0$.

$\int_E g\,d\mu$

1.36 Theorem *Let $(X, \mathfrak{M}, \mu)$ be a measure space, let $\mathfrak{M}^*$ be the collection of all $E \subset X$ for which there exist sets A and $B \in \mathfrak{M}$ such that $A \subset E \subset B$ and $\mu(B - A) = 0$, and define $\mu(E) = \mu(A)$ in this situation. Then $\mathfrak{M}^*$ is a σ-algebra, and μ is a measure on $\mathfrak{M}^*$.*

$E \subseteq \{\mu(A) = 0$, $B \subseteq A$

This extended measure μ is called *complete*, since all subsets of sets of measure 0 are now measurable; the σ-algebra $\mathfrak{M}^*$ is called the *μ-completion* of $\mathfrak{M}$.

$B \in \mathfrak{M}$

(So $\mu(B) = 0$)

1.38 Theorem *Suppose $\{f_n\}$ is a sequence of complex measurable functions defined a.e. on X such that*

$$\sum_{n=1}^{\infty} \int_X |f_n|\,d\mu < \infty. \tag{1}$$

Then the series

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \tag{2}$$

converges for almost all x, $f \in L^1(\mu)$, and

$$\int_X f\,d\mu = \sum_{n=1}^{\infty} \int_X f_n\,d\mu. \tag{3}$$

PROOF Let S_n be the set on which f_n is defined, so that $\mu(S_n^c) = 0$. Put $\varphi(x) = \sum |f_n(x)|$, for $x \in S = \bigcap S_n$. Then $\mu(S^c) = 0$. By (1) and Theorem 1.27,

X

$$\int_S \varphi \, d\mu < \infty. \tag{4}$$

If $E = \{x \in S\colon \varphi(x) < \infty\}$, it follows from (4) that $\mu(E^c) = 0$. The series (2) converges absolutely for every $x \in E$, and if $f(x)$ is defined by (2) for $x \in E$, then $|f(x)| \le \varphi(x)$ on E, so that $f \in L^1(\mu)$ on E, by (4). If $g_n = f_1 + \cdots + f_n$, then $|g_n| \le \varphi$, $g_n(x) \to f(x)$ for all $x \in E$, and Theorem 1.34 gives (3) with E in place of X. This is equivalent to (3), since $\mu(E^c) = 0$. ////

1.39 Theorem

(*a*) *Suppose $f\colon X \to [0, \infty]$ is measurable, $E \in \mathfrak{M}$, and $\int_E f\, d\mu = 0$. Then $f = 0$ a.e. on E.*
(*b*) *Suppose $f \in L^1(\mu)$ and $\int_E f\, d\mu = 0$ for every $E \in \mathfrak{M}$. Then $f = 0$ a.e. on X.*
(*c*) *Suppose $f \in L^1(\mu)$ and*

$$\left| \int_X f\, d\mu \right| = \int_X |f| \, d\mu.$$

Then there is a constant α such that $\alpha f = |f|$ a.e. on X.

Note that (*c*) describes the condition under which equality holds in Theorem 1.33.

PROOF

(*a*) If $A_n = \{x \in E : f(x) > 1/n\}$, $n = 1, 2, 3, \ldots$, then

$$\frac{1}{n}\mu(A_n) \le \int_{A_n} f\, d\mu \le \int_E f\, d\mu = 0,$$

so that $\mu(A_n) = 0$. Since $\{x \in E : f(x) > 0\} = \bigcup A_n$, (*a*) follows.

(*b*) Put $f = u + iv$, let $E = \{x\colon u(x) \ge 0\}$. The real part of $\int_E f\, d\mu$ is then $\int_E u^+ \, d\mu$. Hence $\int_E u^+ \, d\mu = 0$, and (*a*) implies that $u^+ = 0$ a.e. We conclude similarly that

$$u^- = v^+ = v^- = 0 \qquad \text{a.e.}$$

(*c*) Examine the proof of Theorem 1.33. Our present assumption implies that the last inequality in the proof of Theorem 1.33 must actually be an equality. Hence $\int (|f| - u)\, d\mu = 0$. Since $|f| - u \ge 0$, (*a*) shows that $|f| = u$ a.e. This says that the real part of αf is equal to $|\alpha f|$ a.e., hence $\alpha f = |\alpha f| = |f|$ a.e., which is the desired conclusion. ////

1.40 Theorem *Suppose $\mu(X) < \infty$, $f \in L^1(\mu)$, S is a closed set in the complex plane, and the averages*

$$A_E(f) = \frac{1}{\mu(E)} \int_E f\, d\mu$$

lie in S for every $E \in \mathfrak{M}$ with $\mu(E) > 0$. Then $f(x) \in S$ for almost all $x \in X$.

PROOF Let Δ be a closed circular disc (with center at α and radius $r > 0$, say) in the complement of S. Since S^c is the union of countably many such discs, it is enough to prove that $\mu(E) = 0$, where $E = f^{-1}(\Delta)$.

If we had $\mu(E) > 0$, then

$$|A_E(f) - \alpha| = \frac{1}{\mu(E)} \left| \int_E (f - \alpha)\, d\mu \right| \le \frac{1}{\mu(E)} \int_E |f - \alpha|\, d\mu \le r,$$

which is impossible, since $A_E(f) \in S$. Hence $\mu(E) = 0$. ////

1.41 Theorem *Let $\{E_k\}$ be a sequence of measurable sets in X, such that*

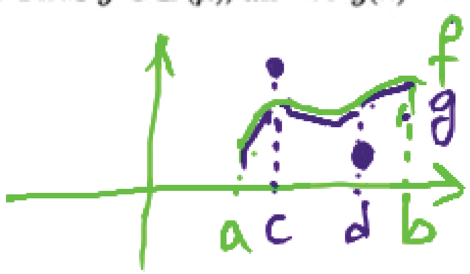
$$\sum_{k=1}^{\infty} \mu(E_k) < \infty. \tag{1}$$

Then almost all $x \in X$ lie in at most finitely many of the sets E_k.

Proof If A is the set of all x which lie in infinitely many E_k, we have to prove that $\mu(A) = 0$. Put

$$g(x) = \sum_{k=1}^{\infty} \chi_{E_k}(x) \qquad (x \in X). \tag{2}$$

For each x, each term in this series is either 0 or 1. Hence $x \in A$ if and only if $g(x) = \infty$. By Theorem 1.27, the integral of g over X is equal to the sum in (1). Thus $g \in L^1(\mu)$, and so $g(x) < \infty$ a.e. ////



$N = \{x : f(x) \neq g(x)\} = \{c, d\}$

$\mu(N) = 0$ ∴ So $f = g$ a.e.

Hence $\int_{[a,b]} f\, d\mu = \int_{[a,b]} g\, d\mu$.

The Lebesgue measure on $\mathbb{R}$ says that $\mu(\{a\}) = 0$

so $\mu(\{a_1, a_2, \dots\}) = \mu\left(\bigcup_{i=1}^{\infty} \{a_i\}\right) = \sum_{i=1}^{\infty} \mu(\{a_i\}) = \sum_{i=1}^{\infty} 0 = 0$ ✓

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For example, if f and g are measurable functions and if

$$\mu(\{x : f(x) \neq g(x)\}) = 0, \tag{1}$$

we say that $f = g$ a.e. $[\mu]$ on X, and we may write $f \sim g$. This is easily seen to be an equivalence relation. The transitivity ($f \sim g$ and $g \sim h$ implies $f \sim h$) is a consequence of the fact that the union of two sets of measure 0 has measure 0.

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$$\int_E f \, d\mu = \int_E g \, d\mu. \tag{2}$$

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Then the series

$$\boxed{f(x) = \sum_{n=1}^{\infty} f_n(x)} \tag{2}$$

converges for almost all x, $f \in L^1(\mu)$, and

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E E

PROOF Let S_n be the set on which f_n is defined, so that $\mu(S_n^c) = 0$. Put $\varphi(x) = \sum |f_n(x)|$, for $x \in S = \bigcap S_n$. Then $\mu(S^c) = 0$. By (1) and Theorem 1.27,

$\mu(\cup S^c) < \sum \mu(S^c) = 0$

$$\int_S \varphi \, d\mu < \infty. \tag{4}$$

$= \sum \int |f_n|$

If $E = \{x \in S: \varphi(x) < \infty\}$, it follows from (4) that $\mu(E^c) = 0$. The series (2) converges absolutely for every $x \in E$, and if $f(x)$ is defined by (2) for $x \in E$, then $|f(x)| \le \varphi(x)$ on E, so that $f \in L^1(\mu)$ on E, by (4). If $g_n = f_1 + \cdots + f_n$, then $|g_n| \le \varphi$, $g_n(x) \to f(x)$ for all $x \in E$, and Theorem 1.34 gives (3) with E in place of X. This is equivalent to (3), since $\mu(E^c) = 0$. ////

$\le \sum_{k=1}^{n} |f_k| \le \sum_{k=1}^{\infty} |f_k| =$

$\int g_n \to \int f$ or $\sum_{k=1}^{n} \int f_k \to \int f$

1.39 Theorem

(a) *Suppose $f: X \to [0, \infty]$ is measurable, $E \in \mathfrak{M}$, and $\int_E f \, d\mu = 0$. Then $f = 0$ a.e. on E.*

(b) *Suppose $f \in L^1(\mu)$ and $\int_E f \, d\mu = 0$ for every $E \in \mathfrak{M}$. Then $f = 0$ a.e. on X.*

(c) *Suppose $f \in L^1(\mu)$ and*

$$\left| \int_X f \, d\mu \right| = \int_X |f| \, d\mu.$$

Then there is a constant α such that $\alpha f = |f|$ a.e. on X.

$\int f = 0$
But $f \neq 0$

Note that (c) describes the condition under which equality holds in Theorem 1.33.

$\{x : f(x) \neq 0\} = \bigcup_{n=1}^{\infty} \underbrace{\{x : f(x) > \tfrac{1}{n}\}}_{A_n}$

$0 \le \mu(\{x : f(x) \neq 0\}) \le \sum_{n=1}^{\infty} \mu(A_n) = 0$

Proof

(a) If $A_n = \{x \in E: f(x) > 1/n\}$, $n = 1, 2, 3, \ldots$, then

$$\underbrace{\frac{1}{n}\mu(A_n)}_{= \int_{A_n} \frac{1}{n} d\mu} \le \int_{A_n} f \, d\mu \le \int_E f \, d\mu = 0,$$

$A_n \subseteq E$

so that $\mu(A_n) = 0$. Since $\{x \in E: f(x) > 0\} = \bigcup A_n$, (a) follows.

(b) Put $f = u + iv$, let $E = \{x: u(x) \ge 0\}$. The real part of $\int_E f \, d\mu$ is then $\int_E u^+ \, d\mu$. Hence $\int_E u^+ \, d\mu = 0$, and (a) implies that $u^+ = 0$ a.e. We conclude similarly that

$$u^- = v^+ = v^- = 0 \qquad \text{a.e.}$$

(2)

$\int f = \int u + i \int v$
$0 \quad 0 \quad 0$

$\mathrm{Re} \int_E f = \int_E (\mathrm{Re} f) = \int_E u^+ - \cancel{\int_E u^-}$

Since $u^- = 0$ on E

(c) Examine the proof of Theorem 1.33. Our present assumption implies that the last inequality in the proof of Theorem 1.33 must actually be an equality. Hence $\int (|f| - u) \, d\mu = 0$. Since $|f| - u \ge 0$, (a) shows that $|f| = u$ a.e. This says that the real part of αf is equal to $|\alpha f|$ a.e., hence $\alpha f = |\alpha f| = |f|$ a.e., which is the desired conclusion. ////

1.40 Theorem *Suppose $\mu(X) < \infty$, $f \in L^1(\mu)$, S is a closed set in the complex plane, and the averages*

$$A_E(f) = \frac{1}{\mu(E)} \int_E f \, d\mu$$

lie in S for every $E \in \mathfrak{M}$ with $\mu(E) > 0$. Then $f(x) \in S$ for almost all $x \in X$.

X

Proof Let Δ be a closed circular disc (with center at α and radius $r > 0$, say) in the complement of S. Since S^c is the union of countably many such discs, it is enough to prove that $\mu(E) = 0$, where $E = f^{-1}(\Delta)$.

If we had $\mu(E) > 0$, then

$$|A_E(f) - \alpha| = \frac{1}{\mu(E)} \left| \int_E (f - \alpha)\, d\mu \right| \le \frac{1}{\mu(E)} \int_E |f - \alpha|\, d\mu \le r,$$

which is impossible, since $A_E(f) \in S$. Hence $\mu(E) = 0$. ////

→ left to students

1.41 Theorem *Let $\{E_k\}$ be a sequence of measurable sets in X, such that*

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty. \tag{1}$$

Then almost all $x \in X$ lie in at most finitely many of the sets E_k.

PROOF If A is the set of all x which lie in infinitely many E_k, we have to prove that $\mu(A) = 0$. Put

→ Left to Students

$$g(x) = \sum_{k=1}^{\infty} \chi_{E_k}(x) \qquad (x \in X). \tag{2}$$

For each x, each term in this series is either 0 or 1. Hence $x \in A$ if and only if $g(x) = \infty$. By Theorem 1.27, the integral of g over X is equal to the sum in (1). Thus $g \in L^1(\mu)$, and so $g(x) < \infty$ a.e. ////

(1) If $\mu(E^c) \neq 0$, then $\underbrace{\int_X \varphi\, d\mu}_{<\infty} \geq \int_{E^c} \varphi\, d\mu = \infty \int_{E^c} 1\, d\mu = \infty \cdot \mu(E^c) = \infty$

since $\varphi(x) = \infty$ on E^c

(2) $u^+ = 0$ a.e.
$u^- = 0$ a.e.
⋮

$\{x : u(x) \neq 0\} \subseteq \{x : u^+(x) \neq 0\} \cup \{x : u^-(x) \neq 0\}$

To prove, let $x \notin \{x : u^+(x) \neq 0\} \cup \{x : u^-(x) \neq 0\}$.
then $u^+(x) = 0 = u^-(x)$. So $u(x) = 0$

$\therefore \mu\{x : u(x) \neq 0\} \leq \underbrace{\mu\{x : u^+(x) \neq 0\}}_{0} + \underbrace{\mu\{x : u^-(x) \neq 0\}}_{0}$

So $\mu\{x : u(x) \neq 0\} = 0$. Hence $u = 0$ a.e.

2.3 Definitions Let X be a topological space, as defined in Sec. 1.2.

(*a*) A set $E \subset X$ is *closed* if its complement E^c is open. (Hence $\varnothing$ and X are closed, finite unions of closed sets are closed, and arbitrary intersections of closed sets are closed.)

(*b*) The *closure* $\bar{E}$ of a set $E \subset X$ is the smallest closed set in X which contains E. (The following argument proves the existence of $\bar{E}$: The collection Ω of all closed subsets of X which contain E is not empty, since $X \in \Omega$; let $\bar{E}$ be the intersection of all members of Ω.)

(*c*) A set $K \subset X$ is *compact* if every open cover of K contains a finite subcover. More explicitly, the requirement is that if $\{V_\alpha\}$ is a collection of open sets whose union contains K, then the union of some finite subcollection of $\{V_\alpha\}$ also contains K.

In particular, if X is itself compact, then X is called a *compact space*.

(*d*) A *neighborhood* of a point $p \in X$ is any open subset of X which contains p. (The use of this term is not quite standardized; some use "neighborhood of p" for any set which contains an open set containing p.)

(*e*) X is a *Hausdorff space* if the following is true: If $p \in X$, $q \in X$, and $p \neq q$, then p has a neighborhood U and q has a neighborhood V such that $U \cap V = \varnothing$.

(*f*) X is *locally compact* if every point of X has a neighborhood whose closure is compact. $\rightarrow \mathbb{R}$ $\quad a \in \mathbb{R} \Rightarrow \overline{(a-1, a+1)} = [a-1, a+1]$ is compact

2.7 Theorem *Suppose U is open in a locally compact Hausdorff space X, $K \subset U$, and K is compact. Then there is an open set V with compact closure such that*

$$:X \to \mathbb{R} \qquad K \subset V \subset \bar{V} \subset U.$$

This condition implies that X is locally compact Hausdorff space.

2.8 Definition Let f be a real (or extended-real) function on a topological space. If

$$f^{-1}((\alpha,\infty)) = \{x : f(x) > \alpha\} \quad \text{(نیم پیوسته پایین)}$$

is open for every real α, f is said to be *lower semicontinuous*. If

$$\{x : f(x) < \alpha\}$$

is open for every real α, f is said to be *upper semicontinuous*.

$$\chi^{-1}((\alpha,\infty)) = \begin{cases} X & \alpha < 0 \\ G & 0 \le \alpha < 1 \\ \ldots \end{cases}$$

(*a*) Characteristic functions of *open* sets are *lower* semicontinuous.
(*b*) Characteristic functions of *closed* sets are *upper* semicontinuous.

what about $\alpha = 0$ & $\alpha = 1$?

The following property is an almost immediate consequence of the definitions:

(*c*) *The supremum of any collection of lower semicontinuous functions is lower semicontinuous. The infimum of any collection of upper semicontinuous functions is upper semicontinuous.*

2.9 Definition The *support* of a complex function f on a topological space X is the closure of the set $\{x \in X : f(x) \neq 0\}$. $: X \to \mathbb{C}$

$$\boxed{\operatorname{supp} f = \overline{\{x : f(x) \neq 0\}} \subseteq X}$$

The collection of all continuous complex functions on X whose support is compact is denoted by $C_c(X)$.

Observe that $C_c(X)$ is a vector space. This is due to two facts:

(*a*) The support of $f + g$ lies in the union of the support of f and the support of g, and any finite union of compact sets is compact.
(*b*) The sum of two continuous complex functions is continuous, as are scalar multiples of continuous functions.

$\operatorname{supp}(f+g) \subseteq \operatorname{supp} f \cup \operatorname{supp} g$

$x \notin \operatorname{supp} f \cup \operatorname{supp} g \Rightarrow f(x) = 0 \;\&\; g(x) = 0 \Rightarrow (f+g)(x) = 0 \Rightarrow x \notin \operatorname{supp}(f+g)$

Corollary *The range of any $f \in C_c(X)$ is a compact subset of the complex plane.*

$$\forall x \in X;\ \left.\begin{array}{l} f(x) = 0 \Rightarrow f(x) \in \{0\} \\ f(x) \neq 0 \Rightarrow x \in K \Rightarrow f(x) \in f(K) \end{array}\right\} \Rightarrow x \in f(K) \cup \{0\}$$

In fact, if K is the support of $f \in C_c(X)$, then $f(X) \subset f(K) \cup \{0\}$. If X is not compact, then $0 \in f(X)$, but 0 need not lie in $f(K)$, as is seen by easy examples.

Why $f(X)$ is closed? compact

2.11 Notation In this chapter the following conventions will be used. The notation

$$K \prec f \tag{1}$$

will mean that K is a compact subset of X, that $f \in C_c(X)$, that $0 \leq f(x) \leq 1$ for all $x \in X$, and that $f(x) = 1$ for all $x \in K$. The notation

$\chi_K \leq f$

$$f \prec V \tag{2}$$

will mean that V is open, that $f \in C_c(X)$, $0 \le f \le 1$, and that the support of f lies in V. The notation

$f \le \chi_V$

$$K \prec f \prec V \tag{3}$$

will be used to indicate that both (1) and (2) hold.

