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C*-Algebras and Hilbert Space Operators

2.1. C*-Algebras

$\equiv\ a \in S \Rightarrow a^* \in S$

We begin by defining a number of concepts that make sense in any algebra with an involution.

$(\lambda a+b)^* = \bar{\lambda} a^* + b^*$

An *involution* on an algebra A is a conjugate-linear map $a \mapsto a^*$ on A, such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. The pair $(A, *)$ is called an *involutive* algebra, or a *$*$-algebra*. If S is a subset of A, we set $S^* = \{a^* \mid a \in S\}$, and if $S^* = S$ we say S is *self-adjoint*. A self-adjoint subalgebra B of A is a *$*$-subalgebra* of A and is a $*$-algebra when endowed with the involution got by restriction. Because the intersection of a family of $*$-subalgebras of A is itself one, there is for every subset S of A a smallest $*$-algebra B of A containing S, called the $*$-algebra *generated* by S.

If I is self-adjoint ideal of A, then the quotient algebra A/I is a $*$-algebra with the involution given by $(a + I)^* = a^* + I$ $(a \in A)$.

We define an involution on $\tilde{A}$ extending that of A by setting $(a, \lambda)^* = (a^*, \bar{\lambda})$. Thus, $\tilde{A}$ is a $*$-algebra, and A is a self-adjoint ideal in $\tilde{A}$.

The subalg. generated by $S = \bigcap B$, $S \subseteq B \leq A$ subalg

$a^* = b - ic$, $b = \frac{a+a^*}{2}$, $c = \frac{a-a^*}{2i}$

An element a in A is *self-adjoint* or *hermitian* if $a = a^*$. For each $a \in A$ there exist unique hermitian elements $b, c \in A$ such that $a = b + ic$ $(b = \frac{1}{2}(a+a^*)$ and $c = \frac{1}{2i}(a-a^*))$. The elements a^*a and aa^* are hermitian.

The set of hermitian elements of A is denoted by A_{sa}.

We say a is *normal* if $a^*a = aa^*$. In this case the $*$-algebra it generates is abelian and is in fact the linear span of all $a^m a^{*n}$, where $m, n \in \mathbf{N}$ and $n + m > 0$.

An element p is a *projection* if $p = p^* = p^2$.

If A is unital, then $1^* = 1$ $(1^* = (11^*)^* = 1)$. If $a \in \mathrm{Inv}(A)$, then

If A is unital, then $1^* = 1$ $(1^* = (11^*)^* = 1)$. If $a \in \operatorname{Inv}(A)$, then $(a^*)^{-1} = (a^{-1})^*$. Hence, for any $a \in A$,

$u^{-1} = u^*$

$$\sigma(a^*) = \sigma(a)^* = \{\bar{\lambda} \in \mathbf{C} \mid \lambda \in \sigma(a)\}.$$

An element u in A is a *unitary* if $u^*u = uu^* = 1$. If $u^*u = 1$, then u is an *isometry*, and if $uu^* = 1$, then u is a *co-isometry*.

If $\varphi\colon A \to B$ is a homomorphism of $*$-algebras A and B and φ preserves adjoints, that is, $\varphi(a^*) = (\varphi(a))^*$ $(a \in A)$, then φ is a *$*$-homomorphism*. If in addition φ is a bijection, it is a *$*$-isomorphism*. If $\varphi\colon A \to B$ is a $*$-homomorphism, then $\ker(\varphi)$ is a self-adjoint ideal in A and $\varphi(A)$ is a $*$-subalgebra of B.

An *automorphism* of a $*$-algebra A is a $*$-isomorphism $\varphi\colon A \to A$. If A is unital and u is a unitary in A, then

$$\operatorname{Ad} u\colon A \to A, \quad a \mapsto uau^*,$$

is an automorphism of A. Such automorphisms are called *inner*. We say elements a, b of A are *unitarily equivalent* if there exists a unitary u of A such that $b = uau^*$. Since the unitaries form a group, this is an equivalence relation on A. Note that $\sigma(a) = \sigma(b)$ if a and b are unitarily equivalent.

A *Banach $*$-algebra* is a $*$-algebra A together with a complete submultiplicative norm such that $\|a^*\| = \|a\|$ $(a \in A)$. If, in addition, A has a unit such that $\|1\| = 1$, we call A a *unital Banach $*$-algebra*.

A *C*-algebra* is a Banach $*$-algebra such that

$= \|1^*\|$

$$\|a^*a\| = \|a\|^2 \qquad (a \in A). \tag{1}$$

A closed $*$-subalgebra of a C*-algebra is obviously also a C*-algebra. We shall therefore call a closed $*$-subalgebra of a C*-algebra a *C*-subalgebra*.

If a C*-algebra has a unit 1, then automatically $\|1\| = 1$, because $\|1\| = \|1^*1\| = \|1\|^2$. Similarly, if p is a non-zero projection, then $\|p\| = 1$.

If u is a unitary of A, then $\|u\| = 1$, since $\|u\|^2 = \|u^*u\| = \|1\| = 1$. Hence, $\sigma(u) \subseteq \mathbf{T}$, for if $\lambda \in \sigma(u)$, then $\lambda^{-1} \in \sigma(u^{-1}) = \sigma(u^*)$, so $|\lambda|$ and $|\lambda^{-1}| \leq 1$; that is, $|\lambda| = 1$. $|\lambda^{-1}| \leq r(u^*) \leq \|u^*\| = 1$

The seemingly mild requirement on a C*-algebra in Eq. (1) is in fact very strong—far more is known about the nature and structure of these algebras than perhaps of any other non-trivial class of algebras. Because of the existence of the involution, C*-algebra theory can be thought of as "infinite-dimensional real analysis." For instance, the study of linear functionals on C*-algebras (and of traces, *cf.* Section 6.2) is "non-commutative measure theory."

2.1.1. *Example.* The scalar field $\mathbf{C}$ is a unital C*-algebra with involution given by complex conjugation $\lambda \mapsto \bar{\lambda}$.

2.1.2. *Example.* If Ω is a locally compact Hausdorff space, then $C_0(\Omega)$ is a C*-algebra with involution $f \mapsto \bar{f}$.

Similarly, all of the following algebras are C*-algebras with involution given by $f \mapsto \bar{f}$:

(a) $\ell^\infty(S)$ where S is a set;
(b) $L^\infty(\Omega, \mu)$ where (Ω, μ) is a measure space;
(c) $C_b(\Omega)$ where Ω is a topological space;
(d) $B_\infty(\Omega)$ where Ω is a measurable space.

2.1.3. *Example.* If H is a Hilbert space, then $B(H)$ is a C*-algebra. We shall see that every C*-algebra can be thought of as a C*-subalgebra of some $B(H)$ (Gelfand–Naimark theorem). We defer to Section 2.3 a fuller consideration of this example.

2.1.4. *Example.* If $(A_\lambda)_{\lambda\in\Lambda}$ is a family of C*-algebras, then the direct sum $\oplus_\lambda A_\lambda$ is a C*-algebra with the pointwise-defined involution, and the restricted sum $\oplus_\lambda^{c_0} A_\lambda$ is a closed self-adjoint ideal (*cf.* Exercise 1.1).

2.1.5. *Example.* If Ω is a non-empty set and A is a C*-algebra, then $\ell^\infty(\Omega, A)$ is a C*-algebra with the pointwise-defined involution. This of course generalises Example 2.1.2 (a). If Ω is a locally compact Hausdorff space, we say a continuous function $f\colon \Omega \to A$ *vanishes at infinity* if, for each $\varepsilon > 0$, the set $\{\omega \in \Omega \mid \|f(\omega)\| \geq \varepsilon\}$ is compact. Denote by $C_0(\Omega, A)$ the set of all such functions. This is a C*-subalgebra of $\ell^\infty(\Omega, A)$.

The following easy result has a surprising and important corollary:

$\sup\{|\lambda| : \lambda \in sp(a)\}$

2.1.1. Theorem. *If a is a self-adjoint element of a C*-algebra A, then $r(a) = \|a\|$.*

$= \|a^*a\|$ $\qquad a^* = a$

Proof. Clearly, $\|a^2\| = \|a\|^2$, and therefore by induction $\|a^{2^n}\| = \|a\|^{2^n}$, so $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \|a\|$. □

well-known $\qquad \{\|a^{2^n}\|^{1/2^n}\}$ is a subsequence of $\{\|a^n\|^{1/n}\}$

2.1.2. Corollary. *There is at most one norm on a $*$-algebra making it a C*-algebra.*

$(A, \|\cdot\|,$
$\|\cdot\|_0$

Proof. If $\|.\|_1$ and $\|.\|_2$ are norms on a $*$-algebra A making it a C*-algebra, then

$$\|a\|_j^2 = \|\underbrace{a^*a}_{\text{self-adj}}\|_j = r(a^*a) = \underbrace{\sup_{\lambda\in\sigma(a^*a)} |\lambda|}_{\text{independent of the given norm}} \qquad (j = 1, 2),$$

so $\|a\|_1 = \|a\|_2$. □

2.1.3. Lemma. *Let A be a Banach algebra endowed with an involution such that $\|a\|^2 \le \|a^*a\|$ $(a \in A)$. Then A is a C*-algebra.*

Proof. The inequalities $\|a\|^2 \le \|a^*a\| \le \|a^*\|\|a\|$ imply that $\|a\| \le \|a^*\|$ for all a. Hence, $\|a\| = \|a^*\|$, and therefore $\|a\|^2 = \|a^*a\|$. □

We associate to each C*-algebra A a certain unital C*-algebra $M(A)$ which contains A as an ideal. This algebra is of great importance in more advanced aspects of the theory, especially in certain approaches to K-theory.

A *double centraliser* for a C*-algebra A is a pair (L, R) of bounded linear maps on A, such that for all $a, b \in A$

$$L(ab) = L(a)b, \quad R(ab) = aR(b) \quad \text{and} \quad R(a)b = aL(b).$$

For example, if $c \in A$ and L_c, R_c are the linear maps on A defined by $L_c(a) = ca$ and $R_c(a) = ac$, then (L_c, R_c) is a double centraliser on A. It is easily checked that for all $c \in A$?

$\|cb\| \le \|c\|\|b\| \Rightarrow \sup_{\|b\|\le 1} \|cb\| \le \|c\|$

$$\boxed{\|c\| = \sup_{\|b\|\le 1} \|cb\|} = \sup_{\|b\|\le 1} \|bc\|,$$

$\|c\| = \|c \underbrace{\frac{c^*}{\|c^*\|}}_{b,\ \|b\|\le 1}\| \overset{5}{=} \frac{\|c^*\|^2}{\|c^*\|} = \|c\|$

and therefore $\|L_c\| = \|R_c\| = \|c\|$.

2.1.4. Lemma. *If (L, R) is a double centraliser on a C*-algebra A, then $\|L\| = \|R\|$.*

Proof. Since $\|aL(b)\| = \|R(a)b\| \le \|R\|\|a\|\|b\|$, we have

$$\|L(b)\| = \sup_{\|a\|\le 1} \|aL(b)\| \le \|R\|\|b\|,$$

and therefore $\|L\| \le \|R\|$. Also, $\|R(a)b\| = \|aL(b)\| \le \|L\|\|a\|\|b\|$ implies

$$\|R(a)\| = \sup_{\|b\|\le 1} \|R(a)b\| \le \|L\|\|a\|,$$

and therefore $\|R\| \le \|L\|$. Thus, $\|L\| = \|R\|$. □

If A is a C*-algebra, we denote the set of its double centralisers by

$M(A)$. We define the norm of the double centraliser (L, R) to be $\|L\| = \|R\|$. It is easy to check $M(A)$ is a closed vector subspace of $B(A) \oplus B(A)$.

$M(A) \underset{\text{closed}}{\subseteq} B(A) \oplus B(A)$ is a Banach space under $\|(T,S)\| = \max\{\|T\|, \|S\|\}$

If (L_1, R_1) and $(L_2, R_2) \in M(A)$, we define their product to be

$$(L_1, R_1)(L_2, R_2) = (L_1 L_2, R_2 R_1).$$

Straightforward computations show that this product is again a double centraliser of A and that $M(A)$ is an algebra under this multiplication.

If $L: A \to A$, define $L^*: A \to A$ by setting $L^*(a) = (L(a^*))^*$. Then L^* is linear and the map $L \mapsto L^*$ is an isometric conjugate-linear map from $B(A)$ to itself such that $L^{**} = L$ and $(L_1 L_2)^* = L_1^* L_2^*$. If (L, R) is a double centraliser on A, so is $(L, R)^* = (R^*, L^*)$. It is easily verified that the map $(L, R) \mapsto (L, R)^*$ is an involution on $M(A)$.

2.1.5. Theorem. *If A is a C^*-algebra, then $M(A)$ is a C^*-algebra under the multiplication, involution, and norm defined above.*

Proof. The only thing that is not completely straightforward that has to be checked is that if $T = (L, R)$ is a double centraliser, then $\|T^*T\| = \|T\|^2$. If $\|a\| \le 1$, then $\|L(a)\|^2 = \|(L(a))^* L(a)\| = \|L^*(a^*) L(a)\| = \|a^* R^* L(a)\| \le \|R^* L\| = \|T^*T\|$, so

$$\|T\|^2 = \sup_{\|a\| \le 1} \|L(a)\|^2 \le \|T^*T\| \le \|T\|^2,$$

and therefore $\|T^*T\| = \|T\|^2$. $\square$

The algebra $M(A)$ is called *multiplier algebra* of A.

The map

$$A \to M(A), \quad a \mapsto (L_a, R_a),$$

is an isometric $*$-homomorphism, and therefore we can, and do, identify A as a C*-subalgebra of $M(A)$. In fact A is an ideal of $M(A)$. Note that $M(A)$ is unital (the double centraliser $(\mathrm{id}_A, \mathrm{id}_A)$ is the unit), so $A = M(A)$ if and only if A is unital.

If A is unital with unit 1, then $(L_1, R_1) = (\mathrm{id}_A, \mathrm{id}_A)$

① $sp(a^*) = \overline{sp(a)}$

$\mu \notin sp(a^*) \Leftrightarrow a^* - \mu 1$ is inv. $\Leftrightarrow \exists b \in A,\ b(a^* - \mu 1) = 1 = \ldots \Leftrightarrow$

$(a - \bar{\mu} 1) b^* = 1 \Leftrightarrow \bar{\mu} \notin sp(a) \Leftrightarrow \mu \notin \overline{sp(a)}$

② $1^* = 1$ Since $a 1^* = (1 a^*)^* = (a^*)^* = a$ (similary $1^* a = a$)

$\Rightarrow \begin{cases} 1^* \text{ is the identity} \Rightarrow 1 = 1^* \\ 1^* = 1 . 1^* = 1 \quad \text{(by Putting } a = 1) \end{cases}$

③ $\|a^* a\| = \|a\|^2 \Rightarrow \|a^*\| = \|a\|$

$\|a\|^2 = \|a^* a\| \leq \|a^*\| \, \cancel{\|a\|} \Rightarrow \|a\| \leq \|a^*\| \; (\Diamond)$

Replacing a by a^* we get from $(\Diamond)$ that

$\|a^*\| \leq \|(a^*)^*\| = \|a\|$. ✓

Note. Let A be a non-unital Banach algebra. Put $\tilde{A} = A \oplus \mathbb{C}$

Then $\tilde{A}$ is a unital Ban alg under $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda \mu)$

$\underbrace{\text{unital}}_{(0,1)}$

In addition $A \underset{\text{isom}}{\simeq} \{(a, 0) \mid a \in A\} \underset{\text{bi-ideal}}{\lhd} \tilde{A}$. So $A \hookrightarrow \tilde{A}$, $a \mapsto (a, 0)$

Even under ...

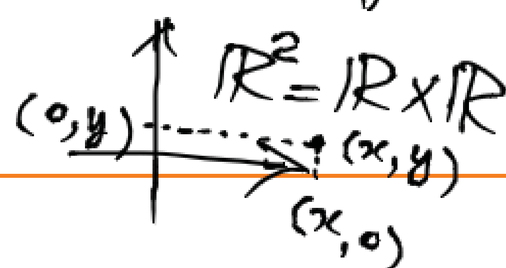
$(a,\lambda)^* = (a^*, \bar{\lambda})$, A is a Banach $*$-alg, but not a C^*-alg.

What is the direct sum in Linear Algebra?

Let X & Y be linear spaces. Then $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$ together with $\lambda(x,y)+(u,v) = (\lambda x + u, \lambda y + v)$ is a linear space, denoted by $X \oplus Y$. It is called external direct sum.

Now let V, W be subspaces of a linear space X such that $V \cap W = \{0\}$. Then $V + W = \{v + w : v \in V, w \in W\}$ is a subspace of X. It is called the inner direct sum denoted by $V \oplus W$.

Th. Each external direct sum $X \times Y$ can be regarded as an inner direct sum.



Proof. Put $V = \{(x,0) \mid x \in X\}$ and $W = \{(0,y) \mid y \in Y\}$. Then $V \cap W = \{0\}$ and $X \times Y = V + W$

$(x,y) = (x,0) + (0,y)$

So $X \times Y = V \oplus W$. $\square$

Th. Any internal direct sum can be regarded as an external direct sum.

Proof. Consider the internal d.s. $V \oplus W$. Then

$$\begin{aligned} \varphi : V \times W &\longrightarrow V \oplus W \\ (x,y) &\longmapsto x+y \end{aligned}$$ is an isomorphism between

linear spaces. So $V \oplus W$ can be identified by $\underbrace{V \times W}_{\text{our External d.s.}}$.

(5) $\underbrace{\|CC^*\|}_{\|C^*\|^2} = \underbrace{\|C^*C\|}_{\|C\|^2}$ ✓

(6) $M(A)$ is closed in $B(A) \oplus B(A)$: Let $\underbrace{(L_n, R_n)}_{\in M(A)} \to \underbrace{(T, S)}_{\in B(A) \oplus B(A)}$. So

$\forall_\varepsilon \exists N \forall n \geq N$; $\underbrace{\|(L_n, R_n) - (T,S)\|}_{\max\{\|L_n - T\|, \|R_n - S\|\}} < \varepsilon$. Hence $\begin{cases} L_n \to T \\ R_n \to S \end{cases}$. Therefore

$\|R_n - S\| \& \|L_n - T\| \leq$

$T(ab) = \lim_n L_n(ab) = \lim_n (L_n(a) b) = (\lim_n L_n(a)) b = T(a) b$. Similarly

$S(ab) = aS(b)$, $aT(b) = R(a) b$. Thus $(T,S) \in M(A)$. ✓

One can prove $\|(a,\lambda)\| = \sup_{b \neq 0} \frac{\|ab+\lambda b\|}{\|b\|}$ is a C^*-norm on $\tilde{A}$.

2.1.6. Theorem. *If A is a C*-algebra, then there is a (necessarily unique) norm on its unitisation $\tilde{A}$ making it into a C*-algebra, and extending the norm of A.*

Proof. Uniqueness of the norm is given by Corollary 2.1.2. The proof of existence falls into two cases, depending on whether A is unital or non-unital.

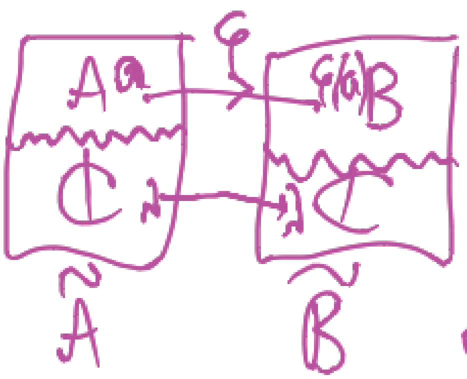
Suppose first that A has a unit e. Then the map φ from $\tilde{A}$ to the direct sum of the C*-algebras A and $\mathbf{C}$ defined by $\varphi(a, \lambda) = (a + \lambda e, \lambda)$ is a $*$-isomorphism. Hence, one gets a norm on $\tilde{A}$ making it a C*-algebra by setting $\|(a, \lambda)\| = \|\varphi(a, \lambda)\|$.

Now suppose A has no unit. If 1 is the unit of $M(A)$, then $A \cap \mathbf{C}1 = 0$. The map φ from $\tilde{A}$ onto the C*-subalgebra $A \oplus \mathbf{C}1$ of $M(A)$ defined by setting $\varphi(a, \lambda) = a + \lambda 1$ is a $*$-isomorphism, so we get a norm on $\tilde{A}$ making it a C*-algebra by setting $\|(a, \lambda)\| = \|\varphi(a, \lambda)\|$. □

If A is a C*-algebra, we shall always understand the norm of $\tilde{A}$ to be the one making it a C*-algebra.

Note that when A is non-unital, $M(A)$ is in general very much bigger than $\tilde{A}$. For instance, it is shown in Section 3.1 that if $A = C_0(\Omega)$, where Ω is a locally compact Hausdorff space, then $M(A) = C_b(\Omega)$.

If $\varphi: A \to B$ is a $*$-homomorphism between $*$-algebras A and B, then it extends uniquely to a unital $*$-homomorphism $\tilde{\varphi}: \underbrace{\tilde{A}}_{A \oplus \mathbb{C}} \to \underbrace{\tilde{B}}_{B \oplus \mathbb{C}}$.



