Duality between X, X^*

4.3 Theorem *Suppose B is the closed unit ball of a normed space X.* $\{x \in X : \|x\| \le 1\}$ *Define*

$x^*: X \xrightarrow[\text{bd}]{\text{lin}} \mathbb{C}$

$$\|x^*\| = \sup\{|\langle x, x^*\rangle| : x \in B\} = \sup_{x \neq 0} \frac{|x^*(x)|}{\|x\|}$$

X^* is the dual of X.

for every $x^* \in X^*$.

$B(X,\mathbb{C}) = X^*$

(*a*) *This norm makes X^* into a Banach space.* Exercise $\mathcal{B}(X,Y)$ is Ban iff so is Y.

(*b*) *Let B^* be the closed unit ball of X^*. For every $x \in X$,*

$\forall x\ \exists x^* \in B^*;\ \langle x, x^*\rangle = \|x\|$

$$\|x\| = \sup\{|\langle x, x^*\rangle| : x^* \in B^*\}.$$

$|\langle x, x^*\rangle| \le \|x^*\|\,\|x\| \le \|x\|$

Consequently, $x^* \to \langle x, x^*\rangle$ is a bounded linear functional on X^*, of norm $\|x\|$.

$T: X \xrightarrow[\text{lin}]{\text{bd}} Y \to \boxed{\|Tx\| \le \|T\|\,\|x\|}$

قانون تابلو

(*c*) *B^* is weak*-compact.*

PROOF. Since $\mathscr{B}(X, Y) = X^*$, when Y is the scalar field, (*a*) is a corollary of Theorem 4.1.

Fix $x \in X$. The corollary to Theorem 3.3 shows that there exists $y^* \in B^*$ such that

$M = \mathbb{C}x$

(1) $$\langle x, y^*\rangle = \|x\|.$$

$f_1: M \to \mathbb{C}$, $\lambda x \mapsto \lambda\|x\|$ → Hahn-Ban Th. $f: X \to \mathbb{C}$, $\|f\| = \|f_1\|$, $f|_M = f_1$

On the other hand,

(2) $$|\langle x, x^*\rangle| \le \|x\|\,\|x^*\| \le \|x\|$$

for every $x^* \in B^*$. Part (*b*) follows from (1) and (2).

Since the open unit ball U of X is dense in B, the definition of $\|x^*\|$ shows that $x^* \in B^*$ if and only if $|\langle x, x^*\rangle| \le 1$ for every $x \in U$. Part (*c*) now follows directly from Theorem 3.15. ////

قضیه آلااغلو

4.5 The second dual of a Banach space The normed dual X^* of a Banach space X is itself a Banach space and hence has a normed dual of its own, denoted by X^{**}. Statement (*b*) of Theorem 4.3 shows that every $x \in X$ defines a unique $\phi x \in X^{**}$, by the equation

$\varphi: X \hookrightarrow X^{**}$, $x \mapsto \varphi x = \hat{x}$

(1) $$\langle x, x^*\rangle = \langle x^*, \phi x\rangle \qquad (x^* \in X^*),$$

$\hat{x}(x^*) = \langle x, x^*\rangle$, $\|\hat{x}\| = \|x\|$

and that

(2) $$\|\phi x\| = \|x\| \qquad (x \in X).$$

It follows from (1) that $\phi: X \to X^{**}$ is linear; by (2), ϕ is an isometry. Since X is now assumed to be complete, $\phi(X)$ is closed in X^{**}.

Thus ϕ is an isometric isomorphism of X onto a closed subspace of X^{**}.

Thus ϕ is an isometric isomorphism of X onto a closed subspace of X^{**}.

Frequently, X is identified with $\phi(X)$; then X is regarded as a subspace of X^{**}.

The members of $\phi(X)$ are exactly those linear functionals on X^* that are continuous relative to its weak*-topology. (See Section 3.14.) Since the norm topology of X^* is stronger, it may happen that $\phi(X)$ is a proper subspace of X^{**}. But there are many important spaces X (for example, all L^p-spaces with $1 < p < \infty$) for which $\phi(X) = X^{**}$; these are called *reflexive*. Some of their properties are given in Exercise 1.

It should be stressed that, in order for X to be reflexive, the existence of *some* isometric isomorphism ϕ of X onto X^{**} is not enough; it is crucial that the identity (1) be satisfied by ϕ.

$$X \times X^* \longrightarrow \mathbb{C}$$
$$(x, x^*) \longmapsto \langle x, x^* \rangle = x^*(x)$$

4.6 Annihilators Suppose X is a Banach space, M is a subspace of X, and N is a subspace of X^*; neither M nor N is assumed to be closed. Their *annihilators* $M^\perp$ and ${}^\perp N$ are defined as follows:

$$M^\perp = \{x^* \in X^* : \langle x, x^* \rangle = 0 \text{ for all } x \in M\},$$

$${}^\perp N = \{x \in X : \langle x, x^* \rangle = 0 \text{ for all } x^* \in N\}.$$

Thus $M^\perp$ consists of all bounded linear functionals on X that vanish on M, and ${}^\perp N$ is the subset of X on which every member of N vanishes. It is clear that $M^\perp$ and ${}^\perp N$ are vector spaces. Since $M^\perp$ is the intersection of the null spaces of the functionals ϕx, where x ranges over M (see Section 4.5), $M^\perp$ is a weak*-closed subspace of X^*. The proof that ${}^\perp N$ is a norm-closed subspace of X is even more direct. The following theorem describes the duality between these two types of annihilators.

4.7 Theorem *Under the preceding hypotheses,*

(*a*) *${}^\perp(M^\perp)$ is the norm-closure of M in X, and* $\;{}^\perp(M^\perp) = \overline{M}$

(*b*) *$({}^\perp N)^\perp$ is the weak*-closure of N in X^*.*

As regards (*a*), recall that the norm-closure of M equals its weak closure, by Theorem 3.12.

hence $M \subseteq {}^\perp(M^\perp)$. So $\overline{M} \subseteq {}^\perp(M^\perp)$ $= {}^\perp(M^\perp)$ by the definition of $M^\perp$

PROOF. If $x \in M$, then $\langle x, x^* \rangle = 0$ for every $x^* \in M^\perp$, so that

$x \in {}^\perp(M^\perp)$! ~~Since $^\perp(M^\perp)$ is norm-closed, it contains the norm-closure $\bar{M}$ of M~~. On the other hand, if $x \notin \bar{M}$ the Hahn-Banach theorem yields an $x^* \in M^\perp$ such that $\underbrace{\langle x, x^* \rangle}_{=\|x\|} \neq 0$. Thus $x \notin {}^\perp(M^\perp)$, and (*a*) is proved.

Similarly, if $x^* \in N$, then $\langle x, x^* \rangle = 0$ for every $x \in {}^\perp N$, so that $x^* \in ({}^\perp N)^\perp$. This weak*-closed subspace of X^* contains the weak*-closure $\tilde{N}$ of N. If $x^* \notin \tilde{N}$, the Hahn-Banach theorem (applied to the locally convex space X^* with its weak*-topology) implies the existence of an $x \in {}^\perp N$ such that $\langle x, x^* \rangle \neq 0$; thus $x^* \notin ({}^\perp N)^\perp$, which proves (*b*). ////

Lemma $M^\perp = \{x^* : x^*|_M = 0\}$ is weak* closed.

weak*-top is the weakest top on X^* such that all $\{\hat{x} : X^* \to \mathbb{C},\ x^* \mapsto x^*(x)\}$ are continuous cts.

Proof.

Let $x^*_\alpha \in M^\perp$ & $x^*_\alpha \xrightarrow{w^*} x^*$ ①

Let $x \in M$. By ①, $\underbrace{x^*_\alpha(x)}_{0} \to x^*(x)$

A net $\{x^*_\alpha\}$ converges to $x^* \in X^*$ in w*-top iff $x^*_\alpha(x) \to x^*(x)\ \forall x \in X$

Hence

$x^*(x) = 0.$

Thus

$x^* \in M^\perp$. □

Lemma (Exercise) $^\perp N$ is closed in the norm topology.

Duality

Now we define the Ban adjoint of $T \in B(X,Y)$ by $\begin{cases} T^*: Y^* \to X^* \\ g \mapsto g \circ T \end{cases}$

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[dr, dotted, "g \circ T"'] & Y \arrow[d, "g"] \\ & \mathbb{C} \end{tikzcd}$$

T^* is bd & $\|T^*\| = \|T\|$, since

$\boxed{\|T^*(g)\| \overset{?}{\leq} M\|g\|}$ Let $\|x\| \leq 1$.

$\|T^*(g)(x)\| = \|g(Tx)\| \leq \|g\| \|Tx\|$

$\leq \|g\| \|T\|$

$\therefore \|T^* g\| = \sup_{\|x\| \leq 1} \|(T^* g)(x)\| \leq \|g\| \|T\|$

$\|T^*\| = \sup \frac{\|T^* g\|}{\|g\|} \leq \|T\|$

References:
1. Functional Analysis (Rudin)
2. C^*-alg & Operator Theory (Murphy)
3. Complete normed alg (Bonsall-Duncan)

Let $x \in X$ & $\|x\| = 1$.
By the Hahn-Ban Th.
$\exists g \in Y^*$; $g(Tx) = \|Tx\|$, $\|g\| = 1$? $\in Y^* \xrightarrow{T^*} X^*$

$X \xrightarrow{T} Y \ni Tx$

$\|Tx\| = |g(Tx)| = |\underbrace{(g \circ T)}_{T^* g}(x)| \leq \|T^*\| \|g\| \|x\|$

$\therefore \|T\| = \sup_{\|x\|=1} \|Tx\| \leq \|T^*\|$

(Rudin's book)

4.9 Theorem *Let M be a closed subspace of a Banach space X.*

(a) *The Hahn-Banach theorem extends each $m^* \in M^*$ to a functional $x^* \in X^*$. Define*

$$\begin{cases} \sigma: M^* \to \frac{X^*}{M^\perp} \\ \sigma m^* = x^* + M^\perp. \end{cases}$$

$M \xrightarrow{m^*} \mathbb{C} \overset{H\text{-}B}{\Longrightarrow} X \xrightarrow{x^*} \mathbb{C}$

$\cancel{\|x^*\| = \|m^*\|}$, $\cancel{x^*|_M = m^*}$

(1) $\sigma(m_1^*) + \sigma(m_2^*) = x_1^* + M + x_2^* + M = (x_1^* + x_2^*) + M = \sigma(m_1^* + m_2^*)$

Then σ is an isometric isomorphism of M^* onto $X^*/M^\perp$.

(b) *Let $\pi: X \to X/M$ be the quotient map. Put $Y = X/M$. For each $y^* \in Y^*$, define*

$$\tau y^* = y^* \pi.$$

Then τ is an isometric isomorphism of Y^* onto $M^\perp$. $x^*|_M = x_1^*|_M$

PROOF. (*a*) If x^* and x_1^* are extensions of m^*, then $x^* - x_1^*$ is in $M^\perp$; hence $x^* + M^\perp = x_1^* + M^\perp$. Thus σ is well defined. A trivial verification shows that σ is linear. (1) Since the restriction of every $x^* \in X^*$ to M is a member of M^*, the range of σ is all of $X^*/M^\perp$.

Fix $m^* \in M^*$. If $x^* \in X^*$ extends m^*, it is obvious that $\sup_{\text{sem}} \frac{\|m^*(y)\|}{\|y\|} = \|m^*\| \overset{(2)}{\leq} \|x^*\|$. ~~The greatest lower bound of the numbers $\|x^*\|$ so obtained is $\|x^* + M^\perp\|$, by the definition of the quotient norm.~~ Hence

Let $z^* \in M^\perp$. Since $z^*|_M = 0$, so $x^* + z^*|_M = x^*|_M = m^*$ … $\|x^* + z^*\| : z^* \in M^\perp\}$

Hence $\|m^*\| \leq \|x^* + z^*\|$.

$$\|m^*\| \leq \|\underbrace{\sigma m^*}_{x^* + M^\perp}\| \leq \|x^*\|. \quad z^* = 0$$

By (2) But Theorem's H-B ~~3.3~~ furnishes an extension x^* of m^* with $\|x^*\| = \|m^*\|$. It follows that $\|\sigma m^*\| = \|m^*\|$. This completes (*a*).

برخسروی ← (*b*) If $x \in X$ and $y^* \in Y^*$, then $\pi x \in Y$; hence $x \to y^* \pi x$ is a continuous linear functional on X which vanishes for $x \in M$. Thus $\tau y^* \in M^\perp$. The linearity of τ is obvious. Fix $x^* \in M^\perp$. Let N be the null space of x^*. Since $M \subset N$, there is a linear functional Λ on Y such that $\Lambda \pi = x^*$. The null space of Λ is $\pi(N)$, a closed subspace of Y, by the definition of the quotient topology in $Y = X/M$. By Theorem 1.18, Λ is continuous, that is, $\Lambda \in Y^*$. Hence $\tau \Lambda = \Lambda \pi = x^*$. The range of τ is therefore all of $M^\perp$.

۹۱/۷/۱۶

It remains to be shown that τ is an isometry.

Let B be the open unit ball in X. Then πB is the open unit ball of $Y = \pi X$. Since $\tau y^* = y^* \pi$, we have

$$\|\tau y^*\| = \|y^* \pi\| = \sup \{ |\langle \pi x, y^* \rangle| : x \in B\}$$
$$= \sup \{ |\langle y, y^* \rangle| : y \in \pi B\} = \|y^*\|$$

for every $y^* \in Y^*$. ////

4.10 Theorem *Suppose X and Y are normed spaces. To each $T \in \mathscr{B}(X, Y)$ corresponds a unique $T^* \in \mathscr{B}(Y^*, X^*)$ that satisfies*

(1) $$\langle Tx, y^* \rangle = \langle x, T^* y^* \rangle$$

for all $x \in X$ and all $y^* \in Y^*$. Moreover, T^* satisfies

(2) $$\|T^*\| = \|T\|.$$

PROOF. If $y^* \in Y^*$ and $T \in \mathscr{B}(X, Y)$, define

(3) $$T^* y^* = y^* \circ T.$$

Being the composition of two continuous linear mappings, $T^* y^* \in X^*$. Also,

$$\langle x, T^* y^* \rangle = (T^* y^*)(x) = y^*(Tx) = \langle Tx, y^* \rangle,$$

which is (1). The fact that (1) holds for every $x \in X$ obviously determines $T^* y^*$ uniquely.

If $y_1^* \in Y^*$ and $y_2^* \in Y^*$, then

$$\langle x, T^*(y_1^* + y_2^*) \rangle = \langle Tx, y_1^* + y_2^* \rangle$$

$$= \langle Tx, y_1^* \rangle + \langle Tx, y_2^* \rangle$$

$$= \langle x, T^* y_1^* \rangle + \langle x, T^* y_2^* \rangle$$

$$= \langle x, T^* y_1^* + T^* y_2^* \rangle$$

for every $x \in X$, so that

$$T^*(y_1^* + y_2^*) = T^* y_1^* + T^* y_2^*. \tag{4}$$

Similarly, $T^*(\alpha y^*) = \alpha T^* y^*$. Thus $T^*: Y^* \to X^*$ is linear. Finally, (*b*) of Theorem 4.3 leads to

$$\|T\| = \sup \{ |\langle Tx, y^* \rangle| : \|x\| \le 1, \|y^*\| \le 1 \}$$

$$= \sup \{ |\langle x, T^* y^* \rangle| : \|x\| \le 1, \|y^*\| \le 1 \}$$

$$= \sup \{ \|T^* y^*\| : \|y^*\| \le 1 \} = \|T^*\|. \qquad ////$$

Exercise. (i) If M is a subspace, then so is $\overline{M}$. (Ban) is a normed space

(ii) If M is a closed subspace of X, then the linear space $\frac{X}{M} = \{x+M : x \in X\}$ endowed with $(x+M)+(y+M) = (x+y)+M$, $\lambda(x+M) = \lambda x + M$ and $\|x+M\| = \inf_{z \in M} \|x+z\|$ is a normed space. (B)

$x+M = y+M \Leftrightarrow x-y \in M$

(iii) If M is a closed subspace of X & $\frac{X}{M}$ is Ban, then X is also Ban. (تمرین کلاسی)

-۹۱,۷,۱۹

4.11 Notation If T maps X into Y, the null space and the range of T will be denoted by $\mathscr{N}(T)$ and $\mathscr{R}(T)$, respectively:

$$\operatorname{Ker} T = \mathscr{N}(T) = \{x \in X: Tx = 0\},$$

$$\mathscr{R}(T) = \{y \in Y: Tx = y \text{ for some } x \in X\}.$$

The next theorem concerns annihilators; see Section 4.6 for the notation.

بین الحاق T در فضای باناخ و الحاق T در فضاهای هیلبرت تفاوت اساسی وجود دارد:

$T^{**} \neq T$ & $(\lambda T)^* \overset{Ban}{=} \lambda T^*$, $(\lambda T)^* \overset{Hil}{=} \bar{\lambda} T^*$ & $T^{**} = T$

4.12 Theorem *Suppose X and Y are Banach spaces, and $T \in \mathscr{B}(X, Y)$. Then*

$$\mathscr{N}(T^*) = \mathscr{R}(T)^{\perp} \quad \text{and} \quad \mathscr{N}(T) = {}^{\perp}\mathscr{R}(T^*).$$

PROOF. In each of the following two columns, each statement is obviously equivalent to the one that immediately follows and/or precedes it.

$y^* \in \mathscr{N}(T^*)$.	$x \in \mathscr{N}(T)$.
$X^* \ni T^*y^* = 0$.	$Tx = 0$.
$\langle x, T^*y^* \rangle = 0$ for all x.	$\langle Tx, y^* \rangle = 0$ for all y^*.
$\langle Tx, y^* \rangle = 0$ for all x.	$\langle x, T^*y^* \rangle = 0$ for all y^*.
$y^* \in \mathscr{R}(T)^{\perp}$.	$x \in {}^{\perp}\mathscr{R}(T^*)$. ////

Corollaries

$\exists f \in X^*;$ $f|_M = 0$ & $f(x) \neq 0 \Longleftarrow x$, M closed

$\overline{R(T)} \neq Y \Rightarrow \exists y \in Y; y \notin \overline{R(T)} \overset{H\text{-}B}{\Longrightarrow} \exists y^* \in Y^*; \underbrace{y^*|_{\overline{R(T)}} = 0}_{y^* \in R(T)^{\perp}}$ & $\underbrace{y^*(y) \neq 0}_{y^* \neq 0}$

(a) $\mathscr{N}(T^*)$ *is weak*-closed in* Y^*.

(b) $\mathscr{R}(T)$ *is dense in* Y *if and only if* T^* *is one-to-one.*

(c) T *is one-to-one if and only if* $\mathscr{R}(T^*)$ *is weak*-dense in* X^*.

Recall that $M^{\perp}$ is weak*-closed in Y^* for every subspace M of Y. In particular, this is true of $\mathscr{R}(T)^{\perp}$. Thus (a) follows from the theorem.

$\overset{\parallel}{N(T^*)}$

As to (b), $\mathscr{R}(T)$ is dense in Y if and only if $\mathscr{R}(T)^{\perp} = \{0\}$; in that case, $\mathscr{N}(T^*) = \{0\}$. $\overset{\parallel}{N(T)}$ $\overset{\parallel}{N(T^*)}$

Likewise, ${}^{\perp}\mathscr{R}(T^*) = \{0\}$ if and only if $\mathscr{R}(T^*)$ is annihilated by no $x \in X$ other than $x = 0$; this says that $\mathscr{R}(T^*)$ is weak*-dense in X^*.

Note that the Hahn-Banach theorem 3.5 was tacitly used in the proofs of (*b*) and (*c*).

$\{x \in X : \|x\| < 1\}$

$\|$

4.13 Theorem *Let U and V be the open unit balls in the Banach spaces X and Y, respectively. If $T \in \mathscr{B}(X, Y)$ and $\delta > 0$, then the implications*

$$(a) \to (b) \to (c) \to (d)$$

hold among the following statements:

(*a*) $\|T^* y^*\| \geq \delta \|y^*\|$ *for every* $y^* \in Y^*$. (T^* in bounded below)

(*b*) $\overline{T(U)} \supset \delta V = \{y \in Y : \|y\| < \delta\}$

(*c*) $T(U) \supset \delta V$.

(*d*) $T(X) = Y$.

Moreover, if (d) holds, then (a) holds for some $\delta > 0$.

PROOF. Assume (*a*), and pick $y_0 \notin \overline{T(U)}$. Since $\overline{T(U)}$ is convex, closed, and balanced, Theorem 3.7 shows that there is a y^* such that $|\langle y, y^* \rangle| \leq 1$ for every $y \in \overline{T(U)}$, but $|\langle y_0, y^* \rangle| > 1$. If $x \in U$, it follows that

$$|\langle x, T^* y^* \rangle| = |\langle \underbrace{Tx}_{\in TU}, y^* \rangle| \leq 1.$$

Thus $\|T^* y^*\| \leq 1$, and now (*a*) gives

$$\delta < \delta |\langle y_0, y^* \rangle| \leq \delta \|y_0\| \|y^*\| \underset{(a)}{\leq} \|y_0\| \|T^* y^*\| \leq \|y_0\|.$$

Consider δ / T

It follows that $y \in \overline{T(U)}$ if $\|y\| \leq \delta$. Thus (*a*) $\to$ (*b*).