2.12 Urysohn's Lemma *Suppose X is a locally compact Hausdorff space, V is open in X, $K \subset V$, and K is compact. Then there exists an $f \in C_c(X)$, such that*

$$K \prec f \prec V. \tag{1}$$

2.13 Theorem *Suppose $V_1, \ldots, V_n$ are open subsets of a locally compact Hausdorff space X, K is compact, and*

$$K \subset V_1 \cup \cdots \cup V_n.$$

Then there exist functions $h_i \prec V_i$ $(i = 1, \ldots, n)$ such that

$$h_1(x) + \cdots + h_n(x) = 1 \qquad (x \in K). \tag{1}$$

Because of (1), the collection $\{h_1, \ldots, h_n\}$ is called a *partition of unity* on K, subordinate to the cover $\{V_1, \ldots, V_n\}$.

The Riesz Representation Theorem

2.14 Theorem *Let X be a locally compact Hausdorff space, and let Λ be a positive linear functional on $C_c(X)$. Then there exists a σ-algebra $\mathfrak{M}$ in X which contains all Borel sets in X, and there exists a unique positive measure μ on $\mathfrak{M}$ which represents Λ in the sense that*

$\Lambda : C_c(X) \to \mathbb{C}$

(a) $\Lambda f = \int_X f \, d\mu$ *for every* $f \in C_c(X)$,

linearity $\Lambda(\alpha f + g) = \alpha \Lambda f + \Lambda g$

and which has the following additional properties:

Positivity: $f \ge 0 \Rightarrow \Lambda f \ge 0$

(b) $\mu(K) < \infty$ *for every compact set* $K \subset X$.

$f \le g \Rightarrow g - f \ge 0 \Rightarrow \Lambda(g-f) \ge 0 \Rightarrow \Lambda g - \Lambda f \ge 0 \Rightarrow \Lambda f \le \Lambda g$

(c) *For every* $E \in \mathfrak{M}$, *we have*

$$\mu(E) = \inf \{\mu(V) : E \subset V,\ V \text{ open}\}.$$

outer regularity

(d) *The relation*

$$\mu(E) = \sup \{\mu(K) : K \subset E,\ K \text{ compact}\}$$

inner regularity

holds for every open set E, and for every $E \in \mathfrak{M}$ with $\mu(E) < \infty$.

(e) *If $E \in \mathfrak{M}$, $A \subset E$, and $\mu(E) = 0$, then $A \in \mathfrak{M}$.*

(e) *If $E \in \mathfrak{M}$, $A \subset E$, and $\mu(E) = 0$, then $A \in \mathfrak{M}$* and so $\mu(A)=0$

Let us begin by proving the uniqueness of μ. If μ satisfies (c) and (d), it is clear that μ is determined on $\mathfrak{M}$ by its values on compact sets. Hence it suffices to prove that $\mu_1(K) = \mu_2(K)$ for all K, whenever μ_1 and μ_2 are measures for which the theorem holds. So, fix K and $\epsilon > 0$. By (b) and (c), there exists a $V \supset K$ with $\mu_2(V) < \mu_2(K) + \epsilon$; by Urysohn's lemma, there exists an f so that $K \prec f \prec V$; hence

$\overset{\shortparallel}{\inf\limits_{V \text{ open}} \mu_2(V)}$

$$\mu_1(K) = \int_X \chi_K \, d\mu_1 \le \int_X f \, d\mu_1 \boxed{= \Lambda f =} \int_X f \, d\mu_2$$

$\chi_K \le f \le \chi_V$

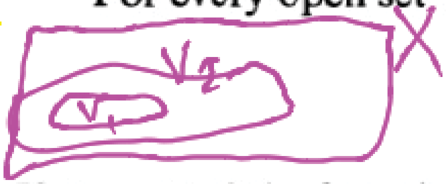
$$\le \int_X \chi_V \, d\mu_2 = \mu_2(V) < \mu_2(K) + \epsilon.$$

Thus $\mu_1(K) \le \mu_2(K)$. If we interchange the roles of μ_1 and μ_2, the opposite inequality is obtained, and the uniqueness of μ is proved.

Incidentally, the above computation shows that (a) forces (b).

Construction of μ and $\mathfrak{M}$

For every open set V in X, define



$$\boxed{\mu(V) = \sup \{\Lambda f : f \prec V\}.} \tag{1}$$

Since $\{\Lambda f : f \prec V_1\} \subseteq \{\Lambda f : f \prec V_2\}$

If $V_1 \subset V_2$, it is clear that (1) implies $\mu(V_1) \le \mu(V_2)$. Hence

$$\boxed{\mu(E) = \inf \{\mu(V) : E \subset V,\ V \text{ open}\},} \tag{2}$$

if E is an open set, and it is consistent with (1) to *define* $\mu(E)$ by (2), for *every* $E \subset X$.

Note that although we have defined $\mu(E)$ for every $E \subset X$, the countable additivity of μ will be proved only on a certain σ-algebra $\mathfrak{M}$ in X.

Let $\mathfrak{M}_F$ be the class of all $E \subset X$ which satisfy two conditions: $\mu(E) < \infty$, and

$$\mu(E) = \sup \{\mu(K) : K \subset E,\ K \text{ compact}\}. \tag{3}$$

$\mathfrak{M}_F$

Finally, let $\mathfrak{M}$ be the class of all $E \subset X$ such that $E \cap K \in \mathfrak{M}_F$ for every compact K.

$E \cap K \subseteq E \Rightarrow \mu(E \cap K) = 0 \Rightarrow E \cap K$

Since $\{\mu(V) : A \subseteq V_{\text{open}}\} \supseteq \{\mu(V) : B \subseteq V_{\text{open}}\}$

Proof that μ and $\mathfrak{M}$ have the required properties

(3) $K \subseteq E \Rightarrow \mu(K) = 0$

It is evident that μ is *monotone*, i.e., that $\mu(A) \le \mu(B)$ if $A \subset B$ and that

$\mu(E) = 0$ implies $E \in \mathfrak{M}_F$ and $\underbrace{E \in \mathfrak{M}}_{R}$. Thus (*e*) holds, and so does (*c*), by definition.

Since the proof of the other assertions is rather long, it will be convenient to divide it into several steps.

Observe that the positivity of Λ implies that Λ is *monotone*: $f \le g$ implies $\Lambda f \le \Lambda g$. This is clear, since $\Lambda g = \Lambda f + \underbrace{\Lambda(g - f)}_{\ge 0}$ and $g - f \ge 0$. This monotonicity will be used in Steps II and X.

STEP I *If $E_1, E_2, E_3, \ldots$ are arbitrary subsets of X, then*

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} \mu(E_i). \tag{4}$$

PROOF We first show that

$$\mu(V_1 \cup V_2) \le \mu(V_1) + \mu(V_2), \tag{5}$$

$\underbrace{\operatorname{supp} g}_{\text{compact}} \subseteq V_1 \cup V_2$

if V_1 and V_2 are open. Choose $g \prec V_1 \cup V_2$. By Theorem 2.13 there are functions h_1 and h_2 such that $h_i \prec V_i$ and $h_1(x) + h_2(x) = 1$ for all x in the support of g. Hence $h_i g \prec V_i$, $g = h_1 g + h_2 g$, and so

$h_i = 0$ on $V_i^c \Rightarrow h_i g = 0$ on V_i^c $\mu(V) = \sup_{k \prec V} \Lambda k$

$$\Lambda g = \Lambda(h_1 g) + \Lambda(h_2 g) \le \mu(V_1) + \mu(V_2). \tag{6}$$

Since (6) holds for *every* $g \prec V_1 \cup V_2$, (5) follows.

If $\mu(E_i) = \infty$ for some i, then (4) is trivially true. Suppose therefore that $\mu(E_i) < \infty$ for every i. Choose $\epsilon > 0$. By (2) there are open sets $V_i \supset E_i$ such that

$\mu(E) = \inf_{E \subseteq V \text{ open}} \mu(V)$

$$\mu(V_i) < \mu(E_i) + 2^{-i}\epsilon \quad (i = 1, 2, 3, \ldots).$$

open open compact $\operatorname{supp} f \subseteq V = \bigcup_{i=1}^{\infty} V_i \Rightarrow \exists n;\ \operatorname{supp} f \subseteq \bigcup_{i=1}^{n} V_i$

Put $V = \bigcup_1^\infty V_i$, and choose $f \prec V$. Since f has compact support, we see that $f \prec V_1 \cup \cdots \cup V_n$ for some n. Applying induction to (5), we therefore obtain

$\sum_{i=1}^{n} \mu(E_i) + \left(\sum_{i=1}^{\infty} 2^{-i}\right)\epsilon$

$$\Lambda f \le \mu(V_1 \cup \cdots \cup V_n) \le \mu(V_1) + \cdots + \mu(V_n) \le \sum_{i=1}^{\infty} \mu(E_i) + \epsilon.$$

Since this holds for every $f \prec V$, and since $\bigcup E_i \subset V$, it follows that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \le \mu(V) \le \sum_{i=1}^{\infty} \mu(E_i) + \epsilon,$$

upper bound on $\{\Lambda f : f \prec V\}$

which proves (4), since ϵ was arbitrary. ////

<u>Note</u>. If $A \subseteq X$ & $B \subseteq X$ are compact, then so is $A \cup B$.

Let $\{G_\alpha\}$ be an open cover for $A \cup B$. So $A \subseteq \bigcup_\alpha G_\alpha$ & $B \subseteq \bigcup_\alpha G_\alpha$. Hence

$\exists\, \alpha_1, \dots, \alpha_n$; $A \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$

$\exists\, \alpha_{n+1}, \dots, \alpha_m$; $B \subseteq \bigcup_{j=n+1}^{m} G_{\alpha_j}$.

Therefore $A \cup B \subseteq \bigcup_{k=1}^{m} G_{\alpha_k}$. $\square$

STEP II *If K is compact, then $K \in \mathfrak{M}_F$ and*

$$\mu(K) = \inf \{\Lambda f : K \prec f\}. \qquad (7)$$

$x \in K \Rightarrow f(x) = 1 > \alpha \Rightarrow x \in V_\alpha$

This implies assertion (*b*) of the theorem.

PROOF If $K \prec f$ and $0 < \alpha < 1$, let $V_\alpha = \{x : f(x) > \alpha\}$. Then $K \subset V_\alpha$, and $\alpha g \le f$ whenever $g \prec V_\alpha$. Hence

$\forall x;\ (\alpha g)(x) = \alpha g(x)$

$$\mu(K) \le \mu(V_\alpha) = \sup \{\Lambda g : g \prec V_\alpha\} \le \alpha^{-1} \Lambda f.$$

$g \prec V_\alpha \Rightarrow \alpha g \le f \Rightarrow \alpha \Lambda g \le \Lambda f$

Let $\alpha \to 1$, to conclude that

$0 \le f \le 1 \Rightarrow \Lambda f \le \Lambda 1 < \infty$

$$\mu(K) \le \Lambda f. \qquad (8)$$

Thus $\mu(K) < \infty$. Since K evidently satisfies (3), $K \in \mathfrak{M}_F$.

If $\epsilon > 0$, there exists $V \supset K$ with $\mu(V) < \mu(K) + \epsilon$. By Urysohn's lemma, $K \prec f \prec V$ for some f. Thus

$$\Lambda f \le \mu(V) < \mu(K) + \epsilon,$$

which, combined with (8), gives (7). ////

STEP III *Every open set satisfies* (3). *Hence $\mathfrak{M}_F$ contains every open set V with $\mu(V) < \infty$.*

$\mu(V) = \sup_{K \subseteq V} \mu(K)$ $\qquad \sup\{\Lambda f : f \prec V\}$

PROOF Let α be a real number such that $\alpha < \mu(V)$. There exists an $f \prec V$ with $\alpha < \Lambda f$. If W is *any* open set which contains the support K of f, then $f \prec W$, hence $\Lambda f \le \mu(W)$. Thus $\Lambda f \le \mu(K)$. This exhibits a compact $K \subset V$ with $\alpha < \mu(K)$, so that (3) holds for V. ////

STEP IV *Suppose $E = \bigcup_{i=1}^{\infty} E_i$, where $E_1, E_2, E_3, \ldots$ are pairwise disjoint members of $\mathfrak{M}_F$. Then*

$\mu(E) - \varepsilon < \mu(K)$

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i). \qquad (9)$$

If, in addition, $\mu(E) < \infty$, then also $E \in \mathfrak{M}_F$.

compact open

PROOF We first show that

for $K_1 \subseteq K_2^c$

$$\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2) \qquad (10)$$

if K_1 and K_2 are disjoint compact sets. Choose $\epsilon > 0$. By Urysohn's lemma, there exists $f \in C_c(X)$ such that $f(x) = 1$ on K_1, $f(x) = 0$ on K_2, and there exists $f \in C_c(X)$ such that $f(x) = 1$ on K_1, $f(x) = 0$ on K_2, and $0 \le f \le 1$. By Step II there exists g such that

$\mu(K) = \inf_{K \prec f} \Lambda f$

$$K_1 \cup K_2 \prec g \quad \text{and} \quad \Lambda g < \mu(K_1 \cup K_2) + \epsilon.$$

∞ Note that $K_1 \prec fg$ and $K_2 \prec (1-f)g$. Since Λ is linear, it follows from (8) that

$\mu(\underset{E}{\cup E_i}) \le \sum \mu(E_i)$

$$\mu(K_1) + \mu(K_2) \le \Lambda(fg) + \Lambda(g - fg) = \Lambda g < \mu(K_1 \cup K_2) + \epsilon.$$

Since ϵ was arbitrary, (10) follows now from Step I. $\mu(K_1 \cup K_2) \le \mu(K_1) + \mu(K_2)$

If $\mu(E) = \infty$, (9) follows from Step I. Assume therefore that $\mu(E) < \infty$, and choose $\epsilon > 0$. Since $E_i \in \mathfrak{M}_F$, there are compact sets $H_i \subset E_i$ with

$\mu(E_i) < \infty$, $\mu(E_i) = \sup_{K \subseteq E_i} \mu(K)$

$$\mu(H_i) > \mu(E_i) - 2^{-i}\epsilon \qquad (i = 1, 2, 3, \ldots). \tag{11}$$

Putting $K_n = H_1 \cup \cdots \cup H_n$ and using induction on (10), we obtain

$H_i \subseteq E_i \subseteq E \Rightarrow K_n = \bigcup_{i=1}^{n} H_i \subseteq E$

$$\mu(E) \ge \mu(K_n) = \sum_{i=1}^{n} \mu(H_i) > \sum_{i=1}^{n} \mu(E_i) - \epsilon. \tag{12}$$

$\sum_{i=1}^{n} 2^{-i}\varepsilon$ $> \sum_{i=1}^{n} \mu(E_i) - \varepsilon$

Since (12) holds for every n and every $\epsilon > 0$, the left side of (9) is not smaller than the right side, and so (9) follows from Step I. $\lim_{n\to\infty} \mu(E) \ge \lim_{n\to\infty} \left(\sum_{i=1}^{n} \mu(E_i) - \varepsilon \right)$

But if $\mu(E) < \infty$ and $\epsilon > 0$, (9) shows that

$\mu(E) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i) \Rightarrow |\mu(E) - \sum_{i=1}^{n} \mu(E_i)| < \varepsilon \;\; \exists N \, \forall n \ge N$

$\underset{n=N}{\Rightarrow} -\varepsilon < \mu(E) - \sum_{i=1}^{N} \mu(E_i) < \varepsilon \Rightarrow$

$$\mu(E) \le \sum_{i=1}^{N} \mu(E_i) + \epsilon \tag{13}$$

$\le \mu(K_n) + 2\varepsilon$

$\mu(E) \ge \sum_{i=1}^{\infty} \mu(E_i) - \varepsilon$, $\varepsilon \to 0$, $\mu(E) \ge \sum_{i=1}^{\infty} \mu(E_i)$

for some N. By (12), it follows that $\mu(E) \le \mu(K_N) + 2\epsilon$, and this shows that E satisfies (3); hence $E \in \mathfrak{M}_F$. ////

STEP V *If $E \in \mathfrak{M}_F$ and $\epsilon > 0$, there is a compact K and an open V such that $K \subset E \subset V$ and $\mu(V - K) < \epsilon$.*

PROOF Our definitions show that there exist $K \subset E$ and $V \supset E$ so that

$\exists V \supseteq E;\ \mu(E) + \frac{\varepsilon}{2} > \mu(V)$ $\qquad$ $\exists K \subseteq E;\ \mu(E) - \frac{\varepsilon}{2} < \mu(K)$

$$\mu(V) - \frac{\epsilon}{2} < \mu(E) < \mu(K) + \frac{\epsilon}{2}.$$

$V \cap K^c$

Since $V - K$ is open, $V - K \in \mathfrak{M}_F$, by Step III. Hence Step IV implies that

closed

$$\mu(K) + \mu(V - K) = \mu(V) < \mu(K) + \epsilon. \qquad ////$$

$K \cup V - K = V$ $\qquad$ $\mu(V-K) = \mu(V) - \mu(K)$ $\quad < \infty$

STEP VI *If $A \in \mathfrak{M}_F$ and $B \in \mathfrak{M}_F$, then $A - B$, $A \cup B$, and $A \cap B$ belong to $\mathfrak{M}_F$* $< \mu(E) < \infty$

PROOF If $\epsilon > 0$, Step V shows that there are sets K_i and V_i such that $K_1 \subset A \subset V_1$, $K_2 \subset B \subset V_2$, and $\mu(V_i - K_i) < \frac{\epsilon}{2}$, for $i = 1, 2$. Since

$$A - B \subset V_1 - K_2 \subset (V_1 - K_1) \cup (K_1 - V_2) \cup (V_2 - K_2), \quad (2)$$

$A \subseteq V_1$, $B^c \subseteq K_2^c$ $\Big\} \Rightarrow A \cap B^c \subseteq V_1 \cap K_2^c$

Step I shows that

$A - B \subseteq A$

$\mu(A-B) \le \mu(A) < \infty$

$$\mu(A - B) \le \frac{\epsilon}{2} + \mu(K_1 - V_2) + \frac{\epsilon}{2}. \tag{14}$$

Since $K_1 - V_2$ is a compact subset of $A - B$, (14) shows that $A - B$ satisfies (3), so that $A - B \in \mathfrak{M}_F$.

Since $A \cup B = (A - B) \cup B$, an application of Step IV shows that $A \cup B \in \mathfrak{M}_F$. Since $A \cap B = A - (A - B)$, we also have $A \cap B \in \mathfrak{M}_F$. ////

STEP VII *$\mathfrak{M}$ is a σ-algebra in X which contains all Borel sets.*

$x \in A \cap K \Rightarrow x \in \bigcup_{i=1} A_i$ & $x \in K$ $\Rightarrow x \in A_j$ & $x \in K \Rightarrow x \in A_j \cap K$

PROOF Let K be an arbitrary compact set in X. $\Rightarrow \big\{ x \in B_1 \cup \dots \cup B_j \Rightarrow x \in \bigcup_{n=1}^{j} B_n$

If $A \in \mathfrak{M}$, then $A^c \cap K = K - (A \cap K)$, so that $A^c \cap K$ is a difference of two members of $\mathscr{M}_F$. Hence $A^c \cap K \in \mathfrak{M}_F$, and we conclude: $A \in \mathfrak{M}$ implies $A^c \in \mathfrak{M}$.

