If X equipped with $d(x,y)=\|x-y\|$ is complete then $(X,\|\cdot\|)$ is called a $\boxed{\text{Banach space.}}$

Examples:

① $\mathbb{C}^n$, $\|(z_1,\cdots,z_n)\|=\left(\sum_{i=1}^{n}|z_i|^2\right)^{\frac{1}{2}}$ Euclidean norm

② $n=1 \longrightarrow \mathbb{C}$, $\|z\|=|z|$

③ $M_{m\times n}(\mathbb{C})$, $\|[a_{ij}]\|_\sigma=\sum_{\substack{1\le i\le m\\1\le j\le n}}|a_{ij}|=\sum_{j=1}^{n}\sum_{i=1}^{m}|a_{ij}|$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn}\end{bmatrix}$$

$\|[a_{ij}]\|_m=\max\{|a_{ij}| : 1\le i\le m \ \& \ 1\le j\le n\}$

$\|[a_{ij}]\|_c=\max_{1\le j\le n}\left(\sum_{i=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}} \longleftarrow \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj}\end{bmatrix}\in\mathbb{C}^n$

$\|[a_{ij}]\|_r=\max_{1\le i\le m}\left(\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}} \longleftarrow [a_{i1}\ \cdots\ a_{in}]\in\mathbb{C}^n$

$\|[a_{ij}]\|_p=\left(\sum_{i,j}|a_{ij}|^p\right)^{\frac{1}{p}}$

$p=1 \rightsquigarrow \|\cdot\|=\|\cdot\|_\sigma$
$p=2 \rightsquigarrow$ Hilbert-Schmidt norm

④ $\ell^\infty=\{\{x_n\} \mid x_n\in\mathbb{C},\ \underbrace{\sup_n|x_n|<\infty}_{\in\mathbb{R}}\}$, $\|\{x_n\}\|=\sup_n|x_n|$

$\lambda\{x_n\}=\{\lambda x_n\}$, $\{x_n\}+\{y_n\}=\{x_n+y_n\}$

$c=\{\{x_n\} \mid \{x_n\} \text{ is convergent}\}\subseteq\ell^\infty$
$\cup\!\!\shortmid$
$c_0=\{\{x_n\} \mid \lim_n x_n=0\}$.

c_0, c are subspaces of ℓ^∞.

⑤ $C(X)=\{f: X\to\mathbb{C} \mid f \ldots$

⑤ $C_b(X) = \{ f : X \to \mathbb{C} \mid f$ is $\underbrace{\text{bounded}}_{\text{bd}}$ & $\underbrace{\text{continuous}}_{\text{cts}} \}$

X → topological space

$(\lambda f + g)(x) = \lambda f(x) + g(x)$ (pointwise) &

$\|f\| = \sup_{x \in X} |f(x)|$

⑥ $C_0(X) = \{ f : X \to \mathbb{C} \mid f$ is cts & $\forall \varepsilon\, \exists K$ compact $\forall x \notin K,\ |f(x)| < \varepsilon \}$, $\|f\| = \sup_{x \in X} |f(x)|$

X → locally compact Hausdorff space

$\leq C_b(X)$

If X is compact, then $C_0(X) = C_b(X) = \underbrace{C(X)}$

$f \in C_b(X)$, Given $\varepsilon > 0$. Put $K := \{ x \in X \mid |f(x)| \geq \varepsilon \}$

$= \underbrace{\underbrace{|f|}_{\text{cts}}^{-1}(\underbrace{[\varepsilon, \infty)}_{\text{closed}})}_{\text{closed}} \subseteq \underbrace{X}_{\text{compact}}$

→ K compact

Hence $f \in C_0(X)$.

⑦ $B(X, Y) = \{ T : X \to Y \mid T$ is bd & linear $\}$

normed space

$(\lambda T + S)(x) = \lambda Tx + Sx$, $\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$

Let $\|x\| \leq 1$.

$\|(T+S)(x)\| = \|Tx + Sx\| \leq \|Tx\| + \|Sx\| \leq \underbrace{\|T\| + \|S\|}_{\text{an upper bound}}$

$\underbrace{\sup_{\|x\| \leq 1} \|(T+S)x\|}_{\|T+S\|} \leq \|T\| + \|S\|$

__Exercise__ $\|T\| \overset{\text{Def.}}{=} \sup_{\|x\| \leq 1} \|Tx\| = \sup \dfrac{\|Tx\|}{\|x\|} = \sup \|Tx\| \overset{?}{=} \sup \|Tx\|$

$x \neq 0$ $\quad \|x\|=1 \quad \underbrace{\|x\|\leq 1}_{\alpha}$

Solution. We shall show that $\alpha = \|T\|$.

$$\underbrace{\{\|Tx\| : \|x\|<1\}}_{\sup\ =\ \alpha} \subseteq \{\|Tx\| : \|x\|\leq 1\}$$

$$\leq \sup = \|T\|$$

$$\alpha \leq \|T\|$$

Let $\|x\|\leq 1$. Assume that $0<\theta<1$ is arbitrary. Then $\|\theta x\| = \theta\|x\| < 1$. Hence

$$\|T(\theta x)\| \leq \alpha$$

$$\theta\|Tx\| \leq \alpha$$

Let $\theta \to 1$, then $\|Tx\| \leq \alpha$.

$$\therefore \underbrace{\sup_{\|x\|\leq 1}\|Tx\|}_{\|T\|} \leq \alpha. \quad \square$$

Ban

Def. $X^* = B(X,\mathbb{C})$ is a Banach space even if X is a normed space.

X' — dual space of X

the space of bd linear functionals

Let $T \in B(X,Y)$. Define $\begin{cases} T' : Y' \to X' \\ g \mapsto T'g = g\circ T \end{cases}$

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[dr, dashed, "g\circ T"'] & Y \arrow[d, "g"] \\ & \mathbb{C} \end{tikzcd}$$

T' is called the Banach adjoint of T.

Theorem T' is linear, is bd & $\|T'\| = \|T\|$

$$T'(\lambda g_1+g_2) \overset{?}{=} \lambda T'g_1+T'g_2 \in X' \quad \checkmark$$

$$[T'(\lambda g_1+g_2)]x = [(\lambda g_1+g_2)\circ T]x = (\lambda g_1+g_2)(Tx)$$
$$= \lambda g_1(Tx)+g_2(Tx) = [\lambda(g_1\circ T)+(g_2\circ T)]x$$
$$= [\lambda T'g_1+T'g_2]x \quad (\forall x\in X)$$

Let $\|g\|\leq 1$. Suppose that $\|x\|\leq 1$. Then

$$\|(g\circ T)(x)\| = \|g(Tx)\| \leq \|g\|\,\|Tx\| \leq \|Tx\| \leq \|T\|\,\|x\| \leq \|T\|$$

$$\|(g\circ T)(x)\| \leq \|T\| \quad (\underset{\|x\|\leq 1}{x\in X})$$

$$\sup_{\|x\|\leq 1}\|(g\circ T)(x)\| = \|g\circ T\| \leq \|T\|$$

$$\|T'g\| \leq \|T\|$$

$$\therefore \Big(\sup_{\|g\|\leq 1}\|T'g\| =\Big) \|T'\| \leq \|T\|.$$

$$\infty > \|T\| = \sup_{x\neq 0}\frac{\|Tx\|}{\|x\|} \Rightarrow \underset{x\neq 0}{\frac{\|Tx\|}{\|x\|}} \leq \|T\| \Rightarrow \boxed{\|Tx\|\leq\|T\|\,\|x\|}$$

قانون تابلو

Lemma (A corollary to the Hahn-Banach Theorem)
If Z is a normed space & $0\neq z_0\in Z$, then
$$\exists h\in Z' ;\ \|h\|=1 \ \&\ |h(z_0)| = \|z_0\|.$$

Let $x\in X$ with $\|x\|\leq 1$. Applying the lemma above to $z_0 = T(x)$ we get $\exists g\in Y'$; $\|g\|=1$ & $|\underbrace{\underbrace{g(Tx)}_{(g\circ T)x}}_{T'g}| = \|Tx\|$

So

$$\|Tx\| = |(T'g)x| \leq \|T'g\|\,\|x\| \leq \|T'g\| \leq \|T'\|\,\|g\| = \|T'\|$$

$\therefore \|Tx\| \leq \|T'\| \quad (x \in X, \|x\| \leq 1)$

$\sup_{\|x\| \leq 1} \|Tx\| \leq \|T'\|$

$\|T\| \leq \|T'\|. \square$

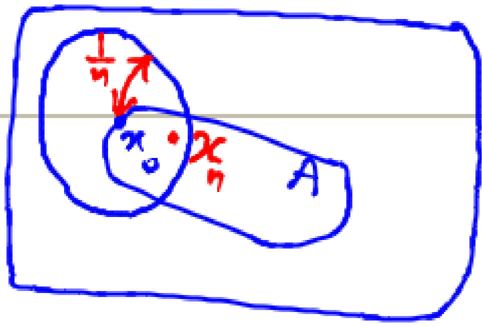
Exercise If $T \& S \in B(X) = B(X,X)$. then $\|TS\| \leq \|T\| \|S\|$.

We know:

Theorem: A functional f on a normed space X is continuous iff $\ker f$ is closed in X.
cts

Lemma. Let (X,d) be a metric space, $A \subseteq X$ & $x \in X$.
Then $x \in \bar{A}$ iff $\exists \{x_n\}$ in A such that $x_n \to x$.

Proof. $(\Rightarrow)$ Let $x \in \bar{A}$. By the definition of a cluster point:
$\forall n;\ N_{\frac{1}{n}}(x) \cap A \neq \emptyset$



Let $x_n \in N_{\frac{1}{n}}(x) \cap A$. Then $0 \leq d(x_n, x) < \frac{1}{n}$. We shall show that $x_n \to x$ as $n \to \infty$:

Let $\varepsilon > 0$ be given. By the Archimedean property of $\mathbb{R}$,
Given $\varepsilon > 0$
$\exists N;\ \frac{1}{N} < \varepsilon$. Then $\forall n \geq N;\ d(x_n, x) < \frac{1}{n} < \frac{1}{N} < \varepsilon$. ////

$n \geq N$

$(\Leftarrow)$ Let $\exists \{x_n\}$ in A s.t. $x_n \to x$. We shall show that $x \in \bar{A}$:
Let $N_r(x)$ be a nbd of x. $\exists N$ ~~$\forall n \geq N$~~: $d(x_N, x) < r$.
So $x_N \in N_r(x) \cap A$. Thus $N_r(x) \cap A \neq \emptyset$. Therefore $x \in \bar{A}$. $\square$

Note: To prove that a set A is closed we should prove that $\bar{A} \subseteq A$ (clearly $A \subseteq \bar{A}$). To show this we assume $x \in \bar{A}$ & $\boxed{\text{show that } x \in A.}$

$\boxed{\exists \{x_n\} \text{ in } A \text{ s.t. } x_n \to x}$

Lemma. Let (X,d) be a complete metric space & $Y \subseteq X$. Then (Y,d) is complete iff Y is closed in X.

<u>Proof</u>. $(\Rightarrow)$ Let (Y,d) be complete. Let $\{y_n\}$ be a sequence in Y s.t. $\exists x \in X$; $y_n \to x$ in (X,d) ?

① $Y \subseteq X$ is a set
② (Y,d) is a metric subspace of X

$\frac{1}{n} \to 0$ in $\mathbb{R}$, but it is not convergent in $(0,1]$ equipped with the Euclidean metric.

Since $\{y_n\}$ is convergent in (X,d), it is a Cauchy sequence

$\forall \varepsilon\ \exists N\ \forall m,n \geq N;\ d(y_n, y_m) < \varepsilon$ (in (X,d))
" (in (Y,d))

Hence $\{y_n\}$ is a Cauchy seq. in (Y,d). Due to (Y,d) is complete

$\exists y \in Y$; $y_n \to y$ in (Y,d). H

$\exists y \in Y;\ y_n \to y$ in (Y,d). Hence

$\forall \varepsilon\ \exists N\ \forall n \geq N;\ d(y_n, y) < \varepsilon$ (in (Y,d))

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ (in (X,d))

Hence $y_n \to y$ in (X,d) ②

①, ② $\Rightarrow x = y \in Y$. ////

($\Leftarrow$) Let Y be a closed subset of (X,d). We shall show that (Y,d) is a complete metric space.