Next, assume (*b*). Take $\delta = 1$, without loss of generality. Then

$\overline{TU} \supseteq V \Rightarrow$ $\overline{T(U)} \supset \bar{V}$. To every $y \in Y$ and every $\varepsilon > 0$ corresponds therefore an

Take closure

$x \in X$ with $\|x\| \leq \|y\|$ and $\|y - Tx\| < \varepsilon$. $y \in Y \Rightarrow \frac{y}{\|y\|} \in V \Rightarrow$

Pick $y_1 \in V$. Pick $\varepsilon_n > 0$ so that $\overline{TU} \ni \frac{y}{\|y\|} \Rightarrow \exists x \in U;$

$\exists N : \frac{1}{2^N} < 1 - \|y_1\|$

$$\sum_{n=1}^{\infty} \varepsilon_n < 1 - \|y_1\|.$$

$\sum_{n=1}^{\infty} \overbrace{\frac{1}{2^{N+n}}}^{\varepsilon_n} = \frac{1}{2^N} < 1 - \|y_1\|$

$\|\frac{y}{\|y\|} - Tx\| < \frac{\varepsilon}{\|y\|} \Rightarrow$

$\|y - T(\|y\| x)\| < \varepsilon$

Assume $n \geq 1$ and y_n is picked. There exists x_n such that $\|x_n\| \leq \|y_n\|$ and $\|y_n - Tx_n\| < \varepsilon_n$. Put

$\underbrace{\|y\|x}_{x'} \rightarrow \|x'\| = \|\|y\| x\| \leq \|y\|$

Unit 1

y_n (2)

$$y_{n+1} = y_n - Tx_n.$$

(3)

By induction, this process defines two sequences $\{x_n\}$ and $\{y_n\}$. Note that

$$\|x_{n+1}\| \le \|y_{n+1}\| = \|y_n - Tx_n\| < \varepsilon_n .$$

Hence

$$\|x_1\| + \sum_{n=1}^{\infty} \|x_{n+1}\|$$

$$\sum_{n=1}^{\infty} \|x_n\| \le \|x_1\| + \sum_{n=1}^{\infty} \varepsilon_n \le \|y_1\| + \sum_{n=1}^{\infty} \varepsilon_n < 1.$$

$\|\sum x_n\| \le \sum \|x_n\|$ 1

It follows that $x = \sum x_n$ is in U (~~see Exercise 23~~) and that

$$Tx = \lim_{N \to \infty} \sum_{n=1}^{N} Tx_n = \lim_{N \to \infty} \sum_{n=1}^{N} (y_n - y_{n+1}) = y_1$$

since $y_{N+1} \to 0$ as $N \to \infty$. Thus $y_1 = Tx \in T(U)$, which proves (*c*).

Note that the preceding argument is just a specialized version of part of the proof of the open mapping theorem 2.11.

That (*c*) implies (*d*) is obvious.

Assume (*d*). By the open mapping theorem, there is a $\delta > 0$ such that $T(U) \supset \delta V$. Hence

$$\begin{aligned} \|T^*y^*\| &= \sup \{ |\langle x, T^*y^* \rangle| : x \in U \} \\ &= \sup \{ |\langle Tx, y^* \rangle| : x \in U \} \\ &\ge \sup \{ |\langle y, y^* \rangle| : y \in \delta V \} = \delta \|y^*\| \end{aligned}$$

for every $y^* \in Y^*$. This is (*a*). ////

Exercise: A normed space X is Ban iff $\forall \{x_n\}$;
(تمرین تکمیلی) $\sum \|x_n\|$ converges in $\mathbb{R}$ $\Longrightarrow$ $\sum x_n$ converges in X
91, ۷, ۱۴

91, ۷, ۱۱

≡ Banach Algebras ≡

An *algebra* is a vector space A together with a bilinear map

$$A^2 \to A, \quad (a, b) \mapsto ab,$$

such that

$$a(bc) = (ab)c \quad (a, b, c \in A).$$

A *subalgebra* of A is a vector subspace B such that $b, b' \in B \Rightarrow bb' \in B$. Endowed with the multiplication got by restriction, B is itself an algebra.

A norm $\|.\|$ on A is said to be *submultiplicative* if

$$\|ab\| \leq \|a\|\|b\| \quad (a, b \in A).$$

In this case the pair $(A, \|.\|)$ is called a *normed algebra*. If A admits a unit 1 ($a1 = 1a = a$, for all $a \in A$) and $\|1\| = 1$, we say that A is a *unital normed algebra*.

A *left* (respectively, *right*) *ideal* in an algebra A is a vector subspace I of A such that

$$a \in A \text{ and } b \in I \Rightarrow ab \in I \quad (\text{respectively, } ba \in I).$$

An *ideal* in A is a vector subspace that is simultaneously a left and a right ideal in A. Obviously, 0 and A are ideals in A, called the *trivial* ideals. A *maximal* ideal in A is a proper ideal (that is, it is not A) that is not contained in any other proper ideal in A. Maximal left ideals are defined similarly.

An ideal I is *modular* if there is an element u in A such that $a - au$ and $a - ua$ are in I for all $a \in A$. It follows easily from Zorn's lemma that every proper modular ideal is contained in a maximal ideal.

If A is unital, then every ideal of A is modular $(u=1)$.

If ω is an element of a locally compact Hausdorff space Ω, and $M_\omega = \{f \in C_0(\Omega) \mid f(\omega) = 0\}$, then M_ω is a modular ideal in the algebra $C_0(\Omega)$. This is so because there is an element $u \in C_0(\Omega)$ such that $u(\omega) = 1$, and hence, $f - uf \in M_\omega$ for all $f \in C_0(\Omega)$. Since M_ω is of codimension one in $C_0(\Omega)$ (as $M \oplus \mathbf{C}u = C_0(\Omega)$), it is a maximal ideal.

Ω $\{\omega\}$ compact open

Urysohn Lemma: If $K \subseteq V$, then (K compact, V open) $\exists f \in \underbrace{C_c(X)}_{\text{cts with compact support}}$; $f|_K = 1$ & $f|_{V^c} = 0$ & $0 \le f \le 1$. $\overline{\{x : f(x) \neq 0\}}$

$C_0(\Omega) \xrightarrow[\text{onto}]{1-1} M_\omega \oplus \mathbb{C}u$, $f \mapsto (f - uf(\omega)) + f(\omega)u$

1.1.1. *Example.* If S is a set, $\ell^\infty(S)$, the set of all bounded complex-valued functions on S, is a unital Banach algebra where the operations are defined pointwise:

$$(f + g)(x) = f(x) + g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$(\lambda f)(x) = \lambda f(x),$$

and the norm is the sup-norm

$$\|f\|_\infty = \sup_{x \in S} |f(x)| < \infty.$$

1) $\{f_n\}$ Cauchy
2) f_n satisfies the Cauchy criterion for uniform convergence
3) $\exists f : f_n \xrightarrow{u} f$
4) $g_n \xrightarrow{\|\cdot\|_\infty} g \Leftrightarrow \forall \varepsilon \exists N \forall n \geq N$ $\forall x \in S$; $|g_n(x) - g(x)| < \varepsilon$; $\|g_n - g\|_\infty \le \varepsilon$
$\Leftrightarrow g_n \xrightarrow{u} g$ $(g_n, g \in \ell^\infty(S))$
5) f_n bd $\Rightarrow f$ bd 6) $f \in \ell^\infty(S)$ 7) $f_n \xrightarrow{\|\cdot\|_\infty} f$ (by (4)) (5)

Note: $C_0(X)$ is the completion of $C_c(X)$ under $\|f\|_\infty = \sup_{x \in X} |f(x)|$

1.1.2. *Example.* If Ω is a topological space, the set $C_b(\Omega)$ of all bounded continuous complex-valued functions on Ω is a closed subalgebra of $\ell^\infty(\Omega)$. Thus, $C_b(\Omega)$ is a unital Banach algebra.

If Ω is compact, $C(\Omega)$, the set of continuous functions from Ω to $\mathbf{C}$, is of course equal to $C_b(\Omega)$.

$C_b(\Omega) \le \ell^\infty(\Omega)$

1.1.3. *Example.* If Ω is a locally compact Hausdorff space, we say that a continuous function f from Ω to $\mathbf{C}$ *vanishes at infinity*, if for each positive number ε the set $\{\omega \in \Omega \mid |f(\omega)| \geq \varepsilon\}$ is compact. We denote the set of such functions by $C_0(\Omega)$. It is a closed subalgebra of $C_b(\Omega)$, and therefore,

$C_0(\Omega) = \{f : \Omega \to \mathbb{C} \mid \forall \varepsilon\ \{x \in \Omega \mid |f(x)| \geq \varepsilon\}$ compact $\}$

such functions by $C_0(\Omega)$. It is a closed subalgebra of $\ell^\infty(\Omega)$, and therefore, a Banach algebra. It is unital if and only if Ω is compact, and in this case $C_0(\Omega) = C(\Omega)$. The algebra $C_0(\Omega)$ is one of the most important examples of a Banach algebra, and we shall see it used constantly in C*-algebra theory (the functional calculus).

is compact

f vanishes at ∞

Exercise $1 \in C_0(\Omega) \Leftrightarrow \Omega$ is compact

Solution:

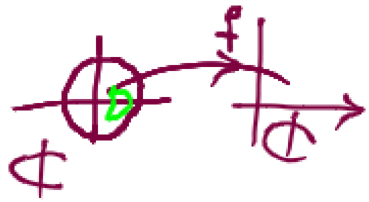
$(\Rightarrow)$ $\{x \in \Omega \mid 1(x) \geq \frac{1}{2}\} = \Omega$ is compact.

$(\Leftarrow)$ $\forall \varepsilon$, $\{x \in \Omega \mid |f(x)| \geq \varepsilon\}$ is a closed subset of Ω, so it is compact.

1.1.4. *Example.* If (Ω, μ) is a measure space, the set $L^\infty(\Omega, \mu)$ of (classes of) essentially bounded complex-valued measurable functions on Ω is a unital Banach algebra with the usual (pointwise-defined) operations and the essential supremum norm $f \mapsto \|f\|_\infty = \inf\{\lambda > 0 : |f(x)| \leq \lambda \text{ a.e.}\}$

1.1.5. *Example.* If Ω is a measurable space, let $B_\infty(\Omega)$ denote the set of all bounded complex-valued measurable functions on Ω. Then $B_\infty(\Omega)$ is a closed subalgebra of $\ell^\infty(\Omega)$, so it is a unital Banach algebra. This example will be used in connection with the spectral theorem in Chapter 2.

1.1.6. *Example.* The set A of all continuous functions on the closed unit disc $\mathbf{D}$ in the plane which are analytic on the interior of $\mathbf{D}$ is a closed subalgebra of $C(\mathbf{D})$, so A is a unital Banach algebra, called the *disc algebra*. This is the motivating example in the theory of function algebras, where many aspects of the theory of analytic functions are extended to a Banach algebraic setting.



All of the above examples are of course *abelian*—that is, $ab = ba$ for all elements a and b—but the following examples are not, in general.

1.1.7. *Example.* If X is a normed vector space, denote by $B(X)$ the set of all bounded linear maps from X to itself (the *operators* on X). It is routine to show that $B(X)$ is a normed algebra with the pointwise-defined operations for addition and scalar multiplication, multiplication given by $(u, v) \mapsto u \circ v$, and norm the *operator norm*:

$$\|u\| = \sup_{x \neq 0} \frac{\|u(x)\|}{\|x\|} = \sup_{\|x\| \leq 1} \|u(x)\|.$$

If X is a Banach space, $B(X)$ is complete and is therefore a Banach algebra.

If $X = \mathbb{C}^n$, then $\varphi : B(\mathbb{C}^n) \cong M_n(\mathbb{C})$

$T \mapsto A_T = [\ \vdots\ ]$ j-th column is Te_j

$e_j = (0, \dots, 0, \underbrace{1}_{j\text{-th}}, 0, \dots, 0)$

1.1.8. *Example.* The algebra $M_n(\mathbf{C})$ of $n \times n$-matrices with entries in $\mathbf{C}$ is identified with $B(\mathbf{C}^n)$. It is therefore a unital Banach algebra. Recall that an *upper triangular* matrix is one of the form

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & \dots & \dots & \lambda_{1n} \\ 0 & \lambda_{22} & \dots & \dots & \lambda_{2n} \\ 0 & 0 & \lambda_{33} & \dots & \lambda_{3n} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \lambda_{nn} \end{pmatrix}$$

(all entries below the main diagonal are zero). These matrices form a subalgebra of $M_n(\mathbf{C})$.

$T_A \longleftarrow\!| A$

$T_A : \mathbb{C}^n \to \mathbb{C}^n$

$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mapsto AX$

Direct Sum

① External Direct sum:

Let X & Y be a vector spaces. $X \times Y$ is a vector space under.

$$\lambda(x_1,y_1)+(x_2,y_2)=(\lambda x_1+x_2,\ \lambda y_1+y_2)$$

$X\times Y$ is denoted by $X\oplus Y$.

(2) Internal Direct Sum:

Let V & W be <u>subspaces</u> of a vector space X such that $V\cap W=\{0\}$. Then $V+W=\langle V\cup W\rangle=\bigcap\limits_{Z\supseteq V\cup W} Z=\{v+w \mid v\in V, w\in W\}$

$V+W$ is denoted by $V\oplus W$.

There is a relationship between in & ex dir sums:

If $X\oplus Y$ is an external direct sum, then

$$X\oplus Y=X\times Y=\underbrace{\{(x,0)\mid x\in X\}}_{\leq X\oplus Y}\oplus\underbrace{\{(0,y)\mid y\in Y\}}_{\leq X\oplus Y}$$ is an internal direct sum.

If $V\oplus W$ is an internal direct sum in X, then

$$\varphi: V\oplus W\longrightarrow V\times W$$
$$u+v\longmapsto(u,v)$$

is an isomorphism of vector spaces.

If H & K are inner product spaces, then $H\oplus K$ is a inner product space under $\langle(x,y),(x',y')\rangle=\langle x,x'\rangle+\langle y,y'\rangle$.

Then $\|(x,y)\|=\sqrt{\|x\|^2+\|y\|^2}$ is Complete if $\|\cdot\|_H$ & $\|\cdot\|_K$ are Complete.

Let $\{(x_n,y_n)\}$ be Cauchy. So

$\forall\varepsilon\ \exists N\ \forall m,n\geq N;$ $\|\underbrace{(x_n,y_n)-(x_m,y_m)}_{(x_n-x_m,\ y_n-y_m)}\|<\varepsilon$

$\|y_n-y_m\|$ & $\|x_n-x_m\|\leq\sqrt{\|x_n-x_m\|^2+\|y_n-y_m\|^2}=\|(x_n,y_n)-(x_m,y_m)\|$

Hence $\{x_n\}$ & $\{y_n\}$ are Cauchy. H&K are complete, so

$\exists x \in H\ \exists y \in K$; $x_n \to x$ & $y_n \to y$. We shall show that

$(x_n, y_n) \longrightarrow (x, y)$:

Let $\varepsilon > 0$ be given.

$$\exists N_1\ \forall n \geq N_1;\ \|x_n - x\| < \frac{\sqrt{\varepsilon}}{2}$$

$$\exists N_2\ \forall n \geq N_2;\ \|y_n - y\| < \frac{\sqrt{\varepsilon}}{3}$$

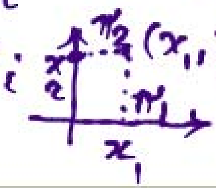
Put $N = \max\{N_1, N_2\}$. Then

$$\forall n \geq N:\ \|(x_n, y_n) - (x, y)\| = \sqrt{\|x_n - x\|^2 + \|y_n - y\|^2}$$

$$< \sqrt{\frac{\varepsilon}{4} + \frac{\varepsilon}{9}} < \varepsilon. \ \square$$

We need two special maps: $\pi_i : H_1 \oplus H_2 \longrightarrow H_i \quad (i = 1, 2)$

$(x_1, x_2) \longmapsto x_i$



projections

$\|\pi_i(x_1, x_2)\| = \|x_i\| \leq \|(x_1, x_2)\|$

$\|\pi_i\| \leq 1$

$\iota_i : H_i \longrightarrow H_1 \oplus H_2$.

$x_i \longmapsto (\cdots, 0, \underbrace{x_i}_{k\text{-th}}, 0, \cdots)$

inclusion $\quad \|\iota_i\| \leq 1$

Let $T_{ij} \in B(H_j, H_i)$ $(1 \leq i, j \leq 2)$. Set $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} : H_1 \oplus H_2 \to H_1 \oplus H_2$

$(x, y) \mapsto (T_{11}x + T_{12}y,\ T_{21}x + T_{22}y)$

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} T_{11}x + T_{12}y \\ T_{21}x + T_{22}y \end{bmatrix}$$

$$\|T(x, y)\| = \|(T_{11}x + T_{12}y,\ T_{21}x + T_{22}y)\| = \sqrt{\|T_{11}x + T_{12}y\|^2 + \|T_{21}x + T_{22}y\|^2}$$

$M = \max\{\|T_{ij}\| : 1 \leq i, j \leq 2\}$ $\quad \|x\|, \|y\| \leq \|(x, y)\|$

$$\leq \sqrt{\left(\|T_{11}\|\,\|x\|+\|T_{12}\|\,\|y\|\right)^2+\left(\|T_{21}\|\cdot\|x\|+\|T_{22}\|\,\|y\|\right)^2}$$

$$\leq \sqrt{8}\,M\,\|(x,y)\|$$

$$\therefore \|T\| \leq \sqrt{8}\,M < \infty$$

Conversely, if $T\in B(H_1\oplus H_2)$, then $T_{ij}=\pi_i T \tau_j$ $(1\leq i,j\leq 2)$

$$\in B(H_j, H_i)$$

$$H_j \xrightarrow{\tau_j} H_1\oplus H_2 \xrightarrow{T} H_1\oplus H_2 \xrightarrow{\pi_i} H_i$$

If we follow the construction in the previous paragraph (in red) we get an operator $\begin{bmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix}$ it is easy to see that $\begin{bmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix}=T$!

$\tau_1+\pi_1$

$$\begin{bmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}\pi_1 T\tau_1 x+\pi_1 T\tau_2 y\\ \pi_2 T\tau_1 x+\pi_2 T\tau_2 y\end{bmatrix}=\begin{bmatrix}\overbrace{\pi_1 T(x,0)+\pi_1 T(0,y)}^{\pi_1 T(x,y)}\\ \underbrace{\pi_2 T(x,0)+\pi_2 T(0,y)}_{\pi_2 T(x,y)}\end{bmatrix}$$

$$= T(x,y).\ \checkmark$$

In general, if $u\in B(X)$, $v\in B(Y)$, then

$$u\oplus v=\begin{bmatrix}u & 0\\ 0 & v\end{bmatrix}\in B(X\oplus Y).$$

$$\boxed{\begin{bmatrix}u & 0\\ 0 & v\end{bmatrix}\begin{bmatrix}x_n\\ y_n\end{bmatrix}=\begin{bmatrix}u x_n\\ v y_n\end{bmatrix}}$$

Exercise 1) $u\oplus v$ is compact iff so are u, v

2) $\sigma(u\oplus v)=\sigma(u)\cup\sigma(v)$

2) $sp(u \oplus v) = sp(u) \cup sp(v)$

$$\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} u-\lambda & 0 \\ 0 & v-\lambda \end{bmatrix}$$

Exercise 1) If $P \in B(X)$ is an idempotent. Then

$X = \operatorname{ran} P \oplus \operatorname{Ker} P$ $\quad x = Px + (x - Px)$

2) If $X = Y \oplus Z$ (internal), then $\exists P \in B(X)$;

$\operatorname{ran} P = Y$ & $\operatorname{Ker} P = Z$ $\quad P: X \longrightarrow X$, $P(y+z) = y$

Exercise. If X & Y are normed spaces, then

$\left(X \oplus_1 Y,\ \|(x,y)\| = \|x\| + \|y\|\right)$ & $\left(X \oplus_\infty Y,\ \|(x,y)\| = \max\{\|x\|, \|y\|\}\right)$

are normed spaces.

$X \oplus_1 Y$ and $X \oplus_\infty Y$ are Ban iff so are X & Y.

We define the *spectrum* of an element a to be the set

$$\sigma(a) = \sigma_A(a) = \{\lambda \in \mathbf{C} \mid \lambda 1 - a \notin \mathrm{Inv}(A)\}.$$

If A is not unital, then $sp(a) = sp_{\tilde{A}}(a, 0)$,

We shall henceforth find it convenient to write $\lambda 1$ simply as λ.

where $\tilde{A} = A \oplus \mathbb{C}$

1.2.1. *Example.* Let $A = C(\Omega)$, where Ω is a compact Hausdorff space. Then $\sigma(f) = f(\Omega)$ for all $f \in A$.

$f - \lambda 1$ is not inv. $\Leftrightarrow$ $\exists x;\ (f-\lambda 1)(x) = 0$ $\Leftrightarrow \exists x;\ f(x) = \lambda$ $\Leftrightarrow \lambda \in f(\Omega)$

$(a,\lambda) + (b,\mu) = (a+b, \lambda+\mu)$

1.2.2. *Example.* Let $A = \ell^\infty(S)$, where S is a non-empty set. Then $\sigma(f) = (f(S))^-$ (the closure in $\mathbf{C}$) for all $f \in A$.

?

$\gamma(a,\lambda) = (\gamma a, \gamma\lambda)$

$(a,\lambda)(b,\mu) = (ab + \mu a + \lambda b, \lambda\mu)$

is a unital Ban alg.

$1_{\tilde{A}} = (0,1)$

$(a,\lambda)(0,1) = (0 + a + 0, \lambda)$

1.2.7. Theorem (Beurling). *If a is an element of a unital Banach algebra A, then*

$$r(a) = \inf_{n \geq 1} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

1.2.2. Theorem. *Let A be a unital Banach algebra and a an element of A such that $\|a\| < 1$. Then $1 - a \in \mathrm{Inv}(A)$ and*

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

Proof. Since $\sum_{n=0}^\infty \|a^n\| \leq \sum_{n=0}^\infty \|a\|^n = (1 - \|a\|)^{-1} < +\infty$, the series $\sum_{n=0}^\infty a^n$ is convergent, to b say, in A, and since $(1-a)(1 + \cdots + a^n) = 1 - a^{n+1}$ converges to $(1-a)b = b(1-a)$ and to 1 as $n \to \infty$, the element b is the inverse of $1 - a$. $\square$

The series in Theorem 1.2.2 is called the *Neumann* series for $(1-a)^{-1}$.