$x \notin B_1 \cup \dots \cup B_{j-1} \Rightarrow x \in B_j \Rightarrow x \in \bigcup_{n=1}^{\infty} B_n$

Next, suppose $A = \bigcup_1^\infty A_i$, where each $A_i \in \mathfrak{M}$. Put $B_1 = A_1 \cap K$, and

$\mu(\bigcup_{n=1}^{\infty} B_n) \le \mu(K) < \infty$

$$B_n = (A_n \cap K) - (B_1 \cup \cdots \cup B_{n-1}) \qquad (n = 2, 3, 4, \ldots). \tag{15}$$

Then $\{B_n\}$ is a disjoint sequence of members of $\mathfrak{M}_F$, by Step VI, and $A \cap K = \bigcup_1^\infty B_n$. It follows from Step IV that $A \cap K \in \mathfrak{M}_F$. Hence $A \in \mathfrak{M}$.

Finally, if C is closed, then $C \cap K$ is compact, hence $C \cap K \in \mathfrak{M}_F$, so $C \in \mathfrak{M}$. In particular, $X \in \mathfrak{M}$.

$\subseteq K$

We have thus proved that $\mathfrak{M}$ is a σ-algebra in X which contains all closed subsets of X. Hence $\mathfrak{M}$ contains all Borel sets in X. ////

STEP VIII *$\mathfrak{M}_F$ consists of precisely those sets $E \in \mathfrak{M}$ for which $\mu(E) < \infty$.* VIII

$\mathfrak{M}_F = \{E \subseteq X \mid \mu(E) < \infty \ \& \ \mu(E) = \sup \mu(K)\} = \{E \subseteq X \mid E \in \mathfrak{M} \ \& \ \mu(E) < \infty\}$

This implies assertion (*d*) of the theorem.

PROOF If $E \in \mathfrak{M}_F$, Steps II and VI imply that $E \cap K \in \mathfrak{M}_F$ for every compact K, hence $E \in \mathfrak{M}$. So $\mathfrak{M}_F \subseteq \{E \in \mathfrak{M} \mid \mu(E) < \infty\}$

Conversely, suppose $E \in \mathfrak{M}$ and $\mu(E) < \infty$, and choose $\epsilon > 0$. There is an open set $V \supset E$ with $\mu(V) < \infty$; by III and V, there is a compact $K \subset V$ with $\mu(V - K) < \frac{\epsilon}{2}$. Since $E \cap K \in \mathfrak{M}_F$, there is a compact set $H \subset E \cap K$ with

$\exists H \subseteq E \cap K$; $\mu(E \cap K) - \epsilon < \mu(H)$

$$\mu(E \cap K) < \mu(H) + \frac{\epsilon}{2}$$

$\mu(E) = \inf \mu(V)$

Since $E \subset (E \cap K) \cup (V - K)$, it follows that $(\exists V;\ \mu(V) < \mu(E) + \gamma < \infty$

(3)

$$\mu(E) \le \mu(E \cap K) + \mu(V - K) < \mu(H) + \cancel{2}\epsilon,$$

which implies that $E \in \mathfrak{M}_F$. ////

$\mu(E) = \sup \mu(H)$ and so $H \subseteq E$

$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$

STEP IX *μ is a measure on $\mathfrak{M}$.*

PROOF The countable additivity of μ on $\mathfrak{M}$ follows immediately from Steps IV and VIII. $E_1, E_2, \ldots$ are disjoint in $\mathfrak{M} \Rightarrow \{ E_i \in \mathfrak{M}_F\ \forall i$, $\forall i;\ \mu(E_i) < \infty$ (iv); $\mu(\bigcup E_i) = \infty = \sum_{i=1}^{\infty} \mu(E_i)$, $\exists i;\ \mu(E_i) = \infty$

STEP X *For every $f \in C_c(X)$, $\Lambda f = \int_X f\, d\mu$.*

$\int f\,d\mu = \int (u+iv)\,d\mu = \int u + i\int v = \Lambda u + i\Lambda v$ linearity of Λ $= \Lambda(u+iv) = \Lambda(f)$

This proves (a), and completes the theorem.

PROOF Clearly, it is enough to prove this for real f. Also, it is enough to prove the *inequality*

(4)

$$\Lambda f \le \int_X f\, d\mu \tag{16}$$

for every real $f \in C_c(X)$. For once (16) is established, the linearity of Λ shows that

If $f \in C_c(X)$ is real,

$$-\Lambda f = \Lambda(-f) \le \int_X (-f)\, d\mu = -\int_X f\, d\mu,$$ then $\operatorname{ran}(f) \subseteq \mathbb{R}$

which, together with (16), shows that equality holds in (16). So $\operatorname{ran} f \subseteq [a, b]$ compact for some a, b.

Let K be the support of a real $f \in C_c(X)$, let $[a, b]$ be an interval which contains the range of f (note the Corollary to Theorem 2.10), choose $\epsilon > 0$, and choose y_i, for $i = 0, 1, \ldots, n$, so that $y_i - y_{i-1} < \epsilon$ and

$\exists n;\ \frac{b-a}{n+1} < \varepsilon$

$$y_0 < a < y_1 < \cdots < y_n = b. \tag{17}$$

Put

$f^{-1}((y_{i-1}, y_i]) = f^{-1}\left(\bigcap_{n=1}^{\infty} (y_{i-1}, y_i + \frac{1}{n})\right) = \bigcap_{n=1}^{\infty} f^{-1}\left((y_{i-1}, y_i + \frac{1}{n})\right)$

$$E_i = \{x : y_{i-1} < f(x) \le y_i\} \cap K \qquad (i = 1, \ldots, n). \tag{18}$$

Since f is continuous, f is Borel measurable, and the sets E_i are therefore disjoint Borel sets whose union is K. There are open sets $V_i \supset E_i$ such that

$$\mu(V_i) < \mu(E_i) + \frac{\epsilon}{n} \qquad (i = 1, \ldots, n) \tag{19}$$

and such that $f(x) < y_i + \epsilon$ for all $x \in V_i$. By Theorem 2.13, there are functions $h_i \prec V_i$ such that $\sum h_i = 1$ on K. Hence $f = \sum h_i f$, and Step II shows that

$$\mu(K) \leq \Lambda(\sum h_i) = \sum \Lambda h_i.$$

Since $h_i f \leq (y_i + \epsilon)h_i$, and since $y_i - \epsilon < f(x)$ on E_i, we have

$$\begin{aligned}
\Lambda f = \sum_{i=1}^{n} \Lambda(h_i f) &\leq \sum_{i=1}^{n} (y_i + \epsilon)\Lambda h_i \\
&= \sum_{i=1}^{n} (|a| + y_i + \epsilon)\Lambda h_i - |a| \sum_{i=1}^{n} \Lambda h_i \\
&\leq \sum_{i=1}^{n} (|a| + y_i + \epsilon)[\mu(E_i) + \epsilon/n] - |a|\mu(K) \\
&= \sum_{i=1}^{n} (y_i - \epsilon)\mu(E_i) + 2\epsilon\mu(K) + \frac{\epsilon}{n}\sum_{i=1}^{n}(|a| + y_i + \epsilon) \\
&\leq \int_X f\,d\mu + \epsilon[2\mu(K) + |a| + b + \epsilon].
\end{aligned}$$

Since ϵ was arbitrary, (16) is established, and the proof of the theorem is complete. ////

Regularity Properties of Borel Measures

2.15 Definition A measure μ defined on the σ-algebra of all Borel sets in a locally compact Hausdorff space X is called a ***Borel measure on X***. If μ is positive, a Borel set $E \subset X$ is *outer regular* or *inner regular*, respectively, if E has property (*c*) or (*d*) of Theorem 2.14. If every Borel set in X is both outer and inner regular, μ is called *regular*.

$\Lambda : C_c(X) \to \mathbb{C}$ is a positive linear functional.

Let f be a real function in $C_c(X)$. Then $f = \underbrace{\frac{|f|+f}{2}}_{\geq 0} - \underbrace{\frac{|f|-f}{2}}_{\geq 0}$

So $\Lambda f = \underbrace{\Lambda\left(\frac{|f|+f}{2}\right)}_{\geq 0} - \underbrace{\Lambda\left(\frac{|f|-f}{2}\right)}_{\geq 0} \in \mathbb{R}$.

① $\sum_{i=1}^{\infty} a_i$ is $\left\{\sum_{i=1}^{n} a_i\right\}_{n=1}^{\infty}$ & its value is $\lim_{n\to\infty} \sum_{i=1}^{n} a_i$.

A sequence is a function on $\mathbb{N}$.

② $V_1 - K_2 \subseteq (V_1 - K_1) \cup (K_1 - V_2) \cup (V_2 - K_2)$

$x \in V_1 - K_2 \Rightarrow \begin{cases} x \in V_1 \\ \& \\ x \notin K_2 \end{cases}$ or: $x \notin K_1 \Rightarrow x \in V_1 - K_1$; $x \in K_1$ or: $x \notin V_2 \Longrightarrow x \in K_1 - V_2$; $\left.\begin{matrix} x \in V_2 \end{matrix}\right\} \Longrightarrow x \in V_2 - K_2$ ✓

Describe the Lebesgue measure : پیروز

on $\mathbb{R}^n$. $(n = 1, 2, \ldots)$

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③ $$x \in E \Rightarrow \begin{cases} x \in E \cap K \Rightarrow x \in (E \cap K) \cup (V - K) & x \in K \\ x \in V - K \Rightarrow x \in (E \cap K) \cup (V - K) & x \notin K \end{cases}$$
$x \in V$

④ Let $\Lambda f \leq \int_X f \, d\mu \quad \forall f \quad (\clubsuit)$.

Let f be a real function. It follows from $(\clubsuit)$ that

$$\underbrace{\Lambda(-f)}_{= -\Lambda(f)} \leq \underbrace{\int_X -f \, d\mu}_{= -\int_X f \, d\mu}. \text{ Hence } \Lambda f \geq \int_X f \, d\mu \quad (\Diamond).$$

By $(\clubsuit)$, $(\Diamond)$, we have $\Lambda f = \int_X f \, d\mu$. $\square$

≡ Banach Spaces ≡

Def. A function $\|\cdot\|: X \to \mathbb{R}$ is called a semi-norm if (i) $x=0 \Rightarrow \|x\|=0$ (ii) $\|\lambda x\| = |\lambda|\,\|x\|$ (iii) $\|x+y\| \le \|x\|+\|y\|$.

If $\|x\|=0 \Rightarrow x=0$, then $\|\cdot\|$ is called a norm.

Exercise. $\forall x;\ \|x\| \geq 0$.

Solution. $\|0\| = \|x-x\| \leq \|x\| + \overbrace{\|-x\|}^{\|x\|} = 2\|x\|$

Then $(X, \|\cdot\|)$ is called a normed space. If $d(x,y) = \|x-y\|$, then d is a metric. If d is complete, then $(X, \|\cdot\|)$ is said to be a Banach space.

Examples 1) $M_n(\mathbb{C})$, $\|[a_{ij}]\|_m = \max_{1\le i,j\le n} |a_{ij}|$, $\|[a_{ij}]\|_p = \left(\sum_{i,j} |a_{ij}|^p\right)^{\frac{1}{p}}$

$\|[a_{ij}]\| = \max_{1\le j\le n}\left(\sum_{i=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}$, $\|[a_{ij}]\|_r = \max_{1\le i\le n}\left(\sum_{j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}$

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

the j-th column

2) $\mathbb{C}^n$, $\|(z_1, \ldots, z_n)\| = \left(\sum_{i=1}^{n} |z_i|^2\right)^{\frac{1}{2}}$

$\{(x_n) \mid \exists m\, \forall n \ge m;\ x_n = 0\}$

3) $c_{00} \subseteq c_0 \subseteq c \subseteq \ell^\infty = \{(x_n) \mid \sup_n |x_n| < \infty\}$

$c_{00} = \{(x_n) \mid \exists m\, \forall n \ge m;\ x_n = 0\}$, $c_0 = \{(x_n) \mid \lim_n x_n = 0\}$, $c = \{(x_n) \mid \{x_n\} \text{ converges}\}$

$\lambda\{x_n\}+\{y_n\}=\{\lambda x_n+y_n\}$

4) $C_c(X)\subseteq C_0(X)\subseteq \ell^\infty(X)$

$C_{00}(X)$

$C_0(X)=\{f:X\to\mathbb{C} \mid f \text{ is cts } \& \ \forall\varepsilon\ \{x:|f(x)|\geq\varepsilon\} \text{ is compact}\}$

locally compact Hausdorff space

Let $f\in C_c(X)$. Then

$$\underset{\text{closed}}{f^{-1}}\big(\underbrace{[\varepsilon,\infty)}_{\text{closed}}\big)=\{x:|f(x)|\geq\varepsilon\}\subseteq \underbrace{\mathrm{supp}(f)}_{\text{compact}}$$

$\{x:|f(x)|\geq\varepsilon\}$ is compact for all $\varepsilon>0$.

$(\lambda f+g)(x)=\lambda f(x)+g(x)$

$\ell^\infty(X)$ = the space of all bd functions

$\|f\|=\sup_{x\in X}|f(x)|$

Note. Example 3 is a special case of Example 4.

$\mathbb{N}$ is a locally compact Hausdorff space when we consider the relative Euclidean top from $\mathbb{R}$.

$C_c(\mathbb{N})=\{f:\mathbb{N}\to\mathbb{C} \mid \mathrm{supp}\, f \text{ is compact}\}$

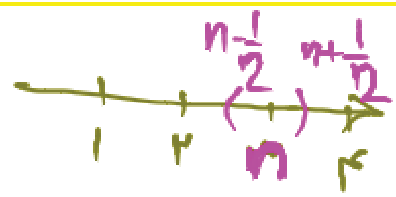
$\mathrm{supp}\, f \subseteq \mathbb{N}$

$\underbrace{\qquad}$ supp f is finite

say, $\{n_1,\dots,n_k\}$

Hence $\forall n > n_k$; $f(n)=0$

$=c_{00}$



$\tau_{\mathbb{N}}=\{\mathbb{N}\cap G \mid G \text{ is open in } \mathbb{R}\}$

$\{n\}=\mathbb{N}\cap(n-\frac{1}{2},n+\frac{1}{2})$ is open, so the top on $\mathbb{N}$ is the discrete top.

$C_0(\mathbb{N})=c_0$

$\ell^\infty(\mathbb{N})=\ell^\infty$

<u>Theorem</u>. Let X & Y be normed spaces. Then $(X\times Y, \|(x,y)\| = \max\{\|x\|,\|y\|\})$ is Ban iff so are X, Y.

<u>Proof</u>. ($\Leftarrow$) Let X, Y be Ban. Let $\{(x_n,y_n)\}$ be a Cauchy sequence in $X\times Y$. So

$$\forall \varepsilon\, \exists N\, \forall m,n \ge N ;\ \|\underbrace{(x_n,y_n)-(x_m,y_m)}_{(x_n-x_m,\ y_n-y_m)}\| < \varepsilon$$

$$\begin{matrix}\|x_n-x_m\| \\ \|y_n-y_m\|\end{matrix} \le \overbrace{\max\{\|x_n-x_m\|,\ \|y_n-y_m\|\}}$$

Hence both $\{x_n\}$ & $\{y_n\}$ are Cauchy sequences. Since X, Y are Ban, $\exists x\in X\ \exists y \in Y;\ x_n \to x$ & $y_n \to y$.

Let $\varepsilon>0$ be given. $\exists N_1\, \forall n \ge N_1;\ \|x_n - x\| < \frac{\varepsilon}{2}$

$\exists N_2\, \forall n \ge N_2;\ \|y_n - y\| < \frac{\varepsilon}{3}$

$\forall n \ge 2N_1 + 71N_2;\ \|x_n - x\| < \frac{\varepsilon}{2}$ & $\|y_n - y\| < \frac{\varepsilon}{3}$

Hence

$$\|(x_n,y_n) - (x,y)\| = \max\{\|x_n - x\|, \|y_n - y\|\} \le \max\{\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{3}\} < \varepsilon$$

$(x_n, y_n) \to (x, y)$.

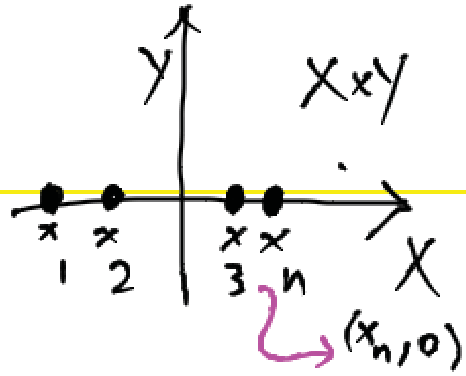
$\left.\begin{matrix} a \le b \\ c \le d \end{matrix}\right\} \Rightarrow \max\{a,c\} \le \max\{b,d\}$

So $X\times Y$ is Ban. ///

Conversely, let $X\times Y$ be a Banach space. We shall show that X is Banach.

Let $\{x_n\}$ be a Cauchy seq. So

$\forall \varepsilon \exists N \forall m,n \geq N;\ \underbrace{\|x_n - x_m\|}_{\|(x_n,0)-(x_m,0)\|} < \varepsilon$



Hence $\{(x_n,0)\}$ is Cauchy in $X \times Y$. Since $X \times Y$ is complete, $\exists (x_0,y_0) \in X \times Y;\ (x_n,0) \to (x_0,y_0)$. So

$$\forall \varepsilon \exists N \forall n \geq N;\ \underbrace{\|(x_n,0)-(x_0,y_0)\|}_{\max\{\|x_n - x_0\|,\, \|y_0\|\}} < \varepsilon$$

$$\|y_0\|,\ \|x_n - x_0\| \leq \max\{\|x_n - x_0\|, \|y_0\|\}$$

Therefore, $x_n \to x_0$.

Note: We indeed showed that $\forall \varepsilon > 0;\ \|y_0\| < \varepsilon$

$\|y_0\|$ از هر عدد مثبتی کمتر است.

If $\|y_0\| \neq 0$, then choose $\varepsilon = \frac{\|y_0\|}{2} > 0$ to get

$$\cancel{\|y_0\|} < \frac{\cancel{\|y_0\|}}{2}$$

$$1 < \frac{1}{2}\ \times.$$

Thus $\|y_0\| = 0$, so $y_0 = 0$.