Let $\{y_n\}$ be a Cauchy sequence in (Y,d). So $\{y_n\}$ is a Cauchy sequence in (X,d). Because of completeness of (X,d), $\exists x \in X;\ y_n \to x$ in (X,d).

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \in Y$

So $x \in \overline{Y} = Y$. Hence

Y is closed

$\forall \varepsilon\ \exists N\ \forall n \geq N;\ d(y_n, x) < \varepsilon$ (in (X,d))

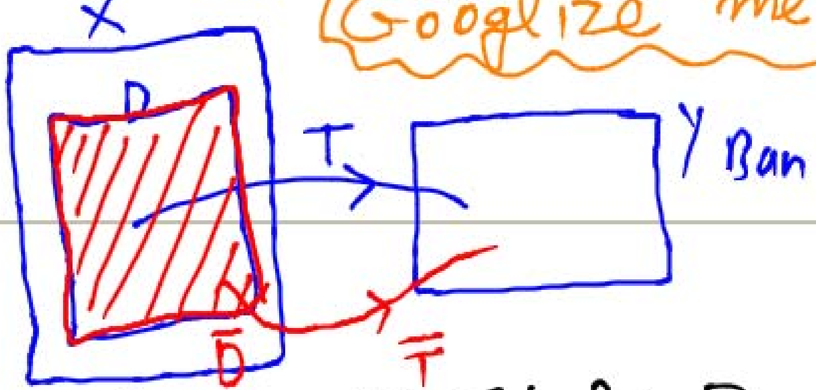
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ (in (Y,d))

Therefore $y_n \to x$ in (Y,d). Thus (Y,d) is complete. □

<u>Th</u> Let $T: D \leq X \to Y$ be a bdd linear map on a subspace D of a normed space X and let Y be a Banach space. Then there exists a unique extension $\overline{T}$ to the closure $\overline{D}$ of D such that $\overline{T}$ is linear and $\|\overline{T}\| = \|T\|$.

(Googlize me (moslehian))

$\overline{T}|_D = T$

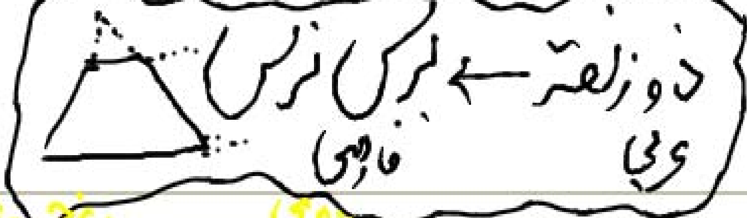


<u>Proof</u>. Let $x \in \overline{D}$. $\exists \{x_n\}$ in D; $x_n \to x$. Hence $\{x_n\}$ is Cauchy, so $\forall \varepsilon \, \exists N \, \forall m, n \geq N$; $\|x_n - x_m\| < \frac{\varepsilon}{\|T\|}$.

Hence $\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \leq \|T\| \|x_n - x_m\| < \|T\| \cdot \frac{\varepsilon}{\|T\|} = \varepsilon$.

Hence $\{Tx_n\}$ is Cauchy in the Banach space Y. So $\exists y \in Y$; $Tx_n \to y$. Let set $\overline{T} : \overline{D} \to Y$

$$x \mapsto \overline{T}x := y$$



$\overline{T}$ is well-defined: This means that the existence of y is independent of choosing the sequence $\{x_n\}$.

Let $x_n \to x$ & $x'_n \to x$, where $x_n, x'_n \in D$. Consider $x_1, x'_1, x_2, x'_2, x_3, x'_3, \ldots$ The even & odd subsequences of above sequence converge to x. So the sequence itself converges to x. If we apply our construction above we get $Tx_1, Tx'_1, Tx_2, Tx'_2, \ldots \to y$

AGAIN: $\lim Tx_n = \lim Tx'_n$

پایان ۲۴ بهمن ۱۳۸۹

Reminder: If $T : D \leq X \longrightarrow Y$ is bd linear map, then

X: normed space, Y: Ban space

$\exists !\, \bar{T} : \bar{D} \longrightarrow Y$ such that $\|\bar{T}\| = \|T\|$

unique linear

Proof. Let $x \in \bar{D}$. $\exists \{x_n\}$ in D; $x_n \longrightarrow x$. Set $\bar{T}x = \lim_n Tx_n$

We want to show that $\bar{T}$ is linear:

Let $x, y \in \bar{D}$, $\lambda \in \mathbb{C}$. $\exists \{x_n\}, \{y_n\}$ in D; $x_n \to x$ & $y_n \to y$

Hence $\lambda x_n + y_n \longrightarrow \lambda x + y \in \bar{D}$

$$0 \leq \|(\lambda x_n + y_n) - (\lambda x + y)\| \leq |\lambda| \underbrace{\|x_n - x\|}_{\to 0} + \underbrace{\|y_n - y\|}_{\to 0}$$

D is a subspace $\Rightarrow$ $\bar{D}$ is a subspace

Let $x, y \in \bar{D}$ & $\lambda \in \mathbb{C}$. $\exists \{x_n\}, \{y_n\}$; $x_n \to x$, $y_n \to y$. Hence $\underbrace{\lambda x_n + y_n}_{\in D} \to \lambda x + y$

So $\lambda x + y \in \bar{D}$. Thus $\bar{D}$ is a subspace

So $\bar{T}(\lambda x + y) = \lim_n T(\lambda x_n + y_n) = \lim_n (\lambda T x_n + T y_n) = \lambda \bar{T}x + \bar{T}y$. ////

To prove the uniqueness of $\bar{T}$, let S be another bd extension of T to $\bar{D}$.

Let $x \in \bar{D}$. $\exists \{x_n\}$ in D s.t. $x_n \to x$. $\bar{T}, S$ are cts on $\bar{D} \supseteq D$, so

$Tx_n = \bar{T}x_n \longrightarrow \bar{T}x$ & $Tx_n = Sx_n \longrightarrow Sx$. Therefore $\bar{T}x = Sx$. $\therefore \bar{T} = S$

$$\{Tx : x \in D \text{ \& } \|x\| \leq 1\} \subseteq \{\bar{T}u : u \in \bar{D} \text{ \& } \|u\| \leq 1\}$$

$$\sup \;\; // \qquad\qquad \leq \sup \;\; //$$

$$\|T\| \leq \|\bar{T}\|$$

کراندار کراندار

$A \subseteq B \subseteq \mathbb{R} \Rightarrow \sup A \leq \sup B$

$\forall b \in B;\ b \leq \sup B$

$\forall a \in A;\ a \leq \sup B$ (اثبات

کران بالای A

$\sup A \leq \sup B$ چون $\sup A$ کوچکترین کران بالای A است (1)

اثبات) فرض کنیم $\sup B < \sup A$، بنا به مشخصه $\sup$: $\exists a \in A;\ \sup B < a \leq \sup A$ اما چون $A \subseteq B$ باید $a \leq \sup B$ که با (1) تناقض دارد.

Now, let $x \in \overline{D}$ & $\|x\| \leq 1$. $\exists \{x_n\}$ in D; $x_n \to x$. So $\overline{T}x = \lim_n Tx_n$.

$$\|\overline{T}x\| = \lim_n \|Tx_n\| \leq \|T\| \lim_n \|x_n\| = \|T\|\|x\| \leq \|T\|$$

$\|Tx_n\| \leq \|T\| \, \|x_n\|$

چرا؟

f: $\|\cdot\|: X \to \mathbb{R}$ is cts:

$$\big| \underbrace{\|x\|}_{f(x)} - \underbrace{\|y\|}_{f(y)} \big| \leq \|x-y\|$$

$$\therefore \|\overline{T}\| = \sup_{x \in \overline{D}, \|x\| \leq 1} \|\overline{T}x\| \leq \|T\|. \quad \square$$

Let $X' = \{ f \mid f: X \to \mathbb{C}$ is bd & linear $\}$. $(X', \|\cdot\|_{\text{operator norm}})$ is a Banach space. Put $X'' = (X')'$.

the dual of X

The map $\hat{\ }: X \overset{\text{linear}}{\hookrightarrow} X''$, $\hat{x}(f) := f(x)$ is an isometry:

$x \longmapsto \hat{x}$

$$\|\hat{x}\| = \sup_{\|f\| \leq 1} \|\hat{x}(f)\| = \sup_{\|f\| \leq 1} |f(x)| \leq \|x\| \sup_{\|f\| \leq 1} \|f\| = \|x\|$$

$$\|x\| = |f(x)| = |\hat{x}(f)| \leq \|\hat{x}\| \, \|f\| \leq \|\hat{x}\|. \quad ////$$

A corollary to the Hahn-Ban theorem

$\forall x \neq 0 \ \exists f \in X'$; $\|f\| = 1$ & $|f(x)| = \|x\|$

The mapping $\hat{\ }$ is 1-1: $\hat{x} = \hat{y} \Rightarrow \hat{x} - \hat{y} = 0 \Rightarrow \widehat{x-y} = 0 \Rightarrow$

$\|x-y\| = \|\widehat{x-y}\| = 0 \Rightarrow x - y = 0 \Rightarrow x = y$. ////

If the map $\hat{\ }$ is surjective, then X is called <u>reflexive</u>.

<u>Exercise</u>: X is reflexive iff so is X'. (تمرین) (Kreyszig book — An Introductory to Functional Analysis)

Example. $\mathbb{C}^n$ is reflexive. (تمرین)

Problem. Is there a Banach space X such that X is isometrically isomorphic to X'', but X is not reflexive. Solution: Yes:

Per Enflo gave a nice counterexample

Let $T: X \to X$ be a (bd) operator & $M \leq X$ (closed) subspace. Then T leaves M invariant if $TM \subseteq M$. Trivially $\{0\}$ & X are invariant under any operator T.

Problem: Does any (bd) operator T have a non-trivial invarian closed subspace M?

P. Enflo answered this question negatively

gave a counterexample (100 Pages) Acta Math. (خبرنامه انجمن ریاضی ایران

Reminder:

Let $T: X \to Y$ be a bd linear operator. Define $T': Y' \to X'$ by $(T'g)(x) = g(Tx)$. Then $\|T'\| = \|T\|$.

Let $M \leq X$. Then $M^\perp = \{f \in X' : f|_M = 0\}$. Similarly if $N \leq X'$. Then ${}^\perp N = \{x \in X : f(x) = 0 \ \forall f \in N\}$.

Lemma 1. $M^\perp$ is a subspace.

Proof. $f, g \in M^\perp$, $\lambda \in \mathbb{C}$. $(\lambda f + g)(x) = \lambda f(x) + g(x) = 0 \ \forall x \in M$

$\therefore \lambda f + g \in M^\perp$. $\square$

Lemma 2. ${}^\perp N$ is a closed subspace

Proof. Clearly $^{\perp}N$ is a subspace.

Let $x_n \in {}^{\perp}N$ & $x_n \to x \in X$.

$\underbrace{f(x_n)}_{0} \longrightarrow f(x) \quad (f \in N)$

$\therefore f(x) = 0 \quad (f \in N)$

$\therefore x \in {}^{\perp}N$. $\square$

Lemma. Let M be a closed subspace of a normed (Ban) space of X. Then $\frac{X}{M} = \{x+M \mid x \in X\}$ is a normed (Ban) space under the quotient norm: $\|x+M\| = \inf_{z \in M} \|x+z\|$.

Proof. ① $\|x+M\| = 0 \Rightarrow \inf_{z\in M}\|x+z\| = 0$. $\exists \{\|x+z_n\|\}$; $\|x+z_n\| \to 0$

Lemma $\inf A \in \bar{A}$ if $A \subseteq \mathbb{R}$

Proof. $\forall r$: $\underbrace{N_r(\beta)}_{(\beta-r,\beta+r)} \cap A \neq \emptyset$ (?)

Since $\beta = \inf A$, $\exists x \in A$; $x < \beta + r$

So $\beta - r < \beta \leq x < \beta + r$ or $x \in (\beta - r, \beta + r) \cap A$

$\therefore N_r(\beta) \cap A \neq \emptyset$.

So $\|z_n - (-x)\| \to 0$. Hence $\underbrace{z_n}_{\in M} \to -x$. Therefore $-x \in \overline{M} = M$, hence $x \in M$, since M is a subspace. Thus $x + M = 0_{\frac{X}{M}}$.