1.2.3. Theorem. *If A is a unital Banach algebra, then* $\mathrm{Inv}(A)$ *is open in A, and the map*

$$\mathrm{Inv}(A) \to A, \quad a \mapsto a^{-1},$$

is differentiable.

$(b-a)a^{-1}$

Proof. Suppose that $a \in \mathrm{Inv}(A)$ and $\|b - a\| < \|a^{-1}\|^{-1}$. Then $\|ba^{-1} - 1\| \leq \|b - a\|\|a^{-1}\| < 1$, so $ba^{-1} \in \mathrm{Inv}(A)$, and therefore, $b \in \mathrm{Inv}(A)$. Thus, $\mathrm{Inv}(A)$ is open in A.

$b = (ba^{-1})a$ $\qquad N_{\|a^{-1}\|^{-1}}(a) \subseteq \mathrm{Inv}(A)$

Note that if $\varphi: A \to B$ is a homomorphism between algebras A and B ~~and B is unital~~, then $\tilde{\varphi}: \tilde{A} \to \tilde{B}$, $a + \lambda \mapsto \varphi(a) + \lambda$, $(a \in A,\ \lambda \in \mathbf{C})$ is the unique unital homomorphism extending φ.

If $\varphi: A \to B$ is a unital homomorphism between unital algebras, then $\varphi(\mathrm{Inv}(A)) \subseteq \mathrm{Inv}(B)$, so $\sigma(\varphi(a)) \subseteq \sigma(a)$ $(a \in A)$.

A *character* on an abelian algebra A is a non-zero homomorphism $\tau: A \to \mathbf{C}$. We denote by $\Omega(A)$ the set of characters on A.

$\tau(1) = \tau(1 \cdot 1) = \tau(1)\tau(1) \Rightarrow \tau(1) = 1$

1.3.3. Theorem. *Let A be a unital abelian Banach algebra.*

(1) *If $\tau \in \Omega(A)$, then $\|\tau\| = 1$.*

(2) *The set $\Omega(A)$ is non-empty, and the map*

$$\tau \mapsto \ker(\tau)$$

is a 1-1 & surjective map between $\Omega(A)$ and maximal modular ideals of A.

$\tau \neq 0$: $\tau(1)=0 \Rightarrow \tau(a)\tau(1)=0 \ \forall a \Rightarrow \tau(a)=0 \Rightarrow \tau=0$ $\forall\sigma$ ✗

$\tau(a-\tau(a)1)=0$

$\Downarrow$

$a-\tau(a)\in \ker\tau$

$\tau(a)\in\sigma(a)$

If $a-\tau(a)1$ is inv., then $\exists b$ $b(a-\tau(a)1)=1$, so $0=\tau(b)\tau(a-\tau(a)1)=\tau(1)=1$ ✗.

$\Downarrow$

$|\tau(a)|\le\|a\|$, $\tau(1)=1$ $\Rightarrow \|\tau\|=1$

1.3.4. Theorem. *Let A be an abelian Banach algebra.*

(1) *If A is unital, then*

$$\sigma(a) = \{\tau(a) \mid \tau \in \Omega(A)\} \quad (a \in A).$$

(2) *If A is non-unital, then*

$$\sigma(a) = \{\tau(a) \mid \tau \in \Omega(A)\} \cup \{0\} \quad (a \in A).$$

(i) $\tau\in\Omega(A) \Rightarrow \exists\, \tilde\tau:\tilde A\to\mathbb{C}$; $\tilde\tau|_A=\tau$ in fact $\tilde\tau(a+\lambda 1)=\tau(a)$ $\Rightarrow \tilde\tau\in\Omega(\tilde A)$

(ii) $\tilde\tau\in\Omega(\tilde A)\Rightarrow \begin{cases}\tilde\tau|_A\neq 0 \Rightarrow \tilde\tau|_A\in\Omega(A)\\ \tilde\tau|_A=0\Rightarrow \tilde\tau:\tilde A\to\mathbb{C};\ \tilde\tau(a+\lambda 1)=\lambda \ (\text{Put } \tilde\tau=\tau_\infty)\end{cases}$

$\lambda\in sp(a)\Rightarrow a-\lambda 1$ is not inv. $\Rightarrow A(a-\lambda 1)\neq A \xRightarrow{\text{Zorn's Lemma}} \exists M;\ A(a-\lambda 1)\subseteq M$ (modular) maximal ideal $\Rightarrow \exists\tau;\ \ker\tau=M \Rightarrow \tau(A(a-\lambda 1))=0 \Rightarrow \tau(a-\lambda 1)=0\Rightarrow \lambda=\tau(a)$

1.3.5. Theorem. *If A is an abelian Banach algebra, then $\Omega(A)$ is a locally compact Hausdorff space. If A is unital, then $\Omega(A)$ is compact.*

Proof. It is easily checked that $\Omega(A) \cup \{0\}$ is weak* closed in the closed unit ball S of A^*. Since S is weak* compact (Banach–Alaoglu theorem), $\Omega(A) \cup \{0\}$ is weak* compact, and therefore, $\Omega(A)$ is locally compact.

If A is unital, then $\Omega(A)$ is weak* closed in S and thus compact. □

Proof (2) of Th. 1.3.4: $\sigma(a)=\sigma_{\tilde A}(a)=\{\tilde\tau(a)\mid\tilde\tau\in\Omega(\tilde A)\}=\{\tau(a)\mid\tau\in\Omega(A)\}\cup\{\tau_\infty(a)\}$, $\tau_\infty(a)=0$

Proof of Th 1.3.5

unital $\Omega(A)=\{\tau\in A' \mid \widehat{ab}(\tau)-\hat a(\tau)\hat b(\tau)=0\}$ $=\underbrace{(\widehat{ab}-\hat a\hat b)}_{\text{cts}}{}^{-1}\underbrace{(\{0\})}_{\text{closed}}$ closed in weak*-top

$\subseteq S=(A')_1 \to$ compact in weak*-top by Ban-Alaoglu Theorem

1.3.6. Theorem (Gelfand Representation). *Suppose that A is an abelian Banach algebra and that $\Omega(A)$ is non-empty. Then the map*

$$A \to C_0(\Omega(A)), \ a \mapsto \hat a,$$

$\hat a(\tau)=\tau(a)$

is a norm-decreasing homomorphism, and

$$r(a) = \|\hat a\|_\infty \qquad (a \in A).$$

If A is unital, $\sigma(a) = \hat a(\Omega(A))$, and if A is non-unital, $\sigma(a) = \hat a(\Omega(A)) \cup \{0\}$, for each $a \in A$.

$f: X\to Y$ function $\Rightarrow f^{-1}(B)=\{x\in X\mid f(x)\in B\}$ & $f(A)=\{f(x)\mid x\in A\}$

If A is not unital, then $\Omega(A)\cup\{0\}$ is still compact, so $\Omega(A)$ is a locally compact Hausdorff space.

$\underbrace{\Omega}(A)\subseteq A'$, $(A', \text{weak}^*\text{-top})$ what is the weak*-top?

maximal ideal space
$\equiv$ character space

Let X be a vector space & $\mathcal{F}_0 = \{P_\alpha\}_{\alpha\in I}$ be a separating family of semi-norms on X. $\exists!$ locally convex top. vector sp. $\tau(X,\mathcal{F}_0)$ on X such that all P_α are continuous:

$B(P_{\alpha_1},\dots,P_{\alpha_n},\varepsilon) = \{x\in X \mid |P_{\alpha_i}(x)|<\varepsilon \ \forall i=1,\dots,n\}$ give a subbasis for a top on X.

Since $\forall\varepsilon\ \exists \underset{B(P_\alpha,\varepsilon)}{\overset{\text{open}}{G}}\ \forall x\in G$; $\quad |P_\alpha(x)-\cancel{P_\alpha(0)}|<\varepsilon$ (*)

P_α is cts at zero. Now $|P_\alpha(x)-P_\alpha(y)|\le P_\alpha(x-y)$ implies that the continuity of P_α at any point $y\in X$: $\forall\varepsilon\ \exists V^{\text{open}} \ni y$, $V = G+y$, $\forall x\in V$: $|P_\alpha(x)-P_\alpha(y)|\le P_\alpha(\underbrace{x-y}_{\in G}) \overset{(*)}{<} \varepsilon$. $\square$

This topology makes X into a l.c.t.v.s.

Next note that if $x_i \xrightarrow{\sigma(X,\mathcal{F}_0)} 0$ then $\forall p\in\mathcal{F}_0$: $p(x_i)\to$

Since all $p\in\mathcal{F}_0$ are cts.

Conversely, if $\forall p\in\mathcal{F}_0$: $px_i\to 0$, then $x_i \xrightarrow{\sigma(X,\mathcal{F}_0)} 0$ since

$$\forall \underset{\overset{\cup}{0}}{S}\in\sigma(X,\mathcal{F}_0)\ \exists i_0\ \forall i\geq i_0;\quad x_i\in S$$

subbasis

$$B(p_{\alpha_1},p_{\alpha_2},\dots,p_{\alpha_n},\varepsilon)$$

$p_{\alpha_1}(x_i)\to 0$, so $\exists i_1\ \forall i\geq i_1$: $|p_{\alpha_1}(x_i)|<\varepsilon$

$\vdots$

$p_{\alpha_n}(x_i)\to 0$, so $\exists i_n\ \forall i\geq i_n$; $|p_{\alpha_n}(x_i)|<\varepsilon$

Put $i_0=\max\{i_1,\dots,i_n\}$. Then $\forall i\geq i_0$;

$$\underbrace{|p_{\alpha_j}(x_i)|<\varepsilon \quad (j=1,\dots,n)}_{x_i\in B(p_{\alpha_1},\dots,p_{\alpha_n},\varepsilon)=S}$$

In general, if X is a l.c.t.v.s., then there exists a separating family of seminorms $\{p_\alpha\}$ which generates the given top on X. These p_α's are the Minkowski functionals of a certain family of balanced absorbing convex subsets

Weak top on a normed space X: $\sigma(X,X')$

$$p_f(x)=|f(x)| \qquad (f\in X')$$

separating seminorms

$\forall x\in X\ \exists p_f$: $p_f(x)\neq 0$ (Hahn-Ban Th)

$$x_\alpha \xrightarrow{\text{weak}} x \iff x_\alpha - x \xrightarrow{\text{weak}} 0 \iff \forall p_f;\ \underbrace{p_f(x_\alpha - x)}_{|f(x_\alpha)-f(x)|} \to 0$$

$$\iff \forall f \in X';\ f(x_\alpha) \to f(x)$$

weak*-top on X' : $\sigma(X', X)$

$$p_x(f) = |f(x)| \qquad (x \in X)$$

$$f_\alpha \xrightarrow{w^*\text{-top}} f \iff f_\alpha - f \xrightarrow{w^*\text{-top}} 0 \iff \forall x;\ \underbrace{p_x(f_\alpha - f)}_{|f_\alpha(x) - f(x)|} \to 0$$

$$\iff \forall x;\ f_\alpha(x) \to f(x) \iff f_\alpha \xrightarrow{p} f$$

p → Pointwise

Strong operator top on $B(H)$: S.O.T

$$p_\xi(T) = \|T\xi\| \qquad (\xi \in H)$$

$$T_\alpha \xrightarrow{SOT} T \iff T_\alpha - T \xrightarrow{SOT} 0 \iff \forall \xi:\ \underbrace{p_\xi(T_\alpha - T)}_{\|T_\alpha\xi - T\xi\|} \to 0$$

$$\iff T_\alpha \xi \to T\xi$$

Weak operator top on $B(H)$: WOT

$$\boxed{x_\alpha \to x \iff \|x_\alpha - x\| \to 0 \ ?}$$

$$p_{\xi,\eta}(T) = |\langle T\xi, \eta\rangle| \qquad (\xi, \eta \in H)$$

$$T_\alpha \xrightarrow{WOT} T \iff \forall \xi, \eta;\ \langle T_\alpha \xi, \eta\rangle \to \langle T\xi, \eta\rangle.$$

[illegible]

$$WOT \leq SOT \leq \|\cdot\|$$

Theorem. Let A be a unital C^*-alg & $a\in A$ be normal. Then $f(\sigma(a))=\sigma(f(a))$ $\forall f\in C(\sigma(a))$.

Proof. $\sigma(f(a))=\{\tau(\underbrace{f(a)}_{\lim_n P_n(a)}) \mid \tau\in\Omega(A)\}=\{\lim_n \underbrace{\tau(P_n(a))}_{P_n(\tau(a))} \mid \tau\in\Omega(A) \text{ for some } \{P_n\}\}$

$$=\{f(\tau(a)) \mid \tau\in\Omega(A)\}=f(\sigma(a)).\ \square$$

≡ Positive elements of a C^*-alg ≡

$$\mathbb{R}^+\subseteq\mathbb{R}\subseteq\mathbb{C} \longleftrightarrow A_+\subseteq A_h\subseteq A$$

Def. $a\in A$ is positive if $\sigma(a)\subseteq\mathbb{R}^{\geq 0}$ & $a=a^*$. Then we write $a\geq 0$

Example $f\in C(X)$ is positive iff $\underbrace{f(X)=\sigma(f)\subseteq\mathbb{R}^{\geq 0}}_{\forall x;\ f(x)\geq 0}$ $\underbrace{\bar f=f}_{\forall x;\ f(x)\in\mathbb{R}}$

Note. $\underset{f\le t_0}{\overset{C(X)\ni}{f\geq 0}} \Rightarrow |f-t_0|\leq t_0$ $\left(\forall x\in X;\ \underbrace{|f(x)-t_0|\leq t_0}_{0=t_0-t_0\leq f(x)\leq t_0+t_0}\right)$

$$|f-t_0|\leq t_0 \Longrightarrow f\geq 0$$

Def. $\underset{a,b\in A_h}{a\leq b} \Leftrightarrow b-a\geq 0$

Example. $\underset{f,g\in C(X)}{f\leq g} \Leftrightarrow g-f\geq 0 \Leftrightarrow \forall x;\ g(x)-f(x)\geq 0 \Leftrightarrow \forall x;\ f(x)\leq g(x)$

Theorem. Let A be a unital C^*-alg. $\forall a\in A_+\ \exists! b\in C^*(a)_+;\ b^2=a$ $\left(b:=a^{\frac{1}{2}}\right)$

Proof. $\varphi: C(\sigma(a)) \underset{\text{isomet}}{\overset{*\text{-isomor}}{\longleftrightarrow}} C^*(a,1)$

$$t \longleftrightarrow a$$

$$1 \longleftrightarrow 1$$

$$\sqrt{t}^2=t \longrightarrow \varphi(\sqrt{t}^2)=\varphi(t)=a$$

$$\underbrace{\varphi(\sqrt{t})}_{b}\varphi(\sqrt{t}) = b^2$$

$$\boxed{\text{Let } c^2 = a\ ,\ c \geq 0}$$

$$f(t) \longleftarrow c$$

$$f(t)=\sqrt{t} \Leftarrow \begin{cases} f(t)^2 = t \\ f(t) \geq 0 \end{cases} \longleftarrow \begin{cases} c^2 = a \\ c \geq 0 \end{cases}$$

$$\dashrightarrow \underbrace{\varphi(f(t))}_{c} = \underbrace{\varphi(\sqrt{t})}_{b} \ . \ \square$$

Th. $\forall a \in A_h \ \exists a_+, a_- \in A_+ ;\ \begin{cases} a = a_+ - a_- \\ |a| = a_+ + a_- \\ a_+ a_- = a_- a_+ = 0 . \end{cases}$

Proof. $C^*(a,1) \xleftrightarrow{*\text{-isom}} C(\sigma(a))$

$$a \longleftrightarrow t$$

$$1 \longleftrightarrow 1$$

$$(a^*a)^{\frac{1}{2}} = |a| \longleftrightarrow |t| = (\bar{t}t)^{\frac{1}{2}}$$

$$\underbrace{f(a)}_{a_+} = \frac{a+|a|}{2} \geq 0 \longleftarrow f(t) = \frac{t+|t|}{2} \geq 0$$

$$\underbrace{g(a)}_{a_-} = \frac{|a|-a}{2} \geq 0 \longleftarrow g(t) = \frac{|t|-t}{2} \geq 0$$

$$\begin{cases} a = a_+ - a_- \\ |a| = a_+ + a_- \\ a_+ a_- = a_- a_+ = 0 \end{cases} \longleftarrow \begin{cases} t = f(t) - g(t) \\ |t| = f(t) + g(t) \\ f(t)g(t) = g(t)f(t) = 0 \end{cases} . \ \square$$

Th. $\forall a \in A_+,\ \|a\| \leq 1 \ \exists u_1, u_2 \in \underbrace{\mathcal{U}(A)}_{\text{unitaries}} ;\ a = \frac{1}{2}(u_1 + u_2)$

Proof. $C^*(a,1) \xleftrightarrow{\text{isomet}} C(\sigma(a))$

$$a \longleftrightarrow t \qquad \iota(t) = t$$

$$1 \longleftrightarrow 1$$

$$\|a\| \leq 1 \longrightarrow \|a\| = \|\iota\| = \sup_{t \in \sigma(a)} |t| \leq 1$$

$t\in\sigma(a) \Downarrow$

$|t|\le 1 \ \forall t\in\sigma(a)$

unitary $u := a + i\sqrt{1-a^2} \longleftarrow t + i\sqrt{1-t^2}$

$u^*u = uu^* = 1 \longleftarrow (t - i\sqrt{1-t^2})(t+i\sqrt{1-t^2}) = t^2 + (1-t^2) = 1$

unitary $v := a - i\sqrt{1-a^2} \longleftarrow t - i\sqrt{1-t^2}$

$v^*v = vv^* = 1$

$$a = \frac{u+v}{2} \longleftarrow t = \frac{t+i\sqrt{1-t^2} + (t - i\sqrt{1-t^2})}{2} \quad \square$$

Corollary. Each element $a\in\mathcal{A}$ is a linear combination of unitaries.

Proof. $a = a_1 + ia_2 \quad (a_1, a_2 \in \mathcal{A}_h)$

$$= (a_{1+} - a_{1-}) + i(a_{2+} - a_{2-})$$

$$= \left(\frac{u_1+v_1}{2} - \frac{u_2+v_2}{2}\right) + i\left(\frac{u_3+v_3}{2} - \frac{u_4+v_4}{2}\right) \square$$

Th. $\forall a,b\in\mathcal{A}_+ ; \quad a+b\ge 0$

Proof. By the functional calculus:

$C^*(a,1) \longleftrightarrow C(\sigma(a))$

$a\ge 0 \longrightarrow f(t)\ge 0 \ \forall t\in\sigma(a)$, $f(t) = t$

$f(t)\le \|f\|$

$\Rightarrow \sup_t |f(t) - \|f\|| \le \|f\|$

$\boxed{\text{always: } a \le \|a\| 1 \longleftarrow t \le \sup_{t\in\sigma(a)} |t| \quad \forall a\in\mathcal{A}_h}$

$\|a - \|a\|\,\| \le \|a\| \longleftarrow$

$\|a - t_0\| \le t_0 \longrightarrow |f(t) - t_0| \le t_0 \ \forall t\in\sigma(a)$, $f(t) = t$

$\Downarrow$

$a\ge 0 \longleftarrow f(t)\ge 0 \ \forall t\in\sigma(a)$

$a \geq 0 \Rightarrow \| a - \|a\| \| \leq \|a\|$

$b \geq 0 \Rightarrow \| b - \|b\| \| \leq \|b\|$

$\Rightarrow \left\| (a+b) - \underbrace{\|a\| + \|b\|}_{t_0} \right\| \leq \| a - \|a\| \| + \| b - \|b\| \| \leq \underbrace{\|a\| + \|b\|}_{t_0} \Rightarrow a+b \geq 0.$ □

Th. **Every abelian C^*-alg $\mathcal{A}$ is isom $*$-isom to $C_0(\Omega)$, where Ω is the character space of $\mathcal{A}$.**

Proof.

$$\mathcal{A} \xrightarrow[\text{isom.}]{*\text{-isom}} C_0(\Omega)$$

$$x \longmapsto \hat{x} \quad ; \quad \hat{x}(\tau) = \tau(x)$$

Sometimes you are dealing with two elements a, b of an arbitrary C^*-alg such that $ab = ba$. Then you may consider abelian C^*-alg generated by $a, b, 1$: $C^*(a,b) = \bigcap_{\substack{a,b \in \mathcal{B} \\ \mathcal{B} \text{ is abelian}}} \mathcal{B}$

Fortunately $C^*(a,b,1) \longleftrightarrow C(\Omega(C^*(a,b)))$

$a \longleftrightarrow f$

$b \longleftrightarrow g$

2.2.4. Theorem. *If a is an arbitrary element of a C^*-algebra A, then a^*a is positive.*

$\boxed{\text{باید عمیق و زیاد بخوانید}}$ $\bar z z = |z|^2 \geq 0$

Proof. First we show that $a = 0$ if $-a^*a \in A^+$. Since $\sigma(-aa^*) \setminus \{0\} = \sigma(-a^*a) \setminus \{0\}$ by Remark 1.2.1, $-aa^* \in A^+$ because $-a^*a \in A^+$. Write $a = b + ic$, where $b, c \in A_{sa}$. Then $a^*a + aa^* = 2b^2 + 2c^2$, so $a^*a = 2b^2 + 2c^2 - aa^* \in A^+$. Hence, $\sigma(a^*a) = \mathbf{R}^+ \cap (-\mathbf{R}^+) = \{0\}$, and therefore $\|a\|^2 = \|a^*a\| = r(a^*a) = 0$. $\sigma(a^*a) = -\sigma(-a^*a)$ [illegible]

Now suppose a is an arbitrary element of A, and we shall show that a^*a is positive. If $b = a^*a$, then b is hermitian, and therefore we can write $b = b^+ - b^-$. If $c = ab^-$, then $-c^*c = -b^-a^*ab^- = -b^-(b^+ - b^-)b^- = (b^-)^3 \in A^+$, so $c = 0$ by the first part of this proof. Hence, $b^- = 0$, so $a^*a = b^+ \in A^+$. □

[illegible]

$a \in A_h \Rightarrow \sigma(a^2) = \sigma(a)^2 = \{\lambda^2 \mid \lambda \in \sigma(a)\} \subseteq \mathbb{R}^{\geq 0} \Rightarrow a^2 \geq 0$, $\sigma(a) \subseteq \mathbb{R}$

2.2.5. Theorem. *Let A be a C^*-algebra.*

$ab^- = 0 \Rightarrow \underbrace{a^*a}_{b = b^+ - b^-}b^- = 0 \Rightarrow {b^-}^2 = 0 \Rightarrow \|{b^-}^2\| = 0 \Rightarrow \|{b^-}^*b^-\| = 0 \Rightarrow \|b^-\|^2 = 0 \Rightarrow b^- = 0$

(1) *The set A^+ is equal to $\{a^*a \mid a \in A\}$.*
(2) *If $a, b \in A_{sa}$ and $c \in A$, then $a \leq b \Rightarrow c^*ac \leq c^*bc$.*
(3) *If $0 \leq a \leq b$, then $\|a\| \leq \|b\|$.* $-9 \leq -3 \not\Rightarrow |-9| \leq |-3|$
(4) *If A is unital and a, b are positive invertible elements, then $a \leq b \Rightarrow 0 \leq b^{-1} \leq a^{-1}$.*

(1) $b \in A^+ \Rightarrow \exists\, \underset{b^{\frac{1}{2}} \geq 0}{a} \in A;\ b = b^{\frac{1}{2}} b^{\frac{1}{2}} = a^*a$ ✓

(2) $a \leq b \Rightarrow b - a \geq 0 \Rightarrow c^*(b-a)c = \underbrace{c^*(b-a)^{\frac{1}{2}}}_{d^*}\underbrace{(b-a)^{\frac{1}{2}}c}_{d} \geq 0 \Longrightarrow c^*ac \leq c^*bc$

(3) $0 \leq a \leq b \xRightarrow{b \leq \|b\|} a \leq \|b\| \Rightarrow \underset{\forall t \in \sigma(a)}{0 \leq t \leq \|b\|} \Rightarrow \sup_{t \in \sigma(a)} |t| \leq \|b\| \Rightarrow \|a\| \leq \|b\|$

multiply by $b^{-\frac{1}{2}}$

(4) $0 \leq a \leq b \Rightarrow 0 \leq b^{-\frac{1}{2}}ab^{-\frac{1}{2}} \leq b^{-\frac{1}{2}}bb^{-\frac{1}{2}} = 1 \Rightarrow (b^{-\frac{1}{2}}ab^{-\frac{1}{2}})^{-1} \geq 1 \Rightarrow b^{\frac{1}{2}}a^{-1}b^{\frac{1}{2}} \geq 1 \Rightarrow a^{-1} \geq b^{-1}$

Proof. Conditions (1) and (2) are implied by Theorem 2.2.4 and the existence of positive square roots for positive elements. To prove Condition (3) we may suppose that A is unital. The inequality $b \leq \|b\|$ is given by the Gelfand representation applied to the C*-algebra generated by 1 and b. Hence, $a \leq \|b\|$. Applying the Gelfand representation again, this time to the C*-algebra generated by 1 and a, we obtain the inequality $\|a\| \leq \|b\|$.