$\tilde{\varphi}(a+\lambda 1) = \varphi(a) + \lambda 1$

$\tilde{\varphi}$ is unique, since if ψ is an extension of φ, then we have

$$\tilde{\varphi}(a+\lambda 1) = \varphi(a) + \lambda 1 = \psi(a) + \lambda \underbrace{\boxed{1}}_{\psi(1)} = \psi(a+\lambda 1)$$

$\varphi(a^*)=\varphi(a)^*$, $\varphi(\lambda a+b)=\lambda\varphi(a)+\varphi(b)$, $\varphi(ab)=\varphi(a)\varphi(b)$

2.1.7. Theorem. *A $*$-homomorphism $\varphi: A \to B$ from a Banach $*$-algebra A to a C^*-algebra B is necessarily norm-decreasing.* $\|\varphi\| \le 1$ or $\|\varphi(a)\| \le \|a\|$

Proof. We may suppose that A, B, and φ are unital (by going to $\tilde{A}, \tilde{B}$, and $\tilde{\varphi}$ if necessary). If $a \in A$, then $\sigma(\varphi a) \subseteq \sigma(a)$, so $\|\varphi a\|^2 = \|\varphi(a)^*\varphi(a)\| = \|\varphi(a^*a)\| = r(\varphi(a^*a)) \le r(a^*a) \le \|a^*a\| \le \|a\|^2$. Hence, $\|\varphi(a)\| \le \|a\|$. □

2.1.8. Theorem. *If a is a hermitian element of a C^*-algebra A, then $\sigma(a) \subseteq \mathbf{R}$.*

self-adj

$e^{ia}\cdot e^{-ia}=1$

Proof. We may suppose that A is unital. Since e^{ia} is unitary, $\sigma(e^{ia}) \subseteq \mathbf{T}$. If $\lambda \in \sigma(a)$ and $b = \sum_{n=1}^{\infty} i^n(a-\lambda)^{n-1}/n!$ then $e^{ia} - e^{i\lambda} = (e^{i(a-\lambda)}-1)e^{i\lambda} = (a-\lambda)be^{i\lambda}$. Since b commutes with a, and since $a - \lambda$ is non-invertible, $e^{ia} - e^{i\lambda}$ is non-invertible. Hence, $e^{i\lambda} \in \mathbf{T}$, and therefore $\lambda \in \mathbf{R}$. Thus, $\sigma(a) \subseteq \mathbf{R}$. □

-f A is not unital, then $\sigma_A(a) := \sigma_{\tilde{A}}(a)$

2.1.9. Theorem. *If τ is a character on a C^*-algebra A, then it preserves adjoints.*

a non-zero hom from A onto $\mathbb{C}$

Proof. If $a \in A$, then $a = b + ic$ where b, c are hermitian elements of A. The numbers $\tau(b)$ and $\tau(c)$ are real because they are in $\sigma(b)$ and $\sigma(c)$ respectively, so $\tau(a^*) = \tau(b - ic) = \tau(b) - i\tau(c) = (\tau(b) + i\tau(c))^- = \tau(a)^-$. □

$\in\mathbb{R}$ $\in\mathbb{R}$

The character space of a unital abelian Banach algebra is non-empty, so this is true in particular for unital abelian C*-algebras. However, there are non-unital, non-zero, abelian Banach algebras for which the character space is empty. Fortunately, this cannot happen in the case of C*-algebras. Let A be a non-unital, non-zero, abelian C*-algebra. Then A contains a non-zero hermitian element, a say. Since $r(a) = \|a\|$ by Theorem 2.1.1, it follows that there is a character τ on $\tilde{A}$ such that $|\tau(a)| = \|a\| \neq 0$. Hence, the restriction of τ to A is a non-zero homomorphism from A to $\mathbf{C}$, that is, a character on A.

We shall now completely determine the abelian C*-algebras. This result can be thought of as a preliminary form of the spectral theorem. It allows us to construct the functional calculus, a very useful tool in the analysis of non-abelian C*-algebras.

2.1.10. Theorem (Gelfand). *If A is a non-zero abelian C^*-algebra, then the Gelfand representation*

$\{\tau : \tau$ is a character on $A\} \subseteq A'$

$\hat{a}$

$$\varphi: A \to C_0(\Omega(A)), \quad a \mapsto \hat{a},$$

$\hat{a}(f) = f(a)$ dual of A

is an isometric $*$-isomorphism.

Proof. That φ is a norm-decreasing homomorphism, such that $\|\varphi(a)\| = r(a)$, is given by Theorem 1.3.6. If $\tau \in \Omega(A)$, then $\varphi(a^*)(\tau) = \tau(a^*) = \tau(a)^- = \varphi(a)^*(\tau)$, so φ is a $*$-homomorphism. Moreover, φ is isometric, since $\|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| = \|\varphi(a^*a)\| = r(a^*a) = \|a^*a\| = \|a\|^2$. Clearly, then, $\varphi(A)$ is a closed $*$-subalgebra of $C_0(\Omega)$ separating the points of $\Omega(A)$, and having the property that for any $\tau \in \Omega(A)$ there is an element $a \in A$ such that $\varphi(a)(\tau) \neq 0$. The Stone–Weierstrass theorem implies, therefore, that $\varphi(A) = C_0(\Omega(A))$. □

9 8 10 φ is onto

Let S be a subset of a C*-algebra A. The C*-algebra *generated* by S is the smallest C*-subalgebra of A containing S. If $S = \{a\}$, we denote by $C^*(a)$ the C*-subalgebra generated by S. If a is a normal, then $C^*(a)$ is abelian. Similarly, if A is unital and a normal, then the C*-subalgebra generated by 1 and a is abelian. 11

Observe that $r(a) = \|a\|$ if a is a normal element of a C*-algebra (apply Theorem 2.1.10 to $C^*(a)$).

$B = \left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{C}\right\} \underset{C^*\text{-subalg}}{\leq} M_2(\mathbb{C}) = A$

The following result is important.

2.1.11. Theorem. *Let B be a C*-subalgebra of a unital C*-algebra A containing the unit of A. Then*

$$\sigma_B(b) = \sigma_A(b) \qquad (b \in B).$$

$\cap$ A

the unit of B differs from the unit of A

We are now going to set up the functional calculus, for which we need to make two easy observations:

If $\theta: \Omega \to \Omega'$ is a continuous map between compact Hausdorff spaces Ω and Ω', then the *transpose* map

$$\theta^t: C(\Omega') \to C(\Omega), \ f \mapsto f\theta,$$

is a unital $*$-homomorphism. Moreover, if θ is a homeomorphism, then θ^t is a $*$-isomorphism.

Our second observation is that a $*$-isomorphism of C*-algebras is necessarily isometric. This is an immediate consequence of Theorem 2.1.7.

2.1.13. Theorem. *Let a be a normal element of a unital C*-algebra A.*

and suppose that z is the inclusion map of $\sigma(a)$ in $\mathbf{C}$. Then there is a unique unital $*$-homomorphism $\varphi: C(\sigma(a)) \to A$ such that $\varphi(z) = a$. Moreover, φ is isometric and $\mathrm{im}(\varphi)$ is the C^*-subalgebra of A generated by 1 and a. 12

Proof. Denote by B the (abelian) C*-algebra generated by 1 and a, and let $\psi: B \to C(\Omega(B))$ be the Gelfand representation. Then ψ is a $*$-isomorphism by Theorem 2.1.10, and so is $\hat{a}^t: C(\sigma(a)) \to C(\Omega(B))$, since $\hat{a}: \Omega(B) \to \sigma(a)$ is a homeomorphism. Let $\varphi: C(\sigma(a)) \to A$ be the composition $\psi^{-1}\hat{a}^t$, so φ is a $*$-homomorphism. Then $\varphi(z) = a$, since $\varphi(z) = \psi^{-1}(\hat{a}^t(z)) = \psi^{-1}(\hat{a}) = a$, and obviously φ is unital. From the Stone–Weierstrass theorem, we know that $C(\sigma(a))$ is generated by 1 and z; φ is therefore the unique unital $*$-homomorphism from $C(\sigma(a))$ to A such that $\varphi(z) = 1$. 13

It is clear that φ is isometric and $\mathrm{im}(\varphi) = B$. □

As in Theorem 2.1.13, let a be a normal element of a unital C*-algebra A, and let z be the inclusion map of $C(\sigma(a))$ in $\mathbf{C}$. We call the unique unital $*$-homomorphism $\varphi: C(\sigma(a)) \to A$ such that $\varphi(z) = a$ the *functional calculus* at a. If p is a polynomial, then $\varphi(p) = p(a)$, so for $f \in C(\sigma(a))$ we may write $f(a)$ for $\varphi(a)$. Note that $f(a)$ is normal.

Let B be the image of φ, so B is the C*-algebra generated by 1 and a. If $\tau \in \Omega(B)$, then $f(\tau(a)) = \tau(f(a))$, since the maps $f \mapsto f(\tau(a))$ and $f \mapsto \tau(f(a))$ from $C(\sigma(a))$ to $\mathbf{C}$ are $*$-homomorphisms agreeing on the generators 1 and z and hence are equal.

2.1.14. Theorem (Spectral Mapping). *Let a be a normal element of a unital C^*-algebra A, and let $f \in C(\sigma(a))$. Then*

$$\sigma(f(a)) = f(\sigma(a)).$$

In an abelian Ban alg
$\sigma(a) = \{\tau(a) : \tau \in \Omega(A)\}$

Moreover, if $g \in C(\sigma(f(a)))$, then

$$(g \circ f)(a) = g(f(a)).$$

Proof. Let B be the abelian C*-subalgebra generated by 1 and a. Then $\sigma(f(a)) = \{\tau(f(a)) \mid \tau \in \Omega(B)\} = \{f(\tau(a)) \mid \tau \in \Omega(B)\} = f(\sigma(a))$.

If C denotes the C*-subalgebra generated by 1 and $f(a)$, then $C \subseteq B$ and for any $\tau \in \Omega(B)$ its restriction τ_C is a character on C. We therefore have $\tau((g \circ f)(a)) = g(f(\tau(a))) = g(\tau_C(f(a))) = \tau_C(g(f(a))) = \tau(g(f(a)))$. Hence, $(g \circ f)(a) = g(f(a))$. The rest left to the PhD students. □

$\tau(f(a)) = \tau(\lim_n P_n(a)) = \lim_n \tau(P_n(a)) = \lim_n P_n(\tau(a))$

$= f(\tau(a)) \in \mathbb{C}$

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2.1.15. Theorem. *Let Ω be a compact Hausdorff space, and for each $\omega \in \Omega$ let δ_ω be the character on $C(\Omega)$ given by evalution at ω; that is, $\delta_\omega(f) = f(\omega)$. Then the map*

$$\Omega \to \Omega(C(\Omega)), \quad \omega \mapsto \delta_\omega,$$

is a homeomorphism.

Proof. This map is continuous because if $(\omega_\lambda)_{\lambda\in\Lambda}$ is a net in Ω converging to a point ω, then $\lim_{\lambda\in\Lambda} f(\omega_\lambda) = f(\omega)$ for all $f \in C(\Omega)$, so the net $(\delta_{\omega_\lambda})$ is weak* convergent to δ_ω. The map is also injective, because if ω, ω' are distinct points of Ω, then by Urysohn's lemma there is a function $f \in C(\Omega)$ such that $f(\omega) = 0$ and $f(\omega') = 1$, and therefore $\delta_\omega \neq \delta_{\omega'}$.

Now we show surjectivity of the map. Let $\tau \in \Omega(C(\Omega))$. Then $M = \ker(\tau)$ is a proper C*-algebra of $C(\Omega)$. Also, M separates the points of Ω, for if ω, ω' are distinct points of Ω, then as we have just seen there is a function $f \in C(\Omega)$ such that $f(\omega) \neq f(\omega')$, so $g = f - \tau(f)$ is a function in M such that $g(\omega) \neq g(\omega')$. It follows from the Stone–Weierstrass theorem that there is a point $\omega \in \Omega$ such that $f(\omega) = 0$ for all $f \in M$. Hence, $(f - \tau(f))(\omega) = 0$, so $f(\omega) = \tau(f)$, for all $f \in C(\Omega)$. Therefore, $\tau = \delta_\omega$. Thus, the map is a continuous bijection between compact Hausdorff spaces and therefore is a homeomorphism. □

If A, B are C^*-alg, then so is $A \oplus B$ under $\max\{\|a\|, \|b\|\} = \|(a,b)\|$:

1. $\|\underbrace{(a,b)^*}_{(a^*,b^*)}(a,b)\| = \|(a^*a, b^*b)\| = \max\{\underbrace{\|a^*a\|}_{\|a\|^2}, \underbrace{\|b^*b\|}_{\|b\|^2}\} = \|(a,b)\|^2$

2. 

$d_Y(f(x), f(y)) = d_X(x,y)$ if f is $\frac{1\text{-}1}{\text{onto}}$.

$(X, \|\cdot\|)$ $\qquad \underset{Y}{\|f(x)\|} := \|x\|$ if f is linear isom.

Exercise $C_0(\Omega)$ is unital iff Ω is compact.

g is the unit of $C_0(\Omega) \Leftrightarrow \underbrace{gf = f}_{g(x)f(x)=f(x)\ \forall x}\ \forall f \overset{\text{Urysohn's Lemma}}{\Leftrightarrow} g = 1$

$C_0(\Omega)$ unital $\Rightarrow 1 \in C_0(\Omega) \overset{\varepsilon=\frac{1}{2}}{\Rightarrow} \underbrace{\{x : |1(x)| \geq \tfrac{1}{2}\}}$ is compact. ✓

Ω compact $\Rightarrow \forall \varepsilon \exists K$ $\underbrace{\{x : |1(x)| \geq \varepsilon\}}_{\text{closed } \subseteq \Omega_{\text{compact}}}^{\Omega}$ is compact $\Rightarrow 1 \in C_0(\Omega)$

In this case we write $C_0(\Omega) = \underset{= C_b(\Omega)}{C(\Omega)}$

(3) $\sigma(\varphi(a)) \subseteq \sigma(a)$ if φ is a hom, since

$\lambda \notin \sigma(a) \Rightarrow a - \lambda 1_A$ is inv. $\Rightarrow \exists b \in A; (a - \lambda 1_A) b = 1_A \xrightarrow[\text{hom}]{\varphi \text{ is unital}} (\varphi(a) - \lambda 1_B)\varphi(b) = 1$
$\Rightarrow \varphi(a) - \lambda 1_B$ is inv $\Rightarrow \lambda \notin \sigma(\varphi(a))$. ✓

Note: $e^a := \sum_{n=0}^{\infty} \frac{a^n}{n!} \in \underset{\text{Banach space}}{A}$, since $\sum_n \left\|\frac{a^n}{n!}\right\|$ is convergent in $\mathbb{R}$

$= \lim_{n \to \infty} \sum_{k=0}^{n} \frac{a^k}{k!}$ $\qquad 0 \leq \left\|\frac{a^n}{n!}\right\| \leq \frac{\|a\|^n}{n!}$, $\sum_{n=0}^{\infty} \frac{\|a\|^n}{n!} = e^{\|a\|} < +\infty$

comparison test

④ If $\underbrace{e^{ia}-e^{i\lambda}}_{} = \underbrace{(a-\lambda)}_{\text{non-inv}}be^{i\lambda}$ is inv., then

$\exists c;\ (a-\lambda)\underbrace{be^{i\lambda}c} = (e^{ia}-e^{i\lambda})c = 1$. So $a-\lambda$ is inv. ✗

⑤ $e^{i\lambda}\in T \overset{?}{\Rightarrow} \lambda\in\mathbb{R}$, Since

$e^{i\lambda}=e^{i(a+ib)}=e^{ia-b} \Rightarrow 1=|\underbrace{e^{i\lambda}}_{\in T}|=|e^{ia-b}|=e^{-b} \Rightarrow \boxed{b=0}$

⑥ If $\underset{\neq 0}{\tau}: A\to\mathbb{C}$ is a linear functional, then

τ is onto, since $\tau(A)\leq\mathbb{C}$, so $\underset{\neq 0}{\dim\tau(A)}\leq\underset{= 1}{\dim\mathbb{C}}$.

Hence $\dim\tau(A)=1$. Thus $\tau(A)=\mathbb{C}$.

⑦ Always $\tau(a)\in\sigma(a)$, since if $\tau(a)\notin\sigma(a)$, then

$\exists b\in A;\ b(a-\tau(a)1)=1$. So $\tau(b)(\underbrace{\tau(a)-\tau(a)}_{0})=\underbrace{\tau(1)}_{1}$ ✗

Note: We have $|\tau(a)|\leq r(a)\leq\|a\|$ ...

$|\tau(a)| \leq \|\tau\| \|a\| \cdot S_0 \|\tau\| = 1$ $\tau(1)=1$

Exercise: If $\varphi : X \to Y$ (X Ban space, Y normed space) isometric isomorphism, then $\varphi(X)$ is closed in Y.

(8) $\forall \tau_1 \neq \tau_2 \in \Omega(A)\ \exists \varphi(a) \in \varphi(A)$; $\varphi(a)(\tau_1) \neq \varphi(a)(\tau_2)$

$\underbrace{\varphi(a)(\tau_1)}_{\tau_1(a)}$, $\underbrace{\varphi(a)(\tau_2)}_{\tau_2(a)}$

$\exists a;\ \tau_1(a) \neq \tau_2(a)$

(9) $\forall \tau \in \Omega(A)\ \exists a \in A$; $\underbrace{\varphi(a)(\tau)}_{\tau(a)} \neq 0$

non-zero $\tau(a) \neq 0$

(10) $B \leq C_0(\Omega)$

*-sub alg

B separates points of Ω

B vanishes nowhere on Ω

Stone–Weierstrass TH $\Longrightarrow \overline{B} = C_0(\Omega)$

(11) If $S \subseteq A$ (subset; A C^*-alg) then $\langle S \rangle = \bigcap B$ (C^*-subalg generated by S) is a C^*-alg, since $\bigcap B$ (closed) is a closed *-subalg of A.

$S \subseteq B \leq A$ (C^*-subalg)

$B \leq A$ (C^*-subalg)

(X,d) complete metric space

$(F \subseteq X, d)$ " " " $\Leftrightarrow F \subseteq X$ closed

(12) $\langle \{a, 1\} \rangle$ is the smallest closed $*$-subalg of A.
 $a = a^*$

~~$a, a^2, \ldots, \lambda a, \mu a^2, \ldots, a^2 + \lambda a^3, \ldots, p(a)$~~

$\{P(a) : P(z)$ is a polynomial in $z\}$ is a $*$-subalg of A.

closed

In fact, $\langle \{a, 1\} \rangle =$ the closure of $\{P(a) : P$ is polynomial$\}$
$a = a^*$

$$\langle \{a,1\} \rangle = \bigcap B$$

$1, a \in B < A$, B C^*-subalg

If a normal, then $\langle \{a, 1\} \rangle = \overline{\{P(a, a^*) : P \text{ is a polynomial in two variables}\}}$

$a^* a = a a^*$

(13) Let $a \in A$ be normal & B be the C^*-subalg generated by $a, 1$. B is abelian. So by the previous

$\exists \varphi$; $\varphi : B \longrightarrow C(\Omega(B))$ is an (isometric) $*$-isom.
$b \longmapsto \hat{b}$

Also, $\Psi : C(\Omega(B)) \longrightarrow C(\sigma(a))$ is an (") " $\sigma(a)$ & $\Omega(B)$ homeom

So $\Psi \circ \varphi : B \longrightarrow C(\sigma(a))$ is an (isometric) $*$-is

$\ldots \varphi : \Sigma \to C(\sigma(a))$ is an (isometric) $*$-isom?

In general, $\alpha : X \to Y$ is homeomorphism between compact Hausdorff spaces, then $\theta : C(Y) \to C(X)$, $f \mapsto f \circ \alpha$ is an isometric $*$-isom between C^*-alg.

$$\begin{array}{ccc} Y & \xrightarrow{f} & \mathbb{C} \\ \uparrow \alpha & \nearrow & \\ X & & \end{array}$$

So we have an isometric $*$-isom between $C(\sigma(a))$ & $B \subseteq A$.

Functional Calculus Λ: $(\psi \circ \varphi)^{-1}$

$$C(\sigma(a)) \hookrightarrow A$$

$$f(z) = z \longmapsto a$$

$$f(z) = 1 \longmapsto 1$$

$$P(z) \longmapsto P(a)$$

$P_n(z) \xrightarrow{\|\cdot\|} \sin z \Rightarrow \Lambda(P_n) \to \Lambda(\sin)$ is the limit of $\{P_n(a)\}$, $\Lambda(P_n) = P_n(a)$, denoted by $\sin a$

Note: In $C(X)$, $f_n \xrightarrow{\|\cdot\|} f \Leftrightarrow \forall \varepsilon \exists N \forall n \geq N;\ \underbrace{\|f_n - f\|}_{\sup_{x \in X} |f_n(x) - f(x)|} < \varepsilon$

$\Leftrightarrow \forall \varepsilon \exists N \forall n \geq N \forall x \in X;\ |f_n(x) - f(x)| < \varepsilon \Leftrightarrow f_n \xrightarrow{u} f$ uniform Convergence

1 Oct. 2013

C^*-alg is an extension of $\mathbb{C}$.

Self-adj elements $\longleftrightarrow \mathbb{R}$

unitaries $\longleftrightarrow \mathbb{T}$

Positive elements $\longleftrightarrow \mathbb{R}^{\geq 0}$

2.2. Positive Elements of C*-Algebras

In this section we introduce a partial order relation on the hermitian elements of a C*-algebra. The principal results are the existence of a unique positive square root for each positive element and Theorem 2.2.4, which asserts that elements of the form a^*a are positive.

2.2.1. ***Remark.*** Let $A = C_0(\Omega)$, where Ω is a locally compact Hausdorff space. Then A_{sa} is the set of real-valued functions in A and there is a natural partial order on A_{sa} given by $f \leq g$ if and only if $f(\omega) \leq g(\omega)$ for all $\omega \in \Omega$. An element $f \in A$ is positive, that is, $f \geq 0$, if and only if f is of the form $f = \bar{g}g$ for some $g \in A$, and in this case f has a unique positive square root in A, namely the function $\omega \mapsto \sqrt{f(\omega)}$. Note that if $f = \bar{f}$ we can also express the positivity condition in terms of the norm: If $t \in \mathbf{R}$, then f is positive if $\|f - t\| \leq t$, and in the reverse direction if $\|f\| \leq t$ and $f \geq 0$, then $\|f - t\| \leq t$. We shall presently define a partial order on an arbitrary C*-algebra that generalises that of $C_0(\Omega)$, and we shall obtain similar, and many other, results. ①

If B has a unit e not equal to the unit 1 of A, then for any $b \in B$ and $\lambda \in \mathbf{C} \setminus \{0\}$ invertibility of $b + \lambda$ in A is equivalent to invertibility of $b + \lambda e$ in B, so $\sigma_A(b) = \sigma_B(b) \cup \{0\}$.