② $\|\lambda(x+M)\| = \|\lambda x + M\| = \inf_{z\in M}\|\lambda x + z\| = |\lambda| \inf_{z \in M}\|x + \frac{z}{\lambda}\|$ (w.l.o.g $\lambda \neq 0$)

$= |\lambda| \inf_{u \in M}\|x+u\| = |\lambda|\|x+M\|$ $\quad \{\frac{z}{\lambda} \mid z \in M\} = M$ (M subspace) $\quad u = \frac{z}{\lambda}$

③ $\|x+M+y+M\| \overset{?}{\leq} \underbrace{\|x+M\|}_{\inf\limits_{z\in M}\|x+z\|} + \underbrace{\|y+M\|}_{\inf\limits_{u\in M}\|y+u\|}$

Let $\varepsilon>0$.

$\exists z\in M;\ \|x+z\| < \|x+M\|+\varepsilon$

$\exists u\in M;\ \|y+u\| < \|y+M\|+\varepsilon$

$\underbrace{\|(x+M)+(y+M)\|}_{\varphi(\varepsilon)} = \|(x+y)+M\| \overset{\inf\ldots}{\leq} \|x+y+\underbrace{(z+u)}_{\in M}\| \leq \|x+z\|+\|y+u\|$

$< \underbrace{\|x+M\|+\varepsilon+\|y+M\|+\varepsilon}_{\psi(\varepsilon)}$

$\lim\limits_{\varepsilon\to 0}\varphi(\varepsilon) \leq \lim\limits_{\varepsilon\to 0}\psi(\varepsilon)$

$\|(x+M)+(y+M)\| \leq \|x+M\|+\|y+M\|$.

Now we prove that $\frac{X}{M}$ is a Banach space if so is X.
$M \to$ closed subspace

Let $\{x_n+M\}$ be a Cauchy seq in $\frac{X}{M}$.

$\forall \varepsilon \exists N \forall m,n \geq N;\ \underbrace{\|(x_n+M)-(x_m+M)\|}_{\inf\limits_{z\in M}\|x_n-x_m+z\|} < \varepsilon$.

Let M be a closed subspace of X. Then $\left(\frac{X}{M}\right)' \cong M^{\perp}$ isometrically isomorphism

Proof Set $\varphi : \left(\frac{X}{M}\right)' \longrightarrow M^{\perp}$

$f \longmapsto \varphi(f) = \underbrace{f \circ \pi}_{\text{linear}}$

$$\begin{array}{ccc} X & \xrightarrow[\text{onto}]{\pi} & \frac{X}{M} \\ & \searrow^{f\circ\pi} & \downarrow f \\ & & \mathbb{C} \end{array}$$

$\|\pi(x)\| = \|x+M\| = \inf_{z\in M}\|x+z\| \leq \|x\|$ norm decreasing $\therefore \|\pi\| \leq 1$

We have

$\forall x \in M;\ \varphi(f)(x) = (f\circ\pi)(x) = f(\pi(x)) = f(\underbrace{0_{\frac{X}{M}}}_{\substack{x+M \\ 0+M}}) = 0$

$\therefore \varphi(f) \in M^{\perp}$

since $\|\varphi(f)\| = \|f\circ\pi\| \leq \|f\|\,\|\pi\| \leq \|f\|$

we conclude that $\|\varphi\| \leq 1$.

Let $x+M \in \frac{X}{M}$, $\|x+M\| = 1$.

$|f(\underbrace{x+M}_{=x+z+M})| = |f(\pi(x+z))| = |\varphi(f)(x+z)| \leq \|\varphi(f)\|\,\|x+z\| \quad \forall z \in M$

(اگر باشد $\frac{|f(x+M)|}{\|\varphi(f)\|} \leq \|x+z\| \quad \forall z \in M$

$\frac{|f(x+M)|}{\|\varphi(f)\|} \leq \inf_{z\in M}\|x+z\| = \|x+M\|$

$\frac{|f(x+M)|}{\|x+M\|} \leq \|\varphi(f)\|$ یعنی

$\|f\| = \sup \quad " \quad \leq \quad "$

Thus $\|\varphi(f)\| = \|f\|$, i.e. φ is an isometry.

So φ is clearly 1-1 (since φ is linear). We shall show that φ is surjective (onto): Let $g \in M^{\perp}$.

Define $\begin{cases} f : \frac{X}{M} \xrightarrow{\text{linear}} \mathbb{C} \\ f(x+M) = g(x) \end{cases}$. Then $\varphi(f) = f\circ\pi = g$.

$$\begin{array}{ccc} X & \xrightarrow{\pi} & \frac{X}{M} \\ & \searrow^{g}\ (\mathrm{c}) & \downarrow f\ ? \\ & & \mathbb{C} \end{array}$$

We should show that $f \in \left(\frac{X}{M}\right)'$:

[illegible] $:\ x+M$

[illegible] $:\ \overset{\|}{y+M}$ $\Big\} \Rightarrow x-y \in M \overset{g\in M^{\perp}}{\Longrightarrow} g(x-y) = 0 \Rightarrow g(x) = g(y) \Rightarrow f(x+M) = f(y+M)$

$f(\lambda(x+M)+(y+M)) = f(\lambda x+y+M) = g(\lambda x+y) = \cdots$

Let $x+M \in \frac{X}{M}$, $\|x+M\|=1$

$$|f(x+M)| = |f(x+z+M)| = |g(x+z)| \leqslant \|g\|\cdot\|x+z\| \quad (z\in M)$$

lower bound $\leftarrow$ $$\frac{|f(x+M)|}{\|g\|} \leqslant \|x+z\| \quad (z\in M)$$

$$\frac{|f(x+M)|}{\|g\|} \leqslant \|x+M\|$$

$$\|f\| = \sup \frac{|f(x+M)|}{\|x+M\|} \leqslant \|g\|. \text{ So } f\in\left(\frac{X}{M}\right)'. \ \square$$

Theorem . $\frac{X'}{M^\perp} \cong M'$

Proof Set $\Psi : \frac{X'}{M^\perp} \xrightarrow{\text{linear}} M'$

$$\begin{cases} \Psi : \frac{X'}{M^\perp} \to M' \\ \Psi(f+M^\perp) = f|_M \in M' \end{cases}$$

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & \frac{X'}{M^\perp} \\ f & \longmapsto & f+M^\perp \\ & & \downarrow \Psi \\ & & f|_M \quad M' \end{array}$$

خوش تعریفی : $f+M^\perp$
(خطی) فی : $g+M^\perp$

$$\left.\begin{array}{l} f+M^\perp \\ \quad\ \ \| \\ g+M^\perp \end{array}\right\} \Rightarrow f-g\in M^\perp \Rightarrow \forall x\in M;\ (f-g)(x)=0 \Rightarrow f|_M = g|_M$$

$f(x) = g(x)$

Let $g\in M^\perp$. So $g(x)=0$ for all $x\in M$.

$$\|\Psi(f+M^\perp)\| = \|f|_M\| = \sup_{\substack{x\in M\\ \|x\|=1}} |f(x)\underbrace{+g(x)}_{0}| \leqslant \sup_{\substack{x\in X\\ \|x\|=1}} |\underbrace{f(x)+g(x)}_{(f+g)(x)}| = \|f+g\|$$

a lower bound

$$\therefore \underbrace{\|\Psi(f+M^\perp)\|}_{\|f|_M\|} \leqslant \inf_{\in M^\perp} \|f+g\| = \|f+M^\perp\|$$

$\therefore \Psi$ is bd.

Let $f\in X'$. By the Hahn-Banach theorem to $f|_M \in M'$ $\exists g\in X'$; $\|g\| = \|f|_M\|$ & $g|_M = f|_M$

So $g-f\in M^\perp$

$$\|f+M^\perp\| = \inf_{h\in M^\perp} \|f+h\| \leqslant \|\cancel{f}+(g-\cancel{f})\| = \|g\| = \|f|_M\| = \|\Psi(f+M^\perp)\|$$

$\therefore \|\Psi(f+M^\perp)\| = \|f+M^\perp\|$, so Ψ is an isometry.

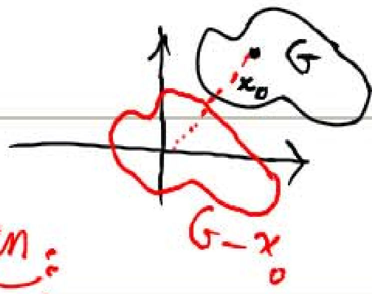
Now we show that Ψ is onto:

Let $h\in M'$. By the Hahn-Banach theorem applied to h, $\exists f \in X^*$; $f|_M = h$ & $\cancel{\|f|_M\| = \|h\|}$. اضافه

$\Psi(f + M^\perp) = f|_M = h$. $\square$

<u>Def</u>. Let X be a vector space endowed with a topology τ such that the operations $+ : X \times X \to X$, $(x, y) \mapsto x+y$ & $\cdot : \mathbb{C} \times X \to X$, $(\lambda, x) \mapsto \lambda x$ are continuous & τ is Hausdorff. Then X is called a <u>topological vector space</u>.

t.v.s.



If G is an open set, then $x_0 \in G$

$0 \in G - x_0 = \{y - x_0 \mid y \in G\}$ is also <u>open</u>:

$T_a : X \to X$, $x \mapsto x + a$ is cts since $x_n \to x \Rightarrow x_n + a \to x + a$, in addition, its inverse $T_a^{-1} = T_{-a}$ is cts, so T_a is a homeomorphism.

$M_\lambda : X \to X$, $x \mapsto \lambda x$ is a homeomorphism $(\lambda \neq 0)$.

Exercise. Let X be a Banach space & M be a closed subspace of X. Then $\frac{X}{M}$ is a Banach space.

Solution. Suppose that $\sum_{n=1}^{\infty}\|x_n+M\|$ is a convergent series. We shall show that $\sum_{n=1}^{\infty}(x_n+M)$ converges in $\frac{X}{M}$ (Then $\frac{X}{M}$ is complete): By the definition of the quotient norm:

$$\exists y_n\in M;\ \|x_n+y_n\|<\|x_n+M\|+\frac{1}{2^n}$$

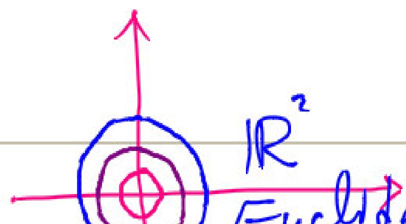
By the comparison criterion, $\sum_{n=1}^{\infty}\|x_n+y_n\|$ is convergent, since $\sum\left(\|x_n+M\|+\frac{1}{2^n}\right)<+\infty$. Due to X is complete, $\sum_{n=1}^{\infty}(x_n+y_n)$ converges to, say $z\in X$. Assume that $s_n=\sum_{k=1}^{n}(x_k+y_k)$ is the n-th partial sum & $t_n=\sum_{k=1}^{n}\underbrace{x_k+M}_{x_k+y_k+M}$.

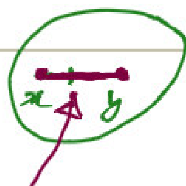
$$\|t_n-(z+M)\|=\left\|\left(\sum_{k=1}^{n}x_k+y_k-z\right)+M\right\|\leq\left\|\sum_{k=1}^{n}(x_k+y_k)-z\right\|=\|s_n-z\|$$

Since $\lim_{n\to\infty}s_n=\sum_{n=1}^{\infty}(x_n+y_n)=z$, we have $t_n\longrightarrow(z+M)$ as $n\to\infty$.
We just proved that $\sum_{n=1}^{\infty}(x_n+M)=z+M$. $\square$

Def. A t.v.s is called locally convex if $\underset{\tilde{x}_0}{0}$ has a local base β whose members are convex

$\forall \underset{\underset{x_0}{\Downarrow}}{G}\in\tau\ \exists V\in\beta;\ x_0\in V\subseteq G$





$\lambda x+(1-\lambda)y$
$0<\lambda<1$

discrete metric

Every subset of X is open.

singleton $\quad \mathcal{B} = \{\{x_0\}\}$

$\mathcal{B}' = \{G \subseteq X \mid x_0 \in G\}$

$\forall \underset{U_{x_0}}{G} \in \tau \,;\; x_0 \in \underbrace{\{x_0\}}_{\text{open}} \subseteq G$

$x_0 \in G \subseteq G$

Note. If $\mathcal{B}$ is a local base at 0, then $\mathcal{B} + x_0 = \{V + x_0 \mid V \in \mathcal{B}\}$ is a local base at x_0.