To prove Condition (4) we first observe that if $c \geq 1$, then c is invertible and $c^{-1} \leq 1$. This is given by the Gelfand representation applied to the C*-subalgebra generated by 1 and c. Now $a \leq b \Rightarrow 1 = a^{-1/2}aa^{-1/2} \leq a^{-1/2}ba^{-1/2} \Rightarrow (a^{-1/2}ba^{-1/2})^{-1} \leq 1$, that is, $a^{1/2}b^{-1}a^{1/2} \leq 1$. Hence, $b^{-1} \leq (a^{1/2})^{-1}(a^{1/2})^{-1} = a^{-1}$. □

2.2.6. Theorem. *If a, b are positive elements of a C^*-algebra A, then the inequality $a \le b$ implies the inequality $a^{1/2} \le b^{1/2}$.* $a\le b \Rightarrow (a^{\frac12})^2 \le (b^{\frac12})^2 \Rightarrow a^{\frac12}\le b^{\frac12}$

Proof. We show $\boxed{a^2 \le b^2 \Rightarrow a \le b}$ and this will prove the theorem. We may suppose that A is unital. Let $t > 0$ and let c, d be the real and imaginary hermitian parts of the element $(t+b+a)(t+b-a)$. Then $=c+id$

$$\begin{aligned} c &= \tfrac{1}{2}((t+b+a)(t+b-a)) + (t+b-a)(t+b+a)) \\ &= t^2 + \underbrace{2tb}_{\ge 0} + \underbrace{b^2 - a^2}_{\ge 0} \\ &\ge \underbrace{t^2}_{>0} \end{aligned}$$

$a + \underbrace{b}_{\ge 0} \ge a$

Consequently, c is both $\xrightarrow{\text{funct. Calculus}}$ invertible and positive. Since $1 + i\overbrace{c^{-1/2}dc^{-1/2}}^{\text{self-adj}} = c^{-1/2}(c+id)c^{-1/2}$ is invertible, therefore $c+id$ is invertible. It follows that $t+b-a$ is left invertible, and therefore invertible, because it is hermitian. Consequently, $\underbrace{-t}_{<0} \notin \sigma(b-a)$. Hence, $\sigma(b-a) \subseteq \mathbf{R}^+$, so $b-a$ is positive, that is, $a \le b$. □

$$\boxed{\begin{array}{c} ab = 1 \ \&\ b \in A_h \\ \Downarrow \\ ba^* = 1 \end{array}}$$

It is not true that $0 \le a \le b \Rightarrow a^2 \le b^2$ in arbitrary C*-algebras. For example, take $A = M_2(\mathbf{C})$. This is a C*-algebra where the involution is given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* = \begin{pmatrix} \bar\alpha & \bar\gamma \\ \bar\beta & \bar\delta \end{pmatrix}.$$

Let p and q be the projections

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad q = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then $p \le p+q$, but $p^2 = p \not\le (p+q)^2 = p+q+pq+qp$, since the matrix

$$q + pq + qp = \tfrac{1}{2}\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

has a negative eigenvalue.

It can be shown that the implication $0 \le a \le b \Rightarrow a^2 \le b^2$ holds only in abelian C*-algebras [Ped, Proposition 1.3.9].

If $\underbrace{a \ge 0}_{\Rightarrow \sigma(a)\subseteq[0,\infty)}$ & $\underbrace{a \text{ is invertible}}_{0\notin\sigma(a)}$, ~~then~~ then $\underbrace{\sigma(a)}_{\text{compact}} \subseteq (0,\infty)$.

Hence $\exists m, M$ $\forall t \in \sigma(a)$; $0 < m \le t \le M$

$\sigma(a) \ni m = \underbrace{\inf_{t\in\sigma(a)} t}_{\ge 0}$ $\qquad$ $M = \underbrace{\sup_{t\in\sigma(a)} t}_{\ge 0} \in \overline{\sigma(a)} = \sigma(a)$

Therefore by functional calculus, a^{-1} exists.

$$\left.\begin{array}{l} a \longrightarrow t \\ b \longleftarrow \frac{1}{t} \\ ab=1 \longleftarrow t\cdot\frac{1}{t}=1 \end{array}\right\} \Rightarrow b=a^{-1}$$

$$\left.\begin{array}{l} a \text{ inv.} \\ 0\le a\le 1 \end{array}\right\} \Rightarrow a^{-1}\ge 1$$

$$a^{\text{inv}} \longrightarrow t$$

$$0\le a\le 1 \longrightarrow 0<t\le 1$$

$$a^{-1}\ge 1 \longleftarrow \frac{1}{t}\ge 1$$

Exercise. $\left.\begin{array}{l} ab=ba \\ a\ge 0,\ b\ge 0 \end{array}\right\} \Rightarrow ab\ge 0$

Proof 1) $A=C^*(a,b) \longleftrightarrow C(\Omega(A))$

$$a \longrightarrow f\ge 0$$

$$b \longrightarrow g\ge 0$$

$$ab\ge 0 \longleftarrow fg\ge 0$$

Proof 2) $\left.\begin{array}{l} \sigma(ab)\underset{ab=ba}{\subseteq}\sigma(a)\sigma(b)\subseteq\mathbb{R}^{\ge 0} \\ (ab)^*=b^*a^*=ba=ab \end{array}\right\} \Rightarrow ab\ge 0$

Proof 3) $ab=ab^{\frac{1}{2}}b^{\frac{1}{2}}=b^{\frac{1}{2}}ab^{\frac{1}{2}}\ge \underbrace{b^{\frac{1}{2}}0\,b^{\frac{1}{2}}}_{0}$

$ab=ba$
$\Rightarrow aP_n(b)=P_n(b)a$
$\Rightarrow af(b)=f(b)a$
$\Rightarrow f\in C(\sigma(b))$

If $a\in A_h$, then $1+ia$ is inv.

$$(1+ia)^{-1} \longleftarrow \frac{1}{1+ti}, \qquad a \longrightarrow t$$

$t\in\mathbb{R}\Rightarrow 1+ti\neq 0$

2.3.1. Theorem. *Let H_1 and H_2 be Hilbert spaces.*

(1) *If $u \in B(H_1, H_2)$, then there is a unique element $u^* \in B(H_2, H_1)$ such that*

$$\langle u(x_1), x_2\rangle = \langle x_1, u^*(x_2)\rangle \qquad (x_1 \in H_1,\ x_2 \in H_2).$$

(2) *The map $u \mapsto u^*$ is conjugate-linear and $u^{**} = u$. Also*

$$\|u\| = \|u^*\| = \|u^*u\|^{1/2}.$$

The proof left to the students.

(1): $y \in \ker u^* \Leftrightarrow u^*y = 0 \Leftrightarrow \langle x, u^*y\rangle = 0\ \forall x \Leftrightarrow \langle ux, y\rangle = 0\ \forall x \Leftrightarrow y \in \operatorname{ran} u^{\perp}$

(2) $\ker u = \operatorname{im}(u^*)^{\perp}$ (by (1)) $\Rightarrow \ker u^{\perp} = \operatorname{im}(u^*)^{\perp\perp} = \overline{\operatorname{im}(u^*)}^{\perp\perp} = \overline{\operatorname{im}(u^*)}$ ($M^{\perp\perp} = M$ if M is closed subsp)

If $u: H_1 \to H_2$ is a continuous linear map between Hilbert spaces, we call the map $u^*: H_2 \to H_1$ the *adjoint* of u. Note that $\ker(u^*) = (\operatorname{im}(u))^{\perp}$, where $\operatorname{im}(u)$ is the range of u, and hence, $(\operatorname{im}(u^*))^{-} = \ker(u)^{\perp}$. (1) (2)

$u\left(\sum_{n=1}^{\infty} \alpha_n e_n\right) = \sum_{n=1}^{\infty} \lambda_n \alpha_n e_n$, $u = \left(\begin{array}{c} ue_j \end{array} \right) = \left(\begin{array}{ccc} & 0 & \\ \cdots & \lambda_j & \cdots \\ & 0 & \end{array} \right)$. the jth row

the jth column

2.3.1. *Example.* Let $(e_n)_{n=1}^{\infty}$ be an orthonormal basis for a Hilbert space H, and suppose that u is an operator diagonal with respect to (e_n), with diagonal sequence (λ_n). Then u^* is also diagonal with respect to (e_n) and its diagonal sequence is $(\bar{\lambda}_n)$. This follows from the observation that $\langle u^*(e_n), e_m\rangle = \langle e_n, u(e_m)\rangle = \langle e_n, \lambda_m e_m\rangle = \bar{\lambda}_m \delta_{nm}$, where δ_{nm} is the Kronecker delta symbol, which implies that $u^*(e_n) = \bar{\lambda}_n e_n$. Since all operators diagonal with respect to the same basis commute, $uu^* = u^*u$; that is, u is normal.

$u^*_{ij} = \langle u^* e_j, e_i\rangle$

$u\left(\sum_{n=1}^{\infty} \alpha_n e_n\right) = \sum_{n=1}^{\infty} \alpha_n e_{n+1}$ or $(\alpha_1, \alpha_2, \ldots) \longmapsto (0, \alpha_1, \alpha_2, \ldots)$

2.3.2. *Example.* Let (e_n) and H be as in the preceding example, but this time let u denote the unilateral shift on this basis, so $u(e_n) = e_{n+1}$ for all $n \geq 1$. The adjoint u^* is the backward shift: $u^*(e_n) = e_{n-1}$ if $n > 1$ and $u^*(e_1) = 0$. It follows that $u^*u = 1$. It is easily seen that u has no eigenvalues. In contrast, u^* has very many, for if $|\lambda| < 1$, then λ is an eigenvalue: Set $x = \sum_{n=1}^{\infty} \lambda^n e_n$ and observe that $x \in H$ because $\sum_{n=1}^{\infty} |\lambda|^{2n} < \infty$, and that $x \neq 0$ and $u^*(x) = \lambda x$. It follows from this, and the fact that $\|u^*\| = \|u\| = 1$, that $\sigma(u) = \sigma(u^*) = \mathbf{D}$.

Incidentally, if $(f_n)_{n=1}^{\infty}$ is an orthonormal basis for another Hilbert space K and v is the unilateral shift on (f_n), so $v(f_n) = f_{n+1}$, then $v = wuw^*$, where $w: H \to K$ is the unitary operator such that $w(e_n) = f_n$ for all $n \geq 1$. From the abstract point of view, the operators u and all $n \geq 1$. From the abstract point of view, the operators u and v are therefore the same, so one can speak of "the" unilateral shift.

2.3.2. Theorem. *Let p, q be projections on a Hilbert space H. Then the following conditions are equivalent:* $p^2 = p = p^*$

(1) $p \leq q$. $\equiv q - p \geq 0$
(2) $pq = p$.
(3) $qp = p$.
(4) $p(H) \subseteq q(H)$.
(5) $\|p(x)\| \leq \|q(x)\| \quad (x \in H)$.
(6) $q - p$ *is a projection.*

(2)⇒(3): $pq = p \Rightarrow (pq)^* = p^* \Rightarrow qp = p$
(3)⇒(2): Similar to (2)⇒(3)
(3)⇒(4) $\forall h \in H$; $p\,h = q(ph) \in q(H)$
(4)⇒(3) $\forall h \in H$, $qp(h) = p(h)$

ran $p \subseteq$ ran q

Proof. Equivalence of Conditions (2),(3), and (4) is clear, as are the implications (2) ⇒ (6) ⇒ (1). We show (1) ⇒ (5) ⇒ (2), and this will prove the theorem. $\langle qx, x\rangle = \langle qx, qx\rangle$

If we assume Condition (1) holds, $\|q(x)\|^2 - \|p(x)\|^2 = \langle (q-p)(x), x\rangle = \|(q-p)^{1/2}(x)\|^2 \geq 0$, so Condition (5) holds.

If now we assume Condition (5) holds, $\|p(1-q)(x)\| \leq \|(q-q^2)(x)\| = 0$, and therefore $p = pq$; that is, Condition (2) holds. $q(1-q)(x)$ □

A continuous linear map $u: H_1 \to H_2$ between Hilbert spaces H_1, H_2 is a *partial isometry* if u is isometric on $\ker(u)^\perp$, that is, $\|u(x)\| = \|x\|$ for all $x \in \ker(u)^\perp$.

$\ker u = \{0\}$ & $u \in B(H)$ partial isom $\Rightarrow H = \ker u \oplus \ker u^\perp \xRightarrow{\ker u = 0} H = \ker u^\perp \Rightarrow u$ is isometry on H

2.3.3. Theorem. *Let H_1, H_2 be Hilbert spaces and $u \in B(H_1, H_2)$. Then the following conditions are equivalent:*

(*) Let $v = u^*u$. So $v^3 = v^2$
$v^3 - v^2 = 0 \Rightarrow sp(v^3 - v^2) = \{0\} \Rightarrow \{\lambda^3 - \lambda^2 \mid \lambda \in sp\, v\} = \{0\}$
$\Rightarrow sp(v) \subseteq \{1, 0\} \Longrightarrow v^2 = v$

(1) $u = uu^*u$.
(2) u^*u *is a projection.*
(3) uu^* *is a projection.*
(4) u *is a partial isometry.*

Funct. Calc.

v	t
$v^2 - v = 0$	$t^2 - t = 0$ on $sp(v)$

$P(1-P) = 0$ if P is projection

Proof. The implication (1) ⇒ (2) is obvious. To show the converse suppose that u^*u is a projection. Then $\|u(x)\|^2 = \langle u(x), u(x)\rangle = \langle u^*u(x), x\rangle = \langle u^*ux, u^*ux\rangle$ $\|u^*u(x)\|^2$ for all $x \in H_1$, so $u(1 - u^*u) = 0$, and therefore $u = uu^*u$.

To show that (2) ⇒ (3), suppose again that u^*u is a projection. Then $(uu^*)^3 = (uu^*)^2$, so $\sigma(uu^*) \subseteq \{0, 1\}$(*). Hence, uu^* is a projection by the functional calculus. Thus, (2) ⇒ (3), and clearly, then, (3) ⇒ (2) by symmetry. $x \in \ker u \Rightarrow u^*u(x) = 0$ / $x \in \ker u^\perp \Rightarrow u^*u(x) = u^*u(u^*uy) = u^*uy = x$

To show that (1) ⇒ (4), suppose that $u = uu^*u$. Then u^*u is the projection onto $\ker(u)^\perp$, since $u^* = u^*uu^*$, and $\ker(u)^\perp = (u^*(H_2))^- =$

$u^*u(H_1)$. Hence, if $x \in \ker(u)^\perp$, then $\|u(x)\|^2 = \langle u^*u(x), x\rangle = \langle x, x\rangle = \|x\|^2$. Thus, u is a partial isometry, so (1) $\Rightarrow$ (4). $\langle ux, ux\rangle$

Finally, we show (4) $\Rightarrow$ (2) (and this will prove the theorem). Suppose that u is a partial isometry. If p is the projection of H_1 on $\ker(u)^\perp$ and $x \in \ker(u)^\perp$, then $\langle u^*u(x), x\rangle = \|u(x)\|^2 = \langle x, x\rangle = \langle p(x), x\rangle$. If $x \in \ker(u)$, then $\langle u^*u(x), x\rangle = 0 = \langle p(x), x\rangle$. Thus, $\langle u^*u(x), x\rangle = \langle p(x), x\rangle$ for all $x \in H_1$. Hence, $u^*u = p$, so (4) $\Rightarrow$ (2). □

Theorem

$T: X \to Y$ conjugate linear, i.e $T(\alpha x + y) = \bar{\alpha} Tx + Ty$, then (i) T is bd & (ii) T is cts are equivalent.

We shall need to view Hilbert spaces as dual spaces. Let H be a Hilbert space and $H_* = H$ as an additive group, but define a new scalar multiplication on H_* by setting $\lambda.x = \bar{\lambda}x$, and a new inner product by setting $\langle x, y\rangle_* = \langle y, x\rangle$. Then H_* is a Hilbert space, and obviously the norm induced by the new inner product is the same as that induced by the old one. If $x \in H$, define $v(x) \in (H_*)^*$ by setting $v(x)(y) = \langle y, x\rangle_* = \langle x, y\rangle$. It is a direct consequence of the Riesz representation theorem that the map

$x_j \xrightarrow{w} x \Leftrightarrow v(x_j) \xrightarrow{w^*} v(x) \Leftrightarrow \forall y \in H;\ v(x_j)(y) \to v(x)(y) \Leftrightarrow \forall y \in H;\ \langle x_j, y\rangle \to \langle x, y\rangle$

$$v: H \to (H_*)^*, \; x \mapsto v(x),$$

is an isometric linear isomorphism, which we use to identify these Banach spaces. The weak* topology on H is called the *weak* topology. A net $(x_\lambda)_{\lambda\in\Lambda}$ converges to a point x in H in the weak topology if and only if $\langle x, y\rangle = \lim_\lambda \langle x_\lambda, y\rangle$ $(y \in H)$. Consequently, the weak topology is weaker than the norm topology, and a bounded linear map between Hilbert spaces is necessarily weakly continuous. The importance to us of the weak topology is the fact that the closed unit ball of H is weakly compact (Banach–Alaoglu theorem).

2.4.1. Theorem. *Let $u: H_1 \to H_2$ be a compact linear map between Hilbert spaces H_1 and H_2. Then the image of the closed unit ball of H_1 under u is compact.*

weak norm

$\tau_1 \subseteq \tau_2$

Proof. Let S be the closed unit ball of H_1. It is weakly compact, and u is weakly continuous, so $u(S)$ is weakly compact and therefore weakly closed. Hence, $u(S)$ is norm-closed, since the weak topology is weaker than the norm topology. Since u is a compact operator, this implies that $u(S)$ is norm-compact. □

2.4.2. Theorem. *Let u be a compact operator on a Hilbert space H. Then both $|u|$ and u^* are compact.*

Proof. Suppose that u has polar decomposition $u = w|u|$ say. Then $|u| = w^*u$, so $|u|$ is compact, and $u^* = |u|w^*$, so u^* is compact. □

2.4.3. Corollary. *If H is any Hilbert space, then $K(H)$ is self-adjoint.*

Thus, $K(H)$ is a C*-algebra, since (as we saw in Chapter 1) $K(H)$ is a closed ideal in $B(H)$.

Exercise. If $u \geq 0 \Rightarrow \langle ux, x\rangle \geq 0 \ \forall x \in H$

Solution. $\langle \underset{\exists v;\, v^*v}{\underbrace{u}} x, x\rangle = \langle v^*vx, x\rangle = \underbrace{\langle vx, vx\rangle}_{\|vx\|^2} \geq 0.$ $\square$

Project. If $\langle ux, x\rangle \geq 0 \ \forall x \in H \Rightarrow u \geq 0$

Solution. $\langle ux, x\rangle = \overline{\langle ux, x\rangle} = \langle x, ux\rangle = \langle u^*x, x\rangle \ \forall x \in H$

$\langle ux, x\rangle \in \mathbb{R}$

$\therefore u = u^*$

.. why $sp(u) \subseteq [0, \infty)$?