Def. A mapping $T: X \to Y$ is called linear if

$T(\lambda x + y) = \lambda Tx + Ty.$

It is called bounded if $\|T\| = \sup_{\|x\| \leq 1} \|Tx\| < \infty$

Exercise. $\longrightarrow$ left to the students

1) $\|T\| = \underbrace{\sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}}_{\alpha} = \sup_{\|x\|=1} \|Tx\|$

$\frac{\|Tx\|}{\|x\|} = \| \underbrace{\frac{1}{\|x\|}}_{\text{scalar}} Tx \| = \| T(\frac{x}{\|x\|}) \| \leq \|T\|$ ← T is linear

$\left\| \frac{x}{\|x\|} \right\| = \frac{\|x\|}{\|x\|} = 1 \leq 1$, $\|T\| = \sup_{\|y\| \leq 1} \|Ty\|$

So $\|T\|$ is an upper bound for $\{ \frac{\|Tx\|}{\|x\|} : x \neq 0 \}$. Hence $\alpha \leq \|T\|$.

Let $\|x\| \leq 1$. Then $\|Tx\| = \frac{\|Tx\|}{1} \leq \frac{\|Tx\|}{\|x\|} \leq \alpha$

Hence $\sup_{\|x\| \leq 1} \|Tx\| \leq \alpha$, so $\|T\| \leq \alpha$. $\square$

2) $\underbrace{\sup_{\|x\| < 1} \|Tx\|}_{\beta} = \|T\| = \sup_{\|x\| \leq 1} \|Tx\|$

$\boxed{A \subseteq B \Rightarrow \sup A \leq \sup B}$

Let $x \in A$. Then $x \in B$. So $x \leq \boxed{\sup B}$ $\therefore \sup A \leq \sup B$

$\beta = \sup \{ \|Tx\| : \|x\| < 1 \} \subseteq \{ \|Tx\| : \|x\| \leq 1 \}$

$\leq \sup \qquad = \|T\|$

Suppose $\|x\| \leq 1$. Let $0 < t < 1$. Then $\|tx\| = t\|x\| < 1$.

So $\|T(tx)\| \leq \beta$. Hence $t\|Tx\| \leq \beta$. ~~By~~ Tending

t to 1 we get $\|Tx\| \leq \beta$. Therefore $\sup\limits_{\|x\| \leq 1} \|Tx\| \leq \beta$. $\square$

3) Always $\boxed{\|Tx\| \leq \|T\| \, \|x\|}$ since $\frac{\|Tx\|}{\|x\|} \leq \alpha = \|T\|$ $\|T\|$

قانون تاطلو

Note. Boundedness in $\begin{cases} \text{Calculus } \|f\| = \sup\limits_{x \in X} |f(x)| < \infty \\ \text{operator Theory } \|T\| = \sup\limits_{\|x\| \leq 1} \|Tx\| < \infty \end{cases}$

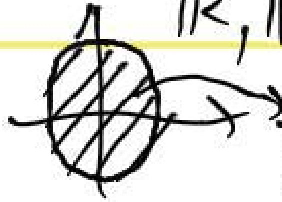
of a map

are different. I mean that if

we have a linear map $\underset{\neq 0}{T} : X \to Y$ & $\sup\limits_{x \in X} \|Tx\| < \infty$, then

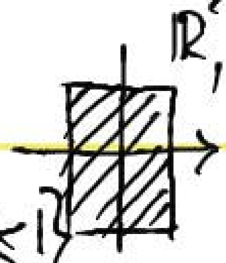
$\exists x_0 ; Tx_0 \neq 0$. Then $\forall n \in \mathbb{N}$; $\|T(nx_0)\| \leq \overset{\|}{M}$. So

$\forall n \in \mathbb{N}$; $n \leq \frac{M}{\|Tx_0\|}$ $\times$.

Example. 

$\mathbb{R}^2$, $\|(x,y)\| = \sqrt{x^2+y^2}$

$B_1(0) = \{(x,y) \mid \sqrt{x^2+y^2} \leq 1\}$



$\mathbb{R}^2$, $\|(x,y)\| = \max\{|x|, |y|\}$

$B_1(0) = \{(x,y) \mid \max\{|x|,|y|\} \leq 1\}$

Th. $T : X \xrightarrow[\text{linear}]{} Y$ is bd iff T is cts at each point of X.

Proof. ($\Rightarrow$) Let $x_0 \in X$. Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{\|T\|}$. Then

$$\forall x;\ \|x - x_0\| < \delta = \frac{\varepsilon}{\|T\|} \Rightarrow \|Tx - Tx_0\| = \|T(x - x_0)\| \leq \|T\|\,\|x - x_0\| < \|T\| \frac{\varepsilon}{\|T\|} = \varepsilon$$

($\Leftarrow$) T is cts on X. So T is cts at origin 0.

By the definition of continuity,

$$\exists \delta\ \forall y;\ \|y - 0\| < \delta \Rightarrow \|Ty - \cancel{T}0\| < 1$$

$$T(0) = T(0+0) = T(0) + T(0), \quad 0 = T(0)$$

Let $\|x\| \leq 1$. Then

$$\frac{\delta}{2}\|x\| \leq \frac{\delta}{2} \Rightarrow \left\|\frac{\delta x}{2}\right\| \leq \frac{\delta}{2} < \delta \Rightarrow \left\|T\left(\frac{\delta x}{2}\right)\right\| < 1 \Rightarrow \|Tx\| < \frac{2}{\delta}$$

$$\therefore \|T\| = \sup_{\|x\| \leq 1} \|Tx\| < \frac{2}{\delta} < \infty. \ \square$$

Exercise T is cts at x_0 iff T is cts at 0.

Exercise. $\|T\| = \sup_{\|x\| < 1} \|Tx\|$ is really a norm.

Hint: $\|x - x_0\| = \|(x - x_0) - 0\|$

<u>Counterexample</u>. There is a metric space whose norm cannot be induced by a norm.

Let S be the set of all complex sequences.

Put $d(\{x_n\}, \{y_n\}) = \sum \frac{|x_n - y_n|}{2^n(1+|x_n - y_n|)} < \infty$, since $\frac{|x_n - y_n|}{2^n(1+|x_n - y_n|)} \leq \left(\frac{1}{2}\right)^n$. There is no norm $\|\cdot\|$ such that $d(x,y) = \|x - y\|$. Otherwise, $\|\{x_n\}\| = d(\{x_n\}, 0) = \sum \frac{|x_n|}{2^n(1+|x_n|)}$

Put $x_n = (1, 0, 0, \dots)$. Then $\underbrace{2\|\{x_n\}\|}_{\frac{1}{2} = \frac{2}{2(1+1)}} \neq \underbrace{\|\{2x_n\}\|}_{\frac{2}{2(1+2)} = \frac{1}{3}}$ ∴ ×.

<u>Def.</u> A linear mapping $f : X \to \mathbb{C}$ is called a <u>linear functional</u>

We denote by X^* the space of all <u>bounded</u> linear functionals on a normed space $(X, \|\cdot\|)$.

ots

the dual of X

(دوگان)

<u>Th</u>. A linear function $f : X \to \mathbb{C}$ is bd iff $\operatorname{Ker} f$ is closed.

$\operatorname{Ker} f = \{x \in X : f(x) = 0\}$

Proof ($\Rightarrow$) If f is cts, then $\ker f = f^{-1}(\underbrace{\{0\}}_{\text{closed}})$ is closed.

($\Leftarrow$) We may assume that $f \neq 0$. Let $\sup\limits_{\|x\|=1} |f(x)| = \infty$. $\forall M \, \exists x_{\|x\|=1}$; $|f(x)| > M$. So

$\forall n \, \exists x_n$, $\|x_n\|=1$; $|f(x_n)| > n$. Since $f \neq 0$, there is $e \in X$; $f(e) \neq 0$

خصوصیت

If $e_0 = \dfrac{e}{f(e)}$, then $f(e_0) = \dfrac{f(e)}{f(e)} = 1$. Put $\boxed{y_n = e_0 - \dfrac{x_n}{f(x_n)}}$.

Then $f(y_n) = f(e_0) - \dfrac{f(x_n)}{f(x_n)} = 1 - 1 = 0$. So $y_n \in \ker(f)$ and

$\|y_n - e_0\| = \left\|\dfrac{x_n}{f(x_n)}\right\| = \dfrac{1}{|f(x_n)|} < \dfrac{1}{n}$. Hence $y_n \to e_0$. So $e_0 \in \overline{\ker f}$

$= \ker f$, since $\ker f$ is closed. Therefore $f(e_0) = 0 \neq 1$ X.

Hence $\sup\limits_{\|x\|=1} |f(x)| < \infty$. Thus f is bd. $\square$ ($\sim P \wedge P$ تناقض)

Exercise 1) $x \in \bar{A} \Leftrightarrow \exists \{x_n\}$ in A such that $x_n \to x$.

2) If (X, d) is a complete metric space & $Y \subseteq X$, then (Y, d) is complete iff Y is closed.

NGI

Th. $(X, \|\cdot\|)$ is Ban iff $\forall \{x_n\}$ in X; $\underbrace{\sum \|x_n\| < \infty}_{\text{in } \mathbb{R}} \Rightarrow \underbrace{\sum_{n=1}^{\infty} x_n \text{ converges}}_{\text{in } X}$

Every absolutely convergent series is convergent.

$0 \leq |x_n| - x_n \leq 2|x_n|$

$\sum |x_n| < \infty \Rightarrow \sum (|x_n| - x_n) < \infty$

$\underset{\sum |x_n| < \infty}{\Rightarrow} \sum \underbrace{\left(|x_n| - (|x_n| - x_n)\right)}_{x_n} < \infty$

Proof. $(\Rightarrow)$ Let X be Ban. Let $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Assume that $s_n = \sum_{k=1}^{n} \|x_k\|$ & $t_n = \sum_{k=1}^{n} x_k$. We shall show that $\{t_n\}$ is convergent.

$\|t_n - t_m\| = \|\sum_{k=m+1}^{n} x_k\| \leq \sum_{k=m+1}^{n} \|x_k\| = |s_n - s_m|$ $n \geq m$

$\{x_n\}$ converges to ...
$\{x_n\}$ is convergent
... convergence of $\{x_n\}$ is ...

Since $\sum_{n=1}^{\infty} \|x_n\|$ is conv., $\{s_n\}$ is conv., so $\{s_n\}$ is Cauchy. Hence

$\forall \varepsilon \exists N \forall n \geq m \geq N$, $\underset{\|t_n - t_m\| \leq}{|s_n - s_m|} < \varepsilon$

So $\{t_n\}$ is Cauchy. Since $(X, \|\cdot\|)$ is Ban, $\exists x$; $t_n \to x$.

So $\sum_{n=1}^{\infty} x_n$ is conv.

Conversely, let $\{x_n\}$ be a Cauchy seq. in X.

$\exists n_1 \; \forall m,n \geq n_1;\; \underbrace{\|x_n - x_m\| < \frac{1}{2^1}}$

in Particular, $\|x_n - x_{n_1}\| < \frac{1}{2^1} \Rightarrow \|x_{n_2} - x_{n_1}\| < \frac{1}{2^1}$

$\{n_k\}_{k\in\mathbb{N}}$

$\exists n_2 > n_1 \; \forall n \geq n_2;\; \|x_n - x_{n_2}\| < \frac{1}{2^2}$

$\vdots$

$\therefore$ there is a subseq. $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{2^k} \qquad \left(\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k < \infty\right)$$

By the Comparison test, $\sum_{k=1}^{\infty}\|x_{n_{k+1}} - x_{n_k}\| < \infty$. By our assumption, $\sum_{k=1}^{\infty}(x_{n_{k+1}} - x_{n_k})$ converges to, say, x. So

$\lim\limits_{N\to\infty} \underbrace{\sum_{k=1}^{N-1}(x_{n_{k+1}} - x_{n_k})}_{x_{n_N} - x_{n_1}} = x$. Hence $\{x_{n_N}\}_{N=1}^{\infty}$ is convergent to $\underbrace{x + x_{n_1}}_{x_0}$.

Now we shall show that $\{x_n\}$ is convergent to x_0:

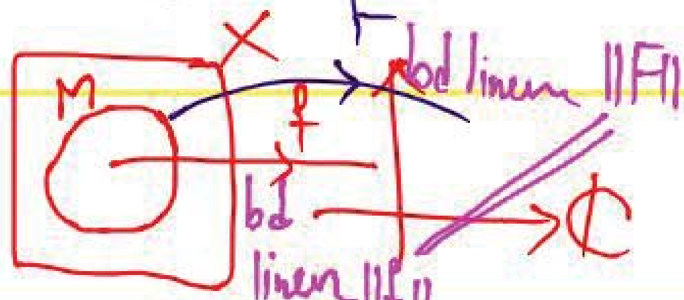
Let $\varepsilon > 0$. $\exists N_0 \; \forall N \geq N_0;\; \|x_{n_N} - x_0\| < \frac{\varepsilon}{2}$.

$\exists M \geq n_{N_0} \; \forall m,n \geq M;\; \|x_n - x_m\| < \frac{\varepsilon}{2}$ (Since $\{x_n\}$ is Cauchy)

$\forall n \geq M$; $\|x_n - x_0\| \leq \|x_n - x_{n_{N_0}}\| + \|x_{n_{N_0}} - x_0\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

So X is Ban. $\square$

<u>Project 2</u>. (Hahn-Banach theorem). Let M be a subspace of $(X, \|\cdot\|)$ & f be a <u>bd</u> linear functional on M. Then f can be extended to a <u>bd</u> linear functional F to X such that $\|F\| = \|f\|$.



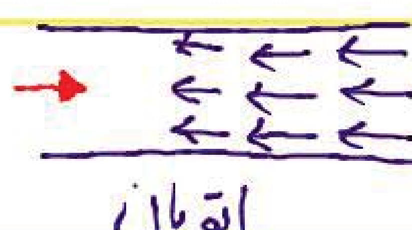
<u>Remark</u>. In linear algebra, $\exists$ a subspace N such that $X = M \oplus N$. Put $F(x) = \begin{cases} f(x) & x \in M \\ 0 & x \in N \end{cases}$. Then F is a linear functional on X, but it may be unbounded.

<u>Note</u>. The space generated by $S \subseteq X$ is $\langle S \rangle$

$\bigcap\limits_{S \subseteq M} M$. In particular, $\langle x_0 \rangle = \mathbb{C}x_0 = \{\lambda x_0 \mid \lambda \in \mathbb{C}\}$

since $\mathbb{C}x_0 = \bigcap\limits_{x_0 \in M} M$.

دوشنبه، ۱۳ آذر ۹۳



Corollary. If $x_0 \in X$, $x_0 \neq 0$, then $\exists f \in X^*$; $f(x_0) = \|x_0\|$ & $\|f\| = 1$

Proof. $\begin{cases} g : \langle x_0 \rangle \to \mathbb{C} \\ g(\lambda x_0) = \lambda \|x_0\| \end{cases}$ is a linear functional

$$g(\lambda x_0 + \mu x_0) = g((\lambda+\mu) x_0) = (\lambda+\mu)\|x_0\| = \lambda\|x_0\| + \mu\|x_0\| = g(\lambda x_0) + g(\mu x_0)$$

$$\|g\| = \sup_{\lambda \neq 0} \frac{|g(\lambda x_0)|}{\|\lambda x_0\|} = \sup_{\lambda \neq 0} \frac{|\lambda|\,\|x_0\|}{|\lambda|\,\|x_0\|} = 1$$

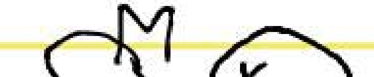
By the Hahn-Banach theorem, $\exists f \in X^*$; $f|_{\langle x_0 \rangle} = g$ & $\|f\| = \underbrace{\|g\|}_{=1}$

$f(x_0) \underset{x_0 \in \langle x_0 \rangle}{=} g(x_0) = g(1 \cdot x_0) = 1\,\|x_0\| = \|x_0\|$. □

Corollary. Let M be a subspace of $(X, \|\cdot\|)$. Then $x_0 \notin \overline{M}$ if and only if $\exists f \in X^*$; $f(x_0) = 1$ & $f|_M = 0$

Proof. ($\Leftarrow$) Let $\exists f \in X^*$; $f(x_0) = 1$ & $f|_M = 0$. We show that $x_0 \notin \overline{M}$. In contrary, assume $x_0 \in \overline{M}$. So $\exists \{x_n\}$ in M such that $x_n \to x_0$. Since f is continuous, $\underbrace{f(x_n)}_{0} \to \underbrace{f(x_0)}_{1}$ ✗.
Hence $x_0 \notin \overline{M}$

($\Rightarrow$) Let $x_0 \notin \overline{M}$. $\exists r_0$; $N(x_0) \cap M = \emptyset$



(?) Let $x_0 \notin M$. $\exists r$, $\underbrace{N_r(x_0) \cap M = \emptyset}$

$\forall x \in M;\ |x - x_0| \geq r$ (♣)

Let $Y = \langle M \cup \{x_0\} \rangle = \{x + \lambda x_0 \mid \lambda \in \mathbb{C},\ x \in M\}$.

$$\alpha(x + \lambda x_0) + (y + \mu x_0) = \alpha x + y + (\alpha\lambda + \mu)x_0$$

Define $\begin{cases} g: Y \to \mathbb{C} \\ g(x + \lambda x_0) = \lambda \end{cases}$. g is linear. g is bounded on Y:

$$r|g(x + \lambda x_0)| = |\lambda|\, r \underset{(♣)}{\leq} |\lambda| \Big\| \underbrace{\tfrac{-x}{\lambda}}_{\in M} - x_0 \Big\| \leq \|x + \lambda x_0\|$$

$$\|g\| = \sup \frac{|g(x + \lambda x_0)|}{\|x + \lambda x_0\|} \leq \frac{1}{r}$$

By the Hahn-Banach theorem, $\exists f: X \xrightarrow{\text{linear}} \mathbb{C}$; $f = g|_Y$ & $\|f\| = \|g\|$. Hence $f(x_0) = g(x_0) = g(0 + 1x_0) = 1$, $f|_M = g|_M = 0$. $\square$

$g(x) = g(x + 0x_0)$

Project. (Banach Open Mapping Theorem).

If $T: X \to Y$ is a bd linear mapping from a Banach space X Onto a Banach space Y, then T is open, i.e., if $U \subseteq X$ is open, then TU is also open in Y. $\boxed{۹۲,۹,۱۸\ [illegible]}$

Corollary. If $T: X \to Y$ is bd, $\boxed{\text{1-1 \& onto}}$ linear mapping between Banach spaces, then T^{-1} is also a bd linear mapping.

Proof. Since T is 1-1 & surjective, so T^{-1} exists as a mapping. In addition, T^{-1} is linear:

$$T^{-1}(\alpha y_1 + y_2) \underset{\Uparrow \text{T is 1-1}}{=} \alpha T^{-1}(y_1) + T^{-1}(y_2)$$

$$T(T^{-1}(\alpha y_1 + y_2)) \overset{?}{=} T(\quad = \quad) = \alpha T(T^{-1}(y_1)) + T(T^{-1}(y_2))$$

$$\alpha y_1 + y_2 \underset{\Uparrow}{=} \alpha y_1 + y_2$$

$\textcircled{?}\ \left.\begin{array}{l} ۲=۳ \\ ۳=۲ \end{array}\right\} \Rightarrow \underbrace{۲+۳}_{۵} = \underbrace{۳+۲}_{۵}$ ✓

این کار را مرتکب نشوید

T maps $\underbrace{\{x : \|x\| < 1\}}_{=A}$ onto an open set in Y, by the open mapping theorem.