Lemma. If W is a nbd of 0 in X (t.v.s.), then there is a nbd U of 0 which is symmetric (i.e. $U = -U$) & $\underbrace{U+U}_{\{x+y \mid x, y \in U\}} \subseteq W$

(Apply the lemma to U. $\exists V_{sym} : V+V \subseteq U$. So $V+V+V+0 \subseteq V+V+V+V \subseteq U+U \subseteq W$. Hence) $\exists S_{open} ;\ \underbrace{S+\cdots+S}_{n} \subseteq W$.

Proof. $+ : X \times X \to X$ is cts at $(0,0)$. so $\exists V_1, V_2$ in X

$V_1 \times V_2 \ni (0,0) \longmapsto 0 \in W$, $\quad (x,y) \mapsto x+y$

such that $V_1 + V_2 \subseteq W$. Put $U = V_1 \cap (-V_1) \cap V_2 \cap (-V_2)$.

Then $U + U \subseteq W$, because of

$$x, y \in U \Rightarrow \begin{cases} x \in V_1 \\ y \in V_2 \end{cases} \Rightarrow x + y \in W.$$

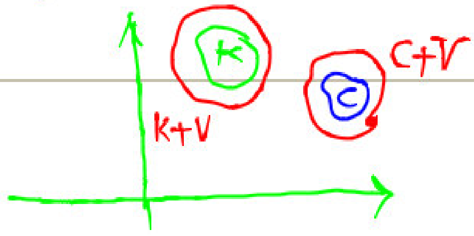
In addition, $x \in U \Rightarrow x \in V_1 \cap (-V_1) \cap V_2 \cap (-V_2) \Rightarrow -x \in (-V_1) \cap V_1 \cap (-V_2) \cap V_2$

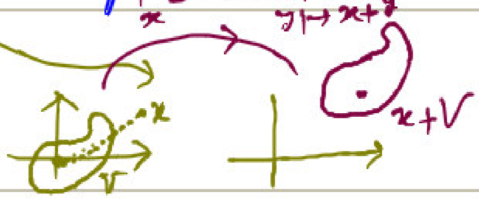
$\Rightarrow -x \in U \qquad \therefore U = -U$. $\square$

Theorem. Suppose K & C are subsets (compact, closed) of a t.v.s. X & $K \cap C = \emptyset$. then 0 has a nbd V such that $(K+V) \cap (C+V) = \emptyset$

$K + V = \bigcup\{x + V \mid x \in K\}$ open; $C + V$ open

$T_x = \cdot + x : X \to X$, $\quad y \mapsto x + y$





Proof. If $K=\emptyset$, then $K+\underset{\text{arbitrary open subset of } X}{V} = \emptyset$ $\left(= \bigcup_{x\in K} x+V\right)$. So $(K+V)\cap(C+V)=\emptyset$.

$\bigcup_{\alpha\in I} A_\alpha$ = the "smallest" subset of X containing all A_α's $=\emptyset$

$=\emptyset$

Let us assume that $K\neq\emptyset$. Consider $x\in K$. By the lemma above, 0 has a symmetric nbd V_x such that $x+V_x+V_x+V_x$ does not intersect C. So

$= \text{why?}$ $x\in C^c$ → open, $0\in C^c - x$, $\exists V_x$; $V_x+V_x+V_x\subseteq C^c-x$, $x+V_x+V_x+V_x\subseteq C^c$

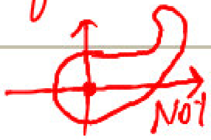
$$(x+V_x+V_x)\cap(C+V_x)=\emptyset$$

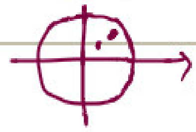
Since K is compact, $\exists x_1,\dots,x_n\in K$; $K\subseteq(x_1+V_{x_1})\cup\dots\cup(x_n+V_{x_n})$ (in fact, $\{x+V_x\}_{x\in K}$ covers K). Put $V=V_{x_1}\cap\dots\cap V_{x_n}$. Then

$$K+V\subseteq\bigcup_{i=1}^{n}(x_i+V_{x_i}+V)\subseteq\bigcup_{i=1}^{n}(x_i+V_{x_i}+V_{x_i})$$

$$\& \left(\bigcup_{i=1}^{n}x_i+V_{x_i}+V_{x_i}\right)\cap\Big(C+\underbrace{\bigcap_{i=1}^{n}V_{x_i}}_{V}\Big)=\emptyset.$$ So $(K+V)\cap(C+V)=\emptyset$ □

Def ① $A\subseteq X$ (vector space) is convex if $\forall x,y\in A\ \forall_{0\le\lambda\le1}$; $\lambda x+(1-\lambda)y\in A$
② $A\subseteq X$ is balanced if $\forall x\in A\ \forall|\lambda|\le1$; $\lambda x\in A$



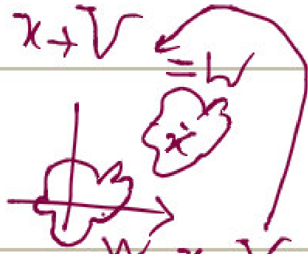


Theorem. ① If $A\subseteq X$ (tvs), then $\overline{A}=\bigcap(A+V)$
V runs through all nbd of 0

$x\in\overline{A}\iff(x+V)\cap A\neq\emptyset\ \forall V$ nbd of 0

$\exists v_1\in V$; $x+v_1\in A\Rightarrow x\in A-v_1$

$x\in A-V$



$\Leftrightarrow x \in A + W \ \forall \underset{\text{nbd}}{W}$ of 0 (since if V is a nbd of 0, then so is $-V$)

$\Leftrightarrow x \in \bigcap_{W} (A+W)$. ////

② If $A \subseteq X$ & $B \subseteq X$, then $\overline{A} + \overline{B} \subseteq \overline{A+B}$ $[1,2] + [-3,-1] = [-2, 1]$

Let $a \in \overline{A}$, $b \in \overline{B}$ & let W be a nbd of $a+b$.

By the continuity of $\underset{(a,b) \mapsto a+b}{+}$, $\exists W_1$ of a $\exists W_2$ of b; $W_1 + W_2 \subseteq W$.

By the definition of the cluster point, $\exists x \in A \cap W_1$ & $\exists y \in B \cap W_2$.

Then $x+y \in (A+B) \cap (W_1 + W_2) \subseteq (A+B) \cap W$ So $a+b \in \overline{A+B}$.

③ If Y is a subspace of X, then so is $\overline{Y}$.

Let $\alpha \in \mathbb{C}$. M_α is a homeomorphism. So $\alpha \overline{Y} = M_\alpha(\overline{Y}) = \overline{M_\alpha(Y)} = \overline{\alpha Y}$

Hence $\alpha \overline{Y} + \overline{Y} = \overline{\alpha Y} + \overline{Y} \underset{②}{\subseteq} \overline{\alpha Y + Y} \subseteq \overline{Y}$. $\underbrace{Y \text{ subspace}}_{\alpha Y + Y \subseteq Y}$

④ If C is convex, then so are $\overline{C}$ & C°.

$\alpha \overline{C} + (1-\alpha)\overline{C} \subseteq \overline{\alpha C} + \overline{(1-\alpha)C} \underset{②}{\subseteq} \overline{\alpha C + (1-\alpha) C} \subseteq \overline{C}$

$\therefore \overline{C}$ is convex

Since $C^\circ \subseteq C$, $\alpha C^\circ + (1-\alpha) C^\circ \subseteq \alpha C + (1-\alpha) C \underset{\text{convex}}{\subseteq} C$

$\underbrace{\alpha C^\circ}_{\substack{C^\circ \text{ open} \\ \& M_\alpha \text{ homeo}\ldots}} + (1-\alpha) C^\circ$ ~~is~~ open, so $\alpha C^\circ + (1-\alpha) C^\circ \subseteq C^\circ$.

$\underset{\text{open}}{V} + \underset{\text{open}}{W} = \bigcup_{x \in V} \underbrace{(x+W)}_{\text{open}}$ $= \underbrace{T_x}_{\text{homeo}}(W)$

$\overline{C} = \bigcap F$, $\underset{\text{closed}}{F \supseteq C}$ & $C^\circ = \bigcup G$, $\underset{\text{open}}{G} \subseteq C$

balanced: $|\lambda| \le 1 \Rightarrow \lambda B \subseteq B \Rightarrow \overline{\lambda B} \subseteq \overline{B} \underset{M_\lambda}{\Rightarrow} \lambda \overline{B} \subseteq \overline{B}$

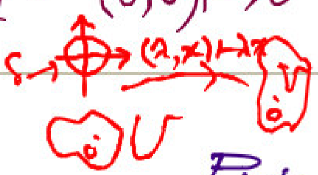
Hence C° is convex.

Similarly ⑤ if B is balanced; then so is $\overline{B}$. If also $0 \in B^\circ$, then B is balanced.

Theorem: In a t.v.s. X every nbd V of 0 contains a balanced nbd W of 0.

$|\alpha| \le 1$ & $x \in W \Rightarrow \alpha x \in W$

Proof. $\mathbb{C} \times X \to X$, $(0,0) \mapsto 0$ under the scalar multiplication & the multip. is cts, so $\exists \delta > 0\ \exists U$ nbh of 0 $\forall \alpha: |\alpha| < \delta \Rightarrow \alpha U \subseteq V$



Put $W = \underbrace{\underbrace{\bigcup_{|\alpha|<\delta} \alpha U}_{\text{open}}}_{\text{open}}$. $U = 1U$

$\forall \beta: |\beta| < 1\ \forall \alpha x \in \bigcup_{|\alpha|<\delta} \alpha U$;

$\beta \cdot \alpha x = (\beta\alpha) x \in \bigcup_{|\gamma|<\delta} \gamma U$

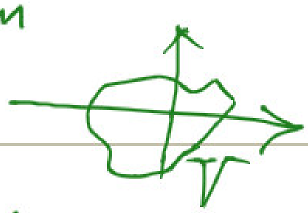
$|\beta\alpha| = |\beta||\alpha| < \delta$

Then W is balanced & open. $\square$

Theorem. Suppose that V is a nbd of 0 in a t.v.s X.

(a) If $0 < r_1 < r_2 < \dots$ & $r_n \to \infty$ as $n \to \infty$, then

$$\bigcup_{n=1}^{\infty} r_n V = X$$



(b) Every compact subset K of X is bd.

$\hookrightarrow$ A subset E of t.v.s. X is said to be bounded if to every nbd V of 0 in X corresponds a number s such that $E \subseteq tV$ for some $t > s$.

Proof (a) Fix $x \in X$. The mapping
$\varphi: \mathbb{C} \to X$ is cts. So
$\alpha \mapsto \alpha x$

$0 \in \varphi^{-1}(V) = \{\alpha \in \mathbb{C} \mid \underbrace{\varphi(\alpha)}_{\alpha x} \in V\}$ is open in $\mathbb{C}$.

Since $\lim_{n\to\infty} \frac{1}{r_n} = \frac{1}{\infty} = 0$ & $\varphi^{-1}(V)$ is a nbh of 0,
we have $\exists N: \frac{1}{r_N} \in \varphi^{-1}(V)$

It follows from the definition of limit

So $\varphi(\frac{1}{r_N}) \in V$ or $\frac{1}{r_N} x \in V$. Hence $x \in r_N V \subseteq \bigcup_{n=1}^{\infty} r_n V$. $\square$

(b) By the theorem above, $\exists \underset{\substack{\text{balanced}\\ \text{nbd of } 0}}{W}$; $W \subseteq V$. By (a)

$K \subseteq X = \bigcup_{n=1}^{\infty} nW$. Since K is compact $\exists n_1 < n_2 < \dots < n_s$;

$K \subseteq n_1 W \cup \dots \cup n_s W = n_s W$

If W is a balanced nbd of 0 & $r < s$, then $\frac{r}{s} < 1$, so $\frac{r}{s} W \subseteq W$. Hence $rW \subseteq sW$.