Polar decomposition:

If $u \in B(H)$, then there is a partial isometry w such that $u = w|u|$ & $w^*u = |u|$. If $\ker w = \underbrace{\ker u}_{\ker |u|}$, then w is unique.

$x \in \ker u \Rightarrow ux = 0 \Rightarrow u^*ux = 0 \Rightarrow |u|^2x = 0 \Rightarrow \langle |u|x, |u|x\rangle = 0 \Rightarrow |u|x = 0 \Rightarrow x \in \ker |u|$

$x \in \ker |u| \Rightarrow |u|x = 0 \Rightarrow |u|^2x = 0 \Rightarrow u^*ux = 0 \Rightarrow \langle u^*ux, x\rangle = 0 \Rightarrow \langle ux, ux\rangle = 0 \Rightarrow ux = 0 \Rightarrow x \in \ker u$

If H is a Hilbert space, we denote by $F(H)$ the set of finite-rank operators on H. It is easy to check that $F(H)$ is a self-adjoint ideal of $B(H)$.

Use Zorn's lemma

So $F(H) \subseteq K(H)$ $u \in F(H) \Rightarrow |u| = w^* u \in F(H) \Rightarrow u^* = |u| w^* \in F(H)$ ($u = w|u|$)

2.4.5. Theorem. *If H is a Hilbert space, then $F(H)$ is dense in $K(H)$.*

Proof. Since $F(H)^-$ and $K(H)$ are both self-adjoint, it suffices to show that if u is a hermitian element of $K(H)$, then $u \in F(H)^-$. Let E be an orthonormal basis of H consisting of eigenvectors of u, and let $\varepsilon > 0$. By Theorem 1.4.11 the set S of eigenvalues λ of u such that $|\lambda| \geq \varepsilon$ is finite. From Theorem 1.4.5 it is therefore clear that the set S' of elements of E corresponding to elements of S is finite. Now define a finite-rank diagonal operator v on H by setting $v(x) = \lambda x$ if $x \in S'$ and λ is the eigenvalue corresponding to x, and setting $v(x) = 0$ if $x \in E \setminus S'$. It is easily checked that $\|v - u\| \leq \sup_{\lambda \in \sigma(u) \setminus S} |\lambda| \leq \varepsilon$. This shows that $u \in F(H)^-$. □

Left to students. $\boxed{\operatorname{rank} T := \dim(\operatorname{ran} T)}$

$T \in F(H) \Leftrightarrow \operatorname{rank} T < \infty \Rightarrow \begin{cases} \operatorname{ran} TS \subseteq \operatorname{ran} T \Rightarrow \operatorname{rank} TS \leq \operatorname{rank} T < \infty \\ \operatorname{rank}(ST) \leq \operatorname{rank} T \end{cases} \Rightarrow \begin{cases} TS \in F(H) \\ ST \in F(H) \end{cases}$

If $\operatorname{ran}(T) = \langle e_1, \dots, e_n \rangle$, then $\operatorname{ran}(ST) = \langle Se_1, \dots, Se_n \rangle$

If x, y are elements of a Hilbert space H we define the operator $x \otimes y$ on H by

$$(x \otimes y)(z) = \langle z, y \rangle x. \Rightarrow \operatorname{ran}(x \otimes y) = \mathbb{C}x = \langle x \rangle$$

Clearly, $\|x \otimes y\| = \|x\|\|y\|$. The rank of $x \otimes y$ is one if x and y are non-zero. If $x, x', y, y' \in H$ and $u \in B(H)$, then the following equalities are readily verified:

$\|x \otimes y\| \leq \|x\| \|y\|$ $\left\|(x \otimes y)\left(\frac{y}{\|y\|}\right)\right\| = \|x\| \|y\|$

$$\begin{aligned} (x \otimes x')(y \otimes y') &= \langle y, x' \rangle (x \otimes y') \\ (x \otimes y)^* &= y \otimes x \\ u(x \otimes y) &= u(x) \otimes y \\ (x \otimes y)u &= x \otimes u^*(y). \end{aligned}$$

$\langle (x \otimes y)\xi, \eta \rangle = \langle \xi, (y \otimes x)\eta \rangle$

$(x \otimes x)^* = x \otimes x$, $(x \otimes x)(x \otimes x) = \langle x, x \rangle (x \otimes x) = x \otimes x \Leftrightarrow \langle x, x \rangle = 1$

The operator $x \otimes x$ is a rank-one projection if and only if $\langle x, x \rangle = 1$, that is, x is a unit vector. Conversely, every rank-one projection is of the form $x \otimes x$ for some unit vector x. Indeed, if $e_1, \dots, e_n$ is an orthonormal set in H, then the operator $\sum_{j=1}^n e_j \otimes e_j$ is the orthogonal projection of H onto the vector subspace $\mathbf{C}e_1 + \cdots + \mathbf{C}e_n$.

$\operatorname{ran} u = \mathbb{C}x$ for some $x \in H$

If $u \in B(H)$ is a rank-one operator and x a non-zero element of its range, then $u = x \otimes y$ for some $y \in H$. For if $z \in H$, then $u(z) = \tau(z)x$ for some scalar $\tau(z) \in \mathbf{C}$. It is readily verified that the map $z \mapsto \tau(z)$ is a bounded linear functional on H, and therefore, by the Riesz representation theorem, there exists $y \in H$ such that $\tau(z) = \langle z, y \rangle$ for all $z \in H$. Therefore, $u(z) = \langle z, y \rangle x$
$u = x \otimes y$. $y \otimes x = (x \otimes y)^* = x \otimes y \Rightarrow x = y$.

2.4.6. Theorem. *If H is a Hilbert space, then $F(H)$ is linearly spanned by the rank-one projections.* $x \otimes x$

$u_1 + i u_2$, $u_1, u_2 \in B(H)_{sa}$

Proof. Let $u \in F(H)$ and we shall show it is a linear combination of rank-one projections. The real and imaginary parts of u are in $F(H)$, since $F(H)$ is self-adjoint, so we may suppose that u is hermitian. Now $u = u^+ - u^-$, and by the polar decomposition $|u| \in F(H)$, so u^+ and u^- belong to $F(H)$. Hence, we may assume that $u \geq 0$. The range $u(H)$ is finite-dimensional, and therefore it is a Hilbert space with an orthonormal basis, $e_1, \ldots, e_n$ say. Let $p = \sum_{j=1}^n e_j \otimes e_j$, so p is the projection of H onto $u(H)$. Then $u = pu = u^{1/2}pu^{1/2} \Rightarrow u = \sum_{j=1}^n x_j \otimes x_j$, where $x_j = u^{1/2}(e_j)$. Now $x_j = \lambda_j f_j$ for some unit vector f_j and scalar λ_j, so $u = \sum_{j=1}^n |\lambda_j|^2 f_j \otimes f_j$, and since the operators $f_j \otimes f_j$ are rank-one projections we are done. □

$u = u^* = up$
$up = pu$
$Q(u)p = pQ(u)$
Stone-Weierstrass
$f(u)p = pf(u)$
$f(t) = t^{\frac{1}{2}}$
$u^{\frac{1}{2}}p = pu^{\frac{1}{2}}$

2.4.7. Theorem. *If H is a Hilbert space and I a non-zero ideal in $B(H)$, then I contains $F(H)$.*

Proof. Let u be a non-zero operator in I. Then for some $x \in H$ we have $u(x) \neq 0$. If p is a rank-one projection, then $p = y \otimes y$ for some unit vector $y \in H$, and clearly there exists $v \in B(H)$ such that $vu(x) = y$ (take $v = (y \otimes u(x))/\|u(x)\|^2$, for instance). Hence, $p = vu(x \otimes x)u^*v^*$, so $p \in I$ as $u \in I$. Thus, I contains all the rank-one projections and therefore by Theorem 2.4.6 it contains $F(H)$. □

$vux \otimes vux = v(ux \otimes ux)v^*$

Proof of (1.1):

① K reduces u:
$u(K) \subseteq K$, $u(K^\perp) \subseteq K^\perp$ (1.1), (1.2)
$\Leftarrow u^*(K) \subseteq K$
$\langle ux, y\rangle = \langle x, u^*y\rangle = 0$, $x \in K^\perp$, $y \in K$

$K = \overline{\langle E\rangle}$, (i) $x \in E \Rightarrow x$ is an eigenvector of u $\Rightarrow ux = \lambda x \in K$

(ii) $x = \sum \alpha_i x_i$, $x_i \in \langle E\rangle$ $\overset{(i)}{\Rightarrow} u(x) \in K$

(iii) $x = \lim_n x_n$, $x_n \in \langle E\rangle$ $\overset{(ii)}{\Rightarrow} ux = \lim_n ux_n \in K$

Proof of (1.2): Let $z \in K$. We shall show that $u^*(z) \in K$:

Let $z \in E$. $\forall x \in E$, $\langle u^*(z), x\rangle = \langle z, \underbrace{ux}_{\exists\lambda;\ \lambda x}\rangle = \bar{\lambda}\langle z, x\rangle = 0$

(E⊊) $\left\{\frac{u^*z}{\|u^*z\|}\right\} \cup E$ an orth. set eigenvectors of u.

$u(u^*z) = u^*(uz) = u^*(\mu z) = \mu(u^*z)$
u^*z is an eigenvector of u

By the maximality of E, $E = \left\{\frac{u^*z}{\|u^*z\|}\right\} \cup E$.

So $\frac{u^*z}{\|u^*z\|} \in E \Rightarrow u^*z \in K$.

② Let $X = Y \oplus Z$, $u \in B(X)$, $u(Y) \subseteq Y$, $u(Z) \subseteq Z$. We have $u|_Y : Y \to Y$, $u|_Z : Z \to Z$ are cts. If u is compact

$y_n \to y \Rightarrow$
$uy_n \to uy \Rightarrow$
$u|_Y(y_n) \to (u|_Y)(y)$

$\{y_n\}$ is bd in Y
$\{y_n\}$ " " " X.
$\{uy_n\}$ has conv. subseq
$\{u|_Y(y_n)\}$ " "

then so are $u|_Y$ & $u|_Z$.

One can see in the case of Hilbert spaces, $H = M \oplus M^\perp$

$(u|_M)^* = u^*|_M$

$\langle u^*|_M x, y\rangle \doteq \langle x, u|_M(y)\rangle$
since $\langle u^* x, y\rangle = \langle x, uy\rangle$

$\Downarrow$

u is normal $\Rightarrow$ $u|_M$ is normal

(3) Any eigenvector of $u|_{K^\perp}$ is an eigenvector of u

since $u|_{K^\perp} x = \mu x \Longrightarrow ux = \mu x$

If $\lambda \in \sigma(u|_{K^\perp})$ is eigenvalue, then $u|_{K^\perp} x = \lambda x$ for some $x \in K^\perp$. Then $\{x\} \cup E \in \Sigma$

By the maximality, $\{x\} \cup E = E$. So $x \in E \cap K^\perp \in K \cap K^\perp = \{0\}$.

$\therefore x = 0$.

Thus λ cannot be an eigenvalue. X.

λ is an eigenvalue of u if $\exists \boxed{x \neq 0}$; $ux = \lambda x$

So $\sigma(u|_{K^\perp}) = \{0\}$. Thus $r(u|_{K^\perp}) = 0$. So $u|_{K^\perp} = 0$

$r(u|_{K^\perp}) = \|u|_{K^\perp}\|$

(4) We have $u|_{K^\perp} = 0$ (so $\|u^*|_{K^\perp}\| = \|u|_{K^\perp}\| = 0$, hence $u^*|_{K^\perp} = 0$).

$K^\perp \quad K^\perp \quad K^\perp \quad K^\perp$

Let $0 \neq x \in K^\perp$. We have $u^* x = 0$ & $\underbrace{ux = 0 = 0x}_{x \text{ is an eigenvector of } u}$.

$\forall y \in E$; $0 = \langle \underbrace{u^* x}_{0}, y \rangle = \langle x, \underbrace{uy}_{\mu y \text{ for some } \mu} \rangle = \bar{\mu} \langle x, y \rangle$. Hence

$\{\frac{x}{\|x\|}\} \cup E \in \sum$. Hence $\frac{x}{\|x\|} \cup E = E$. Therefore

$x \in K \cap K^\perp = \{0\}$ ※. $\square$

(5) $u \neq 0$

$u(z) = \tau(z) x \Rightarrow \begin{cases} u(z_1) = \tau(z_1) x \\ u(z_2) = \tau(z_2) x \\ u(z_1 + z_2) = \tau(z_1 + z_2) x \end{cases} \xrightarrow{u \text{ is linear}} \underbrace{[\tau(z_1) + \tau(z_2) - \tau(z_1 + z_2)]}_{0} x = 0$

$\Downarrow$

$|\tau(z)| \|x\| = \|\tau(z) x\| = \|u(z)\| \leq \|u\| \|z\|$

$\therefore |\tau(z)| \leq \frac{\|u\|}{\|x\|} \|z\|$

(6) $x \otimes y = y \otimes x \Longrightarrow x = y$

Since we may assume that $\|x\| = \|y\| = 1$. Then

$x = \langle y, x \rangle y = |\langle x, y \rangle|^2 x \Rightarrow |\langle x, y \rangle| = 1 = \|x\| \|y\| \Rightarrow$

$y = \langle x, y \rangle x$

$\exists \lambda > 0$; $y = \lambda x$

$x = \langle y, x \rangle y \Downarrow$

$\lambda = 1$

$\Downarrow$

$x = y$

(8) $\operatorname{ran}(\sum_{j=1}^n e_j \otimes e_j) = \mathbb{C} e_1 \oplus \dots \oplus \mathbb{C} e_n$

$(\sum_{j=1}^n e_j \otimes e_j)(x) = \sum_{j=1}^n \langle x, e_j \rangle e_j \in \mathbb{C} e_1 \oplus \dots \oplus \mathbb{C} e_n$

$x \perp y$, $\|x\| = \|y\| = 1 \Rightarrow x, y$ are linearly independent.

Since

$\alpha x + \beta y = 0 \Rightarrow 0 = \langle \alpha x + \beta y, x \rangle = \alpha + \beta \cdot 0 = \alpha$

Similarly, $\mu = 0$.

9) $u(K) \subseteq K \iff uP = PuP$, where P is the projection on K (a closed subspace)

$(\Rightarrow)$ $(PuP)(x) = P(u(\underbrace{\underbrace{P(x)}_{\in K}}_{\in K})) = uP(x)$ $\forall x \in H$

$H = K \oplus K^{\perp}$
$P|_K = id$, $P|_{K^{\perp}} = 0$

$(\Leftarrow)$ $\forall x \in K$; $u(x) = u(Px) = P(u(Px)) \in K$. $\square$

Further,

$u(K^{\perp}) \subseteq K^{\perp} \iff u^*(K) \subseteq K \overset{\text{above}}{\iff} Pu^* = Pu^*P \iff uP = PuP$

Moreover,

K reduces $u \iff \begin{cases} u(K) \subseteq K \\ u(K^{\perp}) \subseteq K^{\perp} \end{cases} \iff \begin{cases} Pu = PuP \\ uP = PuP \end{cases} \iff Pu = uP$

An *approximate unit* for a C*-algebra A is an increasing net $(u_\lambda)_{\lambda\in\Lambda}$ of positive elements in the closed unit ball of A such that $a = \lim_\lambda a u_\lambda$ for all $a \in A$. Equivalently, $a = \lim_\lambda u_\lambda a$ for all $a \in A$.

$a = a.1 = 1.a$

$\|u_\lambda\| \le 1,\ u_\lambda \ge 0$

$a^* = \lim_\lambda a^* u_\lambda = \lim_\lambda (u_\lambda a)^* = (\lim_\lambda u_\lambda a)^*$

$\therefore\ a = \lim_\lambda u_\lambda a$

$a \in A_h \Rightarrow sp(a^2) = \{\lambda^2 : \lambda \in sp\, a\} \subseteq [0,\infty)$, $(a^2)^* = a^2 \Rightarrow a^2 \ge 0$

3.1.1. Example. Let H be a Hilbert space with an orthonormal basis $(e_n)_{n=1}^\infty$. The C*-algebra $K(H)$ is of course non-unital, since $\dim(H) = \infty$. If p_n is the projection onto $\mathbf{C}e_1 + \cdots + \mathbf{C}e_n$, then the increasing sequence (p_n) is an approximate unit for $K(H)$. To see this we need only show that $u = \lim_{n\to\infty} p_n u$ if $u \in F(H)$, since $F(H)$ is dense in $K(H)$. Now if $u \in F(H)$, there exist $x_1, \ldots, x_m, y_1, \ldots, y_m$ in H such that $u = \sum_{k=1}^m x_k \otimes y_k$. Hence, $p_n u = \sum_{k=1}^m p_n(x_k) \otimes y_k$. Since $\lim_{n\to\infty} p_n(x) = x$ for all $x \in H$, therefore for each k,

$P_n \le P_{n+1}$, $\operatorname{ran} P_n \subseteq \operatorname{ran} P_{n+1}$

$x = \sum_{n=1}^\infty \langle x, e_n\rangle e_n = \lim_n \sum_{k=1}^n \langle x, e_k\rangle e_k$, $P_n x$

$$\lim_{n\to\infty} \|p_n(x_k) \otimes y_k - x_k \otimes y_k\| = \lim_{n\to\infty} \|p_n(x_k) - x_k\|\|y_k\| = 0.$$

$(P_n(x_k) - x_k) \otimes y_k$

$P = P^2 = P^*P \Rightarrow P \ge 0$

Hence, $\lim_{n\to\infty} p_n u = u$.

$P = P^* \Rightarrow \|P\| = r(P)$, $P \ne 0$, $\{\max\{0,1\}\} = 1$

$\|P\| = \|P^2\| \le \|P\|\,\|P\|$, $1 \le \|P\|$

Let A be an arbitrary C*-algebra and denote by Λ the set of all positive elements a in A such that $\|a\| < 1$. This set is a poset under the partial order of A_{sa}. In fact, Λ is also upwards-directed; that is, if $a, b \in \Lambda$, then there exists $c \in \Lambda$ such that $a, b \le c$. We show this: If $a \in A^+$, then $1 + a$ is of course invertible in $\tilde{A}$, and $a(1+a)^{-1} = 1 - (1+a)^{-1}$. We claim

$\frac{t}{1+t}$ is op. monot.

$$a, b \in A^+ \text{ and } a \le b \Rightarrow a(1+a)^{-1} \le b(1+b)^{-1}. \tag{1}$$

$a \ge 0 \Leftrightarrow a \ge 0$ & invertb

Indeed, if $0 \le a \le b$, then $1 + a \le 1 + b$ implies $(1+a)^{-1} \ge (1+b)^{-1}$, by Theorem 2.2.5, and therefore $1 - (1+a)^{-1} \le 1 - (1+b)^{-1}$; that is, $a(1+a)^{-1} \le b(1+b)^{-1}$, proving the claim. Observe that if $a \in A^+$, then $a(1+a)^{-1}$ belongs to Λ (use the Gelfand representation applied to the C*-subalgebra generated by 1 and a). Suppose then that a, b are an arbitrary pair of elements of Λ. Put $a' = a(1-a)^{-1}$, $b' = b(1-b)^{-1}$ and $c = (a'+b')(1+a'+b')^{-1}$. Then $c \in \Lambda$, and since $a' \le a' + b'$, we have $a = a'(1+a')^{-1} \le c$, by (1). Similarly, $b \le c$, and therefore Λ is upwards-directed, as asserted.

$a \ge 0 \Rightarrow 1 + a$ inv.
$a \longleftrightarrow t$
$1 + a \longleftrightarrow 1 + t$
$(1+a)^{-1} \longleftrightarrow \frac{1}{1+t}$, $t \in sp(a) \cap [0,\infty)$
$sp(1+a) = \{1 + \lambda \mid \lambda \in sp(a)\} \subseteq [1,\infty)$

3.1.1. Theorem. *Every C*-algebra A admits an approximate unit. Indeed, if Λ is the upwards-directed set of all $a \in A^+$ such that $\|a\| < 1$ and $u_\lambda = \lambda$ for all $\lambda \in \Lambda$, then $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for A (called the canonical approximate unit).*

Proof. From the remarks preceding this theorem, $(u_\lambda)_{\lambda\in\Lambda}$ is an increasing net of positive elements in the closed unit ball of A. Therefore, we need only show that $a = \lim_\lambda u_\lambda a$ for each $a \in A$. Since Λ linearly spans A, we can reduce to the case where $a \in \Lambda$.