Since TA is open & $0 = T0 \in TA$,

$\exists r;\ N_r(0) \subseteq TA$ for $\underset{\{Tx : x \in A\}}{}$

$\forall y; \left(\|y\| < r \Rightarrow y \in TA \Rightarrow \exists x \in A;\ y = Tx \right).$

$\|\underbrace{T^{-1}y}_{x}\| \leq ? \|\underbrace{y}_{Tx}\|$

~~$\|Tx\| \leq \|T\| \|x\|$~~

Let $y \in Y$. Then $\left\| \dfrac{ry}{2\|y\|} \right\| = \dfrac{r}{2} < r$. So $\underbrace{\exists x \in X; \|x\| < 1}_{\exists x \in A}$ & $Tx = \dfrac{ry}{2\|y\|}$

Therefore $\|x\| = \left\| \dfrac{r T^{-1} y}{2\|y\|} \right\| < 1 \Rightarrow \|T^{-1}y\| \leq \dfrac{2}{r}\|y\|$. So T^{-1} is bd. $\square$

(Second Proof not for exam) T is $\frac{1\text{-}1}{\text{onto}}$ & cts. By a theorem in top,
Topological proof

T is open $\Leftrightarrow T^{-1}$ is cts. $\square$ $(T^{-1})^{-1}(V)$

V is open in $X \Leftrightarrow TV$ is open in Y, $TV = (T^{-1})^{-1}(V)$

T is open $\Leftrightarrow$ T^{-1} is cts

<u>Def</u>. Two norms $\|\cdot\|_1$ & $\|\cdot\|_2$ are called equivalent if

$$\exists \alpha, \beta > 0;\ \begin{cases} \|x\|_1 \leq \alpha \|x\|_2 \\ \|x\|_2 \leq \beta \|x\|_1 \end{cases} \quad \text{or } \alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1$$

<u>Note</u>. 1) $I : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is bd iff $\exists \beta;\ \|\underbrace{I(x)}_{x}\|_2 \leq \beta \|x\|_1$

2) $I : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is bi-continuous (both I & I^{-1} are cts; $I^{-1} = I$) iff $\begin{cases} \|x\|_2 \leq \beta \|x\|_1, \\ \|x\|_1 \leq \alpha \|x\|_2 \end{cases}$

for some α, β and all $x \in X$.

Theorem. If $\|\cdot\|_1, \|\cdot\|_2$ are equivalent norms & $(X, \|\cdot\|_1)$ is Ban, then so is $(X, \|\cdot\|_2)$.

Proof. Let $\exists \alpha, \beta$; $\alpha\|x\|_1 \le \|x\|_2 \le \beta\|x\|_1$ $\forall x \in X$.

Let $\{x_n\}$ be a Cauchy seq in $(X, \|\cdot\|_2)$.

$$\left.\begin{array}{l} \|x_n - x_m\|_1 \le \frac{1}{\alpha}\|x_n - x_m\|_2 \\ \forall \varepsilon \exists N \forall m, n \ge N; \quad \|x_n - x_m\|_2 < \alpha\varepsilon \end{array}\right\} \Rightarrow \|x_n - x_m\|_1 < \varepsilon$$

Hence $\{x_n\}$ is Cauchy in $(X, \|\cdot\|_1)$. So that $\exists x \in X$; $x_n \xrightarrow{\|\cdot\|_1} x$

$\hookrightarrow$ complete norm

Therefore

$$\left.\begin{array}{l} \forall \varepsilon \exists N \forall n \ge N; \quad \|x_n - x\|_1 < \frac{\varepsilon}{\beta} \\ \|x_n - x\|_2 \le \beta\|x_n - x\|_1 \end{array}\right\} \Rightarrow \|x_n - x\|_2 < \varepsilon$$

Thus $x_n \xrightarrow{\|\cdot\|_2} x$. $\square$

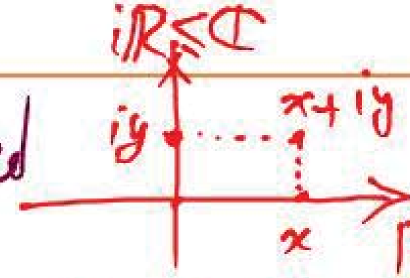
Note. On $M_n(\mathbb{C})$, all norm are equivalent:

$\|[a_{ij}]\|_m = \max|a_{ij}|$, $\|[a_{ij}]\|_\sigma = \sum|a_{ij}|$

$$\|[a_{ij}]\|_m = \max_{i,j}|a_{ij}| \le \sum_{i,j=1}^{n}|a_{ij}| \underset{\beta=1}{=} \|[a_{ij}]\|_\sigma, \quad \|[a_{ij}]\|_\sigma = \sum_{i,j=1}^{n}|a_{ij}| \le n^2 \max_{i,j}|a_{ij}| = \underset{\alpha}{n^2}\|[a_{ij}]\|_m$$

Let X, Y be ~~linear~~ vector spaces. Then $X \times Y$ together with
$\alpha(x,y)+(x',y') = (\alpha x + x', \alpha y + y')$ is a vector space.
Then $X \times Y$ is called the external direct sum of X & Y, denoted by $X \oplus Y$.

If X & Y are subspaces of a linear space Z such that $X \cap Y = \{0\}$, then $X+Y = \{x+y \mid x \in X, y \in Y\}$ is a subspace of Z, denoted by $X \oplus Y$, which is called the internal direct sum of X & Y.



Each vector z of $X+Y$ is uniquely represented as $z = x+y$, where $x \in X$, $y \in Y$, since if $z = x+y = x'+y'$, then $\underbrace{x-x'}_{\in X} = \underbrace{y'-y}_{\in Y}$. So $x-x' = y'-y = 0$. Hence $x = x'$ & $y = y'$.

$\mathbb{C} = \underbrace{\mathbb{R}}_{X} \oplus \underbrace{i\mathbb{R}}_{Y}$, $\mathbb{R} \subseteq \mathbb{C}$

1) If $X \oplus Y$ is internal, then $\begin{cases} \varphi : X \oplus Y \to X \times Y \\ x+y \mapsto (x,y) \end{cases}$ is a vector space isomorphism.

2) If X, Y are linear spaces, put $X_1 = \{(x,0) \mid x \in X\}$, $Y_1 = \{(0,y) \mid y \in Y\}$, then the external direct sum $X \times Y$ is indeed $X_1 \oplus Y_1$, since

$$(x,y) = \underset{\in X_1}{(x,0)} + \underbrace{(0,y)}_{\in Y_1}, \quad X_1 \cap Y_1 = \{0\}$$

Def. If X, Y are normed spaces, then $X \oplus Y$ is a

Def: If X, Y are normed space, then $X \oplus Y$ is a normed spaces under $\|(x,y)\|_p = \left(\|x\|^p + \|y\|^p\right)^{\frac{1}{p}}$

$\|(x,y)\|_\infty = \max\{\|x\|, \|y\|\}$

Th. If X & Y are Banach, then so is $(X \oplus Y, \|\cdot\|_\infty)$.

Proof It is already proved.

If X is a normed space, $X^{**} = (X^*)^*$ is called the second dual of X.

X^* is a normed space

$$\begin{cases} (f+g)(x) = f(x) + g(x) \\ (\alpha f)(x) = \alpha f(x) \\ \|f\| = \sup\limits_{x \neq 0} \dfrac{|f(x)|}{\|x\|} \end{cases}$$

Define $\hat{\ } : X \longrightarrow X^{**}$

$$x \longmapsto \hat{x}, \begin{cases} \hat{x} : X^* \longrightarrow \mathbb{C} \\ \hat{x}(f) = f(x) \end{cases}$$

Then $\hat{\ }$ is a linear map:

$\widehat{x+y} = \hat{x} + \hat{y}$ since $\widehat{x+y}(f) = f(x+y) = f(x) + f(y) = \hat{x}(f) + \hat{y}(f) = (\hat{x} + \hat{y})(f)$

Similarly, $\widehat{\alpha x} = \alpha \hat{x}$.

We show that $\|\hat{x}\| = \|x\|$:

$|\hat{x}(f)| = |f(x)| \leq \|f\| \, \|x\| \Rightarrow \|\hat{x}\| = \sup \dfrac{|\hat{x}(f)|}{\|f\|} \leq \|x\|$.

Let $x \in X$. By the Hahn-Banach theorem $\exists f \in X^*$, $\|f\| = 1$, $f(x) = \|x\|$. $\|x\| = f(x) \underset{f(x) \geq 0}{=} |f(x)| = |\hat{x}(f)| \leq \|\hat{x}\| \underbrace{\|f\|}_{1} = \|\hat{x}\|$ $\square$.

Def. If $\hat{\ }: X \hookrightarrow X^{**}$, $x \mapsto \hat{x}$ is surjective (onto), then X is called reflexive.

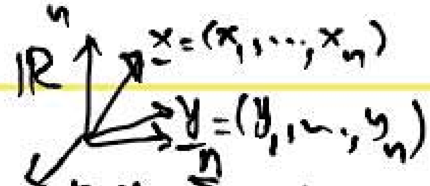
Exercise 1. What can we say about X^* when X is reflexive and conversely? (Kreyszig' book)

Exercise 2 $\mathbb{C}^n$ is reflexive. (Hint: If $\{e_1, \cdots, e_n\}$ is a standard basis for $\mathbb{C}^n$, $e_i := (0, \cdots, 0, 1, 0, 0, \cdots, 0)$ then

$$\left\{ \begin{array}{l} f_j : \mathbb{C}^n \to \mathbb{C} \\ f_j(e_i) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \end{array} \right.$$ is in $(\mathbb{C}^n)^*$ and

$\{f_1, \cdots, f_n\}$ is a basis for $\mathbb{C}^{n*}$.

≡ Hilbert space ≡



$\underline{x} \cdot \underline{y} = \sum_{i=1}^{n} x_i y_i$

Def. Let H be a vector space. Then $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$, $(x, y) \mapsto \langle x, y \rangle$ is called an inner product space if

1) $\langle x, x \rangle \geq 0$ & $\langle x, x \rangle = 0 \Leftrightarrow x = 0$ — without this condition ($\Rightarrow$), $\langle \cdot, \cdot \rangle$ is called semi-inner product.

2) $\langle y, x \rangle = \overline{\langle x, y \rangle}$ conjugate

3) $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$

Then $(H, \langle \cdot, \cdot \rangle)$ is called an inner product space.

Example. On $C([0,1])$ define $\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}\, dx$

$\langle x, y+\lambda z\rangle = \overline{\langle y+\lambda z, x\rangle} = \overline{\langle y,x\rangle+\lambda\langle z,x\rangle} = \overline{\langle y,x\rangle} + \bar\lambda\,\overline{\langle z,x\rangle} = \langle x,y\rangle + \bar\lambda\langle x,z\rangle$

<u>Th</u> (Cauchy-Schwarz inequality). If $(H,\langle\cdot,\cdot\rangle)$ is a semi-inner product space, then $|\langle x,y\rangle|^2 \le \langle x,x\rangle\langle y,y\rangle$

$\boxed{\text{or } |\langle x,y\rangle| \le \|x\|\,\|y\|}$

<u>Proof</u>. $\forall\lambda\in\mathbb{C}$ $0 \le \langle x-\lambda y, x-\lambda y\rangle = \langle x,x\rangle - \bar\lambda\langle x,y\rangle - \lambda\langle y,x\rangle + \lambda\bar\lambda\langle y,y\rangle$.

If $\boxed{\lambda = \dfrac{\langle x,y\rangle}{\langle y,y\rangle}}$, then $0 \le \langle x,x\rangle - \dfrac{\overline{\langle x,y\rangle}\langle x,y\rangle}{\langle y,y\rangle} - \dfrac{\langle x,y\rangle\overline{\langle x,y\rangle}}{\langle y,y\rangle} + \dfrac{\langle x,y\rangle\overline{\langle x,y\rangle}}{\langle y,y\rangle^{\cancel{2}}}\cancel{\langle y,y\rangle}$.

Hence $\dfrac{|\langle x,y\rangle|^2}{\langle y,y\rangle} \le \langle x,x\rangle$, So $|\langle x,y\rangle|^2 \le \langle x,x\rangle\langle y,y\rangle$.

If $(H,\langle\cdot,\cdot\rangle)$ is an inner product space, then $\|x\| := \langle x,x\rangle^{\frac12}$, then $\|\cdot\|$ is really a norm:

$\|x\| = \langle x,x\rangle^{\frac12} \ge 0$

$\|x\| = 0 \Leftrightarrow \langle x,x\rangle = 0 \Leftrightarrow x = 0$

$\|\lambda x\| = \langle\lambda x,\lambda x\rangle^{\frac12} = \left(|\lambda|^2\langle x,x\rangle\right)^{\frac12} = |\lambda|\,\|x\|$

$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$

$= \|x\|^2 + \underbrace{\langle x,y\rangle + \overline{\langle x,y\rangle}}_{=2\,\mathrm{Re}\langle x,y\rangle} + \|y\|^2$

$z+\bar z = (a+bi)+(a-bi) = 2a = 2\,\mathrm{Re}\, z$

$\le 2|\mathrm{Re}\langle x,y\rangle|$ $\quad\forall x;\ x \le |x|$

The Cauchy-Schwarz inequal. $\le 2|\langle x,y\rangle| \le 2\|x\|\,\|y\|$

$|\mathrm{Re}\, z| = |a| \le \sqrt{a^2+b^2} = |z|$

$\therefore \|x+y\|^2 \le (\|x\|+\|y\|)^2$. ✓

If $\|\cdot\|$ on H is complete, then $(H, \langle \cdot, \cdot \rangle)$ is called a Hilbert space.

Counterexamples.

1) $(C[0,1], \|\cdot\|_\infty)$ is a Banach space (why?), but

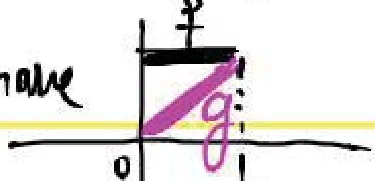
$\hookrightarrow \|f\| = \sup_{x\in X} |f(x)|$

its norm can not be induced by an inner product.

Otherwise, let $\|f\|_\infty = \langle f,f\rangle^{\frac{1}{2}}$ for some $\langle \cdot,\cdot\rangle$ on $C([0,1])$. We know that in an inner product space

$$\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle = \cdots = 2\|x\|^2 + 2\|y\|^2$$

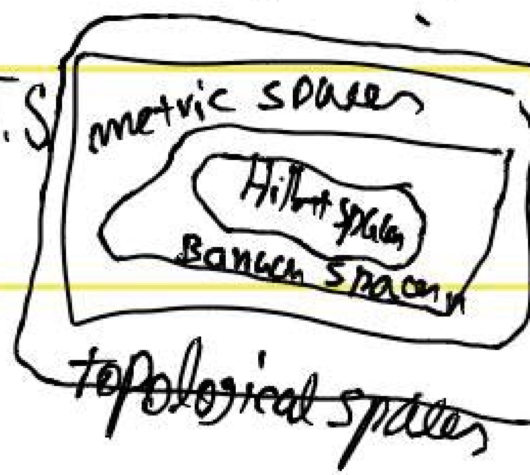
in particular, for $f(t)=1$, $g(t)=t$, we have



$\|f+g\|_\infty = \sup_{t\in[0,1]} |1+t| = 2$, $\|f-g\|_\infty = \sup_{t\in[0,1]} |1-t| = 1$, $\|f\|_\infty = 1$, $\|g\| = 1$

So $\|f+g\|_\infty^2 + \|f-g\|_\infty^2 = 5 \neq 4 = 2\|f\|^2 + 2\|g\|^2$. X.

$H.S \subsetneq B.S. \subsetneq M.S \subsetneq T.S$



2) $\ell^p = \{\{x_n\} \mid x_n \in \mathbb{C}, \underbrace{(\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}}_{\|\{x_n\}\|_p} < \infty\}$

is an inner product space iff $p=2$

If l^p is an inner product space i.e. $\|\underbrace{\{x_n\}}_{x}\|_p = \langle \underbrace{\{x_n\}}_{x}, \underbrace{\{x_n\}}_{x} \rangle$ for some $\langle \cdot, \cdot \rangle$ on l^p, then choose $x = (1,0,0,\cdots)$, $y = (0,1,0,\cdots)$

Then $\|\underbrace{x+y}_{(1,1,0,\cdots)}\|_p^2 + \|\underbrace{x-y}_{(1,-1,0,\cdots)}\|_p^2 = 2^{\frac{2}{p}} + 2^{\frac{2}{p}} = 2^{1+\frac{2}{p}} = 2^2 = 2\|x\|_p^2 + 2\|y\|_p^2$.

So $1 + \frac{2}{p} = 2$, hence $\boxed{p=2}$.

$$\boxed{\frac{۱۲}{۲۴} \overset{?}{=} \frac{۱+۲}{۲+۴} = \frac{۳}{۶} = \frac{۱}{۲}}$$

Conversely, l^2 is an inner product space under

$$\langle \{x_n\}, \{y_n\} \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n} .$$

Then $\|\{x_n\}\|_{\langle,\rangle} = \langle \{x_n\}, \{x_n\} \rangle^{\frac{1}{2}} = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}} = \|\{x_n\}\|_2$

embedding

Note that $\mathbb{C}^n = \{(x_1, x_2, \cdots, x_n) \mid x_i \in \mathbb{C}, 1 \le i \le n\} \xrightarrow[\text{linear}]{\text{1-1}} l^2$ isometry

$$(x_1, \cdots, x_n) \longmapsto (x_1, x_2, \cdots, x_n, 0, 0, \cdots)$$

Euclidean norm $\|(x_1, \cdots, x_n)\| = \left(\sum |x_i|^2\right)^{\frac{1}{2}} = \|(x_1, x_2, \cdots, x_n, 0, 0, \cdots)\|_2$

The above inner product on l^2 is well-defined:

$\forall k;\ \sum_{n=1}^{k} |x_n \overline{y_n}| \le \left(\sum_{n=1}^{k} |x_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{k} |y_n|^2\right)^{\frac{1}{2}}$

Cauchy-Schwarz in $\mathbb{C}^k$

$\boxed{2+2=3.99999\,999\cdots}$

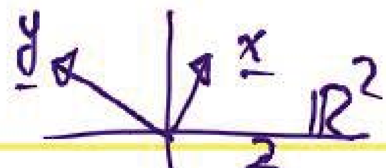
$(\lozenge)$ $\le \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} |y_n|^2\right)^{\frac{1}{2}} < \infty$ since $\{x_n\}, \{y_n\} \in l^2$

Since $\left\{ \sum_{n=1}^{k} |x_n \bar{y}_n| \right\}_{k=1}^{\infty}$ is an increasing seq. and $\left(x_n \geq 0,\ \sum_{n=1}^{k} x_n \leq \sum_{n=1}^{k+1} x_n \right)$

by ($\Diamond$) it is bounded above. Hence it is convergent.
Hence $\sum_{n=1}^{\infty} |x_n \bar{y}_n|$ converges. We know every absolutely convergent seq. is convergent, so $\sum_{n=1}^{\infty} x_n \bar{y}_n$ is convergent.