From these observations and Theorem 2.1.11, it is clear that for any C*-subalgebra B of a C*-algebra A we have $\sigma_B(b) \cup \{0\} = \sigma_A(b) \cup \{0\}$ for all $b \in B$.

$[0,\infty)$

$\parallel$

An element a of a C*-algebra A is *positive* if a is hermitian and $\sigma(a) \subseteq \mathbf{R}^+$. We write $a \geq 0$ to mean that a is positive, and denote by A^+ the set of positive elements of A. By the preceding observation $B^+ = B \cap A^+$ for any C*-subalgebra B of A.

If S is a non-empty set, then an element $f \in \ell^\infty(S)$ is positive in the C*-algebra sense if and only if $f(x) \geq 0$ for all $x \in S$, because $\sigma(f)$ is the closure of the range of f. Hence, if Ω is any locally compact Hausdorff

×

closure of the range of f. Hence, if Ω is any locally compact Hausdorff space, then $f \in C_0(\Omega)$ is positive if and only if $f(\omega) \geq 0$ for all $\omega \in \Omega$.

If a is a hermitian element of a C*-algebra A observe that $C^*(a)$ is the closure of the set of polynomials in a with zero constant term.

2.2.1. Theorem. *Let A be a C*-algebra and $a \in A^+$. Then there exists a unique element $b \in A^+$ such that $b^2 = a$.*

Proof. That there exists $b \in C^*(a)$ such that $b \geq 0$ and $b^2 = a$ follows from the Gelfand representation, since we may use it to identify $C^*(a)$ with $C_0(\Omega)$, where Ω is the character space of $C^*(a)$, and then apply Remark 2.2.1.

Suppose that c is another element of A^+ such that $c^2 = a$. As c commutes with a it must commute with b, since b is the limit of a sequence of polynomials in a. Let B be the (necessarily abelian) C*-subalgebra of A generated by b and c, and let $\varphi: B \to C_0(\Omega)$ be the Gelfand representation of B. Then $\varphi(b)$ and $\varphi(c)$ are positive square roots of $\varphi(a)$ in $C_0(\Omega)$, so by another application of Remark 2.2.1, $\varphi(b) = \varphi(c)$, and therefore $b = c$. □

If A is a C*-algebra and a is a positive element, we denote by $a^{1/2}$ the unique positive element b such that $b^2 = a$.

If c is a hermitian element, then c^2 is positive, and we set $|c| = (c^2)^{1/2}$, $c^+ = \frac{1}{2}(|c| + c)$, and $c^- = \frac{1}{2}(|c| - c)$. Using the Gelfand representation of $C^*(c)$, it is easy to check that $|c|$, c^+ and c^- are positive elements of A such that $c = c^+ - c^-$ and $c^+c^- = 0$.

2.2.2. *Remark.* If a is a hermitian element of the closed unit ball of a unital C*-algebra A, then $1 - a^2 \in A^+$ and the elements

$$u = a + i\sqrt{1 - a^2} \qquad \text{and} \qquad v = a - i\sqrt{1 - a^2}$$

are unitaries such that $a = \frac{1}{2}(u + v)$. Therefore, the unitaries linearly span A, a result that is frequently useful.

2.2.2. Lemma. *Suppose that A is a unital C*-algebra, a is a hermitian element of A and $t \in \mathbf{R}$. Then, $a \geq 0$ if $\|a - t\| \leq t$. In the reverse direction, if $\|a\| \leq t$ and $a \geq 0$, then $\|a - t\| \leq t$.*

Proof. We may suppose that A is the (abelian) C*-subalgebra generated by 1 and a, so by the Gelfand representation $A = C(\sigma(a))$. The result now follows from Remark 2.1.1. □

It is immediate from Lemma 2.2.2 that A^+ is closed in A.

2.2.3. Lemma. *The sum of two positive elements in a C*-algebra is a positive element.*

Proof. Let A be a C*-algebra and a, b positive elements. To show that $a+b \geq 0$ we may suppose that A is unital. By Lemma 2.2.2, $\|a - \|a\|\| \leq \|a\|$ and $\|b - \|b\|\| \leq \|b\|$, so $\|a+b-\|a\|-\|b\|\| \leq \|a-\|a\|\|+\|b-\|b\|\| \leq \|a\|+\|b\|$. By Lemma 2.2.2 again, $a + b \geq 0$. □

2.2.4. Theorem. *If a is an arbitrary element of a C*-algebra A, then a^*a is positive.*

Proof. First we show that $a = 0$ if $-a^*a \in A^+$. Since $\sigma(-aa^*) \setminus \{0\} = \sigma(-a^*a) \setminus \{0\}$ by Remark 1.2.1, $-aa^* \in A^+$ because $-a^*a \in A^+$. Write $a = b + ic$, where $b, c \in A_{sa}$. Then $a^*a + aa^* = 2b^2 + 2c^2$, so $a^*a = 2b^2 + 2c^2 - aa^* \in A^+$. Hence, $\sigma(a^*a) = \mathbf{R}^+ \cap (-\mathbf{R}^+) = \{0\}$, and therefore $\|a\|^2 = \|a^*a\| = r(a^*a) = 0$.

Now suppose a is an arbitrary element of A, and we shall show that a^*a is positive. If $b = a^*a$, then b is hermitian, and therefore we can write $b = b^+ - b^-$. If $c = ab^-$, then $-c^*c = -b^-a^*ab^- = -b^-(b^+ - b^-)b^- = (b^-)^3 \in A^+$, so $c = 0$ by the first part of this proof. Hence, $b^- = 0$, so $a^*a = b^+ \in A^+$. □

If A is a C*-algebra, we make A_{sa} a poset by defining $a \leq b$ to mean $b - a \in A^+$. The relation $\leq$ is translation-invariant; that is, $a \leq b \Rightarrow a + c \leq b + c$ for all $a, b, c \in A_{sa}$. Also, $a \leq b \Rightarrow ta \leq tb$ for all $t \in \mathbf{R}^+$, and $a \leq b \Leftrightarrow -a \geq -b$.

Using Theorem 2.2.4 we can extend our definition of $|a|$: for arbitrary a set $|a| = (a^*a)^{1/2}$.

We summarise some elementary facts about A^+ in the following result.

2.2.5. Theorem. *Let A be a C*-algebra.*

(1) *The set A^+ is equal to $\{a^*a \mid a \in A\}$.*
(2) *If $a, b \in A_{sa}$ and $c \in A$, then $a \leq b \Rightarrow c^*ac \leq c^*bc$.* 5 $\quad 0 \le aa^* = (a^*)^* a^*$
(3) *If $\boxed{0 \leq} a \leq b$, then $\|a\| \leq \|b\|$.* 6 $\quad$ b b
(4) *If A is unital and a, b are positive invertible elements, then $a \leq b \Rightarrow 0 < b^{-1} < a^{-1}$.* 7

$a \geq 0 \Rightarrow a = (a^{\frac{1}{2}})^* a^{\frac{1}{2}}$

Proof. Conditions (1) and (2) are implied by Theorem 2.2.4 and the existence of positive square roots for positive elements. To prove Condition (3) we may suppose that A is unital. The inequality $b \leq \|b\|$ is given by the Gelfand representation applied to the C*-algebra generated by 1 and b. Hence, $a \leq \|b\|$. Applying the Gelfand representation again, this time to the C*-algebra generated by 1 and a, we obtain the inequality $\|a\| \leq \|b\|$.

To prove Condition (4) we first observe that if $c \geq 1$, then c is invertible and $c^{-1} \leq 1$. This is given by the Gelfand representation applied to the C*-subalgebra generated by 1 and c. Now $a \leq b \Rightarrow 1 = a^{-1/2}aa^{-1/2} \leq$

$a^{-1/2}ba^{-1/2} \Rightarrow (a^{-1/2}ba^{-1/2})^{-1} \le 1$, that is, $a^{1/2}b^{-1}a^{1/2} \le 1$. Hence, $b^{-1} \le (a^{1/2})^{-1}(a^{1/2})^{-1} = a^{-1}$. $\square$

2.2.6. Theorem. *If a, b are positive elements of a C^*-algebra A, then the inequality $a \le b$ implies the inequality $a^{1/2} \le b^{1/2}$.*

Proof. We show $a^2 \le b^2 \Rightarrow a \le b$ and this will prove the theorem. We may suppose that A is unital. Let $t > 0$ and let c, d be the real and imaginary hermitian parts of the element $(t+b+a)(t+b-a)$. Then

$$\begin{aligned} c &= \tfrac{1}{2}((t+b+a)(t+b-a)) + (t+b-a)(t+b+a)) \\ &= t^2 + 2tb + b^2 - a^2 \\ &\ge t^2. \end{aligned}$$

Consequently, c is both invertible and positive. Since $1 + ic^{-1/2}dc^{-1/2} = c^{-1/2}(c+id)c^{-1/2}$ is invertible, therefore $c + id$ is invertible. It follows that $t + b - a$ is left invertible, and therefore invertible, because it is hermitian. Consequently, $-t \notin \sigma(b-a)$. Hence, $\sigma(b-a) \subseteq \mathbf{R}^+$, so $b - a$ is positive, that is, $a \le b$. $\square$

It is not true that $0 \le a \le b \Rightarrow a^2 \le b^2$ in arbitrary C*-algebras. For example, take $A = M_2(\mathbf{C})$. This is a C*-algebra where the involution is given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* = \begin{pmatrix} \bar\alpha & \bar\gamma \\ \bar\beta & \bar\delta \end{pmatrix}.$$

Let p and q be the projections

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then $p \le p + q$, but $p^2 = p \not\le (p+q)^2 = p + q + pq + qp$, since the matrix

$$q + pq + qp = \tfrac{1}{2}\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

has a negative eigenvalue.

It can be shown that the implication $0 \le a \le b \Rightarrow a^2 \le b^2$ holds only in abelian C*-algebras [Ped, Proposition 1.3.9].

1 $\sigma(f) = f(\Omega)$, $\forall f \in C_0(\Omega)$ $\quad \lambda \notin \sigma(f) \Leftrightarrow \lambda - f$ is inv $\Leftrightarrow \exists g \in C_0(\Omega); g = \frac{1}{\lambda - f}$

$\Leftrightarrow \forall \omega \in \Omega; f(\omega) \ne \lambda \Leftrightarrow \lambda \notin f(\Omega)$

$f \geq 0$ in $C_0(\Omega) \Leftrightarrow \begin{cases} f = \bar f \\ \sigma(f) \subseteq [0,\infty) \Leftrightarrow f(\Omega) \subseteq [0, \infty). \end{cases}$

x

$\left.\begin{array}{l} \exists t\geq 0;\ \|f-t\|\leq t \\ f=\bar{f} \end{array}\right\} \Longrightarrow \forall\omega\in\Omega;\ \underbrace{|f(\omega)-t|\leq t}_{-t\leq f(\omega)-t\leq t} \Rightarrow \forall\omega,\ f(\omega)\geq 0 \Rightarrow f\geq 0$

$\|f-t\| = \sup_{\omega\in\Omega}|f(\omega)-t|$

$\left.\begin{array}{l} f\geq 0 \\ \|f\|\leq t \end{array}\right\} \Rightarrow \sup_{\omega\in\Omega} f(\omega)\leq t \Rightarrow \forall\omega;\ 0\leq f(\omega)\leq t \Rightarrow \forall\omega;\ \underbrace{-t\leq f(\omega)-t\leq t}_{|f(\omega)-t|\leq t} \Rightarrow \|f-t\|\leq t$

② $\forall a\in A^+\ \exists!\, b\in A^+;\ b^2=a$ (b is denoted by $\underline{a^{\frac{1}{2}}}$).

Positive square root

Let us pass to $C^*(a,1)$.

$$\theta: C^*(a,1) \xleftrightarrow[\text{*-isom}]{\text{isomet}} C(\sigma(a))$$

$$a \longmapsto f(t)=t$$

$$1 \longmapsto g(t)=1$$

Since $a\geq 0$, we have $\begin{cases} a=a^* \\ \sigma(a)\subseteq[0,\infty) \end{cases}$. So $\forall t\in\sigma(a);\ t\geq 0 \Rightarrow f(t)=t$ as positive element in $C(\sigma(a))$

$$b=h(a) \longleftarrow h(t)=\sqrt{t}\in C(\sigma(a))$$

$$b\geq 0 \qquad h\geq 0$$

$$b^2=a \longleftarrow h^2=f$$

Moreover, let $\exists c\in A;\ c\geq 0\ \&\ c^2=a$. So $ca=c.c^2=c^2.c=ac$

Hence c commutes with each polynomial in a. So $c.\underset{\sim}{b}=b.c$

$p_n(t)\xrightarrow{u}\sqrt{t} \leftarrow \exists\{p_n\}\quad \lim_n p_n(a)$ X

Let B be the C^*-subalg of A generated by $b, c, 1$. The C^*-alg B is abelian. By the Gelfand theorem

$$\varphi : B \underset{*\text{-isom}}{\overset{\text{isomet}}{\longleftrightarrow}} C(\Omega(B))$$

$$\left.\begin{array}{l} 0 \le b \longmapsto f_1 \ge 0 \\ 0 \le c \longmapsto f_2 \ge 0 \\ b^2 = a = c^2 \rightsquigarrow f_1^2 = f_2^2 \end{array}\right\} \Rightarrow f_1 = f_2$$

$b = c$

③ $\boxed{c = c^*} \Rightarrow c^2 \ge 0$ (why?)

$$\left\{\begin{array}{l} (c^2)^* = (cc)^* = c^* c^* = c \cdot c = c^2 \\ \sigma(c^2) = \sigma(c)^2 = \{\lambda^2 : \lambda \in \underbrace{\sigma(c)}_{\subseteq \mathbb{R}}\} \subseteq [0, \infty) \end{array}\right.$$

Spectral Theorem $\sigma(f(a)) = f(sp(a))$ cts

$$C^*(c, 1) \underset{*\text{-isom}}{\overset{\text{isomet}}{\longleftrightarrow}} C(\sigma(c))$$

$$c \longmapsto f(t) = t$$

$$\underset{0}{\overset{\vee\!\shortparallel}{c^2}} \longmapsto \underbrace{g(t) = t^2}_{g \ge 0} \ge 0 \; (t \in \sigma(c) \subseteq \mathbb{R})$$

$0 \le c^+ \leftsquigarrow h_1(t) = \max\{t, 0\} \ge 0$

$0 \le c^- \leftsquigarrow h_2(t) = -\min\{t, 0\} \ge 0$

$|c| \leftsquigarrow |t|$

$$c \leq \|c\| \rightsquigarrow f \leq \|f\|, \quad f(t) \leq \|f\|$$

$$c = c^+ - c^- \leftrightsquigarrow t = h_1(t) - h_2(t)$$

$$|c| = c^+ + c^- \leftrightsquigarrow |t| = h_1(t) + h_2(t)$$

$$\|c - \alpha\| \leq \alpha \longleftrightarrow \|f(t) - \alpha\| \leq \alpha$$

$$\|c\| \leq \alpha,\ c \geq 0 \longleftrightarrow f \geq 0,\ \|f\| \leq \alpha$$

Th. Each element $a \in A$ can be represented as a linear combination of 4 positive elements.

Proof. $a = \underbrace{b}_{b^+ - b^-} + i\underbrace{c}_{c^+ - c^-}$. $\square$

Th. $a, b \in A^+ \Rightarrow a + b \in A^+$

$$\left.\begin{array}{l} a \geq 0 \overset{\boxed{a \leq \|a\|}}{\Longrightarrow} \big\| a - \|a\| \big\| \leq \|a\| \\ b \geq 0 \Rightarrow \big\| b - \|b\| \big\| \leq \|b\| \end{array}\right\} \Rightarrow \Big\| \underbrace{(a+b)}_{c} - \underbrace{(\|a\| + \|b\|)}_{\alpha} \Big\|$$

$$= \Big\| (a - \|a\|) + (b - \|b\|) \Big\| \leq \big\| a - \|a\| \big\| + \big\| b - \|b\| \big\| \leq \underbrace{\|a\| + \|b\|}_{\alpha},$$

$\Rightarrow a + b \geq 0$. $\square$

Def. Let $a, b \in A$. We say $a \leq b$ when $b - a \in A^+$.

(5) $a \leq b \Rightarrow b-a \geq 0 \Rightarrow c^*(b-a)c = \underbrace{c^*(b-a)^*}_{d^*}\underbrace{(b-a)c}_{d} \geq 0 \Rightarrow$

$c^*ac \leq c^*bc$

(6) $\underbrace{a}_{\in A} \leq \underbrace{\|a\| \cdot 1}_{\in A}$

$\theta : C^*(1,a) \leftrightarrow C(sp(a))$

$a \longrightarrow f(t) = t \geq 0$

$1 \longrightarrow g(t)=1$

$a \leq \|a\| \cdot 1 \longleftarrow \underbrace{f \leq \|f\| \cdot 1}$

$\underbrace{f(t) \leq \|f\| \cdot 1(t)}$

$\underbrace{f(t) \leq \|f\|}$

$f(t) = |f(t)| = \sup_{t \in sp(a)} |f(t)|$

$0 \leq a \leq b \Rightarrow a \leq \|b\| \Longrightarrow \|a\| \leq \|b\|$ ✓

$b \leq \|b\|$

$a \leq \|b\| \longrightarrow f(t) \leq \|b\| \ \forall t$

$\underbrace{\sup f(t) \leq \|b\|}$

$\|a\| \leq \|b\| \longleftarrow \|f\| \leq \|b\|$

a, b are inv

(7) $\boxed{0 < a} \leq b \Rightarrow 0 < b^{-\frac{1}{2}} a b^{-\frac{1}{2}} \leq 1 \leftarrow$ scaler

means $0 \leq a$ & a is invertible

$\underset{\text{function calculus}}{\Rightarrow} \left(b^{-\frac{1}{2}} a b^{-\frac{1}{2}}\right)^{-1} \geq 1 \Rightarrow b^{\frac{1}{2}} a^{-1} b^{\frac{1}{2}} \geq 1$

$\underset{\text{multiply by } b^{-\frac{1}{2}}}{\Rightarrow} a^{-1} \geq b^{-1}$. $\square$

$\underbrace{a > 0}_{a \geq 0,\ a \text{ inv}} \longrightarrow \underbrace{f(t) = t}_{t \geq 0,\ 0 \notin sp(a)}$

$t > 0$ $g(t) = \frac{1}{t} \in C(sp(a))$

$g(a)$

$\underbrace{\theta^{-1}(fg)}_{\theta^{-1}(f)\theta^{-1}(g)} = \underbrace{\theta^{-1}(1)}_{1} \longleftarrow f \cdot g = 1$

$\underbrace{\theta^{-1}(f)}_{a}\ \underbrace{\theta^{-1}(g)}_{g(a)}$

$\therefore g(a) = a^{-1}$

compact

$sp(a) \subseteq (0, +\infty)$

$\inf sp(a) \in sp(a)$

$sp(a) \subseteq [\alpha, +\infty)$

$\alpha \leq a \longleftarrow \alpha \leq f,\ f(t)=t$

$$\underbrace{\theta^{-1}(f-\alpha\cdot 1)\geq 0}_{\theta^{-1}(f)-\theta^{-1}(\alpha 1)} \xleftarrow[\text{*-hom}]{\theta^{-1}} f-\alpha\cdot 1\geq 0$$

$$\underbrace{\theta^{-1}(f)}_{a}\geq \underbrace{\theta^{-1}(\alpha\cdot 1)}_{\alpha\,\underbrace{\theta^{-1}(1)}_{\underbrace{\alpha\cdot 1}_{\alpha}}}$$

Why any *-homom $\varphi: A\to B$ between C^*-algebras preserves $\leq$?

$a\leq b \Rightarrow b-a\geq 0 \Rightarrow b-a=c^*c \Rightarrow \varphi(b-a)=\varphi(c^*c)=\varphi(c)^*\varphi(c)=\varphi(b)-\varphi(a)\geq 0 \Rightarrow \varphi(a)\leq\varphi(b).\ \square$

Why is the inverse of a *-homom $\theta: A\to B$, a *-hom?

Solution. $\theta(\theta^{-1}(bb'))=bb'=\theta(\theta^{-1}(b))\cdot\theta(\theta^{-1}(b'))$

$=\theta(\theta^{-1}(b)\cdot\theta^{-1}(b')) \underset{\theta \text{ is } 1\text{-}1}{\Longrightarrow} \theta^{-1}(bb')=\theta^{-1}(b)\theta^{-1}(b').\ \square$

We shall need to view Hilbert spaces as dual spaces. Let H be a Hilbert space and $\boxed{H_* = H}$ as an additive group, but define a new scalar multiplication on H_* by setting $\lambda . x = \bar{\lambda} x$, and a new inner product by setting $\langle x, y\rangle_* = \langle y, x\rangle$. Then H_* is a Hilbert space, and obviously the norm induced by the new inner product is the same as that induced by the old one. If $x \in H$, define $v(x) \in (H_*)^*$ by setting $\boxed{v(x)(y)} = \boxed{\langle y, x\rangle_*} = \boxed{\langle x, y\rangle}$. It is a direct consequence of the Riesz representation theorem that the map

$$v\colon H \to (H_*)^*, \; x \mapsto v(x),$$

$\|\langle x, \cdot\rangle\| = \|x\|$

is an isometric linear isomorphism, which we use to identify these Banach spaces. The weak* topology (1) on H is called the *weak* topology. A net $(x_\lambda)_{\lambda\in\Lambda}$ converges to a point x in H in the weak topology if and only if $\langle x, y\rangle = \lim_\lambda \langle x_\lambda, y\rangle$ $(y \in H)$. Consequently, the weak topology is weaker than the norm topology, and a bounded linear map between Hilbert spaces (2) is necessarily weakly continuous. The importance to us of the weak topology is the fact that the closed unit ball of H is weakly compact (Banach–Alaoglu theorem).