For any $t > n_s$, we have

$K \subseteq n_s W \subseteq tW \subseteq tV$. Hence K is bd. □

$d(x,y) = d(x+z, y+z)\ \forall x,y,z$

Theorem (a) If d is a translation invariant metric on a vector space X, then $d(nx, 0) \leq n\, d(x,0)$ $\forall x \in X$ $\forall n$

(b) If $\{x_n\}$ is a sequence in a $\overset{\text{trans. inv.}}{}$ metrizable t.v.s X & if $\lim_n x_n = 0$, then there are $\gamma_n > 0$ such that $\gamma_n \to \infty$ & $\gamma_n x_n \to 0$.

X is a t.v.s with a top τ & $\exists \underset{\text{metric}}{d}$; $\tau = \tau_d$ = the set of all open subsets of (X, d)

(a) $d(nx, 0) \leq \underbrace{d(nx, (n-1)x)}_{d(x,0)} + \underbrace{d((n-1)x, (n-2)x)}_{d(x,0)} + \dots + d(x, 0)$
$= n\, d(x,0)$

(b) Let d be a trans. invarian metric on the space X such that τ_d is the given top on X.

The $X = \{a, b\}$, $(\{a,b\}, \{\{a\}, \emptyset, X\})$ is a t.s. which is Not metrizable

Since $d(x_n, 0) \to 0$, $\exists n_1 < n_2 < \dots$ such that $d(x_{n_k}, 0) < \frac{1}{k^2}$:

$\exists n_1\ \forall n > n_1$; $d(x_n, 0) < \overset{\varepsilon}{\frac{1}{1^2}}$

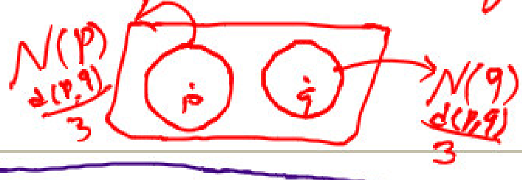
$\therefore d(x_n, 0) < \frac{1}{1^2}$

$\exists n_2 > n_1\ \forall n > n_2$; $d(x_n, 0) < \overset{\varepsilon}{\frac{1}{2^2}}$

$\therefore d(x_{n_2}, 0) < \frac{1}{2^2}$

$\vdots$

Since every metric space is Hausdorff, but the Sirpinski top is Not Hausdorff.



Put $\gamma_n = 1$ if $n < n_1$ & put $\gamma_n = k$ if $n_{k-1} \leq n < n_k$.

For such n, $d(\gamma_n x_n, 0) = d(k x_n, 0) \leq k\, d(x_n, 0) < k \cdot \frac{1}{k^2} = \frac{1}{k}$

So $\gamma_n x_n \to 0$. Note that $\gamma_n \to \infty$. □

Locally convex top. vector spaces (lctvs)

Def. A function $P: X \to \mathbb{R}$ is called a <u>semi-norm</u> on X if
(i) $P(x) \geq 0$ (ii) $P(\lambda x) = |\lambda| P(x)$ (iii) $P(x+y) \leq P(x) + P(y)$

Examples ① Every norm is a seminorm.
② If $f \in X'$ (here X is a normed space), then
$P: X \to \mathbb{R}$ by $P(x) = |f(x)|$

$\forall x \neq 0\ \exists \alpha;\ P_\alpha(x) \neq 0$

Theorem. If X is a l.c.t.v.s, then its topology is generated by a separating family $\{P_\alpha\}$ of seminorms, i.e., $N(P_{\alpha_0}, \varepsilon) = \{x \in X : |P_{\alpha_0}(x)| < \varepsilon\} = P_{\alpha_0}^{-1}((0,\varepsilon))$ is a subbasis, i.e., $N(P_{\alpha_1}, \dots, P_{\alpha_n}, \varepsilon) = \bigcap_{i=1}^{n} P_{\alpha_i}^{-1}((0,\varepsilon))$
$= \{x \in X : P_{\alpha_i}(x) < \varepsilon,\ 1 \leq i \leq n\}$ is a basis, i.e., every open set is a union of a collection of $N(P_{\alpha_1}, \dots, P_{\alpha_n}, \varepsilon)$.

$x, y \in N(P_{\alpha_0}, \varepsilon)$
$0 < \lambda < 1$
$|P_{\alpha_0}(\lambda x + (1-\lambda) y)|$
$\leq \lambda P_{\alpha_0}(x) + (1-\lambda) P_{\alpha_0}(y)$
$< \lambda \varepsilon + (1-\lambda)\varepsilon = \varepsilon$
$\therefore N(P_{\alpha_0}, \varepsilon)$ is convex.

Theorem: $x_\lambda \to x$ in $X \iff \forall \alpha;\ P_\alpha(x_\lambda) \to P_\alpha(x)$.

Reminder: ① If $f: X \to Y$ is a mapping, then $\{f^{-1}(G) \mid G \subseteq Y \text{ is open}\}$ is the weakest top on X such that f is cts.
(X ↓ set, Y ↓ top. sp.)

② If $g: X \to Y$ is a mapping, then $\{V \subseteq Y \mid f^{-1}(V) \text{ is open in } X\}$ is the strongest top on X such that g is cts.
(X ↓ top. sp., Y ↓ set)

Weak top generated by $\mathcal{F}_0$ on a linear space X

Def. Let $\mathcal{F}_0$ be a family of linear functionals $f: X \to \mathbb{C}$.
Then the weakest top on X under which <u>all</u> elements of $\mathcal{F}_0$ are <u>cts</u> is called weak top. generated by $\mathcal{F}_0$, denoted by $\sigma(X, \mathcal{F}_0)$.

In fact, $N(f_1,\dots,f_n,\varepsilon)=\{x\in X : |f_i(x)|<\varepsilon,\ 1\le i\le n\}$ is a typical element of a local basis at origin O.

G open

$\sigma(X,X')\subseteq$ Norm-top

Examples: ① Weak-top on a normed space: $\sigma(X,X')$

$$x_\alpha \xrightarrow{w} x \iff f(x_\alpha)\to f(x) \quad \forall f\in X'$$

② Weak*-top on X': $\sigma(X',\hat{X})$ $\hat{x}: X'\to\mathbb{C}$, $\hat{x}(f)=f(x)$

$$f_\alpha \xrightarrow{w^*} f \iff \underset{\forall \hat{x}\in\hat{X}}{\hat{x}(f_\alpha)\to\hat{x}(f)} \iff f_\alpha(x)\to f(x) \quad \forall x\in X$$

③ $B(H)=\{T: H\to H \mid T \text{ is bd \& linear}\}$

$x^2-4=0$
$x=\pm 2$

$x^2-9=0$
$x=\pm\cancel{3}2$

3.1 $B(H)$ has already a norm: $\|T\|=\sup\limits_{x\neq 0}\dfrac{\|Tx\|}{\|x\|}$

3.2 $\forall x\in H$; $P_x: B(H)\to\mathbb{C}$ is a seminorm.
$P_x(T)=\|Tx\|$ $T_\alpha \xrightarrow{S.O.} T \iff T_\alpha x\to Tx \ \forall x$

The l.c.t.v.s generated by $\{P_x\}_{x\in H}$ is called the Strong Operator Topology on $B(H)$.

$\tau_1\subseteq\tau_2$ on X
$\Updownarrow$
$\forall\{x_\alpha\};\ x_\alpha\xrightarrow{\tau_2}x$
$\Downarrow$
$x_\alpha\xrightarrow{\tau_1}x$

3.3 $\forall x,y\in H$; $q_{x,y}: B(H)\to\mathbb{C}$ is a seminorm.
$q_{x,y}(T)=|\langle Tx,y\rangle|$ $T_\alpha\xrightarrow{W.O.}T \iff \forall x,y;\ \langle T_\alpha x,y\rangle\to\langle Tx,y\rangle$

The l.c.t.v.s. generated by $\{q_{x,y}\}_{x,y\in H}$ is called the Weak Operator topology on $B(H)$.

$$W.O.T \subseteq S.O.T \subseteq \text{Norm Top.}$$

$T_\alpha\xrightarrow{S.O.}T \Rightarrow \|(T_\alpha-T)(x)\|\to 0$

$|\langle T_\alpha x,y\rangle-\langle Tx,y\rangle|\le\|(T_\alpha-T)(x)\|\,\|x\|\Rightarrow\langle T_\alpha x,y\rangle\to\langle Tx,y\rangle$

$\Rightarrow T_\alpha\xrightarrow{W.O.}T$

$T_\alpha\xrightarrow{\|\cdot\|}T\Rightarrow\|T_\alpha x-Tx\|\le\|T_\alpha-T\|\,\|x\|\Rightarrow$

$T_\alpha x\to Tx\ \forall x\Rightarrow T_\alpha\xrightarrow{S.O.}T$

≡ Adjoint of an operator $T \in B(H)$ ≡

$B(H) = \{T : H \to H \mid T \text{ is bd \& linear}\}$

<u>Theorem</u>: $\forall T \in B(H)\ \exists T^* \in B(H);\ \langle Tx, y\rangle = \langle x, T^*y\rangle$

Moreover, $\|T\| = \|T^*\|$ & $\|T^*T\| = \|T\|^2$, $T^{**} = T$, $(\lambda T + S)^* = \bar{\lambda} T^* + S^*$, $(TS)^* = S^* T^*$

<u>Proof</u>. Let $y \in H$ be fixed. Define $f_y : H \to \mathbb{C}$ by $f_y(x) = \langle Tx, y\rangle$

f_y is a bd linear functional:

$$f_y(\lambda x + x') = \langle T(\lambda x + x'), y\rangle = \lambda\langle Tx, y\rangle + \langle Tx', y\rangle$$
$$= \lambda f_y(x) + f(x')$$

$$|f_y(x)| = |\langle Tx, y\rangle| \leq \|Tx\|\,\|y\| \leq \|T\|\,\|x\|\,\|y\| \Rightarrow \|f_y\| \leq \|T\|\,\|y\|. \quad (1)$$

The Riesz representation theorem gives a unique element $z \in H$ such that $f_y(\cdot) = \langle \cdot, z\rangle$ & $\|z\| = \|f_y\|$

Put "T^*y"

$$\langle Tx, y\rangle = f_y(x) = \langle x, T^*y\rangle$$

Thus we have $T^* : H \longrightarrow H$, $y \longmapsto T^*y$. We shall show that $T^* \in B(H)$ & has all required properties:

<u>Lemma</u>. $\langle x, z\rangle = \langle y, z\rangle\ \forall z \Rightarrow x = y$

<u>Proof</u>. $\langle x - y, z\rangle = 0\ \forall z \Rightarrow \langle x - y, x - y\rangle = 0 \Rightarrow x - y = 0 \Rightarrow x = y$ □

Let $y_1, y_2 \in H$ & $\lambda \in \mathbb{C}$.

$$\langle x, T^*(\lambda y_1 + y_2)\rangle = \langle Tx, \lambda y_1 + y_2\rangle = \bar{\lambda}\langle Tx, y_1\rangle + \langle Tx, y_2\rangle$$
$$= \bar{\lambda}\langle x, T^*y_1\rangle + \langle x, T^*y_2\rangle = \langle x, \lambda T^*y_1 + T^*y_2\rangle.$$

By the above lemma, $T^*(\lambda y_1 + y_2) = \lambda T^*y_1 + T^*y_2$

By our above lemma: $T^*(y_1+y_2) = T^*(y_1) + T^*(y_2)$

$\therefore T^*$ is linear

Let $\|y\|=1$. By our construction,

$$\|T^*y\| = \|z\| = \|f_y\| \leq \|T\| \, \|y\| = \|T\|.$$

$$\therefore \|T^*\| = \sup_{\|y\|=1} \|T^*y\| \leq \|T\| < \infty \Rightarrow T^* \in B(H)$$

②

$\forall y$
$\forall x$: $\langle T^*x, y\rangle = \langle x, T^{**}y\rangle \longleftarrow T^{**}$ is the adjoint of T^*

$$\overline{\langle y, T^*x\rangle} = \overline{\langle Ty, x\rangle} = \langle x, Ty\rangle \xrightarrow[\text{Lemma}]{\text{Our}} T^{**}y = Ty, \forall y \Rightarrow T^{**} = T$$

Hence $\|T\| = \|T^{**}\| \underset{③}{\leq} \|T^*\|$

② & ③ $\Rightarrow \|T\| = \|T^*\|$

$$\langle x, (TS)^*y\rangle = \langle (TS)x, y\rangle = \langle T(Sx), y\rangle = \langle Sx, T^*y\rangle = \langle x, S^*(T^*y)\rangle$$

$$= \langle x, (S^*T^*)y\rangle \Longrightarrow (TS)^*y = (S^*T^*)y \ \forall y \Rightarrow (TS)^* = S^*T^*$$

Exercise: $(\lambda S+T)^* = \bar{\lambda} S^* + T^*$.

$$\|T^*T\| \leq \|T^*\| \, \|T\| = \|T\|^2 \quad ④$$

Let $x \in H$.