<u>Def</u>. We say x is orthogonal to y if $\langle x, y \rangle = 0$ and we write $x \perp y$. For a subset $M \subseteq H$, $M^{\perp} := \{ x \in H \mid \langle x, z \rangle = 0\ \forall z \in M \}$.

$M^{\perp}$: perp



$\langle \underline{x}, \underline{y} \rangle = \sum_{i=1}^{2} x_i y_i = 0$

<u>Proposition</u>. 1) $M^{\perp}$ is a closed subspace of H.
2) $M \subseteq M^{\perp\perp}$ 3) $M = M^{\perp\perp}$ if and only if M is closed.
4) $\overline{M}^{\perp} = M^{\perp}$

<u>Proof</u>. 1) Let $x, y \in M^{\perp}$, $\lambda \in \mathbb{C}$.

$$\begin{cases} \langle \lambda x + y, z \rangle = \lambda \underbrace{\langle x, z \rangle}_{\substack{0 \\ \text{since } x \in M^{\perp}}} + \underbrace{\langle y, z \rangle}_{0} = 0 \\ \forall z \in M \end{cases} \Rightarrow \lambda x + y \in M^{\perp}$$

$\overline{M^{\perp}} \stackrel{?}{=} M^{\perp}$. Clearly $M^{\perp} \subseteq \overline{M^{\perp}}$. Let $x \in \overline{M^{\perp}}$. $\exists \{x_n\}$ in $M^{\perp}$ such than $x_n \to x$. We have

$$\forall z \in M;\ \left(\langle x_n, z \rangle = 0\ \forall n \Rightarrow \lim_{n\to\infty} \langle x_n, z \rangle = 0 \Rightarrow \langle \underbrace{\lim_n x_n}_{x}, z \rangle = 0 \right).$$

So $x \in M^\perp$. Thus $\overline{M^\perp} \subseteq M^\perp$. ////

Lemma. $\underbrace{\langle \cdot , z \rangle}_{f} : H \to \mathbb{C}$ is a bounded linear function by the definition of the inner product

$x \mapsto \langle x, z \rangle = f(x)$

Proof. $|\underbrace{\langle x, z \rangle}_{f(x)}| \leq \underbrace{\|z\|}_{M} \|x\|$ C-S inequality. So f is bdd. $\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} \leq \|z\|$.

Moreover, $\frac{|f(z)|}{\|z\|} = \frac{\langle z, z \rangle}{\|z\|} = \frac{\|z\|^2}{\|z\|} = \|z\| \leq \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} \leq \|z\|$.

So $\|f\| = \|z\|$.

<u>Proposition</u> 1) $M \subseteq N \Rightarrow N^{\perp} \subseteq M^{\perp}$

$x \in N^{\perp} \Rightarrow \forall z \in N; \langle x,z\rangle = 0 \xRightarrow{M \subseteq N} \forall z \in M; \langle x,z\rangle = 0 \Rightarrow x \in M^{\perp}$

2) $M \subseteq M^{\perp\perp}$

$x \in M \Rightarrow \forall z \in M^{\perp}; \underbrace{z \perp M}_{\text{So } \langle x,z\rangle = 0} \Longrightarrow \forall z \in M^{\perp}; \langle x,z\rangle = 0 \Rightarrow x \in (M^{\perp})^{\perp}$

3) $\overline{M}^{\perp} = M^{\perp}$

First) It follows from (1) & $M \subseteq \overline{M}$ we have $\overline{M}^{\perp} \subseteq M^{\perp}$

Second) $M^{\perp} \subseteq \overline{M}^{\perp}$: Let $x \in M^{\perp}$. Let $z \in \overline{M}$. So $\exists \{z_n\}$ in M such

that $z_n \to z$. So $\underbrace{\langle x,z_n\rangle}_{=0} \to \langle x,z\rangle \Longrightarrow \langle x,z\rangle = 0$

$0 \leq |\langle x,z_n\rangle - \langle x,z\rangle| = |\langle x, z_n - z\rangle| \leq \|x\| \|z_n - z\|$

$\searrow 0$ $\qquad\qquad \searrow 0$ since $z_n \to z$

$\therefore x \in \overline{M}^{\perp}$

4) M is closed $\Leftrightarrow M = M^{\perp\perp}$

($\Leftarrow$) $M = (M^{\perp})^{\perp}$ is closed since $N^{\perp}$ is always closed.

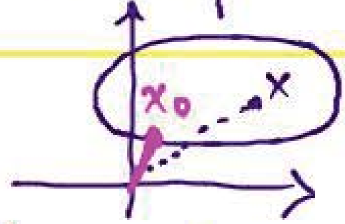
($\Rightarrow$) Exercise

5) $M^{\perp} = M^{\perp\perp\perp}$ since $M^{\perp}$ is closed & for all closed

set N; $N^{\perp\perp} = N$.

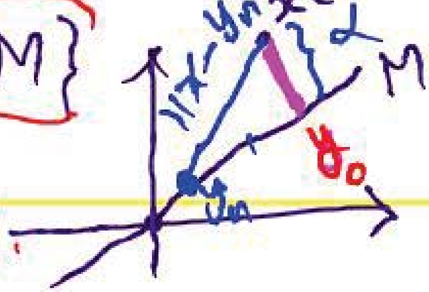
Def. Let V be a vector space. $E \subseteq V$ is convex when
$\forall x, y \ \forall 0 \leq t \leq 1; \ tx + (1-t)y \in E$

Exercise Every nonempty, closed convex set E in a Hilbert space contains a unique element of smallest norm, i.e. $\exists x_0 \in E; \ \|x_0\| = \inf\{\|x\| : x \in E\}$



Th. Let M be a closed subspace of a Hilbert space H & $x \in H$. Then

$$\exists!\, y_0 \in M; \ \|x - y_0\| := \underbrace{\inf \overbrace{\{\|x - y\| : y \in M\}}^{A}}_{\alpha}$$



Proof. Let $\alpha = \inf_{y \in M} \|x - y\|$. So $\exists \{y_n\}$ in M; $\|x - y_n\| \to \alpha$.

In $\mathbb{R}^2$, supaces are lines through O, $\{0\}$, $\mathbb{R}^2$

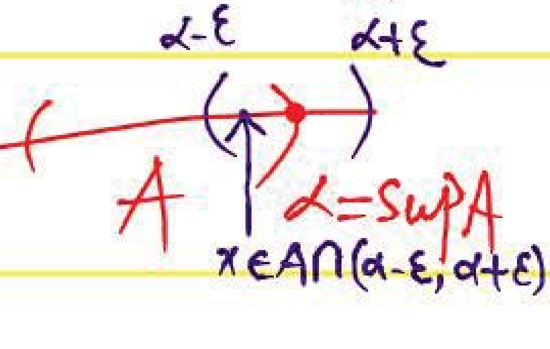
$$0 \leq \|y_n - y_m\|^2 = \|\underbrace{(y_n - x)}_{u} + \underbrace{(x - y_m)}_{v}\|^2$$

$$= 2\|u\|^2 + 2\|v\|^2 - \|u - v\|^2 \quad (\text{Parallelogram law})$$

$$= 2\|\underbrace{y_n - x}_{\in A}\|^2 + 2\|\underbrace{x - y_m}_{\in A}\|^2 - 4\underbrace{\|x - \underbrace{\tfrac{y_m + y_n}{2}}_{\in M}\|}_{\in A}{}^2$$

$$\leq 2\|y_n - x\|^2 + 2\|x - y_m\|^2 - 4\alpha^2 \to 0 \text{ as } m, n \to \infty$$

Exercise. $\sup A, \inf A \in \overline{A}$



So $\{y_n\}$ is Cauchy in H. Since H is Hilbert, $\exists y_0 \in H$; $y_n \to y_0$. So $x - y_n \to x - y_0$. Hence $\|x - y_n\| \to \|x - y_0\|$

Already $\|x-y_n\| \to \alpha$. Hence $\|x-y_0\| = \alpha$. Since $y_n \in M$ & $y_n \to y_0$, we have $y_0 \in \overline{M} \underset{\text{Closed}}{=} M$.

Next, we show that y_0 is unique. Let y_0' be another point of M such that $\|x-y_0'\| = \alpha$. Similar to above

$$0 \le \|y_0'-y_n\|^2 \le 2\|y_0'-x\|^2 + 2\|y_n-x\|^2 - 4\left\|\frac{y_0'+y_n}{2} - x\right\|^2$$
$$\le 2\alpha^2 + 2\|y_n-x\|^2 - 4\alpha^2$$

Taking limits, we get $0 \le \|y_0'-y_0\|^2 \le 2\alpha^2 + 2\alpha^2 - 4\alpha^2 = 0$

$\therefore y_0' = y_0$. $\square$

<u>Exercise</u>. Norm $\|\cdot\|: X \to \mathbb{C}$ is cts on X.

<u>Solution</u>. $\big|\|x\| - \|x_0\|\big| \le \|x-x_0\|$ $\leftarrow$ $\big||a|-|b|\big| \le |a-b| \le |a|+|b|$

Given $\varepsilon > 0$. Choose $\delta = \varepsilon$. Then $\forall x$; $\|x-x_0\| < \delta \Rightarrow \big|\|x\| - \|x_0\|\big| \le \|x-x_0\| < \delta = \varepsilon$ $\square$

Exercise If M is a subspace, then so is $\overline{M}$.

Solution. $\left.\begin{matrix} x, y \in \overline{M} \\ \lambda \in \mathbb{C} \end{matrix}\right\} \Rightarrow \left.\begin{matrix} \exists \{x_n\};\ x_n \xrightarrow{\|\cdot\|} x \\ \exists \{y_n\}\ y_n \xrightarrow{\|\cdot\|} y \end{matrix}\right\} \Rightarrow \underbrace{\lambda x_n + y_n}_{\in M} \xrightarrow{?} \lambda x + y \Rightarrow \lambda x + y \in \overline{M}$

Let $\varepsilon > 0$ be given. Since $x_n \to x$ & $y_n \to y$, we have

$\exists N_1\ \forall n \geq N_1;\ \cancel{\|x_n - x\| < \frac{\varepsilon}{2|\lambda|}}$ $|\lambda| \|x_n - x\| < \frac{\varepsilon}{2}$

$\exists N_2\ \forall n \geq N_2;\ \|y_n - y\| < \frac{\varepsilon}{2}$

Let $N = 2N_1 + 700N_2$. Then

$\forall n \geq N;\ \|(\lambda x_n + y_n) - (\lambda x + y)\| = \|\lambda(x_n - x) + (y_n - y)\| \leq \underbrace{|\lambda| \|x_n - x\|}_{< \frac{\varepsilon}{2}} + \underbrace{\|y_n - y\|}_{< \frac{\varepsilon}{2}} = \varepsilon$

Exercise. Let $x_n \to x$ & $y_n \to y$. Prove that

$$\langle x_n, y_n \rangle - \langle x, y \rangle.$$

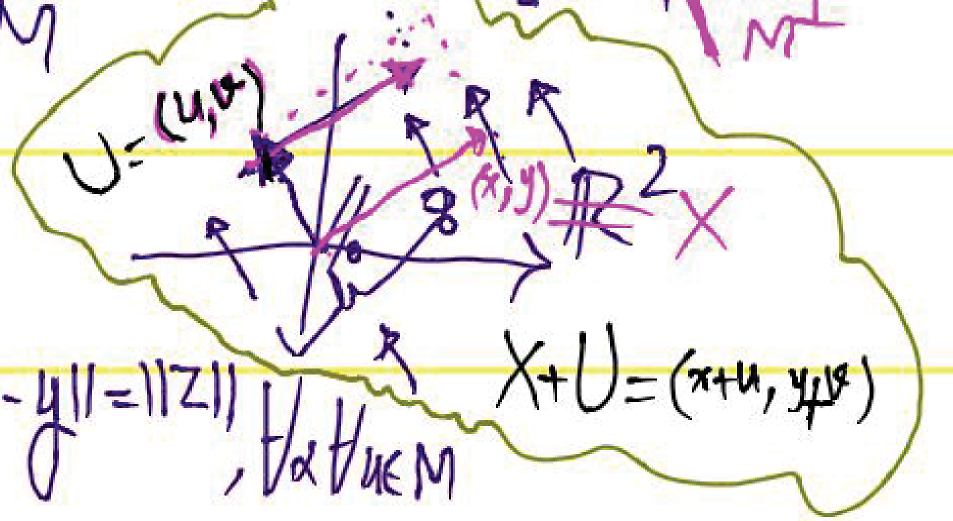
Theorem. If M is a closed subspace of a Hilbert space, then $H = M \oplus M^{\perp}$

$x \in M \cap M^{\perp} = \{0\}$

$\langle x, x \rangle = 0 \Rightarrow x = 0$

Proof. Let $x\in H$. By the previous theorem $\exists!\, y\in M$; $\|x-y\| = \inf_{u\in M}\|x-u\|$.

Put $z = x-y$. We show that $z\in M^\perp$:



$$\|z-\alpha u\| = \|x-\underbrace{y-\alpha u}_{\in M \text{ subspace}}\| \ge \|x-y\| = \|z\| \quad ,\forall\alpha\,\forall u\in M$$

$$\|z-\alpha u\|^2 = \langle z-\alpha u, z-\alpha u\rangle = \|z\|^2 - \underbrace{\bar\alpha\langle z,u\rangle - \alpha\langle u,z\rangle}_{2Re(\alpha\langle u,z\rangle)} + |\alpha|^2\|u\|^2 \ge \|z\|^2$$

So

$\therefore\ 2Re(\alpha\langle u,z\rangle)\le |\alpha|^2\|u\|^2\ \forall\alpha\,\forall u$.

(E.) Assume that $\boxed{\langle u,z\rangle\ne 0}$

Put $\alpha = \frac{t|\langle u,z\rangle|}{\langle u,z\rangle}$ where $t>0$. So $2Re\left(\frac{t|\langle u,z\rangle|\cancel{\langle u,z\rangle}}{\cancel{\langle u,z\rangle}}\right)\le t^2\|u\|^2$ for all $t>0$. Hence $2\cancel{t}|\langle u,z\rangle|\le t^{\cancel{2}}\|u\|^2$.

Hence $0\le|\langle u,z\rangle|\le t\frac{\|u\|^2}{2}$. Let $t\to 0$. Then $|\langle u,z\rangle| = 0$.

Thus $\boxed{\langle u,z\rangle = 0}$. ✗ Thus we must have $\langle u,z\rangle=0\ \forall u$

Therefore $z\in M^\perp$. Hence $H = M+M^\perp$. Since $M\cap M^\perp=\{0\}$, we have $H=M\oplus M^\perp$. □

<u>Exercise</u>. Let $\underset{\text{vector space}}{V} = M + N = \{y+z \mid y \in M, z \in N\}$. Then $V = \underset{M \cap N = \{0\}}{M \oplus N}$

if and only if $\forall x \in V \ \exists! y \in M \ \exists! z \in N;\ x = y + z$

<u>Solution</u>. ($\Rightarrow$) Let $x = y + z = y' + z'$, where $y, y' \in M$, $z, z' \in N$.

$$\underset{M}{\underset{\cap}{y - y'}} = \underset{N}{\underset{\cap}{z' - z}} \xrightarrow[M \cap N = \{0\}]{} y - y' = z' - z = 0 \Rightarrow y = y' \ \& \ z = z'.$$

($\Leftarrow$) $x \in M \cap N \Rightarrow x \in M \ \& \ x \in N \Rightarrow \left.\begin{array}{l} x = \underset{M}{\underset{\cap}{x}} + \underset{N}{\underset{\in}{0}} \\ \shortparallel \\ x = \underset{M}{\underset{\in}{0}} + \underset{N}{\underset{\in}{x}} \end{array}\right\} \xRightarrow{\text{by uniqueness}} x = 0.$

<u>Th</u>. If M is closed $\overset{\text{subspace}}{,}$ then $M^{\perp\perp} = M$

<u>Proof</u>. Since M is a closed subspace, we have

$$\mathcal{H} = M \oplus M^{\perp}$$

We know $M \subseteq M^{\perp\perp}$. Let $x \in M^{\perp\perp} \subseteq H$. So

$\exists! y \in M \ \exists! z \in M^{\perp}$ such that $x = y + z$. Then $\underset{M^{\perp\perp}}{\underset{\cap}{x - y}} = \underset{M^{\perp}}{\underset{\cap}{z}}$

So $x - y = z = 0$. Hence $x = y \in M$. $\square$ $\qquad \{0\} = M^{\perp\perp} \cap M^{\perp}$

Th. Let M be a subspace of H. Then $\underbrace{\overline{M}=H}_{M \text{ is dense in } H}$ iff $M^{\perp}=0$.

Proof. $\overline{M}$ is a closed subspace. So $H=\overline{M}\oplus\overline{M}^{\perp}$.

($\Leftarrow$) Let $M^{\perp}=0$. So $\overline{M}^{\perp}=M^{\perp}=0$. Hence $H=\overline{M}\oplus 0=\overline{M}$.

($\Rightarrow$) Let $\overline{M}=H$. So $\underset{M^{\perp}}{\overset{\shortparallel}{\overline{M}^{\perp}}}=\overline{M}^{\perp}\cap\underbrace{\overline{M}}_{H}=\{0\}$. $\square$

Th. (Proof.) If M is a closed subspace of a Hilbert space, then $H=M\oplus M^{\perp}$. Put $\begin{cases} P: H\to H \\ Px=y\in M\end{cases}$ & $\begin{cases} Q: H\to H \\ Qx=z\in M^{\perp}\end{cases}$.

$$x=y+z, \quad y\in M,\ z\in M^{\perp}$$

Then

P & Q are linear: $P(x+x')=P(\underbrace{y+z+y'+z'}_{(y+y')+(z+z')})=y+y'=Px+Px'$, with $y+y'\in M$, $z+z'\in M^{\perp}$.

$P^2=P$: $P^2x=P(Px)=P(y)=P(\underset{M}{y}+\underset{M^{\perp}}{0})=y=Px$

$Q^2=Q$

$P=I-Q$: $Px=y=x-z=x-Qx=(I-Q)(x)$

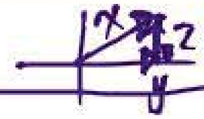
(I: identity operator)

$PQ=QP=0$: $(PQ)x=P(Qx)=Pz=P(0+z)=0=0(x)$

$\operatorname{ran}P = \{\underbrace{Px}_{\in M} : x \in H\} \subseteq M$. Since $y \in M \Rightarrow y = Py \in \operatorname{ran}P$, $M \subseteq \operatorname{ran}P$.