(1) $\sigma(H, H)$ * **

2.4.1. Theorem. *Let $u\colon H_1 \to H_2$ be a compact linear map between Hilbert spaces H_1 and H_2. Then the image of the closed unit ball of H_1 under u is compact.*

weak on $H \equiv$ weak* on H^{**} S

Proof. Let S be the closed unit ball of H_1. It is weakly compact, and u is weakly (3) continuous, so $u(S)$ is weakly compact and therefore weakly closed. Hence, $u(S)$ is norm-closed, since the weak topology is weaker than the norm topology. Since u is a compact operator, this implies that $\overline{u(S)} = u(S)$ is norm-compact. □

S compact $\Leftrightarrow$ $\dim X < +\infty$

2.4.2. Theorem. *Let u be a compact operator on a Hilbert space H. Then both $|u|$ and u^* are compact.*

Proof. Suppose that u has polar decomposition $u = w|u|$ say. Then $|u| = \underbrace{w^*}_{\in B(H)}\underbrace{u}_{\in K(H)}$, so $|u|$ is compact, and $u^* = \underbrace{|u|}_{\in K(H)}\underbrace{w^*}_{\in B(H)}$, so u^* is compact. □

2.4.3. Corollary. *If H is any Hilbert space, then $K(H)$ is self-adjoint.*

Thus, $K(H)$ is a C*-algebra, since (as we saw in Chapter 1) $K(H)$ is a closed ideal in $B(H)$.

An operator u on a Hilbert space H is *diagonalisable* if H admits an orthonormal basis consisting of eigenvectors of u. Diagonalisable operators are necessarily normal, but not all normal operators are diagonalisable. For instance, the bilateral shift is normal (it is a unitary), but it has no eigenvalues.

Let S = [illegible]

L-t-S = Left to Student

2.4.4. Theorem. *If u is a compact normal operator on a Hilbert space H, then it is diagonalisable.* L-t-S

Proof. By Zorn's lemma there is a maximal orthonormal set E of eigenvectors of u. If K is the closed linear span of E, then $H = K \oplus K^\perp$, and K reduces u. The restriction $u_{K^\perp}: K^\perp \to K^\perp$ is compact and normal. An eigenvector of $u_{K^\perp}$ is one for u also, so by maximality of E, the operator $u_{K^\perp}$ has no eigenvectors, and therefore $\sigma(u_{K^\perp}) = \{0\}$ by Theorem 1.4.11. Hence, $\|u_{K^\perp}\| = r(u_{K^\perp})$ (by normality) $= 0$, so $K^\perp = 0$. Thus, $K = H$ and E is an orthonormal basis of eigenvectors of u, so u is diagonalisable.□

If H is a Hilbert space, we denote by $F(H)$ the set of finite-rank operators on H. It is easy to check that $F(H)$ is a self-adjoint ideal of $B(H)$.

2.4.5. Theorem. *If H is a Hilbert space, then $F(H)$ is dense in $K(H)$.*

Proof. Since $F(H)^-$ and $K(H)$ are both self-adjoint, it suffices to show that if u is a hermitian element of $K(H)$, then $u \in F(H)^-$. Let E be an orthonormal basis of H consisting of eigenvectors of u, and let $\varepsilon > 0$. By Theorem 1.4.11 the set S of eigenvalues λ of u such that $|\lambda| \geq \varepsilon$ is finite. From Theorem 1.4.5 it is therefore clear that the set S' of elements of E corresponding to elements of S is finite. Now define a finite-rank diagonal operator v on H by setting $v(x) = \lambda x$ if $x \in S'$ and λ is the eigenvalue corresponding to x, and setting $v(x) = 0$ if $x \in E \setminus S'$. It is easily checked that $\|v - u\| \leq \sup_{\lambda \in \sigma(u) \setminus S} |\lambda| \leq \varepsilon$. This shows that $u \in F(H)^-$. L-t-S □

Left to Students

If x, y are elements of a Hilbert space H we define the operator $x \otimes y$ on H by

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$x \otimes y : H \to H$

$$(x \otimes y)(z) = \langle z, y \rangle x.$$

Find $sp(x \otimes y)$?

Clearly, $\|x \otimes y\| = \|x\|\|y\|$. The rank of $x \otimes y$ is one if x and y are non-zero. If $x, x', y, y' \in H$ and $u \in B(H)$, then the following equalities are readily verified:

$$(x \otimes x')(y \otimes y') = \langle y, x' \rangle (x \otimes y')$$
$$(x \otimes y)^* = y \otimes x$$
$$u(x \otimes y) = u(x) \otimes y$$
$$(x \otimes y)u = x \otimes u^*(y).$$

$(x \otimes x)(x \otimes x) = \langle x, x \rangle x \otimes x = x \otimes x$

The operator $x \otimes x$ is a rank-one projection if and only if $\langle x, x \rangle = 1$, that is, x is a unit vector. Conversely, every rank-one projection is of the form $x \otimes x$ for some unit vector x. Indeed, if $e_1, \ldots, e_n$ is an orthonormal set in H, then the operator $\sum_{j=1}^n e_j \otimes e_j$ is the orthogonal projection of H onto the vector subspace $\mathbf{C}e_1 + \cdots + \mathbf{C}e_n$.

Since $\operatorname{ran}\left(\sum_{j=1}^n e_j \otimes e_j\right) \overset{\supseteq}{\underset{\subseteq}{=}} \mathbb{C}e_1 + \cdots + \mathbb{C}e_n$

$\left(\sum_{j=1}^n e_j \otimes e_j\right)(x) = \sum_{j=1}^n \underbrace{\langle x, e_j \rangle e_j}_{\mathbb{C}e_j}$

For $\lambda_1 e_1 + \cdots + \lambda_n e_n = x$

$\sum_{j=1}^n (e_j \otimes e_j)(\lambda_1 e_1 + \cdots + \lambda_n e_n)$

$= \sum_{j=1}^n \langle \lambda_1 e_1 + \cdots + \lambda_n e_n, e_j \rangle e_j$

$= \sum_{j=1}^n \lambda_j e_j = x$

If $u \in B(H)$ is a rank-one operator and x a non-zero element of its range, then $u = x \otimes y$ for some $y \in H$. For if $z \in H$, then $u(z) = \tau(z)x$ for some scalar $\tau(z) \in \mathbf{C}$. It is readily verified that the map $z \mapsto \tau(z)$ is a bounded linear functional on H, and therefore, by the Riesz representation theorem, there exists $y \in H$ such that $\tau(z) = \langle z, y \rangle$ for all $z \in H$. Therefore, $u = x \otimes y$. (5)

(7) $pu = u = u^* = (pu)^* = u^* p^* = up \Rightarrow pu^n = u^n p \Rightarrow pQ_n(u) = Q_n(u)p$

Take limits

2.4.6. Theorem. *If H is a Hilbert space, then $F(H)$ is linearly spanned by the rank-one projections.*

$Q_n(u) \to u^{\frac{1}{2}} \Leftarrow Q_n(t) \xrightarrow{u} t^{\frac{1}{2}}$

Funct Calculus

Proof. Let $u \in F(H)$ and we shall show it is a linear combination of rank-one projections. The real and imaginary parts of u are in $F(H)$, since $F(H)$ (6) is self-adjoint, so we may suppose that u is hermitian. Now $u = u^+ - u^-$, and by the polar decomposition $|u| \in F(H)$, so u^+ and u^- belong to $F(H)$. Hence, we may assume that $u \geq 0$. The range $u(H)$ is finite-dimensional, and therefore it is a Hilbert space with an orthonormal basis, $e_1, \ldots, e_n$ say. Let $p = \sum_{j=1}^n e_j \otimes e_j$, so p is the projection of H onto $u(H)$. Then $u = pu \overset{u^{\frac{1}{2}}u^{\frac{1}{2}}}{=} u^{1/2}pu^{1/2}$ (7) $\Rightarrow$ (8) $u = \sum_{j=1}^n x_j \otimes x_j$, where $x_j = u^{1/2}(e_j)$. Now $x_j = \lambda_j f_j$ for some unit vector f_j and scalar λ_j, so $u = \sum_{j=1}^n |\lambda_j|^2 f_j \otimes f_j$, and since the operators $f_j \otimes f_j$ are rank-one projections we are done. $\square$

$pu^{\frac{1}{2}} = u^{\frac{1}{2}}p$

2.4.7. Theorem. *If H is a Hilbert space and I a non-zero ideal in $B(H)$, then I contains $F(H)$.*

$F(H) \subseteq I \Rightarrow \overline{F(H)} \subseteq \overline{I} \Rightarrow K(H) \subseteq \overline{I}$

Proof. Let u be a non-zero operator in I. Then for some $x \in H$ we have $u(x) \neq 0$. If p is a rank-one projection, then $p = y \otimes y$ for some unit vector $y \in H$, and clearly there exists $v \in B(H)$ such that $vu(x) = y$ (take $v = (y \otimes u(x))/\|u(x)\|^2$, for instance). Hence, $p = vu(x \otimes x)u^*v^*$, so $p \in I$ as $u \in I$. Thus, I contains all the rank-one projections and therefore by Theorem 2.4.6 it contains $F(H)$. $\square$

If $u: H \to H'$ is a unitary between Hilbert spaces H and H', then the

If $u\colon H \to H'$ is a unitary between Hilbert spaces H and H', then the map

$\forall x\ ux \neq 0$ $u \neq 0$

$\exists x\ ux = 0$

$u = 0$

$$\operatorname{Ad} u\colon K(H) \to K(H'), \quad v \mapsto uvu^*,$$

is a $*$-isomorphism. In fact, all $*$-isomorphisms between $K(H)$ and $K(H')$ are obtained in this way:

2.4.8. Theorem. *Let H and H' be Hilbert spaces and suppose that the map $\varphi\colon K(H) \to K(H')$ is a $*$-isomorphism. Then there exists a unitary $u\colon H \to H'$ such that $\varphi = \operatorname{Ad} u$.*

$\tau_\omega(fg) = (fg)(\omega) = f(\omega)g(\omega) = \tau_\omega(f)\tau_\omega(g)$

$\forall\omega;\ \{\omega\} \subseteq X$, so $\exists f;\ f(\omega) = 1 \Rightarrow \tau_\omega \neq 0$

Let Ω be a locally compact Hausdorff space. For $\omega \in \Omega$, denote by τ_ω the character on $C_0(\Omega)$ given by evaluation at ω: $\tau_\omega(f) = f(\omega)$. If $\omega_1, \ldots, \omega_n$ are distinct points of Ω, then $\tau_{\omega_1}, \ldots, \tau_{\omega_n}$ are linearly independent. For if $\lambda_1\tau_{\omega_1} + \cdots + \lambda_n\tau_{\omega_n} = 0$ and we fix i, then by Urysohn's lemma we may choose $f \in C_0(\Omega)$ such that $f(\omega_i) = 1$ and $f(\omega_j) = 0$ for $j \neq i$. Hence, $0 = \sum_{j=1}^n \lambda_j f(\omega_j) = \lambda_i$.

$\{\omega_i\} \subseteq X - \{\omega_j \mid j \neq i\} \Rightarrow \exists f \in C_c(X);\ f(\omega_i) = 1,\ f(\omega_j) = 0\ \forall j \neq i$

It follows that if $C_0(\Omega)$ is finite-dimensional, then Ω is finite.

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From this observation we show that the projections linearly span an abelian finite-dimensional C*-algebra. We may suppose the algebra is of the form $C_0(\Omega)$ by the Gelfand representation. Then Ω is finite and therefore discrete, so the characteristic functions of the singleton sets span $C_0(\Omega)$.

Suppose now that A is an arbitrary finite-dimensional C*-algebra. It is linearly spanned by its self-adjoint elements, and they in turn are linear combinations of projections by what we have just shown, so it follows that A is the linear span of its projections.

If p is a finite-rank projection on a Hilbert space H, then the C*-algebra $A = pB(H)p$ is finite-dimensional. To see this, write $p = \sum_{j=1}^n e_j \otimes e_j$, where $e_1, \ldots, e_n \in H$. If $u \in B(H)$, then

$$pup = \sum_{j,k=1}^{n} (e_j \otimes e_j)\overbrace{u(e_k \otimes e_k)}^{ue_k \otimes e_k} = \sum_{j,k=1}^{n} \langle u(e_k), e_j\rangle e_j \otimes e_k.$$

Hence, A is in the linear span of the operators $e_j \otimes e_k$ ($j, k = 1, \ldots, n$), and therefore $\dim(A) < \infty$.

$uK \subseteq K \equiv K$ is invariant under u

A closed vector subspace K of H is *invariant* for a subset $A \subseteq B(H)$ if it is invariant for every operator in A. If A is a C*-subalgebra of $B(H)$, it is said to be *irreducible*, or to act *irreducibly* on H, if the only closed vector subspaces of H that are invariant for A are 0 and H. The concept of irreducibility is of great importance in the representation theory of C*-algebras which we shall be taking up in Chapter 5. The following theorem gives a nice connection between irreducibility and the ideal of compact operators.

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x

and will be needed in succeeding chapters.

2.4.9. Theorem. *Let A be a C^*-algebra acting irreducibly on a Hilbert space H and having non-zero intersection with $K(H)$. Then $K(H) \subseteq A$.* Left to students

(1) $\sigma(X^*, X)$ on X^* is the top. generated by the semi-norms $P_x(f) = |f(x)|$ ($x \in X$ is fixed).

An element of the local basis at zero is

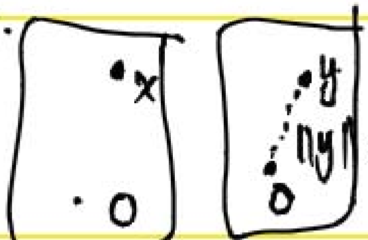
$$N(x_1, \dots, x_n, \varepsilon) = \{f : |f(x_i)| < \varepsilon \ \forall\, 1 \le i \le n\}$$

$$f_n \xrightarrow{w^*} f \iff f_n(x) \to f(x) \ \forall x$$

$\forall G$ open $\exists N(x_1, \dots, x_n, \varepsilon)$; $G \supseteq N(x_1, \dots, x_n, \varepsilon)$

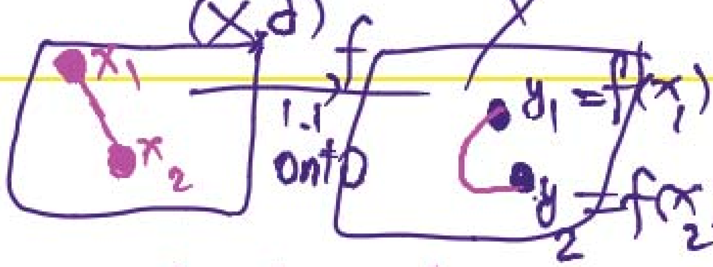
$\vartheta : H \simeq (H_*)^*$

$\{\vartheta^{-1}(G) : G$ is open$\}$ is a top on H.



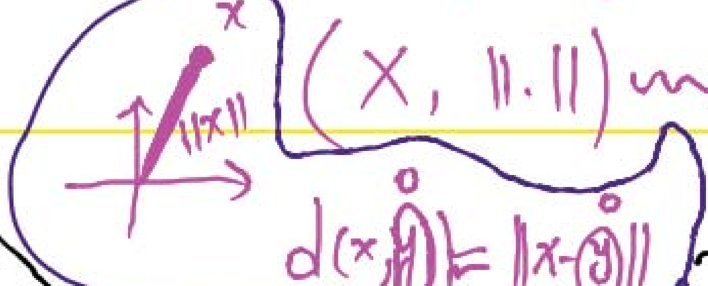
$\|y\| = d(y, 0) = d(x, 0) = \|x\|$

$(X, d) \xrightarrow[\text{1-1 onto}]{f} X$



$y_1 = f(x_1)$, $y_2 = f(x_2)$

$\rho(y_1, y_2) := d(x_1, x_2)$



$(X, \|\cdot\|) \rightsquigarrow (Y, \|\cdot\|')$

$d(x, 0) = \|x - 0\|$

$\|y\|' = \|f(x)\|' := \|x\|$

$x_\alpha \xrightarrow{\text{weak}} x$ in $H \iff \vartheta(x_\alpha) \xrightarrow{w^*} \vartheta(x)$ in $(H_*)^*$

$\iff \vartheta(x_\alpha)(y) \longrightarrow \vartheta(x)(y)\ \forall y \iff \langle x_\alpha, y\rangle \to \langle x, y\rangle\ \forall y$

H has another top: $\underbrace{\text{Norm top}}_{\text{Strong top}}$ $x_n \xrightarrow{\|\cdot\|} x$ in $H \iff \|x_n - x\| \to 0$

<u>Exercise</u> 2 $x_n \xrightarrow{\|\cdot\|} x \Rightarrow x_n \xrightarrow{\text{weakly}} x$

$|\langle x_n, y\rangle - \langle x, y\rangle| = |\langle x_n - x, y\rangle| \le \underbrace{\|x_n - x\|}_{\to 0}\|y\|$

So weak top $\subseteq$ norm top

<u>Def</u>. $T: X \to Y$ (normed spaces) is called <u>compact</u> if $\overline{T(A)}$ is compact for all bounded set $A \subseteq X$.

<u>Th</u>. T is compact iff $\forall \{x_n\}$ bd in X $\exists \{n_k\}\uparrow$; $\{Tx_{n_k}\}$ is converg.

<u>Proof</u> ($\Rightarrow$) Let $\{x_n\}$ be bd. Put $A = \{x_1, x_2, x_3, \dots\}$. $\overline{T(A)}$ is compact. So the sequence $\{Tx_n\}$ in $\overline{T(A)}$ has a convergent subsequence.

($\Leftarrow$) Let A be a bd set. Let $\{y_n\}$ be in $\overline{T(A)}$. $\forall n\ \exists x_n \in A$; $\|Tx_n - y_n\| < \frac{1}{n}$.

Since A is bd, $\{x_n\}$ is bd too. By our assumption $\exists \{Tx_{n_k}\}\ \exists y$; $Tx_{n_k} \to y$. We have $\|y_{n_k} - y\| \le \|y_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - y\| \le \underbrace{\tfrac{1}{n_k}}_{\to 0} + \underbrace{\|Tx_{n_k} - y\|}_{\to 0}$

So $y_{n_k} \to y$. $\square$

in a metric space: K is compact set $\iff$ every seq in K has a conv. subseq.

③ Let $x_\alpha \xrightarrow{w} x$. We want to show that $ux_\alpha \xrightarrow{w} ux$:

$\forall y;\ \langle ux_\alpha, y\rangle = \langle x_\alpha, \underset{H_1}{\underset{\cap}{u^* y}}\rangle \longrightarrow \langle x, u^* y\rangle = \langle ux, y\rangle$. □

④ Some comments on Compact operators:

Th1. $T \in K(X,Y) \Rightarrow T \in B(X,Y)$

Proof. S is the closed unit ball of X $(S = \{x \in X : \|x\| \leq 1\})$

S is bd

$\overline{TS}$ compact

$\overline{TS}$ bd $(\operatorname{diam} TS = \operatorname{diam} \overline{TS})$

TS is bd $\quad (\exists M\ \|Tx\| \leq M\ \forall x_{\|x\| \leq 1})$

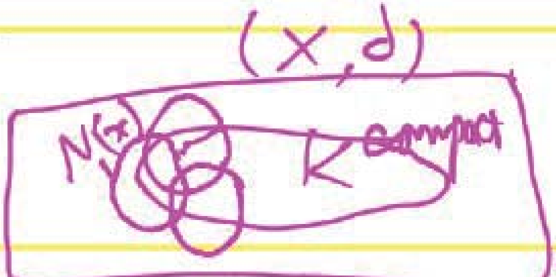
T is bd

Exercise:

A is bd in (X,d) $\overset{\text{Def}}{\Longleftrightarrow}$ $\operatorname{diam} A = \sup_{x,y \in A} d(x,y) < +\infty$ in (X,d) $\Leftrightarrow \exists r;\ \exists x \in X;\ A \subseteq N_r(x)$ in (X,d)

$\Updownarrow$

$\exists M\ \forall x \in A;\ \|x\| \leq M$ in $(X, \|\cdot\|)$



$K \subseteq \bigcup_{x \in K} N(x)$

K compact

$K \subseteq N(x_1) \cup \dots \cup N(x_n)$

$\forall x,y \in K;\ d(x,y) \leq d(x, x_i) + d(x_i, x_j) + d(x_j, y)$

$\leq 2 + \underbrace{\max_{1 \leq i,j \leq n} d(x_i, x_j)}_{\text{constant}}$

Theorem. If $\dim TX<\infty$ & T is bd, then T is compact.