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \underbrace{\langle x, T^*Tx\rangle}_{\geq 0} = |\langle x, T^*Tx\rangle| \overset{\text{C-S-I}}{\leq} \|x\| \, \|T^*Tx\|$$

$$\leq \|x\|^2 \|T^*T\|$$

$$\|T\|^2 = \left(\sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}\right)^2 = \sup_{x\neq 0} \frac{\|Tx\|^2}{\|x\|^2} \leq \|T^*T\| \quad ⑤$$

④, ⑤ $\Longrightarrow \|T^*T\| = \|T\|^2$. $\square$

Theorem: $\overline{\operatorname{ran} T^*} = (\ker T)^\perp \quad \overline{\operatorname{ran} T}^\perp = \ker T^*$

Proof: $y \in \operatorname{ran} T^* \Rightarrow y = T^* x$ for some $x \in H$.

$\Rightarrow \langle y, z\rangle = \langle T^* x, z\rangle = \langle x, \underbrace{Tz}_{=0}\rangle \; \forall z \in \ker T \Rightarrow y \in \ker T^\perp$

$\therefore \operatorname{ran} T^* \subseteq \ker T^\perp \Rightarrow \overline{\operatorname{ran} T^*} \subseteq \overline{\ker T^\perp} = \ker T^\perp$

$z \in \operatorname{ran} T^{*\perp} \Rightarrow \langle z, T^* y\rangle = 0 \; \forall y \in H \Rightarrow \langle Tz, y\rangle = 0 \; \forall y \in H$

$\Rightarrow Tz = 0 \Rightarrow z \in \ker T$

$\overline{\operatorname{ran} T^*}^\perp = \operatorname{ran} T^{*\perp} \subseteq \ker T$

$\ker T^\perp \subseteq \overline{\operatorname{ran} T^*}^{\perp\perp} = \overline{\operatorname{ran} T^*}$ ///

$\ker T^\perp = \overline{\operatorname{ran} T^*} \; \forall T \overset{T \to T^*}{\Rightarrow} \ker T^{*\perp} = \overline{\operatorname{ran} T} \Rightarrow$

$\ker T^* = \ker T^{*\perp\perp} = \overline{\operatorname{ran} T}^\perp = \operatorname{ran} T^\perp$. $\square$

Def. T is self-adjoint if $T = T^*$

T is normal if $TT^* = T^*T$

T is unitary if $TT^* = T^*T = I_H$ identity function

T is isometry if $T^*T = I$

T is co-isometry if $TT^* = I$

T is partial isometry if $TT^*T = T$

T is projection if $T^2 = T = T^*$

Complex Numbers	Operators of $B(H)$
$z \mapsto \bar z$	adjoint operation $T \mapsto T^*$
$\mathbb{R} = \{z \in \mathbb{C} \mid z = \bar z\}$	$B(H)_h$ = self-adjoint operators $(T = T^*)$
$z = a + ib$, $a, b \in \mathbb{R}$ $\frac{z+\bar z}{2} = a = \operatorname{Re} z$ & $b = \operatorname{Im} z = \frac{z - \bar z}{2i}$	$T = T_1 + iT_2$, $T_1, T_2 \in B(H)_h$ $= \frac{T+T^*}{2} + i\frac{T-T^*}{2i}$
$\mathbb{T} = \{z \in \mathbb{C} \mid \underbrace{z\bar z = 1}_{\lvert z\rvert = 1}\}$	$\mathcal{U} = \{T \in B(H) \mid TT^* = T^*T = I\}$
$\mathbb{C}_+ = \{z \in \mathbb{C} \mid z = \bar w w$ for some $w \in \mathbb{C}\}$	$B(H)_+ = \{T \in B(H) \mid T = S^*S$ for some $S \in B(H)\}$

TH1. $\forall T\in B(H)\ \exists\, T_1,T_2\in B(H)_h;\ T=\underbrace{T_1+iT_2}$ ReT, ImT

Proof. $T=T_1+iT_2 \Rightarrow T^*=(T_1+iT_2)^*=T_1^*-iT_2^*=T_1-iT_2$

$$\Rightarrow T+T^*=2T_1 \ \&\ T-T^*=2iT_2 \Rightarrow T_1=\frac{T+T^*}{2},\ T_2=\frac{T-T^*}{2i} \quad \square$$

TH2. U is unitary $\Longleftrightarrow$ U is isometry & Surjective

$\|Ux\|=\|x\|\ \forall x\in H$

$\|Ux-Uy\|=\|x-y\|\ \forall x,y\in H$

Proof. $(\Rightarrow)$ $U^*U=UU^*=I \Rightarrow \begin{cases} ① \ \langle U^*Ux,x\rangle=\langle Ix,x\rangle \Rightarrow \langle Ux,Ux\rangle=\langle x,x\rangle \\ ② \ UU^*y=Iy\ \forall y \Rightarrow U(U^*y)=y\ \forall y \Rightarrow U \text{ is onto} \end{cases}$

$\Rightarrow \|Ux\|^2=\|x\|^2 \Rightarrow \|Ux\|=\|x\|\ \forall x \Leftrightarrow \|Ux-Uy\|=\|U(x-y)\|=\|x-y\|\ \forall x,y$

interesting

$y=0$

U is surjective

$(\Leftarrow)$ Lemma. $T=0 \Leftrightarrow \langle Tx,x\rangle=0\ \forall x\in H$ a $\boxed{\text{Complex}}$ Hilbert space

Proof. $T=0 \Rightarrow Tx=0 \Rightarrow \langle Tx,x\rangle=0\ \forall x\in H$

Let $\langle Tz,z\rangle=0\ \forall z\in H$.

Let $x,y\in H$. So $\langle T(\alpha x+y),\alpha x+y\rangle=0$

$$|\alpha|^2\cancel{\langle Tx,x\rangle}^{0}+\alpha\langle Tx,y\rangle+\bar{\alpha}\langle Ty,x\rangle+\cancel{\langle Ty,y\rangle}^{0}=0$$

$\alpha=1 \Rightarrow \quad \langle Tx,y\rangle+\langle Ty,x\rangle=0$

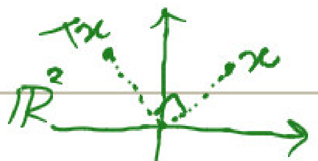
$\alpha=i \Rightarrow \quad i\langle Tx,y\rangle-i\langle Ty,x\rangle=0 \Rightarrow \langle Tx,y\rangle-\langle Ty,x\rangle=0$ Add

$2\langle Tx,y\rangle=0$

$\langle Tx,y\rangle=0$

$\therefore Tx=0\ \forall x$

$\therefore T=0.\ \square$



T is rotation for +90

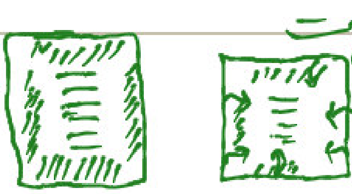
$\langle Tx,x\rangle=0$

Now let U be a surjective isometry.

$\|Ux\|^2=\|x\|^2\ \forall x \Rightarrow \langle Ux,Ux\rangle=\langle x,x\rangle \Rightarrow \langle (U^*U-I)x,x\rangle=0 \xRightarrow[\text{Lemma above}]{} U^*U-I=0 \Rightarrow U^*U=I$

$\longrightarrow U$ is 1-1.

U is 1-1 & surjective, so U is invertible. So. $(U^*U)U^{-1} = I.U^{-1}$



Hence. $U^* = U^{-1}$. Therefore

$$UU^* = UU^{-1} = I. \quad \square$$

TH3. $(T^*)^{-1} = (T^{-1})^*$

Proof. $T^*(T^{-1})^* = (T^{-1}T)^* = I^* = I$

$\langle Ix,y\rangle = \langle x,y\rangle = \langle x, Iy\rangle$, $I = I^*$

$$(T^{-1})^* T^* = \cdots = I. \quad \square$$

TH4. T is normal $\Longleftrightarrow \|Tx\| = \|T^*x\| \ \forall x \in H$

Proof. T is normal $\Longleftrightarrow T^*T = TT^*$

$\Longleftrightarrow \langle T^*Tx, x\rangle = \langle TT^*x, x\rangle \ \forall x$

$\Longleftrightarrow \langle Tx, Tx\rangle = \langle T^*x, T^*x\rangle \ \forall x$

$\Longleftrightarrow \|Tx\|^2 = \|T^*x\|^2 \ \forall x$

$\Longleftrightarrow \|Tx\| = \|T^*x\| \ \forall x$. $\square$

TH5. ① $\underbrace{T \text{ is positive}}_{\substack{T \in B(H)_+ \\ T \geq 0}} \Longleftrightarrow \langle Tx, x\rangle \geq 0 \ \forall x \in H$

② T is self-adjoint $\Longleftrightarrow \langle Tx, x\rangle \in \mathbb{R} \ \forall x \in H$

Proof ① $(\Rightarrow)$ $T \geq 0 \Rightarrow T = S^*S$ for some $S \in B(H) \Rightarrow \langle Tx, x\rangle = \langle S^*Sx, x\rangle$
$= \langle Sx, Sx\rangle = \|Sx\|^2 \geq 0$

② $(\Rightarrow)$ $T \in B(H)_h \Rightarrow T = T^* \Rightarrow \langle Tx, x\rangle = \langle T^*x, x\rangle = \langle x, Tx\rangle$
$= \overline{\langle Tx, x\rangle} \xRightarrow[\substack{b=0 \\ z=a\in\mathbb{R}}]{\substack{z=\bar{z} \\ a+ib=a-ib}} \langle Tx, x\rangle \in \mathbb{R}$. $\square$

TH6. Let M be a closed subspace of H. Then $\exists!$ projection $P: H \to H$ such that $Py = y \ \forall y \in M$ & $Pz = 0 \ \forall z \in M^\perp$. Conversely, for each proj $P \in B(H)$, $\operatorname{ran} P$ is a closed subspace of H. Infact, All Closed Subspaces of H $\overset{\text{correspondence}}{\longleftrightarrow}$ All projections of $B(H)$

Proof. We use $H = M \oplus M^\perp$. Define $P: H \to H$

$\forall x \exists! y \in M \exists! z \in M^\perp ; x = y + z$. $\qquad Px = y \qquad (x = y \oplus z)$

uniqueness

$x = y + z = y' + z'$
$M \ni y - y' = z' - z \in M^\perp$
$M \cap M^\perp = \{0\} \Rightarrow y - y' = z' - z = 0$

$\|x\|^2 = \langle x, x\rangle$
$= \langle y+z, y+z\rangle$
$= \langle y, y\rangle + \cancel{\langle y, z\rangle} + \cancel{\langle z, y\rangle} + \langle z, z\rangle$
$= \|y\|^2 + \|z\|^2$

We have:

① P is linear: $P(x + x') = P((y \oplus z) + (y' \oplus z')) = P((y + y') \oplus (z + z'))$
$= y + y' = Px + Px'$
Similarly, $P(\lambda x) = \lambda Px$

② P is bounded: $\|Px\| = \|y\| \leq \|x\| \quad \therefore \|P\| \leq 1$
$\|x\|^2 = \|y\|^2 + \|z\|^2$

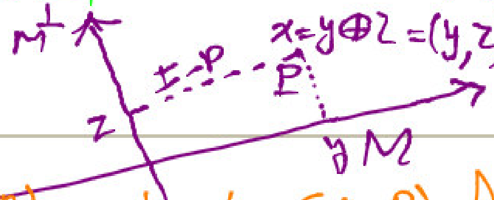
③ $P^2 x = P(Px) = P(y) = P(y \oplus 0) = y = Px \quad \forall x \Rightarrow P^2 = P$ idempotent

$\langle Px, x'\rangle = \langle \underset{M}{y}, y' \oplus \underset{M^\perp}{z}\rangle = \langle y, y'\rangle = \langle y \oplus \underset{M^\perp}{z}, \underset{M}{y'}\rangle = \langle x, y'\rangle = \langle x, \underset{=P^*}{Px'}\rangle \Rightarrow P = P^*$

$\therefore P$ is a projection.