$\lambda_1 \le \lambda_2 \Rightarrow \lambda_1 \le \lambda_2$, u_{λ_1}, u_{λ_2}

$\|a\| < 1 \Rightarrow \|f\| < 1$

Suppose then that $a \in \Lambda$ and that $\varepsilon > 0$. Let $\varphi: C^*(a) \to C_0(\Omega)$ be the Gelfand representation. If $f = \varphi(a)$, then $K = \{\omega \in \Omega \mid |f(\omega)| \ge \varepsilon\}$ is compact, and therefore by Urysohn's lemma there is a continuous function $g: \Omega \to [0,1]$ of compact support such that $g(\omega) = 1$ for all $\omega \in K$. Choose $\delta > 0$ such that $\delta < 1$ and $1 - \delta < \varepsilon$. Then $\|f - \delta g f\| \le \varepsilon$. If $\lambda_0 = \varphi^{-1}(\delta g)$, then $\lambda_0 \in \Lambda$ and $\|a - u_{\lambda_0} a\| \le \varepsilon$. Now suppose that $\lambda \in \Lambda$ and $\lambda \ge \lambda_0$. Then $1 - u_\lambda \le 1 - u_{\lambda_0}$, so $a(1 - u_\lambda)a \le a(1 - u_{\lambda_0})a$. Hence, $\|a - u_\lambda a\|^2 = \|(1-u_\lambda)^{1/2}(1-u_\lambda)^{1/2}a\|^2 \le \|(1-u_\lambda)^{1/2}a\|^2 = \|a(1-u_\lambda)a\| \le \|a(1-u_{\lambda_0})a\| \le \|(1-u_{\lambda_0})a\| \le \varepsilon$. This shows that $a = \lim_\lambda u_\lambda a$. $\square$

$|f(\omega) - \delta g(\omega) f(\omega)| = \begin{cases} (1-\delta)|f(\omega)| < 1 - \delta < \varepsilon & \omega \in K \\ |f(\omega)|\,|1 - \delta g(\omega)| & \omega \notin K \end{cases}$, $0 \le g(\omega) \le 1$

$a, u_\lambda \in A \hookrightarrow \tilde{A} = A \oplus \mathbb{C}$ isometry, $1 - u_\lambda \in \tilde{A}$ & $(1 - u_\lambda)a = a - u_\lambda a$

3.1.2. Theorem. If L is a closed left ideal in a C*-algebra A, then there

3.1.2. Theorem. *If L is a closed left ideal in a C*-algebra A, then there is an increasing net $(u_\lambda)_{\lambda\in\Lambda}$ of positive elements in the closed unit ball of L such that $a = \lim_\lambda au_\lambda$ for all $a \in L$.*

$0 \le \delta g(\omega) \le \delta$

$0 \le 1-\delta \le 1-\delta g(\omega) \le 1$

Proof. Set $B = L \cap L^*$. Since B is a C*-algebra, it admits an approximate unit, $(u_\lambda)_{\lambda\in\Lambda}$ say, by Theorem 3.1.1. If $a \in L$, then $a^*a \in B$, so $0 = \lim_\lambda a^*a(1-u_\lambda)$. Hence, $\lim_\lambda \|a - au_\lambda\|^2 = \lim_\lambda \|(1-u_\lambda)a^*a(1-u_\lambda)\| \le \lim_\lambda \|a^*a(1-u_\lambda)\| = 0$, and therefore $\lim_\lambda \|a - au_\lambda\| = 0$. □

$0 \le u_\lambda \le 1 \Rightarrow 0 \le 1-u_\lambda \le 1 \Rightarrow 0 \le (1-u_\lambda)^{\frac12} \le 1^{\frac12} = 1 \Rightarrow \|(1-u_\lambda)^{\frac12}\| \le 1$

In the preceding proof we worked in the unitisation $\tilde{A}$ of A. We shall frequently do this tacitly.

$0 \le u_\lambda,\ \|u_\lambda\| < 1 \Rightarrow 0 \le u_\lambda \le \|u_\lambda\| \le 1 \Rightarrow 0 \le 1-u_\lambda \le 1 \Rightarrow \|1-u_\lambda\| \le 1$

3.1.3. Theorem. *If I is a closed ideal in a C*-algebra A, then I is self-adjoint and therefore a C*-subalgebra of A. If $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for I, then for each $a \in A$*

$$\inf_{b\in I}\|a+b\| = \|a + I\| = \lim_\lambda \|a - u_\lambda a\| = \lim_\lambda \|a - au_\lambda\|.$$

Proof. By Theorem 3.1.2 there is an increasing net $(u_\lambda)_{\lambda\in\Lambda}$ of positive elements in the closed unit ball of I such that $a = \lim_\lambda au_\lambda$ for all $a \in I$. Hence, $a^* = \lim_\lambda u_\lambda a^*$, so $a^* \in I$, because all of the elements u_λ belong to I. Therefore, I is self-adjoint.

I is right ideal &

Suppose that $(u_\lambda)_{\lambda\in\Lambda}$ is an arbitrary approximate unit of I, that $a \in A$, and that $\varepsilon > 0$. There is an element b of I such that $\|a+b\| < \|a+I\| + \varepsilon/2$. Since $b = \lim_\lambda u_\lambda b$, there exists $\lambda_0 \in \Lambda$ such that $\|b - u_\lambda b\| < \varepsilon/2$ for all $\lambda \ge \lambda_0$, and therefore

$\|a+I\| - \varepsilon$

$(1-u_\lambda)a + (1-u_\lambda)b - (1-u_\lambda)b$

$$\begin{aligned} \|a+I\| \le \|a - u_\lambda a\| &\le \|(1-u_\lambda)(a+b)\| + \|b - u_\lambda b\| \\ &\le \|a+b\| + \|b - u_\lambda b\| \\ &< \|a+I\| + \varepsilon/2 + \varepsilon/2. \end{aligned}$$

$\inf\ \varepsilon \to 0$

It follows that $\|a+I\| = \lim_\lambda \|a - u_\lambda a\|$, and therefore also $\|a+I\| = \|a^* + I\| = \lim_\lambda \|a^* - u_\lambda a^*\| = \lim_\lambda \|a - au_\lambda\|$. □

$= \lim_\lambda \|(a - au_\lambda)^*\|$

Th. $J \triangleleft I \triangleleft A \Rightarrow J \triangleleft A$

3.1.2. Remark. Let I be a closed ideal in a C*-algebra A, and J a closed ideal in I. Then J is also an ideal in A. To show this we need only show that ab and ba are in J if $a \in A$ and b is a positive element of J (since J is a C*-algebra, J^+ linearly spans J). If $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for I, then $b^{1/2} = \lim_\lambda u_\lambda b^{1/2}$ because $b^{1/2} \in I$. Hence, $ab = \lim_\lambda au_\lambda b^{1/2}b^{1/2}$, so $ab \in J$ because $b^{1/2} \in J$, $au_\lambda b^{1/2} \in I$, and J is an ideal in I. Therefore, $a^*b \in J$ also, so $ba \in J$, since J is self-adjoint.

Proof. Let $a \in A$, $b \in J^+$.

$ab = \underbrace{(ab^{\frac12})}_{\in I}\ \underbrace{b^{\frac12}}_{\in J} \in J$

3.1.4. Theorem. *If I is a closed ideal of a C*-algebra A, then the quotient A/I is a C*-algebra under its usual operations and the quotient norm.*

Proof. Let $(u_\lambda)_{\lambda\in\Lambda}$ be a approximate unit for I. If $a \in A$ and $b \in I$, then

$$\begin{aligned} \|a+I\|^2 &= \lim_\lambda \|\overbrace{a - au_\lambda}^{a(1-u_\lambda)}\|^2 \text{ (by Theorem 3.1.3)} \\ &= \lim_\lambda \|(1-u_\lambda)a^*a(1-u_\lambda)\| \\ &\le \lim_\lambda \|(1-u_\lambda)(a^*a+b)(1-u_\lambda)\| + \lim_\lambda \|(1-u_\lambda)b(1-u_\lambda)\| \\ &\le \|a^*a+b\| + \lim_\lambda \|b - u_\lambda b\| \\ &= \|a^*a+b\|. \end{aligned}$$

$\|1-u_\lambda\| \le 1$

$\|(1-u_\lambda)(a^*a+b)(1-u_\lambda)\| \le \|a^*a+b\|$

$\inf_{b\in I}\|a^*a+b\|$

Therefore, $\|a+I\|^2 \le \|a^*a + I\|$. ~~By Lemma 2.1.3,~~ A/I is a C*-algebra. □

So $\le \|a+I\|\,\|a^*+I\| = \|a+I\|^2$

① $\lim P_n u = u\ \forall u \in F(H)$ } $\Rightarrow \lim P_n v = v\ \forall v \in K(H)$

$$\overline{F(H)} = K(H)$$

Let $v \in K(H)$ $F(H)$
Given $\varepsilon > 0$. $\exists u \in F(H)$; $\|u - v\| \le \frac{\varepsilon}{3}$

$$\|P_n v - v\| \le \|P_n v - P_n u\| + \|P_n u - u\| + \|u - v\|$$
$$\le \|v - u\| + \|P_n u - u\| + \frac{\varepsilon}{3}$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \qquad (\exists N_0\ \forall n \ge N_0;\ \|P_n u - u\| < \tfrac{\varepsilon}{3})$$
$$= \varepsilon \qquad (n \ge N_0).$$

③ $a(1+a)^{-1} = 1 - (1+a)^{-1}$

$$\frac{t}{1+t} = \frac{t+1-1}{1+t} = 1 - \frac{1}{1+t}$$

④ $a \ge 0 \Rightarrow a(1+a)^{-1} \in \Lambda$

$$a \longleftrightarrow t \ge 0$$
$$a(1+a)^{-1} \longleftrightarrow \frac{t}{1+t} \ge 0$$

$$\|a(1+a)^{-1}\| < 1 \longleftrightarrow \sup_{t \in sp(a) \subseteq [0,\infty)} \left|\frac{t}{1+t}\right| = \frac{t_0}{1+t_0} < 1$$

$sp(a)$ compact, $\exists t_0 \in sp(a)$

⑤ Λ linearly spans A

$$a = \underbrace{a_1}_{a_1' - a_1''} + a_2 i \qquad a_1, a_2 \in A_{sa}$$

$(a_1', a_1'' \in A_+)$ $\qquad \left(a_1' = 2\|a_1'\| \frac{a_1'}{2\|a_1'\|}\right)$, $\frac{a_1'}{2\|a_1'\|} \in \Lambda$

3.1.5. Theorem. *If $\varphi: A \to B$ is an injective $*$-homomorphism between C^*-algebras A and B, then φ is necessarily isometric.* 1-1

$\varphi: \mathbb{C} \xrightarrow{1-1} M_2(\mathbb{C})$, $\varphi(z) = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}$ is not unital, since $\varphi(1) \neq I$

Proof. It suffices to show that $\|\varphi(a)\|^2 = \|a\|^2$, that is, $\|\varphi(a^*a)\| = \|a^*a\|$. Thus, we may suppose that A is abelian (restrict to $C(a^*a)$ if necessary), and that B is abelian (replace B by $\varphi(A)^-$ if required). Moreover, by extending $\varphi: A \to B$ to $\tilde{\varphi}: \tilde{A} \to \tilde{B}$ if necessary, we may further assume that A, B, and φ are unital.

$\tau_\alpha \circ \varphi \xrightarrow{w^*} \tau \circ \varphi$

If τ is a character on B, then $\tau \circ \varphi$ is one on A. Clearly the map

$\tau_\alpha \xrightarrow{w^*} \tau \Rightarrow \tau_\alpha(b) \to \tau(b)\ \forall b \in B \Rightarrow \tau_\alpha(\varphi(a)) \to \tau(\varphi(a))$

$$\varphi': \Omega(B) \to \Omega(A), \quad \tau \mapsto \tau \circ \varphi,$$

contin

$B \xrightarrow{\tau} \mathbb{C}$, $\varphi \uparrow$ A, $\tau \circ \varphi \in \Omega(A)$

$\tau \circ \varphi = 0$
$(\tau \circ \varphi)(a) = 0\ \forall a \in A$
$\tau(\varphi(a)) = 0\ \forall a \in A$
$\tau(b) = 0\ \forall b \in B$. ✗

is continuous. Hence, $\varphi'(\Omega(B))$ is compact, because $\Omega(A)$ is compact, and therefore $\varphi'(\Omega(B))$ is closed in $\Omega(A)$. If $\varphi'(\Omega(B)) \neq \Omega(A)$, then by Urysohn's lemma there is a non-zero continuous function $f: \Omega(A) \to \mathbf{C}$ such that f vanishes on $\varphi'(\Omega(B))$. By the Gelfand representation, $f = \hat{a}$ for some element $a \in A$. Hence, for each $\tau \in \Omega(B)$, $\tau(\varphi(a)) = \hat{a}(\tau \circ \varphi) = 0$.

so $\|\varphi(a)\| = r(\varphi(a)) = \sup |\tau(\varphi(a))| = 0$ $\quad \in \varphi'(\Omega(B))$

Therefore, $\varphi(a) = 0$, so $a = 0$. But this implies that f is zero, a contradiction. The only way to avoid this is to have $\varphi'(\Omega(B)) = \Omega(A)$. Hence, for each $a \in A$,

$\sup_{\tau \in \Omega(B)} |(\tau \circ \varphi)(a)| =$

$$\|a\| = \|\hat{a}\|_\infty = \sup_{\eta \in \Omega(A)} |\eta(a)| = \sup_{\tau \in \Omega(B)} |\tau(\varphi(a))| = \|\varphi(a)\|.$$

Thus, φ is isometric. □

$\boxed{\varphi'(\Omega(B))}$ τ_0 $\quad \Omega(A)$ $\quad \exists f_0 \in C(\Omega(A))$, $0 \leq f_0 \leq 1$, $f_0| = 1$

$A \xleftrightarrow[\text{*-isom}]{\text{isom}} C(\Omega(A))$
$a \longleftrightarrow \hat{a}$

Consider $f = 1 - f_0$ so $0 \leq f \leq 1$, $f|_{\varphi'(\Omega(B))} = 0$

3.1.6. Theorem. *If $\varphi: A \to B$ is a $*$-homomorphism between C^*-algebras, then $\varphi(A)$ is a C^*-subalgebra of B.*

$\varphi(a) = 0 \Rightarrow a \in \mathrm{Ker}\varphi \Rightarrow a + \mathrm{Ker}\varphi = 0$

Proof. The map

Kerφ is closed ideal.

$$\psi: \underbrace{A/\ker(\varphi)}_{\text{Ban}} \to B, \quad a + \ker(\varphi) \mapsto \varphi(a),$$

is an injective $*$-homomorphism between C*-algebras and is therefore isometric. Its image is $\varphi(A)$, so this space is necessarily complete and therefore closed in B. □

3.1.7. Theorem. *Let B and I be respectively a C^*-subalgebra and a closed ideal in a C^*-algebra A. Then $B + I$ is a C^*-subalgebra of A.*

Proof. We show only that $B + I$ is complete, because the rest is trivial. Since I is complete we need only prove that the quotient $(B + I)/I$ is complete. The intersection $B \cap I$ is a closed ideal in B and the map φ from $B/(B \cap I)$ to A/I defined by setting $\varphi(b + B \cap I) = b + I$ $(b \in B)$ is a $*$-homomorphism with range $(B + I)/I$. By Theorem 3.1.6, $(B + I)/I$ is complete, because it is a C*-algebra. □

Ban $X \Leftarrow \dfrac{X}{M}$ Ban & M Ban

3.1.3. *Remark.* The map

$$\varphi: B/(B \cap I) \to (B + I)/I, \quad b + B \cap I \mapsto b + I,$$

in the preceding proof is in fact clearly a $*$-isomorphism.

X is compact Hausdorff space
so $\forall\, \underset{\text{closed}}{F} \subseteq \underset{\text{open}}{G}$
$\exists\, \underset{\text{open}}{H}$
$F \subseteq H \subseteq \bar{H} \subseteq G$
$F = \varphi'(\Omega(B))$, $G = \{\tau_0\}^c$
Use Urysohn Lemma for F & H

We return to the topic of multiplier algebras, because we can now say a little more about them using the results of this section.

Suppose that I is a closed ideal in a C*-algebra A. If $a \in A$, define L_a and R_a in $B(I)$ by setting $L_a(b) = ab$ and $R_a(b) = ba$. It is a straightforward exercise to verify that (L_a, R_a) is a double centraliser on I and that the map

$$\varphi: A \to M(I), \quad a \mapsto (L_a, R_a),$$

is a $*$-homomorphism. Recall that we identified I as a closed ideal in $M(I)$ by identifying a with (L_a, R_a) if $a \in I$. Hence, φ is an extension of the inclusion map $I \to M(I)$.

$a \mapsto (L_a, R_a)$

If $I_1, I_2, \ldots, I_n$ are sets in A, we define $I_1 I_2 \ldots I_n$ to be the closed linear span of all products $a_1 a_2 \ldots a_n$, where $a_j \in I_j$. If I, J are closed ideals in A, then $I \cap J = IJ$. The inclusion $IJ \subseteq I \cap J$ is obvious. To show the reverse inclusion we need only show that if a is a positive element of $I \cap J$, then $a \in IJ$. Suppose then that $a \in (I \cap J)^+$. Hence, $a^{1/2} \in I \cap J$. If $(u_\lambda)_{\lambda \in \Lambda}$ is an approximate unit for I, then $a = \lim_\lambda (u_\lambda a^{1/2}) a^{1/2}$, and since $u_\lambda a^{1/2} \in I$ for all $\lambda \in \Lambda$, we get $a \in IJ$, as required.

Let I be a closed ideal I in A. We say I is *essential* in A if $aI = 0 \Rightarrow a = 0$ (equivalently, $Ia = 0 \Rightarrow a = 0$). From the preceding observations it is easy to check that I is essential in A if and only if $I \cap J \neq 0$ for all non-zero closed ideals J in A. $(\Rightarrow)$ Let $J \lhd A$, $J \neq 0$ & $I \cap J = 0$. So $\exists b \in J$; $b \neq 0$ & $bI = 0$.

Every C*-algebra I is an essential ideal in its multiplier algebra $M(I)$.

Noncommutative Geometry (topology)

3.1.8. Theorem. *Let I be a closed ideal in a C*-algebra A. Then there is a unique $*$-homomorphism $\varphi: A \to M(I)$ extending the inclusion $I \to M(I)$. Moreover, φ is injective if I is essential in A.* $a \mapsto (L_a, R_a)$ $a \mapsto (L_a, R_a)$

Proof. We have seen above that the inclusion map $I \to M(I)$ admits a $*$-homomorphic extension $\varphi: A \to M(I)$. Suppose that $\psi: A \to M(I)$ is another such extension. If $a \in A$ and $b \in I$, then $\varphi(a)b = \varphi(ab) = ab = \psi(ab) = \psi(a)b$. Hence, $(\varphi(a) - \psi(a))I = 0$, so $\varphi(a) = \psi(a)$, since I is essential in $M(I)$. Thus, $\varphi = \psi$.

Suppose now that I is essential in A and let $a \in \ker(\varphi)$. Then $aI = L_a(I) = 0$, so $a = 0$. Thus, φ is injective. $(L_a, R_a) = \varphi_a = 0$ □

Theorem 3.1.8 tells us that the multiplier algebra $M(I)$ of I is the largest unital C*-algebra containing I as an essential closed ideal.

Let $I \underset{ess}{\lhd} A$. So $\exists!\ \varphi: A \xrightarrow{isom} M(I)$. $*$-isom

Hence $A \hookrightarrow M(I)$. So $M(I)$ in the largest ...

$0 = \|u(x) \otimes x\| = \|u(x)\| \|x\|$ $\therefore \forall x;\ ux = 0$

3.1.2. *Example.* If H is a Hilbert space, then $K(H)$ is an essential ideal in $B(H)$. For if u is an operator in $B(H)$ such that $uK(H) = 0$, then for all $x \in H$ we have $u(x) \otimes x = u(x \otimes x) = 0$, so $u(x) = 0$. By Theorem 3.1.8, the inclusion map $K(H) \to M(K(H))$ extends uniquely to an injective $*$-homomorphism $\varphi: B(H) \to M(K(H))$. We show that φ is surjective, that is, a $*$-isomorphism. Suppose that $(L, R) \in M(K(H))$, and fix a unit vector e in H. The linear map

$L: K(H) \to K(H)$

$$u: H \to H, \quad x \mapsto (L(x \otimes e))(e),$$ ← no composition of operators!

is bounded, since $\|u(x)\| \leq \|L(x \otimes e)\| \leq \|L\| \|x \otimes e\| = \|L\| \|x\|$. If $x, y, z \in H$. then

so $\|u\| \leq \|L\|$

$$(L_u(x \otimes y))(z) = (u(x) \otimes y)(z) \; \leftarrow \langle z, y \rangle ux$$
$$= \langle z, y \rangle (L(x \otimes e))(e)$$

$\exists f_0 : f_0|_F = 1$ & $f_0|_{H^c} = 0$

So $f_0|_{\varphi^{-1}(\Omega(B))} = 1$ & $f_0(\rho) = 0$

$\therefore f = 1 - f_0 \neq 0$

$\left.\begin{array}{l} IJ \subseteq J \\ IJ \subseteq I \end{array}\right\} \Rightarrow IJ \subseteq I \cap J$

$b \in (I \cap J)^+ \Rightarrow b \in I \Rightarrow$ $b \in J$ $b \geq 0$

$b^{\frac{1}{2}} \in I$, $b^{\frac{1}{2}} \in J$ $\Rightarrow b = b^{\frac{1}{2}} b^{\frac{1}{2}} \in IJ$

So $b = 0$. Thus $I \cap J \neq 0$

$(\Leftarrow)$ Let $aI = 0$. So $(AaA)I = 0$

Hence $\overline{AaA} = 0$. Thus $a = 0$. □

$\overline{AaA} \cap I$

$a = \lim_\lambda u_\lambda a u_\lambda$

$\|a - u_\lambda a u_\lambda\| \leq \|a - u_\lambda a\| + \|u_\lambda(a - a u_\lambda)\|$

$M(K(H)) \cong B(H)$

$(u(x\otimes y))(z)$
$$= (L(x\otimes e))((z,y)e) \quad \text{(by linearity)}$$
$$= (L(x\otimes e))(e\otimes y)(z).$$

$L((x\otimes e)(e\otimes y))$

Hence, $L_u(x\otimes y) = L(x\otimes e)(e\otimes y) = L(x\otimes y)$ for all $x, y \in H$. Therefore, $(\varphi(u) - (L,R))K(H) = 0$, so $\varphi(u) = (L,R)$.

$(x\otimes e)(e\otimes y) = \langle e,e\rangle x\otimes y$, $\langle e,e\rangle = 1$

Thus, we may regard $B(H)$ as the multiplier algebra of $K(H)$. This example is the motivating one for the use of the multiplier algebra in K-theory.