Proof: A is a bd set

TA is bd $(\|Tx\|\leq\|T\|\|x\|)$

$\overline{TA}$ is bd

$\exists r\ \overline{TA}\subseteq N^{TX}_r(0)=S^{TX}_r\subseteq TX$

closed

the closed ball in the normed space TX

$\dim TX<\infty$

$\Downarrow$

S^{TX} compact

$\Downarrow$

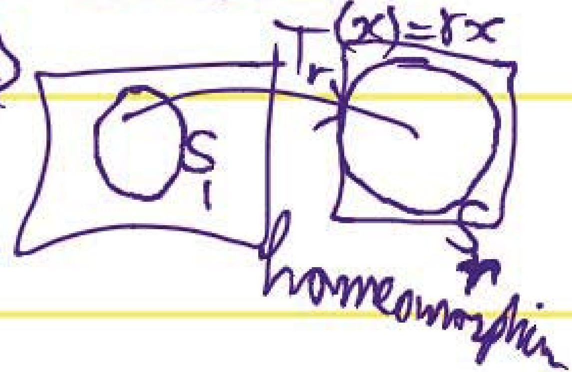
S^{TX}_r compact

X is a normed space

$\dim X<\infty \Leftrightarrow S$ is norm compact

$\dim X<\infty \Rightarrow$ Every subspace of X is closed

$\left.\begin{array}{l}\dim X<\infty\\ T:X\xrightarrow[\text{linear}]{}Y\end{array}\right\}\Rightarrow T$ is bd



$\overline{TA}$ compact

T compact

⑤ u is rank one operator. So $uH=\mathbb{C}x$ for some $x\in H$.

$\forall z\in H;\ u(z)\in\mathbb{C}x$, so $\exists\tau(z)\in\mathbb{C};\ u(z)=\tau(z)x$

Then $\tau: H\to\mathbb{C}$ is bd linear functional:

$\left.\begin{array}{l}u(z)=\lambda x\\ u(z)=\mu x\end{array}\right\}\Rightarrow\lambda x=\mu x\Rightarrow\lambda=\mu$ So τ is well-defined.

$u(z)=\tau(z)x$

$u(z') = \tau(z')x$

$$\left.\begin{array}{l} u(z+z') = \tau(z+z')x \\ \quad \| \\ u(z)+u(z') = \big(\tau(z)+\tau(z')\big)x \end{array}\right\} \Rightarrow \tau(z+z') = \tau(z)+\tau(z'),$$

Let $\|z\| \leq 1$. $\quad u(z) = \tau(z)x$

$$\|u(z)\| = \|\tau(z)x\| = |\tau(z)|\,\|x\| \Rightarrow |\tau(z)| = \frac{\|u(z)\|}{\|x\|} \leq \frac{\|u\|}{\|x\|}$$

So $\|\tau\| \leq \frac{\|u\|}{\|x\|} < \infty$. $\square$

By the Riesz Represen Th $\exists y \in H;\ \tau = \langle \cdot, y \rangle$

Hence $u(z) = \tau(z)x = \langle z, y\rangle x = (x \otimes y)(z) \quad \forall z$

$\therefore\ u = x \otimes y$

⑥ If $u \in F(H)$, then $u^* \in F(H)$

By the polar decomposition, $\underline{u = v|u|}$ & $|u| = v^* u$ $\quad v \to$ partial isometry

So $|u|(H) = v^*\big(\underbrace{u(H)}_{\text{generated by } x_1, \dots, x_n}\big) = \langle v^* x_1, \dots, v^* x_n \rangle$. Hence $|u| \in F(H)$

$u^* = |u|\, v^* \Rightarrow u^*(H) = |u|(v^* H) \subseteq \underbrace{|u|(H)}_{\text{of finite dim}}$. Hence $u^* \in F(H)$

$$\left.\begin{array}{l} u\in F(H) \\ v\in F(H) \end{array}\right\} \Rightarrow \left.\begin{array}{l} u(H) = \langle x_1, \cdots, x_n \rangle \\ v(H) = \langle y_1, \cdots, y_m \rangle \end{array}\right\} \Rightarrow u+v \in F(H)$$

$$(u+v)(H) = \{ux+vx \mid x\in H\} = \langle x_1, \cdots, x_n, y_1, \cdots, y_m \rangle$$

$F(H)$ is an ideal of $K(H)$.

$u\in F(H) \Rightarrow \operatorname{Re} u = \frac{u+u^*}{2} \in F(H)$, $\operatorname{Im} u \in F(H)$, $u = \operatorname{Re}_u + i \operatorname{Im}_u$

$u \in B(H)_h \Rightarrow u_+ = \frac{u+|u|}{2} \in F(H)$, $u_- \in F(H)$

$$u = u_+ - u_-$$

(8) $u = u^{\frac{1}{2}} P u^{\frac{1}{2}} = \sum_{j=1}^{n} u^{\frac{1}{2}} (e_j \otimes e_j) u^{\frac{1}{2}} = \sum \underbrace{(u^{\frac{1}{2}}) e_j}_{x_j} \otimes \underbrace{(u^{\frac{1}{2}})^* e_j}_{u^{\frac{1}{2}} e_j = x_j}$

$x = \|x\| \cdot \frac{x}{\|x\|}$ unit vector

$x_j = \lambda_j f_j$, $\|f_j\| = 1$

$$= \sum \lambda_j (f_j \otimes f_j) \bar{\lambda}_j$$

$$\boxed{\forall x,y\in H \ \exists u\in K(H); \quad ux = y \qquad \frac{(y\otimes x)}{\|x\|^2}(x) = \frac{\cancel{\langle x,x\rangle} y}{\cancel{\langle x,x\rangle}} = y}$$

(9) If $\Omega \supseteq \{\omega_1, \omega_2, \dots\}$, then $C_0(\Omega) \supseteq \{\tau_{\omega_1}, \tau_{\omega_2}, \dots\}$

We observed that every finite subset of $\{\tau_{\omega_1}, \tau_{\omega_2}, \dots\}$ is linearly independent. So $\{\tau_{\omega_1}, \tau_{\omega_2}, \dots\}$ is lin independent. Thus $C_0(\Omega)'$ is infinite dim.

Hence $C_0(\Omega)$ is infinite dim.

<u>Th</u>. If $uK \subseteq K$, then $u^*K^\perp \subseteq K^\perp$ $\quad (K \leq H)$

$\forall x \in K^\perp$, $\langle u^*x, y\rangle = \langle \underbrace{x}_{\in K^\perp}, \underbrace{uy}_{\in K} \rangle = 0 \quad (y \in K) \Rightarrow u^*x \in K^\perp$. $\square$

$u = \lim_{n\to\infty} p_n u$ if $u \in F(H)$, since $F(H)$ is dense in $K(H)$. Now if $u \in F(H)$, there exist $x_1, \dots, x_m, y_1, \dots, y_m$ in H such that $u = \sum_{k=1}^m x_k \otimes y_k$. Hence, $p_n u = \sum_{k=1}^m p_n(x_k) \otimes y_k$. Since $\lim_{n\to\infty} p_n(x) = x$ for all $x \in H$, therefore for each k,

$\{0\} = sp(0) = sp(P^2 - P) = \{\lambda^2 - \lambda : \lambda \in sp(P)\} \Rightarrow \lambda^2 - \lambda = 0 \Rightarrow \lambda = 0, 1$

$\bigcap_n P_n(H) \subseteq P_{n+1}(H)$

$$\lim_{n\to\infty} \|p_n(x_k) \otimes y_k - x_k \otimes y_k\| = \lim_{n\to\infty} \|p_n(x_k) - x_k\| \|y_k\| = 0.$$

$sp(P) \subseteq \{0,1\}$ $\quad \|P\| = r(P) \le 1$ $\quad P_n \le P_{n+1}$

Hence, $\lim_{n\to\infty} p_n u = u$.

Let A be an arbitrary C*-algebra and denote by Λ the set of all positive elements a in A such that $\|a\| < 1$. This set is a poset under the partial order of A_{sa}. In fact, Λ is also upwards-directed; that is, if $a, b \in \Lambda$, then there exists $c \in \Lambda$ such that $a, b \le c$. We show this: If $a \in A^+$, then $1 + a$ is of course invertible in $\tilde{A}$, and $a(1+a)^{-1} = 1 - (1+a)^{-1}$. We claim

$$a, b \in A^+ \text{ and } a \le b \Rightarrow a(1+a)^{-1} \le b(1+b)^{-1}. \tag{1}$$

Indeed, if $0 \le a \le b$, then $1 + a \le 1 + b$ implies $(1+a)^{-1} \ge (1+b)^{-1}$, by Theorem 2.2.5, and therefore $1 - (1+a)^{-1} \le 1 - (1+b)^{-1}$; that is, $a(1+a)^{-1} \le b(1+b)^{-1}$, proving the claim. Observe that if $a \in A^+$, then $a(1+a)^{-1}$ belongs to Λ (use the Gelfand representation applied to the C*-subalgebra generated by 1 and a). Suppose then that a, b are an arbitrary pair of elements of Λ. Put $a' = a(1-a)^{-1}$, $b' = b(1-b)^{-1}$ and $c = (a' + b')(1 + a' + b')^{-1}$. Then $c \in \Lambda$, and since $a' \le a' + b'$, we have $a = a'(1+a')^{-1} \le c$, by (1). Similarly, $b \le c$, and therefore Λ is upwards-directed, as asserted.

3.1.1. Theorem. *Every C^*-algebra A admits an approximate unit. Indeed, if Λ is the upwards-directed set of all $a \in A^+$ such that $\|a\| < 1$ and $u_\lambda = \lambda$ for all $\lambda \in \Lambda$, then $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for A (called the canonical approximate unit).*

$\boxed{u_\lambda = \lambda}$

Proof. From the remarks preceding this theorem, $(u_\lambda)_{\lambda\in\Lambda}$ is an increasing net of positive elements in the closed unit ball of A. Therefore, we need only show that $a = \lim_\lambda u_\lambda a$ for each $a \in A$. Since Λ linearly spans A, we can reduce to the case where $a \in \Lambda$.

$\|a\| \le 1,\ a \mapsto f,\ \|f\| \le 1$

Suppose then that $a \in \Lambda$ and that $\varepsilon > 0$. Let $\varphi: C^*(a) \to C_0(\Omega)$ be the Gelfand representation. If $f = \varphi(a)$, then $K = \{\omega \in \Omega \mid |f(\omega)| \ge \varepsilon\}$ is compact, and therefore by Urysohn's lemma there is a continuous function $g: \Omega \to [0,1]$ of compact support such that $g(\omega) = 1$ for all $\omega \in K$. Choose $\delta > 0$ such that $\delta < 1$ and $1 - \delta < \varepsilon$. Then $\|f - \delta g f\| \le \varepsilon$. If $\lambda_0 =$ $K \subseteq \Omega$

$\ge 0,\ \|\delta g\| < 1$ $\quad |f(\omega) - \delta g(\omega) f(\omega)| \le (1-\delta)|f(\omega)| < \varepsilon$

$\varphi^{-1}(\delta g)$, then $\lambda_0 \in \Lambda$ and $\|a - u_{\lambda_0} a\| \le \varepsilon$. Now suppose that $\lambda \in \Lambda$ and $\lambda \ge \lambda_0$. Then $1 - u_\lambda \le 1 - u_{\lambda_0}$, so $a(1-u_\lambda)a \le a(1-u_{\lambda_0})a$. Hence,

$\|a - u_\lambda a\|^2 = \|(1-u_\lambda)^{1/2}(1-u_\lambda)^{1/2}a\|^2 \le \|(1-u_\lambda)^{1/2}a\|^2 = \|a(1-u_\lambda)a\| \le$

$\|a-u_\lambda a\|^2 = \|(1-u_\lambda)^{1/2}(1-u_\lambda)^{1/2}a\|^2 \leq \|(1-u_\lambda)^{1/2}a\|^2 = \|a(1-u_\lambda)a\| \leq \|a(1-u_{\lambda_0})a\| \leq \|(1-u_{\lambda_0})a\| \leq \varepsilon$. This shows that $a = \lim_\lambda u_\lambda a$. $\square$

$0 \leq u_\lambda$ & $\|u_\lambda\| \leq 1 \Rightarrow 0 \leq 1-u_\lambda \leq 1 \Rightarrow 0 \leq (1-u_\lambda)^{\frac{1}{2}} \leq 1 \Rightarrow \|(1-u_\lambda)^{\frac{1}{2}}\| \leq 1$

$\|ab\| \leq \|a\|\,\|b\|$

3.1.1. *Remark.* If a C*-algebra A is separable, then it admits an approximate unit which is a sequence. For in this case there exist finite sets $F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n \subseteq \ldots$ such that $F = \cup_{n=1}^\infty F_n$ is dense in A. Let $(u_\lambda)_{\lambda\in\Lambda}$ be any approximate unit for A. If $\varepsilon > 0$, and $F_n = \{a_1, \ldots, a_m\}$ say, then there exist $\lambda_1, \ldots, \lambda_m \in \Lambda$ such that $\|a_j - a_j u_\lambda\| < \varepsilon$ if $\lambda \geq \lambda_j$. Choose $\lambda_\varepsilon \in \Lambda$ such that $\lambda_\varepsilon \geq \lambda_1, \ldots, \lambda_m$. Then $\|a - au_\lambda\| < \varepsilon$ for all $a \in F_n$ and all $\lambda \geq \lambda_\varepsilon$. Hence, if n is a positive integer and $\varepsilon = 1/n$, then there exists $\lambda_n = \lambda_\varepsilon \in \Lambda$ such that $\|a - a\lambda_n\| < 1/n$ for all $a \in F_n$. Also, we may obviously choose the λ_n such that $\lambda_n \leq \lambda_{n+1}$ for all n. Consequently, $\lim_{n\to\infty} \|a - au_{\lambda_n}\| = 0$, for all $a \in F$, and since F is dense in A, this also holds for all $a \in A$. Therefore, $(u_{\lambda_n})_{n=1}^\infty$ is an approximate unit for A.

3.1.2. Theorem. *If L is a closed left ideal in a C^*-algebra A, then there is an increasing net $(u_\lambda)_{\lambda\in\Lambda}$ of positive elements in the closed unit ball of L such that $a = \lim_\lambda au_\lambda$ for all $a \in L$.*

$b \in L \cap L^* \Rightarrow b \in L^* \Rightarrow \exists a \in L;\ b = a^* \Rightarrow b^* = a \in L$

$b \in L \Rightarrow b^* \in L^*$

$\Downarrow$

$b^* \in L \cap L^*$

Proof. Set $B = L \cap L^*$. Since B is a C*-algebra, it admits an approximate unit, $(u_\lambda)_{\lambda\in\Lambda}$ say, by Theorem 3.1.1. If $a \in L$, then $a^*a \in B$, so $0 = \lim_\lambda a^*a(1-u_\lambda)$. Hence, $\lim_\lambda \|a - au_\lambda\|^2 = \lim_\lambda \|(1-u_\lambda)a^*a(1-u_\lambda)\| \leq \lim_\lambda \|a^*a(1-u_\lambda)\| = 0$, and therefore $\lim_\lambda \|a - au_\lambda\| = 0$. $\square$

In the preceding proof we worked in the unitisation $\tilde{A}$ of A. We shall frequently do this tacitly.

3.1.3. Theorem. *If I is a closed ideal in a C^*-algebra A, then I is self-adjoint and therefore a C^*-subalgebra of A. If $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for I, then for each $a \in A$*

$$\inf_{z\in I}\|a+z\| = \|a + I\| = \lim_\lambda \|a - u_\lambda a\| = \lim_\lambda \|a - au_\lambda\|.$$

Proof. By Theorem 3.1.2 there is an increasing net $(u_\lambda)_{\lambda\in\Lambda}$ of positive elements in the closed unit ball of I such that $a = \lim_\lambda au_\lambda$ for all $a \in I$. Hence, $a^* = \lim_\lambda u_\lambda a^*$, so $a^* \in I$, because all of the elements u_λ belong to I. Therefore, I is self-adjoint. $\in I$ closed

Suppose that $(u_\lambda)_{\lambda\in\Lambda}$ is an arbitrary approximate unit of I, that $a \in A$, and that $\varepsilon > 0$. There is an element b of I such that $\|a+b\| < \|a+I\| + \varepsilon/2$. Since $b = \lim_\lambda u_\lambda b$, there exists $\lambda_0 \in \Lambda$ such that $\|b - u_\lambda b\| < \varepsilon/2$ for all $\lambda \geq \lambda_0$, and therefore

$$
\begin{aligned}
\|a+I\| - \varepsilon < \|a+I\| \leq \|a - u_\lambda a\| &\leq \|(1-u_\lambda)(a+b)\| + \|b - u_\lambda b\| \\
&\leq \|a+b\| + \|b - u_\lambda b\| \\
&< \|a+I\| + \varepsilon/2 + \varepsilon/2.
\end{aligned}
$$

$\beta - \varepsilon < \alpha < \beta + \varepsilon \Leftrightarrow |\alpha - \beta| < \varepsilon$

It follows that $\|a + I\| = \lim_\lambda \|a - u_\lambda a\|$, and therefore also $\|a + I\| =$

It follows that $\|a+I\| = \lim_\lambda \|a - u_\lambda a\|$, and therefore also $\|a+I\| =$ $\|a^*+I\| = \lim_\lambda \|a^* - u_\lambda a^*\| = \lim_\lambda \|a - au_\lambda\|$. □ 4

3.1.2. *Remark.* Let I be a closed ideal in a C*-algebra A, and J a closed ideal in I. Then J is also an ideal in A. To show this we need only show that ab and ba are in J if $a \in A$ and b is a positive element of J (since J is a C*-algebra, J^+ linearly spans J). If $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for I, then $b^{1/2} = \lim_\lambda u_\lambda b^{1/2}$ because $b^{1/2} \in I$. Hence, $ab = \lim_\lambda au_\lambda b^{1/2}b^{1/2}$, so $ab \in J$ because $b^{1/2} \in J$, $au_\lambda b^{1/2} \in I$, and J is an ideal in I. Therefore, $a^*b \in J$ also, so $ba \in J$, since J is self-adjoint.

$J \triangleleft I \triangleleft A$

3.1.4. Theorem. *If I is a closed ideal of a C*-algebra A, then the quotient A/I is a C*-algebra under its usual operations and the quotient norm.*

Proof. Let $(u_\lambda)_{\lambda\in\Lambda}$ be a approximate unit for I. If $a \in A$ and $b \in I$, then

$$\begin{aligned}
\|a+I\|^2 &= \lim_\lambda \|a - au_\lambda\|^2 \text{ (by Theorem 3.1.3)} \\
&= \lim_\lambda \|(1-u_\lambda)a^*a(1-u_\lambda)\| \quad \text{(C* condition)} \\
&\le \sup_\lambda \|(1-u_\lambda)(a^*a+b)(1-u_\lambda)\| + \lim_\lambda \|(1-u_\lambda)b(1-u_\lambda)\| \\
&\le \|a^*a+b\| + \underbrace{\lim_\lambda \|b - u_\lambda b\|}_{0} \\
&= \|a^*a+b\|.
\end{aligned}$$

4

Therefore, $\|a+I\|^2 \le \|a^*a+I\|$. By Lemma 2.1.3, A/I is a C*-algebra. □

$\le \|a^*+I\|\,\|a+I\| = \|a+I\|^2$

3.1.5. Theorem. *If $\varphi\colon A \to B$ is an injective $*$-homomorphism between C*-algebras A and B, then φ is necessarily isometric.*

Proof. It suffices to show that $\|\varphi(a)\|^2 = \|a\|^2$, that is, $\|\varphi(a^*a)\| = \|a^*a\|$. Thus, we may suppose that A is abelian (restrict to $C(a^*a)$ if necessary), and that B is abelian (replace B by $\varphi(A)^-$ if required). Moreover, by extending $\varphi\colon A \to B$ to $\tilde\varphi\colon \tilde A \to \tilde B$ if necessary, we may further assume that A, B, and φ are unital.

$\exists b \in B;\ \tau(b) \neq 0$

If τ is a character on B, then $\tau\circ\varphi$ is one on A. Clearly the map

$A \cong C(\Omega(A))$, $a \mapsto \hat a$

$$\varphi'\colon \Omega(B) \to \Omega(A),\ \tau \mapsto \tau\circ\varphi,$$

5 compact 6 $\exists \tau_0;$ $\{\tau_0\} \subseteq \Omega(A) / \varphi'(\Omega(B))$

is continuous. Hence, $\varphi'(\Omega(B))$ is compact, because $\Omega(A)$ is compact, and therefore $\varphi'(\Omega(B))$ is closed in $\Omega(A)$. If $\varphi'(\Omega(B)) \neq \Omega(A)$, then by Urysohn's lemma there is a non-zero continuous function $f\colon \Omega(A) \to \mathbf{C}$ such that f vanishes on $\varphi'(\Omega(B))$. By the Gelfand representation, $f = \hat a$ for some element $a \in A$. Hence, for each $\tau \in \Omega(B)$, $\tau(\varphi(a)) = \hat a(\tau\circ\varphi) = 0$

$f(\tau_0) \neq 1$

$1 = \varphi(a_n)$, $1 - \tau(\varphi(a_n)) \not\to 0$, $n \to \infty$, $\Downarrow$, $\exists n_i$

for some element $a \in A$. Hence, for each $\tau \in \Omega(B)$, $\tau(\varphi(a)) = \hat{a}(\tau \circ \varphi) = 0$.