Thus $\operatorname{ran}P = M$.

$$x = \underbrace{y}_{M} + \underbrace{z}_{M^\perp} \Rightarrow \|x\|^2 = \langle y+z, y+z\rangle = \|y\|^2 + \underbrace{\cancel{\langle y, z\rangle}}_{0} + \underbrace{\cancel{\langle z, y\rangle}}_{0} + \|z\|^2$$



$\operatorname{Ker}P = \{x \in H \mid \underbrace{Px}_{y} = 0\} = M^\perp$

$\operatorname{ran}Q = M^\perp$, $\operatorname{Ker}Q = M$

P is bd since, $\|Px\| = \|y\| \leq \|x\| \Rightarrow \sup_{x \neq 0} \frac{\|Px\|}{\|x\|} \leq 1$. $\underset{M \neq 0}{\Rightarrow} \|P\| = 1$

If $x = y \in M$, then $\frac{\|Px\|}{\|x\|} = 1$

Similarly, $\|Q\| = 1$.

$$\boxed{\left.\begin{array}{l}\forall x \in A,\ x \leq \alpha \\ \alpha \in A\end{array}\right\} \Rightarrow \sup A = \alpha}$$

We already observed that $\left\{\begin{array}{l}\langle \cdot, z\rangle : H \to \mathbb{C} \\ x \mapsto \langle x, z\rangle\end{array}\right.$ is a bd linear functional of norm $\|z\|$.

We want to prove the converse.

<u>Theorem</u> (Riesz Theorem). If $\underbrace{f \in H^*}_{\text{the dual of } H}$, then $\exists! z \in H \ \forall_{x \in H}, f(x) = \langle x, z\rangle \ \& \ \|z\| = \|f\|$.

<u>Proof</u>. If $f \equiv 0$, it is enough to choose $z = 0$.

$|\langle x, 0\rangle| \leq \|x\|\,\|0\|$ ⇒ $\langle x, 0\rangle = 0$ (since $\|0\| = 0$)

$\langle x, 0\rangle = 0$

$\langle x, 0\rangle = \langle x, 0+0\rangle = \langle x, 0\rangle + \cancel{\langle x, 0\rangle}$

Hence we may assume $\underbrace{\operatorname{Ker} f}_{\text{a closed subspace}} \neq H$. So $\overline{\operatorname{Ker} f} \neq H$. Thus $\operatorname{Ker} f^{\perp} \neq 0$.

Let $0 \neq z_0 \in \operatorname{Ker} f^{\perp}$. Let $x \in H$. Put $v = \underbrace{f(x)}_{\in \mathbb{C}} z_0 - \underbrace{f(z_0)}_{\in \mathbb{C}} x$. Then

$f(v) = f(x)f(z_0) - f(z_0)f(x) = 0$. So $v \in \operatorname{Ker} f$. Thus

$$\langle v, z_0 \rangle = 0 \Rightarrow f(x)\langle z_0, z_0 \rangle - f(z_0)\langle x, z_0 \rangle = 0 \Rightarrow$$

$$f(x) = \frac{f(z_0)}{\|z_0\|^2}\langle x, z_0 \rangle = \langle x, \underbrace{\frac{\overline{f(z_0)}}{\|z_0\|^2} z_0}_{z} \rangle$$

In addition, $\|f\| = \|z\|$.

Let $f(x) = \langle x, z \rangle = \langle x, z' \rangle$. Then $\langle x, z - z' \rangle = 0 \ \forall x$

So $z - z' = 0$. Hence $z = z'$. $\square$

$$\langle x, w \rangle = 0 \ \forall x \in H \overset{x=w}{\Rightarrow} \langle w, w \rangle = 0 \Rightarrow w = 0$$

<u>Def.</u> We say $\{x_\alpha\}$ in H is an orthogonal family if $\langle x_\alpha, x_\beta \rangle = 0 \ \forall \alpha \neq \beta$. In addition, if $\|x_\alpha\| = 1$, we say that $\{x_\alpha\}$ is an orthonormal family.

Proposition. If $\{x_\alpha\}$ is an orthonormal family, then $\{x_\alpha\}$ is linearly independent.

Proof. Let $\{x_{\alpha_1}, \dots, x_{\alpha_n}\}$ be a finite subset of $\{x_\alpha\}$ & $c_1 x_{\alpha_1} + \dots + c_n x_{\alpha_n} = 0$. Then $\forall j$;

$$0 = \langle 0, x_{\alpha_j} \rangle = \langle c_1 x_{\alpha_1} + \dots + c_n x_{\alpha_n}, x_{\alpha_j} \rangle = c_1 \underbrace{\langle x_{\alpha_1}, x_{\alpha_j} \rangle}_{0} + \dots + c_n \underbrace{\langle x_{\alpha_n}, x_{\alpha_j} \rangle}_{0}$$

$$= 0 + \dots + c_j \underbrace{\langle x_{\alpha_j}, x_{\alpha_j} \rangle}_{\|x_{\alpha_j}\|^2 = 1} + 0 + \dots + 0 = c_j. \quad \square$$

Theorem. If $\{e_n\}$ is an orthonormal seq. in H, then $\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|^2$ (Bessel ineq.)

Proof. Let $x \in H$ & $z = \sum_{k=1}^{n} \langle x, e_k \rangle e_k$. We show that $x - z \perp z$:

$$\langle x - z, z \rangle = \langle x, z \rangle - \langle z, z \rangle = \langle x, \sum_{k=1}^{n} \langle x, e_k \rangle e_k \rangle - \langle \sum_{k=1}^{n} \langle x, e_k \rangle e_k, \sum_{j=1}^{n} \langle x, e_j \rangle e_j \rangle$$

$$= \sum_{k=1}^{n} \overline{\langle x, e_k \rangle} \langle x, e_k \rangle - \underbrace{\sum_{k=1}^{n} \left(\sum_{j=1}^{n} \langle x, e_k \rangle \overline{\langle x, e_j \rangle} \underbrace{\langle e_k, e_j \rangle}_{\text{for } j=k} \right)}_{\sum_{k=1}^{n} \langle x, e_k \rangle \overline{\langle x, e_k \rangle}} = 0$$

عبور ۳

عبور ۲

عبور ① سیگما از خواص ضرب

So $\|x\|^2 = \|(x-z)+z\|^2 = \|x-z\|^2 + \|z\|^2 \geq \|z\|^2 = \langle \sum_{k=1}^{n} \langle x,e_k\rangle e_k, \sum_{j=1}^{n} \langle x,e_j\rangle e_j \rangle$

$= \cdots = \sum_{k=1}^{n} |\langle x,e_k\rangle|^2.$

Hence $\left\{ \sum_{k=1}^{n} |\langle x,e_k\rangle|^2 \right\}$ is an increasing seq, which is bd above. Hence it is convergent and

$$\sum_{k=1}^{\infty} |\langle x,e_k\rangle|^2 \leq \|x\|^2. \quad \Box$$

تمرین (اضافی) سری فوریه و دور

Def. If $\{x_\alpha\}$ is an orthonormal family for H & it is *maximal* then it is called a complete orthonormal family.

$$\{x_\alpha\} \overset{\text{orth}}{\subseteq} \{y_\beta\} \Rightarrow \{x_\alpha\} = \{y_\beta\}$$

Th. Let $\{e_n\}$ be an orthonormal seq, then the following are equivalent:

(i) $\{e_n\}$ is maximal.

(ii) $x = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n, \ \forall x \in H$

(iii) $\langle x, y\rangle = \sum_{n=1}^{\infty} \langle x, e_n\rangle \overline{\langle y, e_n\rangle}, \ \forall x, y \in H$

(iv) $\|x\| = \left(\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2\right)^{\frac{1}{2}}, \ \forall x \in H$

(v) the set of all finite linear combinations of $\{x_n\}$ are dense in H

Project ([illegible])

Then $\{\langle x, e_n\rangle\}$ is called the Fourier coefficients of x with respect to $\{e_n\}$.

Proof : (1)

If X, Y are normed spaces, $T \in B(X, Y)$, then

$$T^*: Y^* \longrightarrow X^*$$
$$g \longmapsto g \circ T$$

is called the dual of T.

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ & \overset{g \circ T}{\searrow} & \downarrow g \\ & & \mathbb{C} \end{array}$$

تعمیم
تعریف
فرض
[illegible]
اصل

[illegible]

Th. T^* is well-defined & $\|T^*\| = \|T\|$.

Proof. $g_1 = g_2 \Rightarrow \forall y \in Y;\ g_1(y) = g_2(y) \Longrightarrow g_1(Tx) = g_2(Tx)\ \forall x \Rightarrow g_1 \circ T = g_2 \circ T$

$$\begin{cases} (g \circ T)(\lambda x + x') = g(T(\lambda x + x')) = g(\lambda Tx + Tx') = \lambda g(Tx) + g(Tx') \\ \qquad = \lambda (g \circ T)(x) + (g \circ T)(x') \\ |(g \circ T)(x)| = |g(Tx)| \underset{g \in Y^*}{\leq} \|g\| \|Tx\| \underset{T \in B(X,Y)}{\leq} \|g\| \|T\| \|x\| \end{cases}$$

$\therefore \|g \circ T\| \leq \|g\| \|T\|$

$\therefore g \circ T \in X^*$

$$\begin{cases} T^*: Y^* \to X^* \\ g \mapsto g \circ T \end{cases} \Rightarrow \|T^* g\| = \|g \circ T\| \leq \|g\| \|T\| \Rightarrow \underbrace{\|T^*\|}_{= \sup_{g \neq 0} \frac{\|T^* g\|}{\|g\|}} \leq \|T\|$$

Let $\|x\| = 1$. By a consequence of the Hahn-Banach theorem $\exists g \in Y^*$ such that $\|g\| = 1$ & $g(Tx) = \|Tx\|$. So $\|Tx\| = \underbrace{g(Tx)}| = |g(Tx)| = |(g \circ T)(x)| \leq \|g \circ T\| \underbrace{\|x\|}_{1} = \|T^* g\| \leq \|T^*\| \|g\|$! Hence $\|Tx\| \leq \|T^*\|$. Taking sup, we get $\|T\| \leq \|T^*\|$ ∎

Closed Graph Theorem. Let $T: \underbrace{D(T)}_{\text{domain of } T} \subseteq X \to Y$ is a linear operator between Banach spaces & $D(T)$ is closed in X. Then T is bounded if and only if $Gr(T) = \{(x, Tx) \mid x \in D(T)\}$ is closed in $X \times Y$ equipped with ~~the product top~~ $\|(x,y)\| = \max\{\|x\|, \|y\|\}$

Proof. ($\Rightarrow$) Let T be bounded & $\underbrace{(x_n, Tx_n)}_{\in Gr(T)} \to (x,y) \in X \times Y$.

So $x_n \to x$ & $Tx_n \to y$. Since T is cts, so $Tx_n \to Tx$. $y = Tx$. Thus $(x,y) = (x, Tx) \in Gr(T)$.

$(x_n, y_n) \to (x,y)$ in $X \times Y$
$\Updownarrow$
$x_n \xrightarrow{\text{in } X} x$ & $y_n \xrightarrow{\text{in } Y} y$

($\Leftarrow$) Since $Gr(T)$ is closed in the Banach space $X \times Y$, $Gr(T)$ is also a Banach space

Let (X, d) be a complete metric space. Then $F \subseteq X$ is closed iff (F, d) is complete.

Define $\varphi: Gr(T) \to D(T)$, $(x, Tx) \longmapsto x$. Clearly φ is linear. In addition, $\|\varphi(x, Tx)\| = \|x\| \leq \max\{\|x\|, \|Tx\|\} = \|(x, Tx)\|$ so φ is bd. φ is also onto, since if $x_0 \in D(T)$, the $\varphi(x_0, Tx_0) = x_0$.

φ is 1-1, since $\varphi(x_1, Tx_1) = \varphi(x_2, Tx_2) \Rightarrow x_1 = x_2 \overset{T \text{ is function}}{\Rightarrow} Tx_1 = Tx_2 \Rightarrow (x_1, Tx_1) = (x_2, Tx_2)$

By a consequence of open mapping theorem, we φ^{-1} is bd

So $\exists M$; $\underbrace{\|\varphi^{-1}(x)\|}_{\|(x,Tx)\|} \leq M\|x\|$. Hence T is bd. $\square$

$\|Tx\| \leq \underbrace{\max\{\|x\|, \|Tx\|\}}$

Proposition. If $T: \mathcal{D}(T) \subseteq X \to Y$ is a bd linear mapping, & $Gr(T)$ is closed, then $\mathcal{D}(T)$ is closed in X.

Proof. Let $\{x_n\}$ be a seq. in $\mathcal{D}(T)$ such that $x_n \to x \in X$. (*)

We shall show that $x \in \mathcal{D}(T)$:

Note that $(x_n, Tx_n) \in Gr(T)$.

$\|(x_n, Tx_n) - (x_m, Tx_m)\| = \|(x_n - x_m, Tx_n - Tx_m)\| = \max\{\|x_n - x_m\|, \underbrace{\|Tx_n - Tx_m\|}_{\leq \|T\|\|x_n - x_m\|}\}$

$= \max\{1, \|T\|\} \|x_n - x_m\|$

Since $\{x_n\}$ is Cauchy (since it is convergent), we conclude that $\{(x_n, Tx_n)\}$ is Cauchy. Due to $X \times Y$ is Banach, $\{(x_n, Tx_n)\}$ is conver in $Gr(T)$.

Thus $\exists (u, v) \in X \times Y$; $(x_n, Tx_n) \to (u, v)$. Since $Gr(T)$ is closed, $(u, v) \in Gr(T)$. Hence $u \in \mathcal{D}(T)$. On the other hand, $x_n \to u$ (**) & $Tx_n \to v$. By (*), (**) $x = u \in \mathcal{D}(T)$. $\square$

Exercise. If $\forall x \in H$; $\langle Tx, x\rangle = 0$, then $T = 0$ $\xrightarrow{}$ Hilbert space over the field $\mathbb{C}$

Th. If $\{e_n\}$ is an orth. seq in H, the T.F.A.E:

(i) $\{e_n\}$ is maximal (ماکزیمال بودن به این معناست که هیچ عنصر دیگری از H را نمی‌توان به $\{e_n\}$ اضافه کرد به طوری که مجموعه حاصل متعامد یکه باقی بماند.)

(ii) the set of all finite linear combinations of $\{e_n\}$ is dense in H

(iii) $x = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n$

$P = \{\sum_{i=1}^{n} \lambda_i e_i \mid \lambda_i \in \mathbb{C}\}$

(iv) $\langle x, y\rangle = \sum \langle x, e_n\rangle \overline{\langle y, e_n\rangle}$

$e_2 + 30e_4 = 0e_1 + 1e_2 + 0e_3 + 30e_4$

(v) $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2$

Proof. (i $\Rightarrow$ ii). Let $\overline{P} \neq H$. By a theorem, $\overline{P}^{\perp} \neq \{0\}$, so (E.) $\exists x_0 \in \overline{P}^{\perp}$, $x_0 \neq 0$. Then $\frac{x_0}{\|x_0\|}$ is orthogonal to each element of P, in particular, $\frac{x_0}{\|x_0\|} \perp e_n$ $\forall n$. Then $\{\frac{x_0}{\|x_0\|}\} \cup \{e_1, e_2, \dots\}$ is orthonormal $\supsetneq \{e_1, e_2, \dots\}$ which is impossible by (i).

(ii $\Rightarrow$ iii) Consider the linear map

$$\begin{cases} \theta : P \longrightarrow \overset{\text{Ban}}{\ell^2} = \{(x_n) \mid \underbrace{(\sum_{n=1}^{\infty} |x_n|^2)^{\frac{1}{2}}}_{= \|(x_n)\|_2} < \infty\} \\ \sum_{i=1}^{n} \lambda_i e_i \longmapsto (\lambda_i)_{i=1}^{n} = (\lambda_1, \dots, \lambda_n, 0, 0, \dots) \end{cases}$$

$$\left\langle \sum_{i=1}^{n}\lambda_i e_i , \sum_{j=1}^{m}\mu_j e_j \right\rangle = \cdots = \sum_{i=1}^{\infty} \lambda_i \overline{\mu_i} = \langle (\lambda_i), (\mu_j) \rangle \quad (n \leq m)$$

$$= \langle \theta(\;), \theta(\;) \rangle . \text{ So } \left\|\theta\left(\sum_{i=1}^{n}\lambda_i e_i\right)\right\| = \|(\lambda_i)\| = \left(\sum|\lambda_i|^2\right)^{\frac{1}{2}}$$

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۸ تا ۱۱:۳۰ کلاس فوق‌العاده

Ban

$T : D \subseteq X \to Y$ is a bd linear map, then T has an extension $\overline{T} : \overline{D} \to Y$ such that $\|\overline{T}\| = \|T\|$. If T is isometry, so is $\overline{T}$ طولپاست

So $\exists\, \overline{\theta} : \underbrace{\overline{P}}_{H} \to \ell^2$ is an isometry

$$\overline{\theta}\Big(\underbrace{\sum_{i=1}^{\infty}\lambda_i e_i}_{x}\Big) = (\lambda_i)_{i=1}^{\infty}$$

$$\langle x, e_i \rangle = \left\langle \sum_{j=1}^{\infty}\lambda_j e_j , e_i \right\rangle = \sum_{j=1}^{\infty} \langle \lambda_j e_j , e_i \rangle = \lambda_i \cancel{\langle e_i, e_i \rangle} = \lambda_i \Rightarrow$$

$$x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i .$$

(iii) $\Rightarrow$ (ii) & (iv) $\Rightarrow$ (v) are clear. We prove (v) $\Rightarrow$ (i):

Assume $\{e_n\}$ is not maximal. So $\exists\, \{e_n\} \subsetneq S$ orthonormal. Let $x_0 \in S \setminus \{e_n\}$.

So $\langle x_0, e_n \rangle = 0 \;\forall n$. Hence $\|x_0\| = \left(\sum_{n=1}^{\infty} |\langle x_0, e_n\rangle|^2\right)^{\frac{1}{2}} = 0$.

So $x_0 = 0$. Here $\|x_0\| = 0$. On the other hand $\|x_0\| = 1$. $\times$.

2.8 Definition Let f be a real (or extended-real) function on a topological space. If

$$f^{-1}((\alpha,\infty)) = \{x: f(x) > \alpha\}$$

$\{x : \sup_i f_i(x) > \alpha\} = \bigcup_i \{x : f_i(x) > \alpha\}$

$\Downarrow$

$\sup_i f_i$ is l.s.cts if so are f_i's.

is open for every real α, f is said to be *lower semicontinuous*. If

$$\{x: f(x) < \alpha\}$$

is open for every real α, f is said to be *upper semicontinuous*.

A real function is obviously continuous if and only if it is both upper and lower semicontinuous.

The simplest examples of semicontinuity are furnished by characteristic functions:

(*a*) Characteristic functions of *open* sets are *lower* semicontinuous.
(*b*) Characteristic functions of *closed* sets are *upper* semicontinuous.

The following property is an almost immediate consequence of the definitions:

(*c*) *The supremum of any collection of lower semicontinuous functions is lower semicontinuous. The infimum of any collection of upper semicontinuous functions is upper semicontinuous.*

5.8 The Banach-Steinhaus Theorem *Suppose X is a Banach space, Y is a normed linear space, and $\{\Lambda_\alpha\}$ is a collection of bounded linear transformations of X into Y, where α ranges over some index set A. Then either there exists an $M < \infty$ such that*

$$\|\Lambda_\alpha\| \le M \qquad \sup_\alpha \|\Lambda_\alpha\| < \infty \tag{1}$$

for every $\alpha \in A$, or

$$\sup_{\alpha \in A} \|\Lambda_\alpha x\| = \infty \tag{2}$$

for all x belonging to some dense G_δ in X.