④ $I - P$ is a proj: $(I-P)(I-P) = I - P - \cancel{P} + \cancel{P^2} = I - P$
$(I-P)^* = I^* - P^* = I - P$
$\Rightarrow I - P$ is a proj.

Infact $I - P: H \to H$
$x = y \oplus z \longmapsto (I-P)(x) = x - Px = x - y = z$



⑤ $\operatorname{ran} P = M$ & $\ker(I-P)$, $\operatorname{ran}(I - P) = M^\perp$, $\ker(I - P) = M$

$y \in M \Rightarrow Py = P(y \oplus 0) = y \xRightarrow{P|_M = id} y \in \operatorname{ran} P \quad \therefore M \subseteq \operatorname{ran} P$ ▮ $y' \in \operatorname{ran} P \Rightarrow y' = Px = P(\underset{M}{y} \oplus z) = y \in M$
$\therefore \operatorname{ran} P \subseteq M$

⑥ $\|x\|^2 = \|Px\|^2 + \|(I-P)x\|^2 \longleftarrow \|x\|^2 = \|y\|^2 + \|z\|^2$, $x = y \oplus z$

⑦ $P(I-P) = P - P^2 = P - P = 0 \Rightarrow P \perp (I-P)$

P & $I-P$ are orthogonal projections.

⑧ $\ker P = M^\perp$

First proof)

$x \in \ker P \Rightarrow P\underbrace{x}_{y \oplus z} = 0 \Rightarrow x = z \in M^\perp$; y

$\therefore \ker P \subseteq M^\perp$

$z' \in M^\perp \Rightarrow P(z') = P(0 \oplus z') = 0 \Rightarrow z' \in \ker P$

$\therefore M^\perp \subseteq \ker P$

Second proof) $\ker P^\perp = \overline{\operatorname{ran} P^*} = \overline{\operatorname{ran} P} = \overline{M} = M \Rightarrow \ker P = M^\perp$

Conversely, Let $q \in B(H)$ be a projection. Then $\operatorname{ran} q$ is a closed subspace: Let $qx_n \to x$. Since $q: H \to H$ is cts,

clear

$$\left.\begin{array}{l} q(qx_n) \to qx \\ \underbrace{}_{q^2 x_n} \\ \underbrace{}_{qx_n} \to x \end{array}\right\} \Rightarrow x = qx \in \operatorname{ran} q$$

$\square$

TH9. Let P, q be projections in $B(H)$. Then the following assertions are equivalent:

① $P \le q$ ② $PH \subseteq qH$ ③ $Pq = qP = P$ ④ $\|Px\| \le \|qx\|\ \forall x$

Proof. Definition: Let $A, B \in B(H)_h$. We say $A \le B \Leftrightarrow B - A \ge 0$

$\Updownarrow$

$\langle Ax, x\rangle \le \langle Bx, x\rangle\ \forall x \Leftrightarrow \langle (B-A)x, x\rangle \ge 0\ \forall x$

①$\Leftrightarrow$④ : $P \le q \Leftrightarrow \langle \underset{p^2}{P}x, x\rangle \le \langle \underset{q^2}{q}x, x\rangle \Leftrightarrow \langle Px, \underset{p}{P^*}x\rangle \le \langle qx, \underset{q}{q^*}x\rangle \Leftrightarrow \|Px\|^2 \le \|qx\|^2$

④$\Rightarrow$③ : $\|Px\| \le \|qx\|\ \forall x \Longrightarrow \|P(1-q)x\| \le \|q(1-q)x\| \Rightarrow P(1-q)x = 0 \overset{\forall x}{\Rightarrow} P(1-q) = 0$

$\Rightarrow P = Pq \Rightarrow \underset{P}{P^*} = (Pq)^* = q^* p^* = qP \Rightarrow P = Pq = qP$

③ $\Rightarrow$ ②: $p = pq = qp \Rightarrow p(H) = qp(H) \subseteq q(H)$

② $\Rightarrow$ ③: Let $p(H) \subseteq q(H)$. Then $q\underbrace{px}_{\in p(H) \subseteq q(H)} = px \;\; \forall x \Rightarrow qp = p \Rightarrow$

$\underbrace{p^*}_{p} = \underbrace{p^* q^*}_{pq} \Rightarrow p = pq = qp.$

② $\Rightarrow$ ①: Let $pH \subseteq qH$. ... $\langle px, x \rangle \leq \langle qx, x \rangle \; \forall x \Rightarrow p \leq q. \square$

weak and strong convergence in Hilbert spaces

<u>Def</u>. We say ① $x_n \xrightarrow{s} x$ strongly if $x_n \xrightarrow{\|\cdot\|} x$. ② $x_n \xrightarrow{w} x$ weakly if $f(x_n) \to f(x)$ $\forall f \in H' = \{\langle \cdot, y\rangle \mid y \in H\}$. Hence
$\underbrace{H'}_{\text{dual of } H}$

$$x_n \xrightarrow{w} x \text{ iff } \langle x_n, y\rangle \longrightarrow \langle x, y\rangle \ \forall y \in H$$

<u>Theorem</u>. If $x_n \xrightarrow{s} x$, then $x_n \xrightarrow{w} x$

<u>Proof</u>. $0 \leq |\langle x_n, y\rangle - \langle x, y\rangle| = |\langle x_n - x, y\rangle| \leq \|x_n - x\|\,\|y\|$. □ ($\to 0$, $\to 0$)

<u>Theorem</u>. If $x_n \xrightarrow{w} x$, then $\{x_n\}$ is bounded in H, that is $\exists M \forall n$; $\|x_n\| \leq M$.

<u>Proof</u>. $x_n \xrightarrow{w} x \Rightarrow f(x_n) \to f(x)$ $\forall f \in H' \Rightarrow \hat{x}_n(f) \to \hat{x}(f)$

$\Rightarrow \{\hat{x}_n\}$ is a seq of bd linear maps $\hat{x}_n : \underbrace{H'}_{\text{Ban}} \longrightarrow \underbrace{\mathbb{C}}_{\text{Ban}}$ such that $\sup_n |\hat{x}_n(f)| < \infty$ $\forall f \in H'$ $\xRightarrow{\text{Banach-Steinhause Theorem}}$ $\sup_n \|\hat{x}_n\| < \infty \Rightarrow \sup_n \|x_n\| < \infty$. □

Every convergent sequence is bd.

≡ Banach Algebras ≡

<u>Def</u>. An algebra is a <u>ring</u> $\mathcal{A}$ that is a <u>vector space</u> & $a \cdot (\lambda b) = (\lambda a) \cdot b = \lambda (a \cdot b)$.

(ring: $+, \cdot$; vector space: $+$, scalar)

<u>Def</u>. A normed (Banach) algebra $\mathcal{A}$ is an algebra that is a normed (Banach) space under a norm $\|\cdot\|$ such that
$$\|a \cdot b\| \leq \|a\|\,\|b\|.$$

Examples:

① $\mathbb{C}$ $\|\cdot\| := |\cdot|$, ordinary addition, mult and scalar mult.

② $M_n(\mathbb{C})$ together with matrix $\begin{cases} \text{add} \\ \text{mult} \\ \text{s. mult} \end{cases}$ and the operator norm:

$A \in M_n(\mathbb{C}) \Rightarrow T_A : \mathbb{C}^n \to \mathbb{C}^n$
$X \mapsto AX$ ($n\times 1$ column matrix)

$\Rightarrow \|A\| :=$ the operator norm of T_A $= \sup_{X \neq 0} \frac{\|TX\|}{\|X\|}$

In fact, $M_n(\mathbb{C}) \cong B(\mathbb{C}^n)$

③ $\underbrace{B(X)}_{\text{normed space}} := \{T : X \to X \mid T \text{ is bd \& linear}\}$

& $\|T\| =$ the operator norm. Our action are:

$(T+S)(x) = Tx + Sx$
$(\lambda T)(x) = \lambda Tx$
$(TS)(x) = T(S(x)) = T(Sx)$ composition

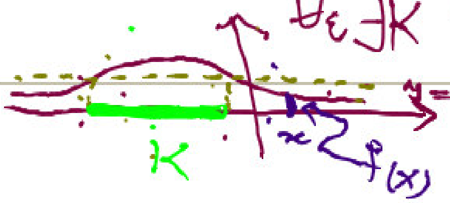
④ $C_b(X) := \{f : X \to \mathbb{C} \mid f \text{ is bd \& cts}\}$, X Hausdorff top. space

$\|f\| :=$ the supremum norm $= \sup_{t \in X} |f(t)|$

$\cup$

$C_0(X) = \{f : X \to \mathbb{C} \mid f \text{ is cts \& } f \text{ vanishes at infinity}\}$, X locally compact Hausdorff sp

$\forall \varepsilon\, \exists K$ compact $\forall x \notin K;\ |f(x)| < \varepsilon$



$C_0(X) = C_b(X)$ iff X is compact
$\hookrightarrow C(X) \hookleftarrow$

Our operations:

$(f+g)(x) = f(x) + g(x)$
$(\lambda f)(x) = \lambda f(x)$
$(fg)(x) = f(x) \cdot g(x)$ the mult. in $\mathbb{C}$

$$C_c(X) = \{f : X \to \mathbb{C} \mid f \text{ is cts } \& \ \underbrace{supp(f)}_{= \overline{\{x \in X \mid f(x) \neq 0\}}} \text{ is compact}\}$$

$$\overline{C_c(X)} = C_0(X)$$

Theorem. If X is compact, then $C_c(X) = C_0(X)$
$$\qquad \shortparallel \qquad C_b(X)$$

Proof. Let $f \in C_b(X)$. Then $supp(f) = \underbrace{\overline{\{x : f(x) \neq 0\}}}_{\text{closed}} \subseteq \underbrace{X}_{\text{Hausdorff compact}}$

is compact. So $f \in C_c(X)$.

$$\therefore C_b(X) \subseteq C_c(X).$$

Let $f \in C_c(\overset{\text{locally compact}}{X})$.

$f : \overset{\text{compact}}{X} \xrightarrow{\text{cts}} \mathbb{C}$

$\Rightarrow \underbrace{f(X)}_{\text{compact}} \subseteq \mathbb{C}$

$\Rightarrow f(X)$ closed & bd

$\Rightarrow \sup_{t \in X} |f(t)| < \infty$

$$X = \{x : f(x) = 0\} \cup \{x : f(x) \neq 0\}$$
$$\subseteq \{x : f(x) = 0\} \cup supp(f)$$
$$f(X) \subseteq f(\{x : f(x) = 0\} \cup supp(f)) = f(\{x : f(x) = 0\}) \cup f(supp\, f)$$
$$= \underbrace{\underbrace{\{0\}}_{bd} \cup \underbrace{\underset{\text{cts}}{f}(\underset{\text{compact}}{supp\, f})}_{bd}}_{bd}$$

$\square$

Definition. Let $a \in \mathcal{A}$ ← a unital algebra (normed alg). The spectrum $sp(a)$ of a is defined to be $\{\lambda \in \mathbb{C} \mid a - \lambda \underbrace{\mathbf{1}}_{\text{the unit of alg}} \text{ is not invertible}\}$ $\sigma(a)$

Remark. In $M_n(\mathbb{C})$, $\lambda \in sp(\underbrace{A}_{\in M_n}) \Leftrightarrow \underbrace{A - \lambda I}_{\in B(\mathbb{C}^n)}$ is not invertible

$\Leftrightarrow$ ($A \in B(\mathbb{C}^n)$ is 1-1 iff is onto) $A - \lambda I$ is not 1-1 $\Leftrightarrow \exists x \in \mathbb{C}^n, x \neq 0; (A - \lambda I)x = 0 \Leftrightarrow \exists x \in \mathbb{C}^n, x \neq 0;$

$Ax = \lambda I x = \lambda x \Leftrightarrow \lambda$ is an eigenvalue of A

thus $sp(A)$ = the set of all eigenvalues of A

Theorem. $sp(a)$ is a non-empty compact subset of $\mathbb{C}$.
a → an element of a Banach alg $\mathcal{A}$

Lemma. If $\mathcal{A}$ is a unital Ban alg, $x \in \mathcal{A}$ and $\|x\| < 1$,
$\exists 1; 1.x = x.1 = x$
then $1 - x$ is invertible and $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$

Proof. Let s_n be the partial sum of $\sum_{n=0}^{\infty} x^n$, i.e. $s_n = \sum_{k=0}^{n} x^k$.