7 $\varphi'(\Omega(B))$ $\tau(\varphi(a))\neq 0$ $\Downarrow$ $\tau\circ\varphi\neq 0$

Therefore, $\varphi(a) = 0$, so $a = 0$. But this implies that f is zero, a contradiction. The only way to avoid this is to have $\varphi'(\Omega(B)) = \Omega(A)$. Hence, for each $a \in A$,

$$\|a\| = \|\hat{a}\|_\infty = \sup_{\tau \in \Omega(A)} |\tau(a)| = \sup_{\tau \in \Omega(B)} |\underbrace{\tau(\varphi(a))}_{\widehat{\varphi(a)}(\tau)}| = \|\varphi(a)\|.$$

$= \|\widehat{\varphi(a)}\|$

Thus, φ is isometric. $\square$

3.1.6. Theorem. *If $\varphi\colon A \to B$ is a $*$-homomorphism between C^*-algebras, then $\varphi(A)$ is a C^*-subalgebra of B.*

Proof. The map

$$\Psi: A/\ker(\varphi) \to B,\ a + \ker(\varphi) \mapsto \varphi(a),$$

is an injective $*$-homomorphism between C*-algebras and is therefore isometric. Its image is $\varphi(A)$, so this space is necessarily complete and therefore closed in B. $\square$

$\varphi(A) = \operatorname{ran}\varphi = \operatorname{ran}\Psi$ is C^*-subalg of B

3.1.7. Theorem. *Let B and I be respectively a C^*-subalgebra and a closed ideal in a C^*-algebra A. Then $B + I$ is a C^*-subalgebra of A.*

$= \{b + x \mid b \in B \,\&\, x \in I\}$

Proof. We show only that $B + I$ is complete, because the rest is trivial. Since I is complete we need only prove that the quotient $(B + I)/I$ is complete. The intersection $B \cap I$ is a closed ideal in B and the map φ from $B/(B \cap I)$ to A/I defined by setting $\varphi(b + B \cap I) = b + I$ ($b \in B$) is a $*$-homomorphism with range $(B + I)/I$. By Theorem 3.1.6, $(B + I)/I$ is complete, because it is a C*-algebra. $\square$

$(b+x)(c+y) = \underbrace{bc}_{\in B} + \underbrace{(xc + by + xy)}_{\in I} \in B+I$

$b + x \in B + I$ where $b \in B$ & $x \in I \Rightarrow (b+x)^* = b^* + x^* \in B + I$

If $I_1, I_2, \ldots, I_n$ are sets in A, we define $I_1 I_2 \ldots I_n$ to be the closed linear span of all products $a_1 a_2 \ldots a_n$, where $a_j \in I_j$. If I, J are closed ideals in A, then $I \cap J = IJ$. The inclusion $IJ \subseteq I \cap J$ is obvious. To show the reverse inclusion we need only show that if a is a positive element of $I \cap J$, then $a \in IJ$. Suppose then that $a \in (I \cap J)^+$. Hence, $a^{1/2} \in I \cap J$. If $(u_\lambda)_{\lambda \in \Lambda}$ is an approximate unit for I, then $a = \lim_\lambda (u_\lambda a^{1/2}) a^{1/2}$, and since $u_\lambda a^{1/2} \in I$ for all $\lambda \in \Lambda$, we get $a \in IJ$, as required.

$a \in I \cap J$, $a \geq 0$; $a \in I$, $a \in J$ $\Downarrow$ $a^{\frac{1}{2}} \in I$, $a^{\frac{1}{2}} \in J$ $\Downarrow$ $a = a^{\frac{1}{2}} a^{\frac{1}{2}} \in IJ$

Let I be a closed ideal I in A. We say I is *essential* in A if $aI = 0 \Rightarrow a = 0$ (equivalently, $Ia = 0 \Rightarrow a = 0$). From the preceding observations it is easy to check that I is essential in A if and only if $I \cap J \neq 0$ for all non-zero closed ideals J in A.

8

Every C*-algebra I is an essential ideal in its multiplier algebra $M(I)$

Every C^*-algebra I is an essential ideal in its multiplier algebra $M(I)$.

3.1.8. Theorem. *Let I be a closed ideal in a C^*-algebra A. Then there is a unique $*$-homomorphism $\varphi: A \to M(I)$ extending the inclusion $I \to M(I)$. Moreover, φ is injective if I is essential in A.*

Proof. We have seen above that the inclusion map $I \to M(I)$ admits a $*$-homomorphic extension $\varphi: A \to M(I)$. Suppose that $\psi: A \to M(I)$ is another such extension. If $a \in A$ and $b \in I$, then $\varphi(a)b = \varphi(ab) = ab = \psi(ab) = \psi(a)b$. Hence, $(\varphi(a) - \psi(a))I = 0$, so $\varphi(a) = \psi(a)$, since I is essential in $M(I)$. Thus, $\varphi = \psi$.

Suppose now that I is essential in A and let $a \in \ker(\varphi)$. Then $aI = L_a(I) = 0$, so $a = 0$. Thus, φ is injective. $\square$

(2) Let $\lim_n P_n u = u$ $\forall u \in F(H)$. Assume that $v \in K(H)$. Given $\varepsilon > 0$

So $\exists u \in F(H)$; $\|u - v\| < \frac{\varepsilon}{3}$ (since $\overline{F(H)} = K(H)$).

$$\|P_n v - v\| \leq \underbrace{\|P_n\|}_{\leq 1} \underbrace{\|v - u\|}_{< \frac{\varepsilon}{3}} + \underbrace{\|P_n u - u\|}_{< \frac{\varepsilon}{3} \; \exists N \forall n \geq N} + \underbrace{\|v - u\|}_{< \frac{\varepsilon}{3}}$$

So $\forall n \geq N$; $\|P_n v - v\| < \varepsilon$. Hence $\lim_n P_n v = v$. $\square$

Fourier Series $\sum_{n=1}^{\infty} \lambda_n e_n = x$

(3) $\lim_{n \to \infty} P_n x = x$ $\left(P_n(H) = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n \right)$ since $\lim_{n\to\infty} P_n x = \lim_{n\to\infty} P_n \sum_{j=1}^{\infty} \lambda_j e_j = \lim_{n\to\infty} \sum_{j=1}^{n} \lambda_j e_j$

$\Delta = \{ \underbrace{\{\text{[illegible]}\}}_{A_{\{\text{[illegible]}\}}}, \underbrace{\emptyset}_{A_\emptyset}, \underbrace{\mathbb{N}}_{A_{\mathbb{N}}}, \underbrace{\{1, 2, \text{[illegible]}\}}_{A_{\{1,2,\text{[illegible]}\}}} \} = \{A_\alpha\}_{\alpha \in \Delta}$

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$\|a\| \leq 1$, $a = a^*$ $\Rightarrow$ $\sup|t| \leq 1 \Rightarrow t \leq |t| \leq \sup|t| \leq 1 \Rightarrow a \leq 1$.

$\mathcal{B}$ is a closed $*$-subalg of $\mathcal{A}$ $\Rightarrow$ $\mathcal{B}$ is $*$-alg, $\mathcal{B}$ is closed $\Rightarrow$ $\mathcal{B}$ is Banach $\Rightarrow$ C^*-alg

$\|b^*b\| = \|b\|^2$ (since $b \in \mathcal{A}$)

(4) $\|a^* + I\| = \inf\limits_{b \in I} \|a^* + b\| = \inf\limits_{\cancel{b \in I}\; b^* \in I} \|a + b^*\| = \inf\limits_{c \in I} \|a + c\| = \|a + I\|$

$$\boxed{\text{If } M \text{ \& } \tfrac{X}{M} \text{ are Ban, the so is } X.}$$

(5) $\tau_\alpha \xrightarrow{w^*\text{-top}} \tau \Rightarrow \tau_\alpha(b) \to \tau(b) \Rightarrow \underbrace{(\tau_\alpha \circ \varphi)(a)}_{\tau_\alpha(\varphi(a))} \to \underbrace{(\tau \circ \varphi)(a)}_{\tau(\varphi(a))} \Rightarrow \tau_\alpha \circ \varphi \xrightarrow{w^*} \tau \circ \varphi$

$\tau_\alpha \in \Omega(\mathcal{B})$, $\forall b \in \mathcal{B}$, $\tau\circ\varphi \in \Omega(\mathcal{A}) \subseteq \mathcal{A}'$

(6) $\varphi'(\Omega(\mathcal{B}))$ is w^*-Compact $\Rightarrow \varphi'(\Omega(\mathcal{B}))$ w^*-closed $\Rightarrow \varphi'(\Omega(\mathcal{B})) \subseteq \Omega(A)$ $\xrightarrow{\text{norm-Compact}}$ $\varphi'(\Omega(\mathcal{B}))$ norm-Compact

w^*-top $\subseteq$ norm-top

is norm-closed

(7) $\tau(\varphi(a)) = 0 \;\forall \tau \Rightarrow \widehat{\varphi(a)}(\tau) = 0 \;\forall \tau \Rightarrow \widehat{\varphi(a)} = 0 \Rightarrow \varphi(a) = 0$

$\|\varphi(a)\| = \|\widehat{\varphi(a)}\|$

(8) $\underbrace{I \overset{ess}{\trianglelefteq} A}_{aI = 0 \Rightarrow a = 0} \iff \forall J \trianglelefteq A,\ I \cap J \neq \{0\}$

$J \neq \{0\}$

$(\Leftarrow)$ $a \in A$, $aI = \{0\} \Rightarrow (\overline{AaA})I \subseteq \overline{A\underbrace{aI}_{0}} = \{0\} \Rightarrow \overline{AaA} = \{0\} \Rightarrow u_\lambda a u_\lambda = 0$ $\forall \lambda$

$(\overline{AaA})I = \overline{AaA} \cap I$, where $\overline{AaA}$ is an ideal of A.

$\Rightarrow \lim\limits_\lambda u_\lambda a u_\lambda = 0$

$$\|u_\lambda a u_\lambda - a\| \le \underbrace{\|u_\lambda a u_\lambda - a u_\lambda\|}_{\le \|u_\lambda a - a\| \to 0} + \|a u_\lambda - a\| \to 0$$

$a = \lim\limits_\lambda u_\lambda a u_\lambda = 0$

$(\Rightarrow)$ $I \overset{\text{ess}}{\triangleleft} A$ ($aI = 0 \Rightarrow a = 0$). Let $J \triangleleft A$, $J \neq \{0\}$. We shall prove that $\{0\} \neq I \cap J$.

Since $J \neq \{0\}$, $\exists\, b \in J$; $b \neq 0$. If $I \cap J (= IJ) = \{0\}$, then

$IJ = 0$. So $bI = 0$. Hence $b = 0$ ※. $\square$

Hereditary C*-Subalgebras

A C*-subalgebra B of a C*-algebra A is said to be *hereditary* if for $a \in A^+$ and $b \in B^+$ the inequality $a \leq b$ implies $a \in B$. Obviously, 0 and A are hereditary C*-subalgebras of A, and any intersection of hereditary C*-subalgebras is one also. The hereditary C*-subalgebra *generated* by a subset S of A is the smallest hereditary C*-subalgebra of A containing S. موروثی

3.2.1. *Example.* If p is a projection in a C*-algebra A, the C*-subalgebra pAp is hereditary. For, assuming $0 \leq b \leq pap$, then $0 \leq (1-p)b(1-p) \leq (1-p)pap(1-p) = 0$, so $(1-p)b(1-p) = 0$. Hence, $\|b^{1/2}(1-p)\|^2 = \|(1-p)b(1-p)\| = 0$, so $b(1-p) = 0$. Therefore, $b = pbp \in pAp$.

$p^2 - p = 0$

$b^{\frac{1}{2}}(1-p) = 0 \to b = bp \Rightarrow b = b^* = pb \Rightarrow b = pb = p(bp)$

The correspondence between hereditary C*-subalgebras and closed left ideals in the following theorem is very useful.

3.2.1. Theorem. *Let A be a C*-algebra.*

(1) *If L is a closed left ideal in A, then $L \cap L^*$ is a hereditary C*-subalgebra of A. The map $L \mapsto L \cap L^*$ is a bijection from the set of closed left ideals of A onto the set of hereditary C*-subalgebras of A.*

(2) *If L_1, L_2 are closed left ideals of A, then $L_1 \subseteq L_2$ if and only if $L_1 \cap L_1^* \subseteq L_2 \cap L_2^*$.*

(3) *If B is a hereditary C*-subalgebra of A, then the set*

$$L(B) = \{a \in A \mid a^*a \in B\}$$

is the unique closed left ideal of A corresponding to B.

Proof. If L is a closed left ideal of A, then clearly $B = L \cap L^*$ is a C*-subalgebra of A. Suppose that $a \in A^+$ and $b \in B^+$ and $a \leq b$. By Theorem 3.1.2 there is an increasing net $(u_\lambda)_{\lambda \in \Lambda}$ in the closed unit ball of L^+ such that $\lim_\lambda bu_\lambda = b$. Now $0 \leq (1-u_\lambda)a(1-u_\lambda) \leq (1-u_\lambda)b(1-u_\lambda)$, so $\|a^{1/2} - a^{1/2}u_\lambda\|^2 = \|(1-u_\lambda)a(1-u_\lambda)\| \leq \|(1-u_\lambda)b(1-u_\lambda)\| \leq \|b - bu_\lambda\|$. Hence, $a^{1/2} = \lim_\lambda a^{1/2}u_\lambda$, so $a^{1/2} \in L$, since $u_\lambda \in L$ ($\lambda \in \Lambda$). Therefore, $a \in B$, so B is hereditary in A.

$a = a^{\frac{1}{2}}a^{\frac{1}{2}}$

Suppose now that L_1, L_2 are closed left ideals of A. It is evident that $L_1 \subseteq L_2 \Rightarrow L_1 \cap L_1^* \subseteq L_2 \cap L_2^*$. To show the reverse implication, suppose that $L_1 \cap L_1^* \subseteq L_2 \cap L_2^*$ and let $(u_\lambda)_{\lambda \in \Lambda}$ be an approximate unit for $L_1 \cap L_1^*$, and $a \in L_1$. Then $\lim_\lambda \|a - au_\lambda\|^2 = \lim_\lambda \|(1-u_\lambda)a^*a(1-u_\lambda)\| \leq \lim_\lambda \|a^*a(1-u_\lambda)\| = 0$, since $a^*a \in L_1 \cap L_1^*$. It follows that $\lim_\lambda au_\lambda = a$. Therefore, $a \in L_2$, since $u_\lambda \in L_1 \cap L_1^* \subseteq L_2$. This proves Condition (2).

Now let B be a hereditary C*-subalgebra of A and let $L = L(B)$. If $a, b \in L$, $(a+b)^*(a+b) \leq (a+b)^*(a+b) + (a-b)^*(a-b) = 2a^*a + 2b^*b \in B$

$a, b \in L$, $(a+b)^*(a+b) \le (a+b)^*(a+b) + (a-b)^*(a-b) = 2a^*a + 2b^*b \in B$, so $a + b \in L$. If $a \in A$ and $b \in L$, then $(ab)^*(ab) = b^*a^*ab \le \|a\|^2 b^*b \in B$, so $ab \in L$. Similarly, L is closed under scalar multiplication. Thus, L is a left ideal, and it is obviously closed, since B is closed. If $b \in B$, then $b^*b \in B$, so $b \in L$. Hence, $B \subseteq L \cap L^*$. If $0 \le b \in L \cap L^*$, then $b^2 \in B$, so $b \in B$, and therefore $L \cap L^* \subseteq B$. Hence, $L \cap L^* = B$. This proves Condition (3), and Condition (1) follows directly. $\square$

b^*b

closed B

$\left\{ \begin{matrix} a \in L \Rightarrow a_n^* a_n \in B \\ a_n \to a \quad a_n^* a_n \to a^* a \end{matrix} \right\} \Rightarrow a^*a \in B \Downarrow a \in L$

$(b^2)^{\frac{1}{2}}$

3.2.2. Theorem. *Let B be a C^*-subalgebra of a C^*-algebra A. Then B is hereditary in A if and only if $bab' \in B$ for all $b, b' \in B$ and $a \in A$.*

Proof. If B is hereditary, then by Theorem 3.2.1 $B = L \cap L^*$ for some closed left ideal L of A. Hence, if $b, b' \in B$ and $a \in A$, we have $b(ab') \in L$ and $b'^*(a^*b^*) \in L$, so $bab' \in B$.

$\boxed{x \in L^* \Leftrightarrow x^* \in L}$ $\in L$

Conversely, suppose B has the property that $bab' \in B$ for all $b, b' \in B$ and $a \in A$. If $(u_\lambda)_{\lambda \in \Lambda}$ is an approximate unit for B and $a \in A^+$, $b \in B^+$, and $a \le b$, then $0 \le (1 - u_\lambda)a(1 - u_\lambda) \le (1 - u_\lambda)b(1 - u_\lambda)$, and therefore $\|a^{1/2} - a^{1/2}u_\lambda\| \le \|b^{1/2} - b^{1/2}u_\lambda\|$. Since $b^{1/2} = \lim_\lambda b^{1/2}u_\lambda$, therefore, $a^{1/2} = \lim_\lambda a^{1/2}u_\lambda$, so $a = \lim_\lambda u_\lambda a u_\lambda \in B$. Thus, B is hereditary. $\square$

$a = \lim a u_\lambda$

$\|a - u_\lambda a u_\lambda\| = \underbrace{\|a - a u_\lambda\|}_{\to 0} + \underbrace{\|a u_\lambda - u_\lambda a u_\lambda\|}_{\le \|a - u_\lambda a\| \to 0}$

The following corollary is obvious.

3.2.3. Corollary. *Every closed ideal of a C^*-algebra is a hereditary C^*-subalgebra.*

$B = L \cap \underbrace{L^*}_{=L} = L$ $\quad \left\{ \begin{matrix} b'ab \in L \ \forall b, b' \in L \ \forall a \in A \\ L \text{ is a } C^*\text{-subalg of } A. \end{matrix} \right.$

3.2.4. Corollary. *If A is a C^*-algebra and $a \in A^+$, then $(aAa)^-$ is the hereditary C^*-subalgebra of A generated by a.*

$\bigcap_{a \in B \subseteq A \text{ hereditary}} B$

Proof. The only thing we show is that $a \in (aAa)^-$, because the rest is routine. If $(u_\lambda)_{\lambda \in \Lambda}$ is an approximate unit for A, then $a^2 = \lim_\lambda a u_\lambda a$, so $a^2 \in (aAa)^-$. Since $(aAa)^-$ is a C*-algebra, $a = \sqrt{a^2} \in (aAa)^-$ also. $\square$

(1)

In the separable case, every hereditary C*-subalgebra is of the form in the preceding corollary:

3.2.5. Theorem. *Suppose that B is a separable hereditary C^*-subalgebra of a C^*-algebra A. Then there is a positive element $a \in B$ such that $B = (aAa)^-$.*

Proof. Since B is a separable C*-algebra, it admits a sequential approximate unit, $(u_n)_{n=1}^\infty$ say (*cf.* Remark 3.1.1). Set $a = \sum_{n=1}^\infty u_n/2^n$. Then $a \in B^+$, so B contains $(aAa)^-$. Since $u_n/2^n \le a$, and $(aAa)^-$ is hereditary by Corollary 3.2.4, therefore $u_n \in (aAa)^-$. If $b \in B$, then

$b = \lim_{n\to\infty} u_n b u_n$, and $u_n b u_n \in (aAa)^-$, so $b \in (aAa)^-$. This shows that $B = (aAa)^-$. □

If the separability condition is dropped in Theorem 3.2.5, the result may fail. To see this let H be a Hilbert space, and suppose that u is a positive element of $B(H)$ such that $K(H) = (uB(H)u)^-$. If $x \in H$, then $x \otimes x = \lim_{n\to\infty} uv_n u$ for a sequence (v_n) in $B(H)$, and therefore x is in the closure of the range of u. This shows that $H = (u(H))^-$, and therefore H is separable, since the range of a compact operator is separable (*cf.* Remark 1.4.1). Thus, if H is a non-separable Hilbert space, then the hereditary C*-subalgebra $K(H)$ of $B(H)$ is not of the form $(uB(H)u)^-$ for any $u \in B(H)^+$.

3.2.6. Theorem. *Suppose that B is a hereditary C*-subalgebra of a unital C*-algebra A, and let $a \in A^+$. If for each $\varepsilon > 0$ there exists $b \in B^+$ such that $a \leq b + \varepsilon$, then $a \in B$.*

$\overset{\shortparallel}{(b^{\frac{1}{2}})^2}$

Proof. Let $\varepsilon > 0$. By the hypothesis there exists $b_\varepsilon \in B^+$ such that $a \leq b_\varepsilon^2 + \varepsilon^2$, so $a \leq (b_\varepsilon + \varepsilon)^2$. Hence, $(b_\varepsilon + \varepsilon)^{-1} a (b_\varepsilon + \varepsilon)^{-1} \leq 1$, and therefore $\|(b_\varepsilon + \varepsilon)^{-1} a (b_\varepsilon + \varepsilon)^{-1}\| \leq 1$. Using the fact that $1 - b_\varepsilon(b_\varepsilon + \varepsilon)^{-1} = \varepsilon(b_\varepsilon + \varepsilon)^{-1}$, we get

$$\begin{aligned}\|a^{1/2} - a^{1/2} b_\varepsilon (b_\varepsilon + \varepsilon)^{-1}\|^2 &= \varepsilon^2 \|a^{1/2}(b_\varepsilon + \varepsilon)^{-1}\|^2 \\ &= \varepsilon^2 \|(b_\varepsilon + \varepsilon)^{-1} a (b_\varepsilon + \varepsilon)^{-1}\| \\ &\leq \varepsilon^2.\end{aligned}$$

If $b \geq 0$, then $b + \varepsilon$ is invertible.

Hence,

$$a^{1/2} = \lim_{\varepsilon \to 0} a^{1/2} b_\varepsilon (b_\varepsilon + \varepsilon)^{-1},$$

Solution (1) $\left\{ \begin{array}{l} b \geq 0 \longleftrightarrow t \geq 0 \\ b+\varepsilon \text{ is inv.} \longleftarrow t+\varepsilon > 0 \text{ invertible} \end{array} \right.$

and therefore also

$$a^{1/2} = \lim_{\varepsilon \to 0} (b_\varepsilon + \varepsilon)^{-1} b_\varepsilon a^{1/2},$$

by taking adjoints. Thus,

$$a = \lim_{\varepsilon \to 0} (b_\varepsilon + \varepsilon)^{-1} b_\varepsilon a b_\varepsilon (b_\varepsilon + \varepsilon)^{-1}.$$

Solution (2) $\underset{\Downarrow \atop b+\varepsilon \text{ is inv.}}{0 \notin sp(b+\varepsilon)} = sp(b) + \varepsilon \subseteq [\varepsilon, \infty)$

Now $b_\varepsilon(b_\varepsilon + \varepsilon)^{-1} \in B$, and therefore $(b_\varepsilon + \varepsilon)^{-1} b_\varepsilon a b_\varepsilon (b_\varepsilon + \varepsilon)^{-1} \in B$, since B is hereditary in A. It follows that $a \in B$. □

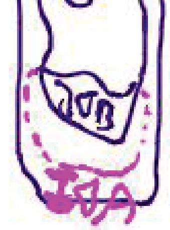
We briefly indicate the connection between the ideal structure of a C*-algebra and its hereditary C*-subalgebras in the following results, but we shall defer to Chapter 5 a fuller consideration of this matter.