3.1.3. *Example.* If Ω is a locally compact Hausdorff space, then it is easy to check that $C_0(\Omega)$ is an essential ideal (1) in the C*-algebra $C_b(\Omega)$. Therefore, by Theorem 3.1.8 there is a unique injective $*$-homomorphism $\varphi: C_b(\Omega) \to M(C_0(\Omega))$ extending the inclusion $C_0(\Omega) \to M(C_0(\Omega))$. We show that φ is surjective, that is, a $*$-isomorphism. To see this, it suffices to show that if $g \in M(C_0(\Omega))$ is positive, then it is the range of φ. If $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for $C_0(\Omega)$, then for each $\omega \in \Omega$ the net of real numbers $(gu_\lambda(\omega))$ is increasing and bounded above by $\|g\|$, and therefore it converges to a number, $h(\omega)$ say. The function

$M(C_0(\Omega)) \cong C_b(\Omega)$

$\lambda \le \mu \Rightarrow u_\lambda \le u_\mu \Rightarrow g(u_\lambda) \le g(u_\mu) \Rightarrow gu_\lambda(\omega) \le gu_\mu(\omega) \ \forall \omega$

$|gu_\lambda(\omega)| \le \|gu_\lambda\| \le \|g\|\|u_\lambda\| \le \|g\|$

$$h: \Omega \to \mathbf{C}, \quad \omega \mapsto h(\omega) = \lim_\lambda gu_\lambda(\omega)$$

is bounded. Moreover, if $f \in C_0(\Omega)$, then $hf = gf$, since $f = \lim_\lambda fu_\lambda$. To see that h is continuous, let $(\omega_\mu)_{\mu\in M}$ be a net in Ω converging to a point ω. Let K be a compact neighbourhood of ω in Ω. To show that $h(\omega) = \lim_\mu h(\omega_\mu)$, we may suppose $\omega_\mu \in K$ for all indices μ (there exists μ_0 such that $\omega_\mu \in K$ for all indices $\mu \ge \mu_0$, so, if neccessary, replace the net $(\omega_\mu)_{\mu\in M}$ by the net $(\omega_\mu)_{\mu\ge\mu_0}$). Use Urysohn's lemma to choose a function $f \in C_0(\Omega)$ such that $f = 1$ on K. Since $fh \in C_0(\Omega)$,

$(gf)(\omega) = \lim_\lambda gfu_\lambda(\omega) = f(\omega)\lim_\lambda gu_\lambda(\omega) = f(\omega)h(\omega) = (fh)(\omega)$

$$h(\omega) = fh(\omega) = \lim_\mu fh(\omega_\mu) = \lim_\mu h(\omega_\mu).$$

$\omega_\mu \to \omega$

K, $\omega_\mu \ \forall \mu \ge \mu_0$

Therefore, h is continuous, so $h \in C_b(\Omega)$. For f an arbitrary function in $C_0(\Omega)$ we have $\varphi(h)f = \varphi(hf) = hf = gf$, so $(\varphi(h) - g)C_0(\Omega) = 0$. Consequently, $g = \varphi(h)$.

(1) $C_0(\Omega) \trianglelefteq C_b(\Omega)$ (es)

Let $f \in C_b(\Omega)$ & $f\,C_0(\Omega) = 0$. Assume $f \ne 0$. So $\exists \omega_0 \in \Omega$; $f(\omega_0) \ne 0$. By Urysohn's Lemma $\exists g \in C_0(\Omega)$; $g(\omega_0) = 1$. Hence $(fg)(\omega_0) \ne 0$. So $fg \ne 0$. Thus $f\,C_0(\Omega) \ne 0$. ✗

$\therefore f = 0$. $\square$

N.B. $\overline{C_c(X)}^{\|\cdot\|_\infty} = C_0(\Omega)$

$\overline{C_c(X)}^{\|\cdot\|_p} = L^p(\Omega)$

Def A C^*-subalg B of a C^*-alg A is called hereditary if $\forall a\in A_+\ \forall b\in B_+$; $a\le b \Rightarrow a\in B$

Th $\{\text{closed left ideals of } A\} \xleftrightarrow[\text{onto}]{1-1} \{\text{Hereditary } C^*\text{-subalg of } A\}$

$$L \longmapsto L\cap L^*$$

$$L=\{a\in A : a^*a\in B\} \longleftarrow\!\shortmid\ B$$

$$L_1\subseteq L_2 \Longleftrightarrow \theta(L_1)\subseteq\theta(L_2)$$

شیار، مرادخواه (طلب شما)

Def. A linear functional $\tau: A \to \mathbb{C}$ is called positive if $\tau(A_+) \subseteq \mathbb{R}^{\geq 0}$

Ex. (1) Any character is positive: $\tau(a^*a) = \tau(a^*)\tau(a) = \overline{\tau(a)}\tau(a) = |\tau(a)|^2 \geq 0$.

(2) $\tau: M_n \to \mathbb{C}$ by $\tau([a_{ij}]) = \sum a_{ii}$ is positive.

Let A be a C*-algebra and τ a positive linear functional on A. Then the function

$\sigma: A^2 \to \mathbb{C}$, $\sigma(b,a) = \overline{\sigma(a,b)}$

$$A^2 \to \mathbf{C}, \ (a, b) \mapsto \tau(b^*a),$$

is a positive sesquilinear form on A. Hence, $\tau(b^*a) = \tau(a^*b)^-$ and $|\tau(b^*a)| \leq \tau(a^*a)^{1/2}\tau(b^*b)^{1/2}$. Moreover, the function $a \mapsto \tau(a^*a)^{1/2}$ is a semi-norm on A.

Suppose now only that τ is a linear functional on A and that M is an element of $\mathbf{R}^+$ such that $|\tau(a)| \leq M$ for all positive elements of the closed unit ball of A. Then τ is bounded with norm $\|\tau\| \leq 4M$. We show this: First suppose that a is a hermitian element of A such that $\|a\| \leq 1$. Then a^+, a^- are positive elements of the closed unit ball of A, and therefore $|\tau(a)| = |\tau(a^+) - \tau(a^-)| \leq 2M$. Now suppose that a is an arbitrary element of the closed unit ball of A, so $a = b + ic$ where b, c are its real and imaginary parts, and $\|b\|, \|c\| \leq 1$. Then $|\tau(a)| = |\tau(b) + i\tau(c)| \leq 4M$.

$\tau(A_+) \subseteq \mathbb{R}^{\geq 0} \Rightarrow \tau(A_{sa}) \subseteq \mathbb{R}$ since $\tau(a) = \tau(a^+ - a^-) = \tau(a^+) - \tau(a^-) \in \mathbb{R}$

(2) $a \leq b$, $b - a \geq 0 \Rightarrow \tau(b-a) \geq 0 \Rightarrow \tau(a) \leq \tau(b)$

3.3.1. Theorem. *If τ is a positive linear functional on a C*-algebra A, then it is bounded.*

$\sup_{a \geq 0, \|a\| \leq 1} |\tau(a)| = +\infty$

Proof. If τ is not bounded, then by the preceding remarks $\sup_{a \in S} \tau(a) = +\infty$, where S is the set of all positive elements of A of norm not greater then 1. Hence, there is a sequence (a_n) in S such that $2^n \leq \tau(a_n)$ for all $n \in \mathbf{N}$. Set $a = \sum_{n=0}^{\infty} a_n/2^n$, so $a \in A^+$. Now $1 \leq \tau(a_n/2^n)$ and therefore $N \leq \sum_{n=0}^{N-1} \tau(a_n/2^n) = \tau(\sum_{n=0}^{N-1} a_n/2^n) \leq \tau(a)$. Hence, $\tau(a)$ is an upper bound for the set $\mathbf{N}$, which is impossible. This shows that τ is bounded. $\square$

$\sum_{n=0}^{N-1} 1$

(1) $a_n \geq 0$, $a_n \xrightarrow{\|\cdot\|} a \Rightarrow \forall n\ \|a_n - \|a_n\|\| \leq \|a_n\| \Rightarrow \|a - \|a\|\| \leq \|a\| \Rightarrow a \geq 0$

3.3.2. Theorem. *If τ is a positive linear functional on a C*-algebra A, then $\tau(a^*) = \tau(a)^-$ and $|\tau(a)|^2 \leq \|\tau\|\tau(a^*a)$ for all $a \in A$.*

Proof. Let $(u_\lambda)_{\lambda \in \Lambda}$ be an approximate unit for A. Then

$$\tau(a^*) = \lim_\lambda \tau(a^*u_\lambda) = \lim_\lambda \tau(u_\lambda a)^- = \tau(a)^-.$$

Also, $|\tau(a)|^2 = \lim_\lambda |\tau(u_\lambda a)|^2 \leq \sup_\lambda \tau(u_\lambda^2)\tau(a^*a) \leq \|\tau\|\tau(a^*a)$. $\square$

$|\tau(b^*a)|^2 \leq \tau(b^*b)\tau(a^*a) \leq \sup_b \tau(b^*b)\tau(a^*a)$

(3) $|\tau(u_\lambda^2)| \leq \|\tau\| \|u_\lambda^2\| \leq \|\tau\|$

3.3.3. Theorem. *Let τ be a bounded linear functional on a C*-algebra A. The following conditions are equivalent:*

(1) *τ is positive.*
(2) *For each approximate unit $(u_\lambda)_{\lambda \in \Lambda}$ of A, $\|\tau\| = \lim_\lambda \tau(u_\lambda)$.*
(3) *For some approximate unit $(u_\lambda)_{\lambda \in \Lambda}$ of A, $\|\tau\| = \lim_\lambda \tau(u_\lambda)$.*

Replace τ by $\frac{\tau}{\|\tau\|}$

Proof. We may suppose that $\|\tau\| = 1$. First we show the implication (1) $\Rightarrow$ (2) holds. Suppose that τ is positive, and let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit of A. Then $(\tau(u_\lambda))_{\lambda\in\Lambda}$ is an increasing net in $\mathbf{R}$, so it converges to its supremum, which is obviously not greater than 1. Thus, $\lim_\lambda \tau(u_\lambda) \leq 1$. Now suppose that $a \in A$ and $\|a\| \leq 1$. Then $|\tau(u_\lambda a)|^2 \leq \tau(u_\lambda^2)\tau(a^*a) \leq \tau(u_\lambda)\tau(a^*a) \leq \lim_\lambda \tau(u_\lambda)$, so $|\tau(a)|^2 \leq \lim_\lambda \tau(u_\lambda)$. Hence, $1 \leq \lim_\lambda \tau(u_\lambda)$. Therefore, $1 = \lim_\lambda \tau(u_\lambda)$, so (1) $\Rightarrow$ (2). by taking limits

$u_\lambda \leq u_\mu$ (4) $\Downarrow$ $\tau(u_\lambda) \leq \tau(u_\mu)$

$u_\lambda \leq 1$, $u_\lambda^{\frac{1}{2}} u_\lambda u_\lambda^{\frac{1}{2}} \leq u_\lambda^{\frac{1}{2}} 1 u_\lambda^{\frac{1}{2}}$, $u_\lambda^2 \leq u_\lambda$

That (2) $\Rightarrow$ (3) is obvious.

Now we show that (3) $\Rightarrow$ (1). Suppose that $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit such that $1 = \lim_\lambda \tau(u_\lambda)$. Let a be a self-adjoint element of A such that $\|a\| \leq 1$ and write $\tau(a) = \alpha + i\beta$ where α, β are real numbers. To show that $\tau(a) \in \mathbf{R}$, we may suppose that $\beta \leq 0$. If n is a positive integer, then

$\beta \geq 0 \Rightarrow \tau(-a) = -\alpha + i(-\beta)$, $-\beta \leq 0$ $\Downarrow$ $\tau(-a) \in \mathbf{R}$ $\Downarrow$ $\tau(a) \in \mathbf{R}$

$$
\begin{aligned}
\|a - inu_\lambda\|^2 &= \|(a + inu_\lambda)(a - inu_\lambda)\| \\
&= \|a^2 + n^2u_\lambda^2 - in(au_\lambda - u_\lambda a)\| \\
&\leq 1 + n^2 + n\|au_\lambda - u_\lambda a\|,
\end{aligned}
$$

triangle inequality

so

$$
|\tau(a - inu_\lambda)|^2 \leq 1 + n^2 + n\|au_\lambda - u_\lambda a\|.
$$

However, $\lim_\lambda \tau(a - inu_\lambda) = \tau(a) - in$, and $\lim_\lambda au_\lambda - u_\lambda a = 0$, so in the limit as $\lambda \to \infty$ we get $= a - a$

$$
|\alpha + i\beta - in|^2 \leq 1 + n^2.
$$

$|z|^2 = \bar{z}z$

The left-hand side of this inequality is $\alpha^2 + \beta^2 - 2n\beta + n^2$, so if we cancel and rearrange we get

$$
-2n\beta \leq 1 - \beta^2 - \alpha^2 \text{ or } n \leq \frac{1-\beta^2-\alpha^2}{-2\beta}
$$

Since β is not positive and this inequality holds for all positive integers n, β must be zero. Therefore, $\tau(a)$ is real if a is hermitian.

Now suppose that a is positive and $\|a\| \leq 1$. Then $u_\lambda - a$ is hermitian and $\|u_\lambda - a\| \leq 1$, so $\tau(u_\lambda - a) \leq 1$. But then $1 - \tau(a) = \lim_\lambda \tau(u_\lambda - a) \leq 1$, and therefore $\tau(a) \geq 0$. Thus, τ is positive and we have shown (3) $\Rightarrow$ (1). $\square$

(5') $\|u_\lambda - a\| = \|u_\lambda - au_\lambda\| \leq \|u_\lambda\|\|1 - a\| \leq 1 \cdot 1 = 1$

3.3.4. Corollary. *If τ is a bounded linear functional on a unital C*-algebra, then τ is positive if and only if $\tau(1) = \|\tau\|$.*

Proof. The sequence which is constantly 1 is an approximate unit for the C*-algebra. Apply Theorem 3.3.3. $\square$

$u_\lambda = 1 \ \forall\lambda$

3.3.5. Corollary. *If τ, τ' are positive linear functionals on a C*-algebra, then $\|\tau + \tau'\| = \|\tau\| + \|\tau'\|$.*

Proof. If $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for the algebra, then $\|\tau + \tau'\| = \lim_\lambda(\tau + \tau')(u_\lambda) = \lim_\lambda \tau(u_\lambda) + \lim_\lambda \tau'(u_\lambda) = \|\tau\| + \|\tau'\|$. $\square$

A *state* on a C*-algebra A is a positive linear functional on A of norm one. We denote by $S(A)$ the set of states of A.

3.3.6. Theorem. *If a is a normal element of a non-zero C^*-algebra A, then there is a state τ of A such that $\|a\| = |\tau(a)|$.*

If $f: X \to \mathbb{R}$ cts, X compact, then $\exists p,q \in X \; \forall x \in X$ $|f(p)| \leq |f(x)| \leq |f(q)|$

Proof. We may assume that $a \neq 0$. Let B be the C*-algebra generated by 1 and a in $\tilde{A}$. Since B is abelian and $\hat{a}$ is continuous on the compact space $\Omega(B)$, there is a character τ_2 ($\|\tau_2\|=1$) on B such that $\|a\| = \|\hat{a}\|_\infty = |\tau_2(a)|$. By the Hahn–Banach theorem, there is a bounded linear functional τ_1 on $\tilde{A}$ extending τ_2 and preserving the norm, so $\|\tau_1\| = 1$. Since $\tau_1(1) = \tau_2(1) = 1$, τ_1 is positive by Corollary 3.3.4. If τ denotes the restriction of τ_1 to A, then τ is a positive linear functional on A such that $\|a\| = |\tau(a)|$. Hence, $\|\tau\|\|a\| \geq |\tau(a)| = \|a\|$, so $\|\tau\| \geq 1$, and the reverse inequality is obvious. Therefore, τ is a state of A. $\square$

$\|\tau\| \leq \|\tau_1\| = \|\tau_2\| = 1$ $\quad$ $= |\tau_2(a)| = |\tau_1(a)|$

3.3.7. Theorem. *Suppose that τ is a positive linear functional on a C^*-algebra A.*

(1) *For each $a \in A$, $\tau(a^*a) = 0$ if and only if $\tau(ba) = 0$ for all $b \in A$.*
(2) *The inequality*

$$\tau(b^*a^*ab) \leq \|a^*a\|\tau(b^*b)$$

holds for all $a, b \in A$.

$a^*a \leq \|a^*a\|$

$b^*(a^*a)b \leq b^*\|a^*a\|b$

Proof. Condition (1) follows from the Cauchy–Schwarz inequality. $|\tau(ba)|^2 = |\tau((b^*)^*a)|^2 \leq \tau(bb^*)\underbrace{\tau(a^*a)}_{0}$

To show Condition (2), we may suppose, using Condition (1), that $\tau(b^*b) > 0$. The function

$$\rho: A \to \mathbf{C}, \; c \mapsto \tau(b^*cb)/\tau(b^*b),$$

is positive and linear, so if $(u_\lambda)_{\lambda \in \Lambda}$ is any approximate unit for A, then

$$\|\rho\| = \lim_\lambda \rho(u_\lambda) = \lim_\lambda \tau(b^*u_\lambda b)/\tau(b^*b) = \tau(b^*b)/\tau(b^*b) = 1.$$

Hence, $\rho(a^*a) \leq \|a^*a\|$, and therefore $\tau(b^*a^*ab) \leq \|a^*a\|\tau(b^*b)$. $\square$

We turn now to the problem of extending positive linear functionals.

3.3.8. Theorem. *Let B be a C^*-subalgebra of a C^*-algebra A, and suppose that τ is a positive linear functional on B. Then there is a positive linear functional τ' on A extending τ such that $\|\tau'\| = \|\tau\|$.*

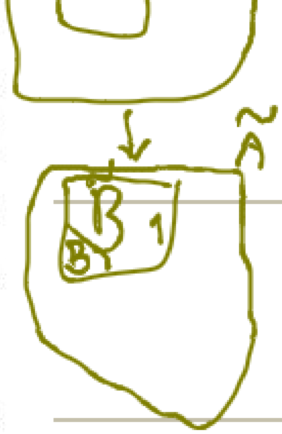
$\tilde{B} = B \oplus \mathbb{C}$

Proof. Suppose first that $A = \tilde{B}$. Define a linear functional τ' on A by setting $\tau'(b + \lambda) = \tau(b) + \lambda\|\tau\|$ ($b \in B$, $\lambda \in \mathbf{C}$). Let $(u_\lambda)_{\lambda \in \Lambda}$ be an approximate unit for B. By Theorem 3.3.3, $\|\tau\| = \lim_\lambda \tau(u_\lambda)$. Now suppose that $b \in B$ and $\mu \in \mathbf{C}$. Then $|\tau'(b + \mu)| = |\lim_\lambda \tau(bu_\lambda) + \mu \lim_\lambda \tau(u_\lambda)| = |\lim_\lambda \tau((b + \mu)(u_\lambda))| \leq \sup_\lambda \|\tau\|\|(b + \mu)u_\lambda\| \leq \|\tau\|\|b + \mu\|$, since $\|u_\lambda\| \leq 1$. Hence, $\|\tau'\| \leq \|\tau\|$, and the reverse inequality is obvious. Thus, $\|\tau'\| = \|\tau\| = \tau'(1)$, so τ' is positive by Corollary 3.3.4. This proves the theorem in the case $A = \tilde{B}$.

A

Now suppose that A is an arbitrary C*-algebra containing B as a

C*-subalgebra. Replacing B and A by $\tilde{B}$ and $\tilde{A}$ if necessary, we may suppose that A has a unit 1 which lies in B. By the Hahn–Banach theorem, there is a functional $\tau' \in A^*$ extending τ and of the same norm. Since $\tau'(1) = \tau(1) = \|\tau\| = \|\tau'\|$, it follows as before from Corollary 3.3.4 that τ' is positive. □



In the case of hereditary C*-subalgebras, we can strengthen the above result—we can even write down an "expression" for τ':

3.3.9. Theorem. *Let B be a hereditary C*-subalgebra of a C*-algebra A. If τ is a positive linear functional on B, then there is a unique positive linear functional τ' on A extending τ and preserving the norm. Moreover, if $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for B, then*

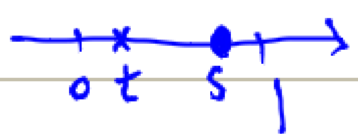
$$\tau'(a) = \lim_\lambda \tau(u_\lambda a u_\lambda) \qquad (a \in A).$$

5) Let $0 \leq T, S \leq I$. We want to show that $\|T-S\| \leq 1$:

$$\|T-S\| = \sup_{\|x\|=1} \underbrace{\langle (T-S)x, x\rangle}_{\langle Tx,x\rangle - \langle Sx,x\rangle} \leq 1$$

$0 \leq \langle Tx, x\rangle \leq \langle Ix, x\rangle = 1$

$0 \leq \langle Sx, x\rangle \leq 1$



$|t-s| \leq 1$

3.3.9. Theorem. *Let B be a hereditary C^*-subalgebra of a C^*-algebra A. If τ is a positive linear functional on B, then there is a unique positive linear functional τ' on A extending τ and preserving the norm. Moreover, if $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for B, then*

$0 \le a \le b \Rightarrow a \in B$, $b \in B$

$$\tau'(a) = \lim_\lambda \tau(\overbrace{u_\lambda a u_\lambda}^{\in B}) \quad (1) \quad (a \in A).$$

$B \xrightarrow{\tau} \mathbb{C} \Rightarrow A \xrightarrow{\tau'} \mathbb{C}$

2 $\Downarrow$

$\tilde{A} \xrightarrow{\tau'} \mathbb{C}$

$\|\tau'\| = \|\tau\|$

Proof. Of course we already have existence, so we only prove uniqueness. Let τ' be a positive linear functional on A extending τ and preserving the norm. We may in turn extend τ' in a norm-preserving fashion to a positive functional (also denoted τ') on $\tilde{A}$. Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for B. Then $\lim_\lambda \tau(u_\lambda) = \|\tau\| = \|\tau'\| = \tau'(1)$, so $\lim_\lambda \tau'(1-u_\lambda) = 0$. Thus, for any element $a \in A$,

$$\begin{aligned}
|\tau'(a) - \underbrace{\tau}_{\tau'}(u_\lambda a u_\lambda)| &\le |\tau'(a - u_\lambda a)| + |\tau'(u_\lambda a - u_\lambda a u_\lambda)| \\
&\le \tau'((1-u_\lambda)^2)^{1/2}\tau'(a^*a)^{1/2} \\
&\qquad + \tau'(a^*u_\lambda^2 a)^{1/2}\tau'((1-u_\lambda)^2)^{1/2} \\
&\le (\tau'(1-u_\lambda))^{1/2}\tau'(a^*a)^{1/2} + \tau'(a^*a)^{1/2}(\tau'(1-u_\lambda))^{1/2}.
\end{aligned}$$

$|\tau(b^*a)| \le \tau(b^*b)^{\frac{1}{2}}\tau(a^*a)^{\frac{1}{2}}$

$u_\lambda^2 \le 1$

$\Downarrow$

$a^* u_\lambda^2 a \le a^*a$

Since $\lim_\lambda \tau'(1-u_\lambda) = 0$, these inequalities imply $\lim_\lambda \tau(u_\lambda a u_\lambda) = \tau'(a)$. □

Let Ω be a compact Hausdorff space and denote by $C(\Omega, \mathbf{R})$ the real Banach space of all real-valued continuous functions on Ω. The operations on $C(\Omega,\mathbf{R})$ are the pointwise-defined ones and the norm is the sup-norm. The Riesz–Kakutani theorem asserts that if $\tau: C(\Omega,\mathbf{R}) \to \mathbf{R}$ is a bounded real-linear functional, then there is a unique real measure $\mu \in M(\Omega)$ such that $\tau(f) = \int f\,d\mu$ for all $f \in C(\Omega,\mathbf{R})$. Moreover, $\|\mu\| = \|\tau\|$, and μ is positive if and only if τ is positive; that is, $\tau(f) \ge 0$ for all $f \in C(\Omega,\mathbf{R})$ such that $f \ge 0$. The Jordan decomposition for a real measure $\mu \in M(\Omega)$ asserts that there are positive measures $\mu^+, \mu^- \in M(\Omega)$ such that $\mu = \mu^+ - \mu^-$ and $\|\mu\| = \|\mu^+\| + \|\mu^-\|$. We translate this via the Riesz–Kakutani theorem into a statement about linear functionals: If $\tau: C(\Omega,\mathbf{R}) \to \mathbf{R}$ is a bounded real-linear functional, then there exist positive bounded real-linear functionals $\tau_+, \tau_-: C(\Omega,\mathbf{R}) \to \mathbf{R}$ such that $\tau = \tau_+ - \tau_-$ and $\|\tau\| = \|\tau_+\| + \|\tau_-\|$. We are now going to prove an analogue of this result for C*-algebras.