PROOF Put

$$\varphi(x) = \sup_{\alpha \in A} \|\Lambda_\alpha x\| \qquad (x \in X) \tag{3}$$

and let

$$V_n = \{x: \varphi(x) > n\} \qquad (n = 1, 2, 3, \ldots). \tag{4}$$

Since each Λ_α is continuous and since the norm of Y is a continuous function on Y (an immediate consequence of the triangle inequality, as in the proof of Theorem 4.6), each function $x \to \|\Lambda_\alpha x\|$ is continuous on X. Hence φ is lower semicontinuous, and each V_n is open.

5.6 Baire's Theorem *If X is a complete metric space, the intersection of every countable collection of dense open subsets of X is dense in X.*

Two cases happen: (i) If one of the sets V_n is not dense in X, then $\exists x_0 \in X$; $x_0 \notin \bar{V}_N$. $\exists r$; $N_r(x_0) \cap V_N = \emptyset$ or equivalently, if $\|z - x_0\| < r \Rightarrow z \notin V_N$ (or equiv. $\varphi(x + x_0) \leq N$ or equiv. $\|\Lambda_\alpha(x + x_0)\| \leq N \ \forall \alpha$).

We have $\|\Lambda_\alpha x\| = \|\Lambda_\alpha(x + x_0) - \Lambda_\alpha(x_0)\| \leq \|\Lambda_\alpha(x + x_0)\| + \|\Lambda_\alpha(0 + x_0)\| \leq N + N = 2N$ ($\|x\| < r$)

So $\forall x \in X$; $\|x\| = 1$: $\left\|\frac{rx}{2}\right\| = \frac{r}{2} < r \Rightarrow \left\|\Lambda_\alpha\left(\frac{r}{2}x\right)\right\| < 2N \Rightarrow \|\Lambda_\alpha(x)\| \leq \frac{4N}{r}$.

$\therefore \|\Lambda_\alpha\| = \sup_{\|x\|=1} \|\Lambda_\alpha x\| \leq \frac{4N}{r}$, so $\sup_\alpha \|\Lambda_\alpha\| < \infty$.

(ii) All V_n's are dense in X. By the Baire theorem $\bigcap_{n=1}^\infty V_n$ is dense in X.

$\forall x \in \bigcap_{n=1}^\infty V_n$; $(\forall n; x \in V_n) \Rightarrow (\forall n; \underbrace{\varphi(x)}_{\sup_\alpha \|\Lambda_\alpha x\|} > n) \Rightarrow \sup_\alpha \|\Lambda_\alpha x\| = \infty$. $\square$

Application 1

L^p-SPACES

3.1 Definition A real function φ defined on a segment (a, b), where $-\infty \leq a < b \leq \infty$, is called *convex* if the inequality

$$\varphi((1 - \lambda)x + \lambda y) \leq (1 - \lambda)\varphi(x) + \lambda\varphi(y) \qquad (1)$$

holds whenever $a < x < b$, $a < y < b$, and $0 \leq \lambda \leq 1$.

to be open, is important.

3.2 Theorem *If φ is convex on (a, b) then φ is continuous on (a, b).*

3.4 Definition If p and q are positive real numbers such that $p + q = pq$, or equivalently

$p > 1 \Rightarrow \frac{1}{q} = 1 - \frac{1}{p} < 1 \Rightarrow q > 1$

$$\frac{1}{p} + \frac{1}{q} = 1, \qquad (1)$$

then we call p and q a pair of *conjugate exponents*. It is clear that (1) implies $1 < p < \infty$ and $1 < q < \infty$. An important special case is $p = q = 2$.

As $p \to 1$, (1) forces $q \to \infty$. Consequently 1 and ∞ are also regarded as a pair of conjugate exponents. Many analysts denote the exponent conjugate to p by p', often without saying so explicitly.

3.5 Theorem *Let p and q be conjugate exponents, $1 < p < \infty$. Let X be a measure space, with measure μ. Let f and g be measurable functions on X, with range in $[0, \infty]$. Then*

→ Holder inequality

$$\int_X fg\,d\mu \le \underbrace{\left\{\int_X f^p\,d\mu\right\}^{1/p}}_{A}\underbrace{\left\{\int_X g^q\,d\mu\right\}^{1/q}}_{B} \tag{1}$$

Hölder inequality

and

Minkowski inequality

$$\left\{\int_X (f+g)^p\,d\mu\right\}^{1/p} \le \left\{\int_X f^p\,d\mu\right\}^{1/p} + \left\{\int_X g^p\,d\mu\right\}^{1/p}. \tag{2}$$

PROOF Let A and B be the two factors on the right of (1). If $A = 0$, then $f = 0$ a.e. (by Theorem 1.39); hence $fg = 0$ a.e., so (1) holds. If $A > 0$ and $B = \infty$, (1) is again trivial. So we need consider only the case $0 < A < \infty$, $0 < B < \infty$. Put

$$F = \frac{f}{A}, \qquad G = \frac{g}{B}. \tag{3}$$

$F^p = \frac{f^p}{A^p} \Rightarrow \int F^p = \frac{1}{A^p}\int f^p = \frac{A^p}{A^p} = 1$

This gives

$$\int_X F^p\,d\mu = \int_X G^q\,d\mu = 1. \tag{4}$$

$F(x) = e^{\frac{s}{p}}$

$\Uparrow$

$\ln F(x) = \frac{s}{p}$

$\Uparrow$

$s = p \ln F(x)$

If $x \in X$ is such that $0 < F(x) < \infty$ and $0 < G(x) < \infty$, there are real numbers s and t such that $F(x) = e^{s/p}$, $G(x) = e^{t/q}$. Since $1/p + 1/q = 1$, the convexity of the exponential function implies that

$$e^{s/p + t/q} \le p^{-1}e^s + \underbrace{q^{-1}}_{(1-p^{-1})}e^t. \tag{5}$$

It follows that

$$F(x)G(x) \le p^{-1}F(x)^p + q^{-1}G(x)^q \tag{6}$$

for every $x \in X$. Integration of (6) yields

$$\int_X FG\,d\mu \le p^{-1} + q^{-1} = 1, \tag{7}$$

$\int \frac{fg}{AB} \le 1 \Rightarrow \int fg \le AB$

by (4); inserting (3) into (7), we obtain (1).

To prove (2), we write

$$(f+g)^p = f\cdot(f+g)^{p-1} + g\cdot(f+g)^{p-1}. \tag{8}$$

Hölder's inequality gives

$$\int f\cdot(f+g)^{p-1} \le \left\{\int f^p\right\}^{1/p}\left\{\int (f+g)^{(p-1)q}\right\}^{1/q}. \tag{9}$$

$\frac{1}{p} + \frac{1}{q} = 1$

Let (9′) be the inequality (9) with f and g interchanged. Since $(p-1)q = p$, addition of (9) and (9′) gives

$$\int (f+g)^p \le \left\{\int (f+g)^p\right\}^{1/q}\left[\left\{\int f^p\right\}^{1/p} + \left\{\int g^p\right\}^{1/p}\right]. \tag{10}$$

Clearly, it is enough to prove (2) in the case that the left side is greater than 0 and the right side is less than ∞. The convexity of the function t^p for $0 < t < \infty$ shows that

$\lambda = \frac{1}{2}$

$f(\frac{1}{2}x + \frac{1}{2}y) \le \frac{1}{2}f(x) + \frac{1}{2}f(y)$

$$\left(\frac{f+g}{2}\right)^p \le \frac{1}{2}(f^p + g^p).$$

$\int\left(\frac{f+g}{2}\right)^p \le \frac{1}{2}\int f^p + \frac{1}{2}\int g^p < \infty$

Hence the left side of (2) is less than ∞, and (2) follows from (10) if we divide by the first factor on the right of (10), bearing in mind that $1 - 1/q = 1/p$. This completes the proof. ////

The L^p-spaces

In this section, X will be an arbitrary measure space with a positive measure μ.

3.6 Definition If $0 < p < \infty$ and if f is a complex measurable function on X, define

$$\|f\|_p = \left\{\int_X |f|^p \, d\mu\right\}^{1/p} \tag{1}$$

and let $L^p(\mu)$ consist of all f for which

$$\|f\|_p < \infty. \tag{2}$$

We call $\|f\|_p$ the L^p-norm of f.

If μ is Lebesgue measure on R^k, we write $L^p(R^k)$ instead of $L^p(\mu)$, as in Sec. 2.21. If μ is the counting measure on a set A, it is customary to denote the corresponding L^p-space by $\ell^p(A)$, or simply by ℓ^p, if A is countable. An element of ℓ^p may be regarded as a complex sequence $x = \{\xi_n\}$, and

$$\|x\|_p = \left\{\sum_{n=1}^{\infty} |\xi_n|^p\right\}^{1/p}.$$

$\mu(A) = \begin{cases} \text{Card}\,A & A \text{ finite set} \\ \infty & A \text{ infinite set} \end{cases}$

Let $\mathbb{N}$ be equipped with the counting measure & $f \geq 0$

$\int_{\mathbb{N}} f\,d\mu = \sum_{n=1}^{\infty} \int_{\{n\}} f\,d\mu = \sum_{n=1}^{\infty} f(n)\underbrace{\mu(\{n\})}_{1} = \sum_{n=1}^{\infty} \underbrace{f(n)}_{x_n}$, $\mathbb{N} = \bigcup_{n\in\mathbb{N}}\{n\}$

$\ell^p = \{\{x_n\} \mid \int_{\mathbb{N}} |f|^p d\mu < \infty\}$, $\sum |x_n|^p$

3.7 Definition Suppose $g: X \to [0, \infty]$ is measurable. Let S be the set of all real α such that

$$\mu(\underbrace{g^{-1}((\alpha, \infty])}_{\{x \mid g(x) > \alpha\}}) = 0. \tag{1}$$

$\equiv g(x) \leq \alpha$ a.e.

If $S = \varnothing$, put $\beta = \infty$. If $S \neq \varnothing$, put $\beta = \inf S$. Since

$$g^{-1}((\beta, \infty]) = \bigcup_{n=1}^{\infty} g^{-1}\left(\left(\beta + \frac{1}{n}, \infty\right]\right), \tag{2}$$

$\subseteq g^{-1}(\alpha_n, \infty)$, $\alpha_n \in S$

and since the union of a countable collection of sets of measure 0 has measure 0, we see that $\beta \in S$. We call β the *essential supremum* of g.

If f is a complex measurable function on X, we define $\|f\|_\infty$ to be the essential supremum of $|f|$, and we let $L^\infty(\mu)$ consist of all f for which $\|f\|_\infty < \infty$. The members of $L^\infty(\mu)$ are sometimes called *essentially bounded* measurable functions on X. If f is bd, then $\|f\|_\infty = \sup_{x\in X} f(x)$

It follows from this definition that the inequality $|f(x)| \leq \lambda$ holds for almost all x if and only if $\lambda \geq \|f\|_\infty$.

$\inf S < \lambda \Rightarrow \exists \alpha \in S;\ \alpha < \lambda \Rightarrow f^{-1}(\alpha, \infty) \supseteq f^{-1}(\lambda, \infty) \Rightarrow \mu(f^{-1}(\lambda,\infty)) = 0 \Rightarrow |f(x)| \leq \lambda$ a.e.

$|f(x)| \leq \lambda$ a.e. $\Rightarrow \lambda \in S \Rightarrow \lambda \geq \inf S = \|f\|_\infty$.

3.8 Theorem *If p and q are conjugate exponents, $1 \leq p \leq \infty$, and if $f \in L^p(\mu)$ and $g \in L^q(\mu)$, then $fg \in L^1(\mu)$, and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q. \tag{1}$$

PROOF For $1 < p < \infty$, (1) is simply Hölder's inequality, applied to $|f|$ and $|g|$. If $p = \infty$, note that

$$\underbrace{|f(x)g(x)|}_{=|f(x)||g(x)|} \leq \|f\|_\infty |g(x)| \quad \text{a.e.} \tag{2}$$

for almost all x; integrating (2), we obtain (1). If $p = 1$, then $q = \infty$, and the same argument applies. ////

3.9 Theorem *Suppose* $1 \le p \le \infty$, *and* $f \in L^p(\mu)$, $g \in L^p(\mu)$. *Then* $f + g \in L^p(\mu)$, *and*

$$\|f + g\|_p \le \|f\|_p + \|g\|_p. \tag{1}$$

PROOF For $1 < p < \infty$, this follows from Minkowski's inequality, since

$$\int_X |f + g|^p \, d\mu \le \int_X (|f| + |g|)^p \, d\mu.$$

For $p = 1$ or $p = \infty$, (1) is a trivial consequence of the inequality $|f + g| \le |f| + |g|$. $\rightsquigarrow \int|f+g| \le \int(|f|+|g|) \le \int|f| + \int|g|$ ////

$\hookrightarrow |f+g| \le \|f\|_\infty + \|g\|_\infty$ a.e. $\Rightarrow \|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$.

3.11 Theorem $L^p(\mu)$ *is a complete metric space, for* $1 \le p \le \infty$ *and for every positive measure* μ.

① Theorem. Let $\{x_n\}$ be a sequence in a Hilbert space H and $x_0 \in H$ such that $\langle x_n, x\rangle \to \langle x_0, x\rangle$ $\forall x \in H$; (We say $\{x_n\}$ weakly tends to x_0)

Then $\{x_n\}$ is bd, i.e, $\exists M$ th; $\|x_n\| \le M$.

Proof: put $\{\underbrace{\langle x_n, \cdot\rangle}_{\Lambda_n}\}_{n=1}^{\infty}$. $\forall x \in H$; $\{\Lambda_n(x)\}$ is convergent, so it is bd. By the Ban.-Stein. it is uniformly bounded, so $\exists M \forall n$; $\underbrace{\|\Lambda_n\|}_{\|x_n\|} \le M$. □

$x_n \xrightarrow{\|\cdot\|} x \Rightarrow \|x_n - x\| \to 0 \Rightarrow \forall y: |\langle x_n, y\rangle - \langle x, y\rangle| = |\langle x_n - x, y\rangle| \le \underbrace{\|x_n - x\|}_{\to 0}\|y\| \Rightarrow \forall y, \langle x_n, y\rangle \to \langle x, y\rangle$

$\Rightarrow x_n \xrightarrow{\text{weakly}} x$

An immediate consequence of the Banach-Stein theorem is that if $\{\Lambda_\alpha\}$ is $\underbrace{\text{pointwise bounded}}_{\forall x;\ \{\Lambda_\alpha x\} \text{ is bd}}$, then $\{\Lambda_\alpha\}$ is $\underbrace{\text{uniformly bd}}_{\exists M \forall \alpha;\ \|\Lambda_\alpha\| \le M}$

$\exists M_x \forall \alpha \|\Lambda_\alpha x\| \le M_x$ $\qquad$ $\exists M \forall x \forall \alpha;\ \|\Lambda_\alpha x\| \le M$

3.11 Theorem *$L^p(\mu)$ is a complete metric space, for $1 \le p \le \infty$ and for every positive measure μ.*

L^P is the set of all equivalence classes:

$[f] = \{g \mid g \sim f\}$

PROOF Assume first that $1 \le p < \infty$. Let $\{f_n\}$ be a Cauchy sequence in $L^p(\mu)$. There is a subsequence $\{f_{n_i}\}$, $n_1 < n_2 < \cdots$, such that

$$\|f_{n_{i+1}} - f_{n_i}\|_p < 2^{-i} \qquad (i = 1, 2, 3, \ldots). \tag{1}$$

n_{i+1}

Put

$$g_k = \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|, \quad g = \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}|. \tag{2}$$

$\|g_k\|_p \le \sum_{i=1}^{k} \|f_{n_{i+1}} - f_{n_i}\|_p < \sum 2^{-i} < 1$

Since (1) holds, the Minkowski inequality shows that $\|g_k\|_p < 1$ for $k = 1, 2, 3, \ldots$. Hence an application of Fatou's lemma to $\{g_k^p\}$ gives $\|g\|_p \le 1$. In particular, $g(x) < \infty$ a.e., so that the series

g^p

$\lim \int g_k^p \ge \int \lim g_k^p$

$g \sim f \Leftrightarrow g = f$ a.e.

$$f_{n_1}(x) + \sum_{i=1}^{\infty} (f_{n_{i+1}}(x) - f_{n_i}(x)) \tag{3}$$

$\|f\|_p = 0 \Rightarrow \int |f|^p = 0$

converges absolutely for almost every $x \in X$. Denote the sum of (3) by $f(x)$, for those x at which (3) converges; put $f(x) = 0$ on the remaining set of measure zero. Since

$\Rightarrow |f|^p = 0$ a.e. $\Rightarrow$

$$f_{n_1} + \sum_{i=1}^{k-1} (f_{n_{i+1}} - f_{n_i}) = f_{n_k}, \tag{4}$$

$f = 0$ a.e. $\Rightarrow [f] = [0]$

we see that

$$f(x) = \lim_{i \to \infty} f_{n_i}(x) \qquad \text{a.e.} \tag{5}$$

Having found a function f which is the pointwise limit a.e. of $\{f_{n_i}\}$, we now have to prove that this f is the L^p-limit of $\{f_n\}$. Choose $\epsilon > 0$. There exists an N such that $\|f_n - f_m\|_p < \epsilon$ if $n > N$ and $m > N$. For every $m > N$, Fatou's lemma shows therefore that

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$$\int_X |f - f_m|^p \, d\mu \le \liminf_{i \to \infty} \int_X |f_{n_i} - f_m|^p \, d\mu \le \epsilon^p. \tag{6}$$

$\|f_{n_i} - f_m\|_p^p$

کلاس f یعنی رده توابعی که تقریباً همه جا با f برابرند

We conclude from (6) that $f - f_m \in L^p(\mu)$, hence that $f \in L^p(\mu)$ [since $f = (f - f_m) + f_m$], and finally that $\|f - f_m\|_p \to 0$ as $m \to \infty$. This completes the proof for the case $1 \le p < \infty$.

In $L^\infty(\mu)$ the proof is much easier. Suppose $\{f_n\}$ is a Cauchy sequence in $L^\infty(\mu)$, let A_k and $B_{m,n}$ be the sets where $|f_k(x)| > \|f_k\|_\infty$ and where $|f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty$, and let E be the union of these sets, for $k, m, n = 1, 2, 3, \ldots$. Then $\mu(E) = 0$, and on the complement of E the sequence $\{f_n\}$ converges uniformly to a bounded function f. Define $f(x) = 0$ for $x \in E$. Then $f \in L^\infty(\mu)$, and $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. ////

$L^P =$ مجموعه این کلاس ها

$\|[f]\| = \int |f|^p \, d\mu$

$\forall x;\ f_n(x)$ is conv to $f(x)$

$\forall k\ |f_k(x)| \le \sup_k \|f_k\|$

$E = (\bigcup_k A_k) \cup (\bigcup_{m,n} B_{m,n})$

$\mu(E) \le \sum \mu(A_k) + \sum \mu(B_{m,n}) = 0$

Last statement: درس بخوانید