3.2.7. Theorem. *Let B be a hereditary C*-subalgebra of a C*-algebra A, and let J be a closed ideal of B. Then there exists a closed ideal I of A such that $J = B \cap I$.*

> closed linear span

such that $J = B \cap I$. closed linear span

Proof. Let $I = AJA$. Then I is a closed ideal of A. Since J is a C*-algebra, $J \overset{2}{=} J^3$, and since B is hereditary in A, we have $B \cap I = BIB$ (both of these assertions follow easily from the existence of approximate



3.2.2. Example. If H is a Hilbert space, then the C*-algebra $K(H)$ is simple. For if I is a closed non-zero ideal of $K(H)$, it is also an ideal of $B(H)$ (*cf.* Remark 3.1.2), so I contains the ideal $F(H)$ by Theorem 2.4.7, and therefore $I = K(H)$.

It is not true that C*-subalgebras of simple C*-algebras are necessarily simple. For instance, if p, q are finite-rank non-zero projections on a Hilbert space H such that $pq = 0$, then $A = \mathbf{C}p + \mathbf{C}q$ is a non-simple C*-subalgebra of the simple C*-algebra $K(H)$ (the closed ideal $Ap = \mathbf{C}p$ of A is non-trivial).

3.2.8. Theorem. *Every hereditary C*-subalgebra of a simple C*-algebra is simple.*

Proof. Let B be a hereditary C*-subalgebra of a simple C*-algebra A. If J is a closed ideal of B, then $J = B \cap I$ for some closed ideal I of A by Theorem 3.2.7. Simplicity of A implies that $I = 0$ or A, and therefore $J = 0$ or B. □

3.3. Positive Linear Functionals

For abelian C*-algebras we were able completely to determine the structure of the algebra in terms of the character space, that is, in terms of the one-dimensional representations. For the non-abelian case this is quite inadequate, and we have to look at representations of arbitrary dimension. There is a deep inter-relationship between the representations and the positive linear functionals of a C*-algebra. Representations will be defined and some aspects of this inter-relationship investigated in the next section. In this section we establish the basic properties of positive linear functionals.

If $\varphi: A \to B$ is a linear map between C*-algebras, it is said to be *positive* if $\varphi(A^+) \subseteq \cancel{\varphi(}B^+\cancel{)}$. In this case $\varphi(A_{sa}) \subseteq B_{sa}$, and the restriction map $\varphi: A_{sa} \to B_{sa}$ is increasing.

Every $*$-homomorphism is positive.

3.3.1. Example. Let $A = C(\mathbf{T})$ and let m be normalised arc length measure on $\mathbf{T}$. Then the linear functional

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}$$

$$C(\mathbf{T}) \to \mathbf{C}, \quad f \mapsto \int f\, dm,$$

is positive (and not a homomorphism).

is positive (and not a homomorphism).

$A \geq 0 \Rightarrow \langle Ax, x\rangle \geq 0 \ \forall x$

3.3.2. *Example.* Let $A = M_n(\mathbf{C})$. The linear functional

$\Rightarrow \forall j;\ \underbrace{\langle Ae_j, e_j\rangle}_{a_{jj}} \geq 0$

$$\mathrm{tr}\colon A \to \mathbf{C},\quad (a_{ij}) \mapsto \sum_{i=1}^{n} a_{ii},$$

is positive. It is called the *trace*. Observe that there are no non-zero $*$-homomorphisms from $M_n(\mathbf{C})$ to $\mathbf{C}$ if $n > 1$.

Let A be a C*-algebra and τ a positive linear functional on A. Then the function

Cauchy-Schwarz ineq.

$$A^2 \to \mathbf{C},\ (a, b) \mapsto \tau(b^*a),$$

(3)

is a positive sesquilinear form on A. Hence, $\tau(b^*a) = \tau(a^*b)^-$ and $|\tau(b^*a)| \leq \tau(a^*a)^{1/2}\tau(b^*b)^{1/2}$. Moreover, the function $a \mapsto \tau(a^*a)^{1/2}$ is a semi-norm on A.

$a \geq 0,\ \|a\| \leq 1$

Suppose now only that τ is a linear functional on A and that M is an element of $\mathbf{R}^+$ such that $|\tau(a)| \leq M$ for all positive elements of the closed unit ball of A. Then τ is bounded with norm $\|\tau\| \leq 4M$. We show this: First suppose that a is a hermitian element of A such that $\|a\| \leq 1$. Then a^+, a^- are positive elements of the closed unit ball of A, and therefore $|\tau(a)| = |\tau(a^+) - \tau(a^-)| \leq 2M$. Now suppose that a is an arbitrary element of the closed unit ball of A, so $a = b + ic$ where b, c are its real and imaginary parts, and $\|b\|, \|c\| \leq 1$. Then $|\tau(a)| = |\tau(b) + i\tau(c)| \leq 4M$.

$\leq |\tau(a^+)| + |\tau(a^-)|$

3.3.1. Theorem. *If τ is a positive linear functional on a C*-algebra A, then it is bounded.*

Proof. If τ is not bounded, then by the preceding remarks $\sup_{a \in S} \tau(a) = +\infty$, where S is the set of all positive elements of A of norm not greater then 1. Hence, there is a sequence (a_n) in S such that $2^n \leq \tau(a_n)$ for all $n \in \mathbf{N}$. Set $a = \sum_{n=0}^{\infty} a_n/2^n$, so $a \in A^+$. Now $1 \leq \tau(a_n/2^n)$ and therefore $N \leq \sum_{n=0}^{N-1} \tau(a_n/2^n) = \tau(\sum_{n=0}^{N-1} a_n/2^n) \leq \tau(a)$. Hence, $\tau(a)$ is an upper bound for the set $\mathbf{N}$, which is impossible. This shows that τ is bounded. □

3.3.2. Theorem. *If τ is a positive linear functional on a C*-algebra A, then $\tau(a^*) = \tau(a)^-$ and $|\tau(a)|^2 \leq \|\tau\|\tau(a^*a)$ for all $a \in A$.*

Proof. Let $(u_\lambda)_{\lambda \in \Lambda}$ be an approximate unit for A. Then

$$\tau(a^*) = \lim_{\lambda} \tau(a^*u_\lambda) = \lim_{\lambda} \tau(u_\lambda a)^- = \tau(a)^-.$$

Also, $|\tau(a)|^2 = \lim_\lambda |\tau(u_\lambda a)|^2 \le \sup_\lambda \tau(u_\lambda^2)\tau(a^*a) \le \|\tau\|\tau(a^*a)$. □

$|\tau(u_\lambda a)|^2 \le \tau(u_\lambda^2)\tau(a^*a) \le \tau(a^*a)\sup_\lambda \tau(u_\lambda)$

$a \ge 0$

① $\overline{aAa}$ is a hereditary C^*-subalg of A:

$\left.\begin{array}{l} axa \in aAa \\ aya \in aAa \end{array}\right\} \Rightarrow (axa)(aya) = a\underbrace{(xa^2y)}_{\in A}a \in aAa, \ldots \Rightarrow aAa$ is subalg

$\Rightarrow \overline{aAa}$ subalg

$x \in \overline{aAa} \Rightarrow ax_na \to x \Rightarrow (ax_na)^* \to x^* \Rightarrow \underbrace{ax_n^*a}_{aAa} \to x^* \Rightarrow x^* \in \overline{aAa}$

$\therefore \overline{aAa}$ is a C^*-alg

$\left.\begin{array}{l} x, x' \in \overline{aAa} \\ b \in A \end{array}\right. \Rightarrow ax_na \to x,\ ax_n'a \to x' \Big\} \Rightarrow \underbrace{\overbrace{ax_na\, b\, ax_n'a}}_{\in aAa} \longrightarrow xbx' \in \overline{aAa}$

$a \in \overline{aAa}$, since $a \ge 0$ $\quad au_\lambda \to a \Rightarrow \underbrace{au_\lambda a}_{aAa} \to a^2 \Rightarrow a^2 \in \overline{aAa} \Rightarrow a \in \overline{aAa}$ ($\hookrightarrow C^*$-alg)

② $J = J^3$

J^3 = closed linear span $\{abc \mid a, b, c \in J\}$

$\text{span}\{abc \mid a,b,c \in J\} \subseteq J \Leftarrow \sum \lambda_j \underbrace{\overset{\in B}{\hat{a}_j} \overset{\in B}{b_j} c_j}_{\in J \subseteq B}$

$\Downarrow$

$\overline{\text{span} \ldots} \subseteq \overline{J} = J$

$\underbrace{\qquad\qquad}_{J^3}$

$a \in J \Rightarrow a = \lim \underbrace{u_\lambda a u_\lambda}_{\in J^3} \in J^3$. ✓

$\{u_\lambda\}$ is approx unit for J

(3) $\langle \cdot , \cdot \rangle : A \times A \to \mathbb{C}$ is sesquilinear

linear, conjugate linear

If $\langle a, a \rangle \geq 0$, $\forall a$, then $\langle \cdot , \cdot \rangle$ is called positive.

$$\langle a, b \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \langle a + i^k b, a + i^k b \rangle \Rightarrow \overline{\langle b, a \rangle} = \langle a, b \rangle$$

$$\Downarrow$$

$$\overline{\tau(a^* b)} = \tau(b^* a)$$

3.3.3. Theorem. *Let τ be a bounded linear functional on a C^*-algebra A. The following conditions are equivalent:*

(1) *τ is positive.*
(2) *For each approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ of A, $\|\tau\| = \lim_\lambda \tau(u_\lambda)$.*
(3) *For some approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ of A, $\|\tau\| = \lim_\lambda \tau(u_\lambda)$.*

we can replace τ by $\frac{\tau}{\|\tau\|}$.

$u_\lambda \le u_\mu$
$\Downarrow$
$\tau(u_\lambda) \le \tau(u_\mu)$

Proof. We may suppose that $\|\tau\| = 1$. First we show the implication (1) $\Rightarrow$ (2) holds. Suppose that τ is positive, and let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit of A. Then $(\tau(u_\lambda)_\lambda)_{\lambda\in\Lambda}$ is an increasing net in $\mathbf{R}$, so it converges to its supremum, which is obviously not greater than 1. Thus, $\lim_\lambda \tau(u_\lambda) \le 1$. Now suppose that $a \in A$ and $\|a\| \le 1$. Then $|\tau(u_\lambda a)|^2 \le \tau(u_\lambda^2)\tau(a^*a) \overset{(i)}{\le} \tau(u_\lambda)\tau(a^*a) \le \lim_\lambda \tau(u_\lambda)$, so $|\tau(a)|^2 \le \lim_\lambda \tau(u_\lambda)$. Hence, $1 \le \lim_\lambda \tau(u_\lambda)$. Therefore, $1 = \lim_\lambda \tau(u_\lambda)$, so (1) $\Rightarrow$ (2).

(i) $u_\lambda \le 1$ in $\tilde{A}$ $\xRightarrow{\text{multiply by } u_\lambda^{\frac{1}{2}}}$ $u_\lambda^2 \le u_\lambda$ in A

$\sup_{\|a\|\le 1} |\tau(a)|^2 = \|\tau\|^2 = 1$

That (2) $\Rightarrow$ (3) is obvious.

Now we show that (3) $\Rightarrow$ (1). Suppose that $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit such that $1 = \lim_\lambda \tau(u_\lambda)$. Let a be a self-adjoint element of A such that $\|a\| \le 1$ and write $\tau(a) = \alpha + i\beta$ where α, β are real numbers. To show that $\tau(a) \in \mathbf{R}$, we may suppose that $\beta \le 0$. If n is a positive integer, then

If $\beta > 0$, then $-a = -\alpha + i(-\beta)$, $-\beta \le 0$
$\|-a\| = \|a\| \le 1$, $-a \in A_{sa}$
So we can replace a by $-a$.

قسمت (ثابت= تابع

$$\begin{aligned}\|a - inu_\lambda\|^2 &= \|(a + inu_\lambda)(a - inu_\lambda)\| \\ &= \|a^2 + n^2u_\lambda^2 - in(au_\lambda - u_\lambda a)\| \\ &\le 1 + n^2 + n\|au_\lambda - u_\lambda a\|,\end{aligned}$$

so

$$|\tau(a - inu_\lambda)|^2 \le 1 + n^2 + n\|au_\lambda - u_\lambda a\|.$$

However, $\lim_\lambda \tau(a - inu_\lambda) = \underbrace{\tau(a)}_{\alpha+i\beta} - in$, and $\lim_\lambda au_\lambda - u_\lambda a = 0$, so in the limit as $\lambda \to \infty$ we get

$$\alpha^2+\beta^2-2n\beta+n^2 = |\alpha + \underbrace{i\beta - in}_{i(\beta-n)}|^2 \le 1 + n^2.$$

The left-hand side of this inequality is $\alpha^2 + \beta^2 - 2n\beta + n^2$, so if we cancel and rearrange we get

$$-2n\beta \le 1 - \beta^2 - \alpha^2.$$

$\|c - d\| \le \max\{\|c\|, \|d\|\}$
$c \ge 0,\ d \ge 0$

Since β is not positive and this inequality holds for all positive integers n, β must be zero. Therefore, $\tau(a)$ is real if a is hermitian.

Now suppose that a is positive and $\|a\| \le 1$. Then $u_\lambda - a$ is hermitian and $\|u_\lambda - a\| \le 1$, so $\tau(u_\lambda - a) \le 1$. But then $1 - \tau(a) = \lim_\lambda \tau(u_\lambda - a) \le 1$, and therefore $\tau(a) \ge 0$. Thus, τ is positive and we have shown (3) $\Rightarrow$ (1).□

3.3.4. Corollary. *If τ is a bounded linear functional on a unital C^*-algebra, then τ is positive if and only if $\tau(1) = \|\tau\|$.*

Proof. The sequence which is constantly 1 is an approximate unit for the C^*-algebra. Apply Theorem 3.3.3. □

C^*-algebra. Apply Theorem 3.3.3. □

3.3.5. Corollary. *If τ, τ' are $\boxed{\text{positive}}$ linear functionals on a C^*-algebra, then $\|\tau + \tau'\| = \|\tau\| + \|\tau'\|$.* $=$ $\|\tau+\tau'\| \le \|\tau\| + \|\tau'\|$

Proof. If $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for the algebra, then $\|\tau+\tau'\| = \lim_\lambda(\tau+\tau')(u_\lambda) = \lim_\lambda \tau(u_\lambda) + \lim_\lambda \tau'(u_\lambda) = \|\tau\| + \|\tau'\|$. □

A *state* on a C*-algebra A is a positive linear functional on A of norm one. We denote by $S(A)$ the set of states of A.

compact

3.3.6. Theorem. *If a is a normal element of a non-zero C^*-algebra A, then there is a state τ of A such that $\boxed{\|a\| = |\tau(a)|}$.*

$B \to C(\Omega(B))$

$a \mapsto \hat{a}$

Proof. We may assume that $a \neq 0$. Let B be the C*-algebra generated by 1 and a in $\tilde{A}$. Since B is abelian and $\hat{a}$ is continuous on the compact space $\Omega(B)$, there is a character τ_2 on B such that $\|a\| = \|\hat{a}\|_\infty = |\tau_2(a)|$. By the Hahn–Banach theorem, there is a bounded linear functional τ_1 on $\tilde{A}$ extending τ_2 and preserving the norm, so $\|\tau_1\| = 1$. Since $\tau_1(1) = \tau_2(1) = 1$, τ_1 is positive by Corollary 3.3.4. If τ denotes the restriction of τ_1 to A, then τ is a positive linear functional on A such that $\|a\| = |\tau(a)|$. Hence, $\|\tau\|\|a\| \ge |\tau(a)| = \|a\|$, so $\|\tau\| \ge 1$, and the reverse inequality is obvious. Therefore, τ is a state of A. □

$\|\hat{a}\| = r(a)$ $\|$ $\|a\|$

$=\|\tau_2\|$

τ_1 is character

character

$\|\tau\| \le \|\tau_1\| = \|\tau_2\| \overset{\downarrow}{=} 1$ $= |\tau_2(a)| = |\tau_1(a)|$

3.3.7. Theorem. *Suppose that τ is a positive linear functional on a C^*-algebra A.*

(1) *For each $a \in A$, $\tau(a^*a) = 0$ if and only if $\tau(ba) = 0$ for all $b \in A$.*
(2) *The inequality*

$$\tau(b^*a^*ab) \le \|a^*a\|\tau(b^*b)$$

holds for all $a, b \in A$.

$0 \le |\tau(ba)| \le \tau(a^*a)\tau(bb^*)$

Proof. Condition (1) follows from the Cauchy–Schwarz inequality.

To show Condition (2), we may suppose, using Condition (1), that $\tau(b^*b) > 0$. The function

$$\rho: A \to \mathbf{C}, \ c \mapsto \tau(b^*cb)/\tau(b^*b),$$

is positive and linear, so if $(u_\lambda)_{\lambda\in\Lambda}$ is any approximate unit for A, then

$$\|\rho\| = \lim_\lambda \rho(u_\lambda) = \lim_\lambda \tau(b^*u_\lambda b)/\tau(b^*b) = \tau(b^*b)/\tau(b^*b) = 1.$$

Hence, $\rho(a^*a) \le \|a^*a\|$, and therefore $\tau(b^*a^*ab) \le \|a^*a\|\tau(b^*b)$. □

We turn now to the problem of extending positive linear functionals.

3.3.8. Theorem. *Let B be a C^*-subalgebra of a C^*-algebra A, and suppose that τ is a positive linear functional on B. Then there is a positive* pose that τ is a positive linear functional on B. Then there is a positive *linear functional τ' on A extending τ such that $\|\tau'\| = \|\tau\|$.*

Proof. Suppose first that $A = \tilde{B}$. Define a linear functional τ' on A by setting $\tau'(b + \lambda) = \tau(b) + \lambda\|\tau\|$ ($b \in B$, $\lambda \in \mathbf{C}$). Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for B. By Theorem 3.3.3, $\|\tau\| = \lim_\lambda \tau(u_\lambda)$. Now suppose that $b \in B$ and $\mu \in \mathbf{C}$. Then $|\tau'(b + \mu)| = |\lim_\lambda \tau(bu_\lambda) + \mu \lim_\lambda \tau(u_\lambda)| = |\lim_\lambda \tau((b + \mu)(u_\lambda))| \leq \sup_\lambda \|\tau\|\|(b + \mu)u_\lambda\| \leq \|\tau\|\|b + \mu\|$, since $\|u_\lambda\| \leq 1$. Hence, $\|\tau'\| \leq \|\tau\|$, and the reverse inequality is obvious. Thus, $\|\tau'\| = \|\tau\| = \tau'(1)$, so τ' is positive by Corollary 3.3.4. This proves the theorem in the case $A = \tilde{B}$.

$|\tau(b)| = |\tau'(b+0)| \leq \|\tau'\| \underbrace{\|b+0\|}_{\|b\|}$

$\tau: B \to \mathbb{C} \rightsquigarrow \tau': \tilde{B} \to \mathbb{C}$

$\tau'(b+\lambda) = \tau(b) + \lambda\|\tau\|$

$\tau'(1) = \tau(0) + 1\|\tau\| = \|\tau\|$

Now suppose that A is an arbitrary C*-algebra containing B as a C*-subalgebra. Replacing B and A by $\tilde{B}$ and $\tilde{A}$ if necessary, we may suppose that A has a unit 1 which lies in B. By the Hahn–Banach theorem, there is a functional $\tau' \in A^*$ extending τ and of the same norm. Since $\tau'(1) = \tau(1) = \|\tau\| = \|\tau'\|$, it follows as before from Corollary 3.3.4 that τ' is positive. In fact $B \xrightarrow{\tau} \mathbb{C} \Rightarrow \tilde{B} \xrightarrow{\tau'} \mathbb{C} \Rightarrow \tilde{A} \xrightarrow{\tau''} \mathbb{C}$ ✓ □

In the case of hereditary C*-subalgebras, we can strengthen the above result—we can even write down an "expression" for τ':

3.3.9. Theorem. *Let B be a hereditary C*-subalgebra of a C*-algebra A. If τ is a positive linear functional on B, then there is a unique positive linear functional τ' on A extending τ and preserving the norm. Moreover, if $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for B, then*

$$\tau'(a) = \lim_\lambda \tau(u_\lambda a u_\lambda) \qquad (a \in A).$$

Proof. Of course we already have existence, so we only prove uniqueness. Let τ' be a positive linear functional on A extending τ and preserving the norm. We may in turn extend τ' in a norm-preserving fashion to a positive functional (also denoted τ') on $\tilde{A}$. Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for B. Then $\lim_\lambda \tau(u_\lambda) = \|\tau\| = \|\tau'\| = \tau'(1)$, so $\lim_\lambda \tau'(1 - u_\lambda) = 0$. Thus, for any element $a \in A$,

$$\begin{aligned}
|\tau'(a) - \tau(u_\lambda a u_\lambda)| &\leq |\tau'(a - u_\lambda a)| + |\tau'(u_\lambda a - u_\lambda a u_\lambda)| \\
&\leq \tau'((1 - u_\lambda)^2)^{1/2} \tau'(a^*a)^{1/2} \\
&\qquad + \tau'(a^* u_\lambda^2 a)^{1/2} \tau'((1 - u_\lambda)^2)^{1/2} \\
&\leq (\tau'(1 - u_\lambda))^{1/2} \tau'(a^*a)^{1/2} + \tau'(a^*a)^{1/2} (\tau'(1 - u_\lambda))^{1/2}.
\end{aligned}$$

Since $\lim_\lambda \tau'(1 - u_\lambda) = 0$, these inequalities imply $\lim_\lambda \tau(u_\lambda a u_\lambda) = \tau'(a)$.□

Let A be a C*-algebra. If τ is a bounded linear functional on A, then

$$\|\tau\| = \sup_{\|a\| \leq 1} |Re(\tau(a))|. \tag{1}$$

For if $a \in A$ and $\|a\| \leq 1$, then there is a number $\lambda \in \mathbf{T}$ such that $\lambda\tau(a) \in \mathbf{R}$, so $|\tau(a)| = |Re(\tau(\lambda a))| \leq \|\tau\|$, which implies Eq. (1).