Then $\|s_n - s_m\| = \|\sum_{k=m+1}^{n} x^k\| \leq \sum_{k=m+1}^{n} \|x\|^k = |t_n - t_m|$, (*)

where $t_n = \sum_{k=0}^{n} \|x\|^k$. Due to $\sum_{n=0}^{\infty} \|x\|^n$ ($\in \mathbb{R}$) converges (a geometric series with $\|x\| < 1$)

$\{t_n\}$ is convergent and so Cauchy. (*) shows that $\{s_n\}$ Cauchy and is convergent, since $\mathcal{A}$ is Banach

Thus $\sum_{n=0}^{\infty} x^n$ converges in $\mathcal{A}$. (Hence $x^n \to 0$ as $n \to \infty$)
$x^1, x^2, \ldots$

$(1-x)(\underbrace{1 + x + \cdots + x^n}_{\sum_{k=0}^{n} x^k}) = 1 - x^{n+1} \Rightarrow (1-x)\sum_{n=0}^{\infty} x^n = 1 - 0 = 1 = \sum_{n=0}^{\infty} x^n (1-x)$

→ subsequence of $\{x^n\}$ so $x^{n+1} \to 0$ as $n \to \infty$
$x^2, x^3, x^4, \ldots$

$\therefore (1-x)^{-1} = \sum_{n=0}^{\infty} x^n$. $\square$

Def. Let $x \in \mathcal{A} \rightarrow$ unital Ban alg. $r(x) = \sup_{\lambda \in sp(x)} |\lambda|$ is called the spectral radius.

Theorem. $r(x) \leq \|x\|$ $\forall x \in \mathcal{A} \rightarrow$ unital Ban. alg

Proof. Let $\|x\| < r(x) = \sup_{\lambda \in sp(x)} |\lambda|$. By the definition of supremum, $\exists \lambda_0 \in sp(x)$; $\|x\| < |\lambda_0| \leq \sup_{\lambda \in sp(x)} |\lambda|$.

So $\left\| \frac{x}{\lambda_0} \right\| = \frac{\|x\|}{|\lambda_0|} < 1$. By the above lemma, $\therefore 1 - \frac{x}{\lambda_0}$ is invertible. Hence $\lambda_0 \left(1 - \frac{x}{\lambda_0}\right) = \lambda_0 1 - x$ is invertible. Therefore $\lambda_0 \notin sp(x)$ $\times$. Thus $r(x) \leq \|x\|$. $\square$

Def. Let $\mathcal{A}$ be a non-unital normed alg. A unitization of $\mathcal{A}$ is a unital normed algebra $\mathcal{B}$ containing $\mathcal{A}$ as an ideal.

Theorem. A non-unital normed alg has a unitization

Proof Let $\mathcal{A}_+ = \mathcal{A} \oplus \mathbb{C}$: $\begin{cases} (a,\lambda) + (b,\mu) = (a+b, \lambda+\mu) \\ \theta(a,\lambda) = (\theta a, \theta\lambda) \\ (a,\lambda)(b,\mu) := (ab + \lambda b + \mu a, \lambda\mu) \end{cases}$

set $\|(a,\lambda)\| := \|a\| + |\lambda|$. Then $(\mathcal{A}_+, \|\cdot\|)$ is a normed alg:

$$\|(a,\lambda)(b,\mu)\| = \|(ab+\lambda b+\mu a, \lambda\mu)\| = \|ab+\lambda b+\mu a\| + |\lambda\mu|$$
$$\leq \|ab\|+\|\lambda b\|+\|\mu a\|+|\lambda||\mu| \leq \|a\|\|b\|+|\lambda|\|b\|+|\mu|\|a\|+|\lambda||\mu|$$
$$= (\|a\|+|\lambda|)(\|b\|+|\mu|) = \|(a,\lambda)\|\|(b,\mu)\|$$

Note that $(a,\lambda)(0,1) = (a.0+\lambda.0+1.a, \lambda.1) = (a,\lambda)$
$= (0,1)(a,\lambda)$

So $\mathcal{A}_+$ has the unit $(0,1)$.

The mapping $\varphi : \mathcal{A} \longrightarrow \mathcal{A}_+$ is an isometric & monomorphism.
$a \longmapsto (a,0)$

$\|\varphi(a)\| = \|(a,0)\| = \|a\|+|0| = \|a\|$, $\left\{\begin{array}{l}\varphi(ab)=(ab,0)=(a,0)(b,0)=\varphi(a)\varphi(b)\\ \varphi(\lambda a)=(\lambda a,0)=\lambda(a,0)=\lambda\varphi(a)\\ \varphi(a+b)=(a+b,0)=(a,0)+(b,0)=\varphi(a)+\varphi(b)\end{array}\right.$

alg homomorphism

Hence $\mathcal{A}$ can be identified with $\varphi(A)=\{(a,0) \mid a\in A\} \subseteq \mathcal{A}_+$.

Moreover, $\mathcal{A} \underset{\text{ideal}}{\triangleleft} \mathcal{A}_+$: $\left\{\begin{array}{l}(a,0)(b,\lambda) = (ab+\lambda a+0b, 0.\lambda) = (c,0)\in\mathcal{A}\\ (b,\lambda)(a,0)\in\mathcal{A}.\ \square\end{array}\right.$

Completion of a $\left\{\begin{array}{l}\text{metric space}\\ \text{normed space}\\ \text{inner product space}\\ \text{normed alg}\end{array}\right.$

Motivation. $\mathbb{Q}$ equipped with the Euclidean metric is not complete: $\forall n\ \exists p_n\in\mathbb{Q}$; $\sqrt{2} < p_n < \sqrt{2}+\frac{1}{n}$

Hence $p_n \xrightarrow{\text{in } \mathbb{R}} \sqrt{2}\notin\mathbb{Q}$. Hence $\{p_n\}$ in $\mathbb{R}$ is convergent and so is Cauchy in $\mathbb{R}$. Therefore $\{p_n\}$ is Cauchy in $\mathbb{Q}$ but $\{p_n\}$ does not converge to any point $\mathbb{Q}$.

(Since if $p_n \xrightarrow{\text{in } \mathbb{Q}} p\in\mathbb{Q}$ then $p_n \xrightarrow{\text{in } \mathbb{R}} p$. On the other hand, $p_n \xrightarrow{\text{in } \mathbb{R}} \sqrt{2}$. Hence $\sqrt{2} = p\in\mathbb{Q}$, X.) Thus $\mathbb{Q}$ is an incomplete metric space. In fact, there exists a complete metric space containing $\mathbb{Q}$ as a <u>dense</u> subset. Indeed, this space is nothing than $\mathbb{R}$.

<u>Theorem</u>. Let (X,d) be an incomplete metric space. Then $\exists! (\tilde{X},\tilde{d})$ ~~that~~ is complete & $\exists \varphi: X \to \tilde{X}$ that is 1-1 & isometry and $\overline{\varphi(X)} = \tilde{X}$.

<u>Proof</u>. Let $\hat{X}$ = the set of all Cauchy sequences in X.

Define a relation $\sim$ as: $\{p_n\} \sim \{q_n\} \Leftrightarrow \lim_n d(p_n,q_n) = 0$.

Clearly $\sim$ is an equivalence relation. Put

$$[p_n] := \{\{q_n\} \mid \{p_n\} \sim \{q_n\}\}$$

$$\tilde{X} := \{[p_n] \mid \{p_n\} \text{ is Cauchy}\} \;\&\; \tilde{d}([p_n],[q_n]) = \lim_n d(p_n,q_n)$$

One can show that $(\tilde{X},\tilde{d})$ is a complete metric space.

Now define $\varphi: X \longrightarrow \tilde{X}$

$$x \longmapsto \underbrace{[\{p_n\}]}_{[x]} \text{ where } p_n = x \;\forall n$$

$$\tilde{d}(\underbrace{[x]}_{\varphi(x)}, \underbrace{[y]}_{\varphi(y)}) = \lim_n d(\underbrace{p_n}_{x}, \underbrace{q_n}_{y}) = d(x,y) \qquad x,y \in X$$

Hence φ is an isometry. In fact, $\{[x] \mid x \in X\} = \varphi(X)$ is dense in $\tilde{X}$. ////

Next assume that $(X, \|\cdot\|)$ is a normed space.

$$\underset{\text{normed space}}{(X,\|\cdot\|)} \to \underset{d(x,y)=\|x-y\|}{(X,d)} \text{ incomplete metric space} \to (\tilde{X},\tilde{d}) \text{ complete metric space}$$

$\|x\| = d(x,0)$

$\tilde{X} = \tilde{X}$

$\tilde{x} = [p_n]$

$\|\tilde{x}\|^{\sim} = \lim_n \|p_n\|$

$\|\tilde{x}\|^{\sim} = \tilde{d}(\tilde{x},\tilde{0})$

$(\tilde{X}, \|\cdot\|^{\sim})$ Ban space

$\tilde{x} = \tilde{x}$ space

$(X, \langle \cdot, \cdot \rangle)$ inner product space $\longrightarrow$ $(X, \|\cdot\|)$ normed space $\longrightarrow$ $(\tilde{X}, \|\cdot\|^{\sim})$ Banach space

$\|x\| := \langle x, x \rangle^{\frac{1}{2}}$

$\bar{X} = \tilde{X}$

$\langle \tilde{x}, \tilde{y} \rangle^{\sim} := \lim_n \langle p_n, q_n \rangle$ $\quad \tilde{x} = [p_n]$, $\tilde{y} = [q_n]$

$(\tilde{X}, \langle \cdot, \cdot \rangle^{\sim})$ Hilbert space

$\langle \tilde{x}, \tilde{y} \rangle^{\sim} := \frac{1}{4} \sum_{k=0}^{3} \| \tilde{x} + i^k \tilde{y} \|^{\sim 2}$

X is a normed alg $\longrightarrow$ $(X, \|\cdot\|)$ normed space, X alg, $\|xy\| \leq \|x\| \|y\|$ $\longrightarrow$ $(\tilde{X}, \|\cdot\|^{\sim})$ Banach space

$\tilde{x} \cdot \tilde{y} = [p_n] \cdot [q_n] = [p_n q_n]$

$\|\tilde{x} \cdot \tilde{y}\|^{\sim} = \lim \|p_n q_n\| \leq \lim \|p_n\| \lim \|q_n\| = \|\tilde{x}\|^{\sim} \cdot \|\tilde{y}\|^{\sim}$

<u>Def</u>. A C^*-alg is a Banach alg $\mathcal{A}$ having an involution $* : \mathcal{A} \to \mathcal{A}$, $a \mapsto a^*$ $\left(a^{**} = a,\ (a + \lambda b)^* = a^* + \bar{\lambda} b^*,\ (ab)^* = b^* a^* \right)$ such that $\|a^* a\| = \|a\|^2$ (C^*-condition).

$\boxed{\text{THE END}}$

$B(H)$: $(T + \lambda S)(x) = Tx + \lambda Sx$, $(TS)(x) = T(Sx)$, $\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$; $T \mapsto T^*$; $\langle Tx, y \rangle = \langle x, T^* y \rangle$

$C(X)$: $(f + \lambda g)(x) = f(x) + \lambda g(x)$, $(fg)(x) = f(x) g(x)$, $\|f\| = \sup_{x \in X} |f(x)|$; $f \mapsto \bar{f}$; $\bar{f}(x) = \overline{f(x)}$

M_n: $[a_{ij}] + \lambda [b_{ij}] = [a_{ij} + \lambda b_{ij}]$, $[a_{ij}][b_{ij}] = \left[\sum_{k=1}^{n} a_{ik} b_{kj}\right]$, $\|[a_{ij}]\|$ = operator norm; $[a_{ij}] \mapsto [\bar{a}_{ji}]$