$C(\Omega,\mathbb{R})'$

1-1 $\updownarrow$ onto

$M(\Omega)$

$\mu = \mu^+ - \mu^-$

$\int f\,d\mu = \int f\,d\mu^+ - \int f\,d\mu^-$

$\tau = \tau_+ - \tau_-$

Let A be a C*-algebra. If τ is a bounded linear functional on A, then

$$\|\tau\| = \sup_{\|a\|\le 1} |Re(\tau(a))|. \tag{1}$$

For if $a \in A$ and $\|a\| \le 1$, then there is a number $\lambda \in \mathbf{T}$ such that $\lambda\tau(a) \in \mathbf{R}$, so $|\tau(a)| = |Re(\tau(\lambda a))| \le \|\tau\|$, which implies Eq. (1).

$\tau \in A'$

$\Downarrow$

$\tau^*(a) = \overline{\tau(a^*)}$

Def

If $\tau \in A^*$, we define $\tau^* \in A^*$ by setting $\tau^*(a) = \tau(a^*)^-$ for all $a \in A$. Note that $\tau^{**} = \tau$, $\|\tau^*\| = \|\tau\|$, and the map $\tau \mapsto \tau^*$ is conjugate-linear.

We say a functional $\tau \in A^*$ is *self-adjoint* if $\tau = \tau^*$. For any bounded linear functional τ on A, there are unique self-adjoint bounded linear functionals τ_1 and τ_2 on A such that $\tau = \tau_1 + i\tau_2$ (take $\tau_1 = (\tau + \tau^*)/2$ and $\tau_2 = (\tau - \tau^*)/2i$).

$\tau = \tau^* \Leftrightarrow \tau$ is self-adj

The condition $\tau = \tau^*$ is equivalent to $\tau(A_{sa}) \subseteq \mathbf{R}$, and therefore if τ is self-adjoint, the restriction $\tau' : A_{sa} \to \mathbf{R}$ of τ is a bounded real-linear functional. Moreover, $\|\tau\| = \|\tau'\|$; that is,

$$\|\tau\| = \sup_{\substack{a \in A_{sa} \\ \|a\| \leq 1}} |\tau(a)|.$$

For if $a \in A$, we have $\mathrm{Re}(\tau(a)) = \tau(\mathrm{Re}(a))$, so

$$\|\tau\| = \sup_{\|a\| \leq 1} |\mathrm{Re}(\tau(a))| \leq \sup_{\substack{b \in A_{sa} \\ \|b\| \leq 1}} |\tau(b)| \leq \|\tau\|.$$

Left to PhD students!

We denote by A^*_{sa} the set of self-adjoint functionals in A^*, and by A^*_+ the set of positive functionals in A^*.

We adopt some temporary notation for the proof of the next theorem: If X is a real-linear Banach space, we denote its dual (over $\mathbf{R}$) by $X^\natural$.

The space A_{sa} is a real-linear Banach space and it is an easy exercise to verify that A^*_{sa} is a real-linear vector subspace of A^* and that the map $A^*_{sa} \to A^\natural_{sa}$, $\tau \mapsto \tau'$, is an isometric real-linear isomorphism. We shall use these observations in the proof of the following result.

3.3.10. Theorem (Jordan Decomposition). *Let τ be a self-adjoint bounded linear functional on a C^*-algebra A. Then there exist positive linear functionals τ_+, τ_- on A such that $\tau = \tau_+ - \tau_-$ and $\|\tau\| = \|\tau_+\| + \|\tau_-\|$.*

The Gelfand–Naimark Representation

isomt.
$A \hookrightarrow B(H)$
$*$-isom
(but not surjective)

In this section we introduce the important GNS construction and prove that every C*-algebra can be regarded as a C*-subalgebra of $B(H)$ for some Hilbert space H. It is partly due to this concrete realisation of the C*-algebras that their theory is so accessible in comparison with more general Banach algebras.

A *representation* of a C*-algebra A is a pair (H, φ) where H is a Hilbert space and $\varphi : A \to B(H)$ is a $*$-homomorphism. We say (H, φ) is *faithful* if φ is injective.

C*-alg

If $(H_\lambda, \varphi_\lambda)_{\lambda \in \Lambda}$ is a family of representations of A, their *direct sum* is the representation (H, φ) got by setting $H = \oplus_\lambda H_\lambda$, and $\varphi(a)((x_\lambda)_\lambda) = (\varphi_\lambda(a)(x_\lambda))_\lambda$ for all $a \in A$ and all $(x_\lambda)_\lambda \in H$. It is readily verified that (H, φ) is indeed a representation of A. If for each non-zero element $a \in A$ there is an index λ such that $\varphi_\lambda(a) \neq 0$, then (H, φ) is faithful.

$\oplus_\alpha \varphi_\alpha : A \to B(\oplus_\alpha H_\alpha)$
$a \mapsto \oplus_\alpha \varphi_\alpha(a)$

Recall now that if H is an inner product space (that is, a pre-Hilbert

$\oplus \varphi_\alpha(a) : \oplus H_\alpha \to \oplus H_\alpha$

space), then there is a unique inner product on the Banach space completion $\hat{H}$ of H extending the inner product of H and having as its associated norm the norm of $\hat{H}$. We call $\hat{H}$ endowed with this inner product the *Hilbert space completion* of H. $(x)_\alpha \mapsto (\varphi(a)(x)_\alpha)_\alpha$

With each positive linear functional, there is associated a representation. Suppose that τ is a positive linear functional on a C*-algebra A. Setting $a_\alpha \to a \Rightarrow a_\alpha^* a_\alpha \to a^* a$ $\quad a\in A,\ b \in N_\tau \Rightarrow |\tau(b^*a^*ab)| \le \|a^*a\| \underbrace{\tau(b^*b)}_{0} \Rightarrow ab \in N_\tau$

$\Rightarrow \tau(a_\alpha^* a_\alpha) \to \tau(a^*a) \Rightarrow \tau(a^*a) = 0 \Rightarrow a \in N_\tau$

$$N_\tau = \{a \in A \mid \tau(a^*a) = 0\},$$

it is easy to check (using Theorem 3.3.7) that N_τ is a closed left ideal of A and that the map

4
$$(A/N_\tau)^2 \to \mathbf{C},\ (a + N_\tau, b + N_\tau) \mapsto \tau(b^*a)$$

www.ams.org
Notices-2013
I. Gelfand

$\langle a+N_\tau, a+N_\tau\rangle = \tau(a^*a) = 0 \Rightarrow a \in N_\tau \Rightarrow a+N_\tau = 0$

5
is a well-defined inner product on A/N_τ. We denote by H_τ the Hilbert completion of A/N_τ. $\overline{A/N_\tau} = H_\tau$

If $a \in A$, define an operator $\varphi(a) \in B(A/N_\tau)$ by setting

$\varphi(a): A/N_\tau \to A/N_\tau \subseteq H_\tau$

$$\varphi(a)(b + N_\tau) = ab + N_\tau.$$

$\langle ab+N_\tau, ab+N_\tau\rangle$

The inequality $\|\varphi(a)\| \le \|a\|$ holds since we have $\|\varphi(a)(b + N_\tau)\|^2 = \tau(b^*a^*ab) \le \|a\|^2\tau(b^*b) = \|a\|^2\|b + N_\tau\|^2$ (the latter inequality is given by Theorem 3.3.7). The operator $\varphi(a)$ has a unique extension to a bounded operator $\varphi_\tau(a)$ on H_τ. The map

$T: D \subseteq X \to Y$
$\Downarrow$
$\overline{T}: \overline{D} \to Y$
$\|\overline{T}\| = \|T\|$

5
$$\varphi_\tau: A \to B(H_\tau),\ a \mapsto \varphi_\tau(a),$$

is a $*$-homomorphism (this is an easy exercise).

The representation (H_τ, φ_τ) of A is the *Gelfand–Naimark–Segal* representation (or *GNS representation*) associated to τ.

If A is non-zero, we define its *universal* representation to be the direct sum of all the representations (H_τ, φ_τ), where τ ranges over $S(A)$.

$0 = \varphi_\tau(b^4) = \varphi_\tau(b)^4$ self-adj $\Downarrow$ $\varphi_\tau(b) = 0$

3.4.1. Theorem (Gelfand–Naimark). *If A is a C*-algebra, then it has a faithful representation. Specifically, its universal representation is faithful.*

$\oplus (H_\tau, \varphi_\tau)$ τ is state

Proof. Let (H, φ) be the universal representation of A and suppose that a is an element of A such that $\varphi(a) = 0$. By Theorem 3.3.6 there is a state τ on A such that $\|a^*a\| = \tau(a^*a)$. Hence, if $b = (a^*a)^{1/4}$, then $\|a\|^2 = \tau(a^*a) = \tau(b^4) = \|\varphi_\tau(b)(b + N_\tau)\|^2 = 0$ (since $\varphi_\tau(b^4) = \varphi_\tau(a^*a) = 0$, so $\varphi_\tau(b) = 0$). Hence, $a = 0$, and φ is injective. □

$\langle b^2+N_\tau, b^2+N_\tau\rangle$ $\quad \oplus \varphi_\tau(a) = 0$

The Gelfand–Naimark theorem is one of those results that are used all of the time. For the present we give just two applications.

The first application is to matrix algebras. If A is an algebra, $M_n(A)$

denotes the algebra of all $n \times n$ matrices with entries in A. (The operations are defined just as for scalar matrices.) If A is a $*$-algebra, so is $M_n(A)$, where the involution is given by $(a_{ij})^*_{i,j} = (a^*_{ji})_{i,j}$. Algebraic structure

شنبه
91,10,5

If $\varphi: A \to B$ is a $*$-homomorphism between $*$-algebras, its *inflation* is the $*$-homomorphism (also denoted φ)

$$\varphi: M_n(A) \to M_n(B),\ (a_{ij}) \mapsto (\varphi(a_{ij})).$$

If H is a Hilbert space, we write $H^{(n)}$ for the orthogonal sum of n copies of H. If $u \in M_n(B(H))$, we define $\varphi(u) \in B(H^{(n)})$ by setting

6

$$\varphi(u)(x_1, \ldots, x_n) = (\sum_{j=1}^{n} u_{1j}(x_j), \ldots, \sum_{j=1}^{n} u_{nj}(x_j)),$$

for all $(x_1, \ldots, x_n) \in H^{(n)}$. It is readily verified that the map

$$\varphi: M_n(B(H)) \to B(H^{(n)}),\ u \mapsto \varphi(u), \quad 7$$

is a $*$-isomorphism. We call φ the *canonical* $*$-isomorphism of $M_n(B(H))$ onto $B(H^{(n)})$, and use it to identify these two algebras. If v is an operator in $B(H^{(n)})$ such that $v = \varphi(u)$ where $u \in M_n(B(H))$, we call u the *operator matrix* of v. We define a norm on $M_n(B(H))$ making it a C*-algebra by setting $\|u\| \overset{\text{Def}}{:=} \|\varphi(u)\|$. The following inequalities for $u \in M_n(B(H))$ are easy to verify and are often useful:

قانون تابلو

$$\|u_{ij}\| \le \|u\| \le \sum_{k,l=1}^{n} \|u_{kl}\| \qquad (i, j = 1, \ldots, n).$$

3.4.2. Theorem. *If A is a C*-algebra, then there is a unique norm on $M_n(A)$ making it a C*-algebra.*

G-N Th.
$A \hookrightarrow B(H)$

Proof. Let the pair (H, φ) be the universal representation of A, so the $*$-homomorphism $\varphi: M_n(A) \to M_n(B(H))$ is injective. We define a norm on $M_n(A)$ making it a C*-algebra by setting $\|a\| = \|\varphi(a)\|$ for $a \in M_n(A)$ (completeness can be easily checked using the inequalities preceding this theorem). Uniqueness is given by Corollary 2.1.2. $\square$

$M_n(A) \hookrightarrow M_n(B(H))$
is a C*-alg

3.4.1. *Remark.* If A is a C*-algebra and $a \in M_n(A)$, then

$$\|a_{ij}\| \le \|a\| \le \sum_{k,l=1}^{n} \|a_{kl}\| \qquad (i, j = 1, \ldots, n).$$

These inequalities follow from the corresponding inequalities in $M_n(B(H))$.

Matrix algebras play a fundamental role in the K-theory of C*-algebras. The idea is to study not just the algebra A but simultaneously all of the matrix algebras $M_n(A)$ over A also.

Whereas it seems that the only way known of showing that matrix algebras over general C*-algebras are themselves normable as C*-algebras is to use the Gelfand–Naimark representation, for our second application of this representation alternative proofs exist, but the proof given here has the virtue of being very "natural."

3.4.3. Theorem. *Let a be a self-adjoint element of a C*-algebra A. Then $a \in A^+$ if and only if $\tau(a) \geq 0$ for all positive linear functionals τ on A.*

Proof. The forward implication is plain. Suppose conversely that $\tau(a) \geq 0$ for all positive linear functionals τ on A. Let (H, φ) be the universal representation of A, and let $x \in H$. Then the linear functional

$\oplus (H_\tau, \varphi_\tau)$
$\tau \in S(A)$

$$\tau: A \to \mathbf{C}, \quad b \mapsto \langle \varphi(b)(x), x \rangle,$$

9

is positive, so $\tau(a) \geq 0$; that is, $\langle \varphi(a)(x), x \rangle \geq 0$. Since this is true f $x \in H$, and since $\varphi(a)$ is self-adjoint, therefore $\varphi(a)$ is a positive ope on H. Hence, $\varphi(a) \in \varphi(A)^+$, so $a \in A^+$, because the map $\varphi: A \to \varphi($ a $*$-isomorphism.

$\varphi^{-1}(\varphi(a))$

isomet

$\varphi: A \hookrightarrow B(H) \Rightarrow \varphi: A \xrightarrow[*\text{-isom}]{\text{isom}} \varphi(A) \Rightarrow \varphi^{-1}: \varphi(A) \xrightarrow[*\text{-isom}]{} A$

(1) B is hd C^*-subalg of $A \iff \forall a \in A\ \forall b, b' \in B;\ bab' \in B$

Proof. $(\Rightarrow)$ $\exists L$ left ideal of A; $B = L \cap L^*$

$\forall a \in A\ \forall b, b' \in B;\quad bab' = (\underbrace{ba}_{\in A})\underbrace{b'}_{\in L} \in L$

$(bab')^* = (\underbrace{b'^* a^*}_{\in A})\underbrace{b^*}_{\in L} \in L \Rightarrow bab' \in L^*$ $\Big\} \Rightarrow bab' \in B$

$(\Leftarrow)$ Let $0 \leq a \leq b \in B$. Let (u_λ) be an approx identity for B.

We have $0 \leq \|(1-u_\lambda)\underbrace{a}_{a^{\frac12}a^{\frac12}}(1-u_\lambda)\| \leq \|(1-u_\lambda)\underbrace{b}_{b^{\frac12}b^{\frac12}}(1-u_\lambda)\| \Rightarrow \|a^{\frac12}(1-u_\lambda)\|^2 \leq$

$\|b^{\frac12}(1-u_\lambda)\| \quad \Rightarrow \lim a^{\frac12}u_\lambda = a^{\frac12} \Rightarrow a = a^{\frac12}a^{\frac12} = \lim u_\lambda a u_\lambda \in B$

$\in B$ λ $\searrow_0$ $\Rightarrow$ λ λ $\Rightarrow$ λ λ $\in B$

$\hookrightarrow \lim u a^{\frac{1}{2}} = a^{\frac{1}{2}}$

(2) Corollary. A is a hd C^*-subalg of $\tilde{A}$.

Every closed ideal is hd in the given C^*-alg — previous theorem

(3) Let $\varphi_1 : A \to B(H_1)$ & $\varphi_2 : A \to B(H_2)$ be $*$-repn. Define $\varphi_1 \oplus \varphi_2 : A \to B(H_1 \oplus H_2)$

$$a \mapsto (\varphi_1 \oplus \varphi_2)(a) : H_1 \oplus H_2 \longrightarrow H_1 \oplus H_2$$

$$(x_1, x_2) \mapsto (\varphi_1(a)(x_1), \varphi_2(a)(x_2))$$

(4) If M is a subspace of a normed space X, then $\frac{X}{M}$ is a vector space & $\|x+M\| = \inf_{z \in M} \|x+z\|$ is a semi-norm. If M is closed, then $\|\cdot\|$ is a norm on $\frac{X}{M}$. If M is a two-sided ideal, then $\frac{X}{M}$ is an alg under $(x+M)(y+M) = xy+M$

(5) φ is $*$-map: $\varphi_\tau(a^*) = \varphi_\tau(a)^*$ or equivalently
$\tau: A \to B(H_\tau)$ $\langle \varphi_\tau(a^*)x, y\rangle = \langle \varphi_\tau(a)^*x, y\rangle$ $\forall x,y \in H_\tau$.

Since $\frac{A}{N_\tau}$ is dense in H_τ it enough to show that

$$\langle \varphi_\tau(a^*)(b+N_\tau), c+N_\tau\rangle = \langle \varphi_\tau(a)^*(b+N_\tau), c+N_\tau\rangle$$

$$\underbrace{\langle a^*b+N_\tau, c+N_\tau\rangle}_{\tau(c^*(a^*b))} = \underbrace{\langle b+N_\tau, ac+N_\tau\rangle}_{\tau((ac)^*b)}$$

$$\tau(c^*(a^*b)) = \tau((ac)^*b) \checkmark$$

(6) Given $(H_i, \langle\cdot,\cdot\rangle_i)_{i\in I}$. We can introduce the direct sum of H_is:

$$\bigoplus_{i\in I} H_i = \left\{ (x_i) \;\middle|\; \left(\sum_{i\in I} \|x_i\|^2\right)^{\frac{1}{2}} < +\infty \right\}$$

$\sum_{i\in I} \|x_i\|^2 = \sup_{\substack{F \text{ finite subset of } I}} \sum_{i\in F} \|x_i\|^2$

all x_i are zero except a countable number of x_is

(In particular, $H_1 \oplus H_2 = \{(x_1, x_2) : x_1 \in H_1 \;\&\; x_2 \in H_2\}$)

Define $\langle (x_i), (y_i)\rangle = \sum_{i\in I} \langle x_i, y_i\rangle \in \mathbb{C}$

the Series is absolutely convergent

$$\sum_{i=1}^{n} |\langle x_i, y_i \rangle| \le \sum_{i=1}^{n} \|x_i\| \|y_i\| \le \left(\sum_{i=1}^{n} \|x_i\|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \|y_i\|^2\right)^{\frac{1}{2}}$$

$$\le \left(\sum_i \|x_i\|^2\right)^{\frac{1}{2}} \left(\sum_i \|y_i\|^2\right)^{\frac{1}{2}} < \infty$$

(In particular, $\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle$)

If H_i is complete, then so is $\bigoplus H_i$.

(7) $\varphi : M_n(B(H)) \xleftrightarrow[\text{(algebraic)}]{*\text{-isomorphic}} B(H^{(n)})$

$H^{(n)} = \underbrace{H \oplus \cdots \oplus H}_{n}$

$$u = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ \vdots & & & \\ u_{n1} & \cdots & & u_{nn} \end{bmatrix}_{n \times n} \longmapsto \varphi(u) : H^{(n)} \to H^{(n)}$$

$u_{ij} = \pi_i u e_j$

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mapsto u\underline{x} = \begin{bmatrix} \sum_{j=1}^{n} u_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} u_{nj} x_j \end{bmatrix}$$

(8) $\varphi : X \longrightarrow (Y, d)$ is a 1-1 & onto map, then (X set)

$\rho(x_1, x_2) := d(\varphi(x_1), \varphi(x_2))$ is a metric on X. ...

(9) Note: Every $*$-hom $\varphi : A \to B$ is positive: $\varphi(a^*a) = \varphi(a)^* \varphi(a) \geq 0$