$(\tau\rho)^* = \overline{\tau\rho(a^*)}$

If $\tau \in A^*$, we define $\tau^* \in A^*$ by setting $\boxed{\tau^*(a) = \tau(a^*)^-}$ for all $a \in A$. Note that $\tau^{**} = \tau$, $\|\tau^*\| = \|\tau\|$, and the map $\tau \mapsto \tau^*$ is conjugate-linear.

$= \overline{\rho(a^*)\tau(a^*)}$

We say a functional $\tau \in A^*$ is *self-adjoint* if $\tau = \tau^*$. For any bounded linear functional τ on A, there are unique self-adjoint bounded linear functionals τ_1 and τ_2 on A such that $\tau = \tau_1 + i\tau_2$ (take $\tau_1 = (\tau + \tau^*)/2$ and $\tau_2 = (\tau - \tau^*)/2i$).

$= \rho^*\tau^*(a)$

The condition $\tau = \tau^*$ is equivalent to $\tau(A_{sa}) \subseteq \mathbf{R}$, and therefore if τ is self-adjoint, the restriction $\tau' \colon A_{sa} \to \mathbf{R}$ of τ is a bounded real-linear functional. Moreover, $\|\tau\| = \|\tau'\|$; that is,

$\therefore (\tau\rho)^* = \rho^*\tau^*$

$$\|\tau\| = \sup_{\substack{a \in A_{sa} \\ \|a\| \leq 1}} |\tau(a)|.$$

For if $a \in A$, we have $\mathrm{Re}(\tau(a)) = \tau(\mathrm{Re}(a))$, so

$$\|\tau\| = \sup_{\|a\| \leq 1} |\,\mathrm{Re}(\tau(a))| \leq \sup_{\substack{b \in A_{sa} \\ \|b\| \leq 1}} |\tau(b)| \leq \|\tau\|.$$

We denote by A^*_{sa} the set of self-adjoint functionals in A^*, and by A^*_+ the set of positive functionals in A^*.

We adopt some temporary notation for the proof of the next theorem: If X is a real-linear Banach space, we denote its dual (over $\mathbf{R}$) by $X^\natural$.

The space A_{sa} is a real-linear Banach space and it is an easy exercise to verify that A^*_{sa} is a real-linear vector subspace of A^* and that the map $A^*_{sa} \to A^\natural_{sa}$, $\tau \mapsto \tau'$, is an isometric real-linear isomorphism. We shall use these observations in the proof of the following result.

3.3.10. Theorem (Jordan Decomposition). *Let τ be a self-adjoint bounded linear functional on a C*-algebra A. Then there exist positive linear functionals τ_+, τ_- on A such that $\tau = \tau_+ - \tau_-$ and $\|\tau\| = \|\tau_+\| + \|\tau_-\|$.*

3.4. The Gelfand–Naimark Representation

-Segal

In this section we introduce the important GNS construction and prove that every C*-algebra can be regarded as a C*-subalgebra of $B(H)$ for some Hilbert space H. It is partly due to this concrete realisation of the C*-algebras that their theory is so accessible in comparison with more general Banach algebras.

A *representation* of a C*-algebra A is a pair (H, φ) where H is a Hilbert space and $\varphi: A \to B(H)$ is a $*$-homomorphism. We say (H, φ) is *faithful* if φ is injective. $\equiv$ 1-1

If $(H_\lambda, \varphi_\lambda)_{\lambda \in \Lambda}$ is a family of representations of A, their *direct sum* is the representation (H, φ) got by setting $H = \oplus_\lambda H_\lambda$, and $\varphi(a)((x_\lambda)_\lambda) = (\varphi_\lambda(a)(x_\lambda))_\lambda$ for all $a \in A$ and all $(x_\lambda)_\lambda \in H$. It is readily verified that (H, φ) is indeed a representation of A. If for each non-zero element $a \in A$ there is an index λ such that $\varphi_\lambda(a) \neq 0$, then (H, φ) is faithful.

Recall now that if H is an inner product space (that is, a pre-Hilbert space), then there is a unique inner product on the Banach space completion $\hat{H}$ of H extending the inner product of H and having as its associated norm the norm of $\hat{H}$. We call $\hat{H}$ endowed with this inner product the *Hilbert space completion* of H.

With each positive linear functional, there is associated a representation. Suppose that τ is a positive linear functional on a C*-algebra A. Setting

$$N_\tau = \{a \in A \mid \tau(a^* a) = 0\}, \qquad |\tau(a^*b^*ba)| \le \|b^*b\|\tau(a^*a)$$

it is easy to check (using Theorem 3.3.7) that N_τ is a closed left ideal of A and that the map

$$A/N_\tau \times A/N_\tau \quad (A/N_\tau)^2 \to \mathbf{C}, \ (a + N_\tau, b + N_\tau) \mapsto \tau(b^* a),$$

$$(a+N_\tau, a+N_\tau) = 0 \Rightarrow \tau(a^*a) = 0 \Rightarrow a \in N_\tau \Rightarrow a + N_\tau = 0_{A/N_\tau}$$

is a well-defined inner product on A/N_τ. We denote by H_τ the Hilbert completion of A/N_τ.

If $a \in A$, define an operator $\varphi(a) \in B(A/N_\tau)$ by setting

$$\varphi(a): A/N_\tau \longrightarrow A/N_\tau$$

$$\varphi(a)(b + N_\tau) = ab + N_\tau. \qquad = \|ab + N_\tau\|^2$$

The inequality $\|\varphi(a)\| \le \|a\|$ holds since we have $\|\varphi(a)(b + N_\tau)\|^2 = \tau(b^* a^* ab) \le \|a\|^2 \tau(b^* b) = \|a\|^2 \|b + N_\tau\|^2$ (the latter inequality is given by Theorem 3.3.7). The operator $\varphi(a)$ has a unique extension to a bounded operator $\varphi_\tau(a)$ on H_τ. The map

$$: \frac{A}{N_\tau} \to \overline{\frac{A}{N_\tau}} \Rightarrow \underbrace{\overline{\varphi(a)}}_{\varphi_\tau(a)} \ \underbrace{\overline{\frac{A}{N_\tau}}}_{H_\tau} \longrightarrow H_\tau$$

$$\varphi_\tau: A \to B(H_\tau), \ a \mapsto \varphi_\tau(a),$$

is a $*$-homomorphism (this is an easy exercise).

is a $*$-homomorphism (this is an easy exercise).

The representation (H_τ, φ_τ) of A is the *Gelfand–Naimark–Segal* representation (or *GNS representation*) associated to τ.

If A is non-zero, we define its *universal* representation to be the direct sum of all the representations (H_τ, φ_τ), where τ ranges over $S(A)$.

3.4.1. Theorem (Gelfand–Naimark). *If A is a C^*-algebra, then it has a faithful representation. Specifically, its universal representation is faithful.*

Proof. Let (H, φ) be the universal representation of A and suppose that a is an element of A such that $\varphi(a) = 0$. By Theorem 3.3.6 there is a state τ on A such that $\|a^*a\| = \tau(a^*a)$. Hence, if $b = (a^*a)^{1/4}$, then $\|a\|^2 = \tau(a^*a) = \tau(b^4) = \|\varphi_\tau(b)(b + N_\tau)\|^2 = 0$ (since $\varphi_\tau(b^4) = \varphi_\tau(a^*a) = 0$, so $\varphi_\tau(b) = 0$). Hence, $a = 0$, and φ is injective. $\square$

The Gelfand–Naimark theorem is one of those results that are used all of the time. For the present we give just two applications.

The first application is to matrix algebras. If A is an algebra, $M_n(A)$ denotes the algebra of all $n \times n$ matrices with entries in A. (The operations are defined just as for scalar matrices.) If A is a $*$-algebra, so is $M_n(A)$, where the involution is given by $(a_{ij})^*_{i,j} = (a^*_{ji})_{i,j}$.

If $\varphi: A \to B$ is a $*$-homomorphism between $*$-algebras, its *inflation* is the $*$-homomorphism (also denoted φ)

$$\varphi: M_n(A) \to M_n(B), \ (a_{ij}) \mapsto (\varphi(a_{ij})).$$

If H is a Hilbert space, we write $H^{(n)}$ for the orthogonal sum of n copies of H. If $u \in M_n(B(H))$, we define $\varphi(u) \in B(H^{(n)})$ by setting

$$\varphi(u)(x_1, \ldots, x_n) = \left(\sum_{j=1}^n u_{1j}(x_j), \ldots, \sum_{j=1}^n u_{nj}(x_j)\right),$$

for all $(x_1, \ldots, x_n) \in H^{(n)}$. It is readily verified that the map

$$\varphi: M_n(B(H)) \to B(H^{(n)}), \ u \mapsto \varphi(u),$$

is a $*$-isomorphism. We call φ the *canonical* $*$-isomorphism of $M_n(B(H))$ onto $B(H^{(n)})$, and use it to identify these two algebras. If v is an operator in $B(H^{(n)})$ such that $v = \varphi(u)$ where $u \in M_n(B(H))$, we call u the *operator matrix* of v. We define a norm on $M_n(B(H))$ making it a C*-algebra by setting $\|u\| = \|\varphi(u)\|$. The following inequalities for $u \in M_n(B(H))$ are easy to verify and are often useful:

$$\|u_{ij}\| \le \|u\| \le \sum_{k,l=1}^n \|u_{kl}\| \qquad (i, j = 1, \ldots, n).$$

$$\sum_{k,l=1}$$

3.4.2. Theorem. *If A is a C^*-algebra, then there is a unique norm on $M_n(A)$ making it a C^*-algebra.*

Proof. Let the pair (H, φ) be the universal representation of A, so the $*$-homomorphism $\varphi: M_n(A) \to M_n(B(H))$ is injective. We define a norm on $M_n(A)$ making it a C*-algebra by setting $\|a\| = \|\varphi(a)\|$ for $a \in M_n(A)$ (completeness can be easily checked using the inequalities preceding this theorem). Uniqueness is given by Corollary 2.1.2. □

3.4.1. *Remark.* If A is a C*-algebra and $a \in M_n(A)$, then

$$\|a_{ij}\| \leq \|a\| \leq \sum_{k,l=1}^{n} \|a_{kl}\| \qquad (i, j = 1, \ldots, n).$$

These inequalities follow from the corresponding inequalities in $M_n(B(H))$.

Matrix algebras play a fundamental role in the K-theory of C*-algebras. The idea is to study not just the algebra A but simultaneously all of the matrix algebras $M_n(A)$ over A also.

Whereas it seems that the only way known of showing that matrix algebras over general C*-algebras are themselves normable as C*-algebras is to use the Gelfand–Naimark representation, for our second application of this representation alternative proofs exist, but the proof given here has the virtue of being very "natural."

3.4.3. Theorem. *Let a be a self-adjoint element of a C^*-algebra A. Then $a \in A^+$ if and only if $\tau(a) \geq 0$ for all positive linear functionals τ on A.*

Proof. The forward implication is plain. Suppose conversely that $\tau(a) \geq 0$ for all positive linear functionals τ on A. Let (H, φ) be the universal representation of A, and let $x \in H$. Then the linear functional

$$\tau: A \to \mathbf{C}, \quad b \mapsto \langle \varphi(b)(x), x \rangle,$$

is positive, so $\tau(a) \geq 0$; that is, $\langle \varphi(a)(x), x \rangle \geq 0$. Since this is true for all $x \in H$, and since $\varphi(a)$ is self-adjoint, therefore $\varphi(a)$ is a positive operator on H. Hence, $\varphi(a) \in \varphi(A)^+$, so $a \in A^+$, because the map $\varphi: A \to \varphi(A)$ is a $*$-isomorphism. □

In general, if B a C^*-subalg of A, whose units are the same & $b \in B$, then $sp_B(b) = sp_A(b)$. So

$B_+ = A_+ \cap B$

Let $x_n \in \mathbb{R}$, $x_n \leq M$ for all n & $\lim x_n$ exists. Then $\underbrace{\lim x_n}_{=l} \leq M$

<u>Proof</u>. Let $l > M$. By the difinition of limit, we have

$$\exists N\ \forall n \geq N;\ \underbrace{|x_n - l| < l - M}_{-\cancel{l}+M < x_n - \cancel{l} < l - M} \Rightarrow M < x_n\ \forall n \geq N$$

$\Rightarrow M < x_N$.

On the other hand $x_N \leq M$. ✗.

① If $(H_i)_i$ be a family of Hilbert spaces. Put

$$\oplus H_i = \{ (x_i) \in \prod_i H_i \mid \sum_i \|x_i\|^2 < \infty \}$$

Cartesian product

$\langle (x_i),(y_i)\rangle = \sum_i \langle x_i, y_i\rangle$ is well-defined:

$$\sum_{i=1}^n |\langle x_i,y_i\rangle| \leq \sum_{i=1}^n \|x_i\|\|y_i\| \leq \left(\sum_{i=1}^n \|x_i\|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^n \|y_i\|^2\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_i \|x_i\|^2\right)^{\frac{1}{2}}\left(\sum_i \|y_i\|^2\right)^{\frac{1}{2}} < \infty$$

So the increasing seq $\left\{\sum_{i=1}^n |\langle x_i,y_i\rangle|\right\}$ is bounded above, so it is convergent. Hence $\sum_i |\langle x_i,y_i\rangle|$ converges. So $\sum_i \langle x_i,y_i\rangle$ is convergent.

Thus $\|(x_i)\| = \langle (x_i),(x_i)\rangle^{\frac{1}{2}} = \left(\sum_i \|x_i\|^2\right)^{\frac{1}{2}}$

If $T_i \in B(H_i)$ and $\sup_i \|T_i\| < \infty$, then $\oplus T_i : \oplus H_i \to \oplus H_i$, $(x_i) \mapsto (T_i x_i)$ is a bounded linear operator:

$$\|(\oplus T_i)(x_i)\| = \|(T_i x_i)\| = \left(\sum_i \|T_i x_i\|^2\right)^{\frac{1}{2}} \leq \Big(\sum_i \underbrace{\|T_i\|^2\|x_i\|^2}_{\leq (\sup \|T_i\|)^2 \|x_i\|^2}\Big)^{\frac{1}{2}}$$

$$\leq \sup_i \|T_i\| \left(\sum_i \|x_i\|^2\right)^{\frac{1}{2}} = \left(\sup_i \|T_i\|\right)\|(x_i)\|$$

So $\underbrace{\sup \frac{\|\oplus T_i(x_i)\|}{\|(x_i)\|}}_{\|\oplus T_i\|} \leq \sup_i \|T_i\|$

$\left(\sum \|T_i x_i\|^2\right)^{\frac{1}{2}}$

$\forall \varepsilon \exists i_0 : \sup_i \|T_i\| - \varepsilon < \|T_{i_0}\| - \frac{\varepsilon}{2} < \|T_{i_0}(x_0)\| \quad (\|x_0\| = 1)$

Put $\underset{\oplus H_i}{\underset{\cap}{(x_i^0)}} = (\underbrace{0, \cdots, 0, x_0}_{\text{the } i_0\text{th entry}}, 0, \cdots)$. Then $\|(x_i^0)\| = 1$ &

$\sup_i \|T_i\| - \varepsilon < \|T_{i_0}(x_0)\| = \|T_{i_0}(x_0)\| = \frac{\|\oplus T_i(x_i^0)\|}{\|(x_i^0)\|}$

$\therefore \|\oplus T_i\| = \sup_i \|T_i\|$

For $i \in \{1, 2\} = I$, $T_1 \in B(H_1)$, $T_2 \in B(H_2)$, $\begin{cases} T_1 \oplus T_2 : H_1 \oplus H_2 \to H_1 \oplus H_2 \\ (x_1, x_2) \mapsto (T_1 x_1, T_2 x_2) \end{cases}$

or $\begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \underset{H_1 \oplus H_2}{\underset{\cap}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}} = \underset{H_1 \oplus H_2}{\underset{\cap}{\begin{bmatrix} T_1 x_1 \\ T_2 x_2 \end{bmatrix}}}$ in the matrix language.

Let $A, B \in M_n(\mathbb{C})$, Put $\langle A, B\rangle = tr(B^*A)$. Then

$$\|A\|_2^2 = tr(A^*A) = tr\left([b_{ij}]\,[a_{ij}]\right) = tr\Big[\sum_{k=1}^{n} \overset{\bar{a}_{ki}}{b_{ik}}\, a_{kj}\Big]_{i,j}$$

$$A = [a_{ij}] \Rightarrow A^* = \big[\overset{b_{ij}}{\bar{a}_{ji}}\big] \qquad = \sum_{i=1}^{n}\Big(\sum_{k=1}^{n} \underbrace{\bar{a}_{ki}\, a_{ki}}_{|a_{ki}|^2}\Big)$$

$$tr([a_{ij}]) = \sum_{i=1}^{n} a_{ii}$$

$$= \sum_{i,j=1}^{n} |a_{ij}|^2$$

$\underbrace{\text{2-Schatten norm}}_{\text{Hilbert-Schmidt}}$

$\underline{\text{Th}}$. If $D \subseteq X^{\text{normed space}}$ & $T : D \to Y^{\text{Ban}}$ is a bd linear operator, then $\exists!\ \tilde{T} : \overline{D} \to Y$ such that $\|\tilde{T}\| = \|T\|$ & $\tilde{T}|_D = T$.

Completion

Let (X,d) be a metric space, which is not complete. Define a relation $\sim$ on the set of all Cauchy sequences by $\{x_n\} \sim \{y_n\} \Leftrightarrow \lim_n d(x_n, y_n) = 0$. The relation $\sim$ is equivalent:

$\{x_n\} \sim \{x_n\} \Leftarrow \lim_n d(x_n, x_n) = \lim_n 0 = 0$

$\{x_n\} \sim \{y_n\} \Rightarrow \lim_n d(x_n, y_n) = 0 \Rightarrow \lim_n d(y_n, x_n) = 0 \Rightarrow \{y_n\} \sim \{x_n\}$

$\{x_n\} \sim \{y_n\}$ & $\{y_n\} \sim \{z_n\} \Rightarrow 0 \le d(x_n, z_n) \le \underbrace{d(x_n, y_n)}_{\to 0} + \underbrace{d(y_n, z_n)}_{\to 0}$

$\Longrightarrow \lim_n d(x_n, z_n) = 0 \Rightarrow \{x_n\} \sim \{z_n\}$

Let $[x_n] = \{\{y_n\} \mid \{x_n\} \sim \{y_n\}\}$. Put $\tilde{X} = \{[x_n] \mid \{x_n\} \text{ is Cauchy in } X\}$ & $\tilde{d}([x_n], [y_n]) := \lim_n d(x_n, y_n)$. In fact, $\tilde{d}$ is a metric on $\tilde{X}$.

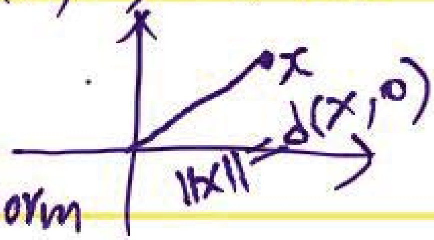
One can show that $(\tilde{X}, \tilde{d})$ is complete.

The mapping $\iota: X \longrightarrow \tilde{X}$ is 1-1 & isometry:

$x \mapsto [x]$ (we consider the const. seq. with term x)

$\tilde{d}([x], [y]) = \lim_n d(\underbrace{x_n}_{x}, \underbrace{y_n}_{y}) = d(x, y)$. In addition, $\iota(X)$ is dense in $\tilde{X}$.

Now let $(X,\|\cdot\|)$ be a normed space. We know X equipped with $d(x,y)=\|x-y\|$ is a metric space. Let $(\tilde{X},\tilde{d})$ be its completion as above. Put $\|[x_n]\|^{\sim}=\tilde{d}([x_n],[0])$



One can show that $\|\cdot\|^{\sim}$ is a complete norm on $\tilde{X}$, $\|x\|=\|[x]\|^{\sim}$, and $\iota: X \hookrightarrow \tilde{X}$ is an isometric isomorphism onto a dense subspace of $\tilde{X}$.

Another way for defining $\|\cdot\|^{\sim}$ is as follows:

$$[x_n]\in\tilde{X} \Longrightarrow \|[x_n]\|^{\sim} := \lim_n \|x_n\|.$$

Finally, let $(X,\langle\cdot,\cdot\rangle)$ be an inner product space. Then $(X, \|x\|=\langle x,x\rangle^{\frac{1}{2}})$ is a normed space. As above $\exists$ a Banach space $(\tilde{X},\|\cdot\|^{\sim})$ such that X can be embedded in $\tilde{X}$. Put $\langle [x_n],[y_n]\rangle^{\sim} := \lim_n \langle x_n,y_n\rangle$. Then $(\tilde{X},\langle\cdot,\cdot\rangle^{\sim})$ is a Hilbert space. Another way for defining $\langle\cdot,\cdot\rangle^{\sim}$ is:

$$\langle [x_n],[y_n]\rangle^{\sim} := \frac{1}{4}\sum_{k=0}^{3} \|[x_n]+i^k[y_n]\|^{\sim 2}$$ (Polarization identity). $\square$