# LINEAR CONTROL SYSTEMS

Ali Karimpour
Professor
Ferdowsi University of Mashhad

# Lecture 3

## External and Internal Description Model. (SS and TF Description)

*Topics to be covered include:*

- External Description.(TF Model)
- Internal Model Description. (SS Model)

# TF model properties

1- It is available just for linear  time invariant systems. (LTI)

2- It is derived by zero initial condition.

3- It just shows the relation between input and output so it may lose some information.

4- It is just dependent on the structure of system and it is independent to the input value and type.

5- It can be used to show delay systems but SS can not.

# Laplace transform table

| $f(t) \qquad (t \geq 0)$ | $\mathcal{L}[f(t)]$ |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $\delta_D(t)$ | $1$ |
| $t$ | $\frac{1}{s^2}$ |
| $t^n \qquad n \in \mathbb{Z}^+$ | $\frac{n!}{s^{n+1}}$ |
| $e^{\alpha t} \qquad \alpha \in \mathbb{C}$ | $\frac{1}{s-\alpha}$ |
| $te^{\alpha t} \qquad \alpha \in \mathbb{C}$ | $\frac{1}{(s-\alpha)^2}$ |
| $\cos(\omega_o t)$ | $\frac{s}{s^2+\omega_o^2}$ |
| $\sin(\omega_o t)$ | $\frac{\omega_o}{s^2+\omega_o^2}$ |
| $e^{\alpha t}\sin(\omega_o t+\beta)$ | $\frac{(\sin\beta)s+\omega_o^2\cos\beta-\alpha\sin\beta}{(s-\alpha)^2+\omega_o^2}$ |
| $t\sin(\omega_o t)$ | $\frac{2\omega_o s}{(s^2+\omega_o^2)^2}$ |
| $t\cos(\omega_o t)$ | $\frac{s^2-\omega_o^2}{(s^2+\omega_o^2)^2}$ |
| u(t) - u(t-τ) | $\frac{1-e^{-s\tau}}{s}$ |

# Laplace transform properties

| $f(t)$ | $\mathcal{L}[f(t)]$ | Names |
|---|---|---|
| $\sum_{i=1}^{l} a_i f_i(t)$ | $\sum_{i=1}^{l} a_i F_i(s)$ | Linear combination |
| $\frac{dy(t)}{dt}$ | $sY(s) - y(0^-)$ | Derivative Law |
| $\frac{d^k y(t)}{dt^k}$ | $s^k Y(s) - \sum_{i=1}^{k} s^{k-i} \left. \frac{d^{i-1}y(t)}{dt^{i-1}} \right\|_{t=0^-}$ | High order derivative |
| $\int_{0^-}^{t} y(\tau)d\tau$ | $\frac{1}{s}Y(s)$ | Integral Law |
| $y(t-\tau)\, u(t-\tau)$ | $e^{-s\tau}Y(s)$ | Delay |
| $ty(t)$ | $-\frac{dY(s)}{ds}$ | |
| $t^k y(t)$ | $(-1)^k \frac{d^k Y(s)}{ds^k}$ | |
| $\int_{0^-}^{t} f_1(\tau) f_2(t-\tau)d\tau$ | $F_1(s)F_2(s)$ | Convolution |
| $\lim_{t\to\infty} y(t)$ | $\lim_{s\to 0} sY(s)$ | Final Value Theorem |
| $\lim_{t\to 0^+} y(t)$ | $\lim_{s\to\infty} sY(s)$ | Initial Value Theorem |
| $f_1(t)f_2(t)$ | $\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F_1(\zeta)F_2(s-\zeta)d\zeta$ | Time domain product |
| $e^{at} f_1(t)$ | $F_1(s-a)$ | Frequency Shift |

# Laplace transform properties

| | | |
|---|---|---|
| $\lim_{t\to\infty} y(t)$ | $\lim_{s\to 0} sY(s)$ | Final Value Theorem |
| $\lim_{t\to 0^+} y(t)$ | $\lim_{s\to\infty} sY(s)$ | Initial Value Theorem |

**Example 1:** Check F.V.T and I.V.T for following:

$a)\ Y(s)=\dfrac{3}{s(s+3)}$ $\quad Y(s)=\dfrac{1}{s}+\dfrac{-1}{s+3} \quad y(t)=1-e^{-3t}\ \ t>0$

$b)\ Y(s)=\dfrac{2s}{s^2+1}$ Only I.V.T is valid.

$c)\ Y(s)=\dfrac{1}{s-1}$ Only I.V.T is valid.

$d)\ Y(s)=\dfrac{s}{s+1}$ F.V.T is valid and I.V.T is valid.

# Poles of Transfer function

Transfer function $$G(s)=\frac{n(s)}{d(s)}$$

Pole: *s=p* is a value of s that the absolute value of TF is infinite.

How to find it? Let *d(s)=0* and find the roots

Why is it important? It shows the behavior of the system.

Why?????

# Poles of Transfer function

**Example 2:** Poles and their physical meaning.

Transfer function $$G(s)=\frac{s+6}{s^2+5s+6}$$

$s^2+5s+6=0$ → $p_1= -2,\ p_2=-3$ MATLAB roots([1 5 6])

$$\text{Step Response of } c(s)=\frac{s+6}{s(s^2+5s+6)}=\frac{1}{s}+\frac{-2}{s+2}+\frac{1}{s+3}$$

$$c(t)=1u(t)-2e^{-2t}u(t)+1e^{-3t}u(t)$$

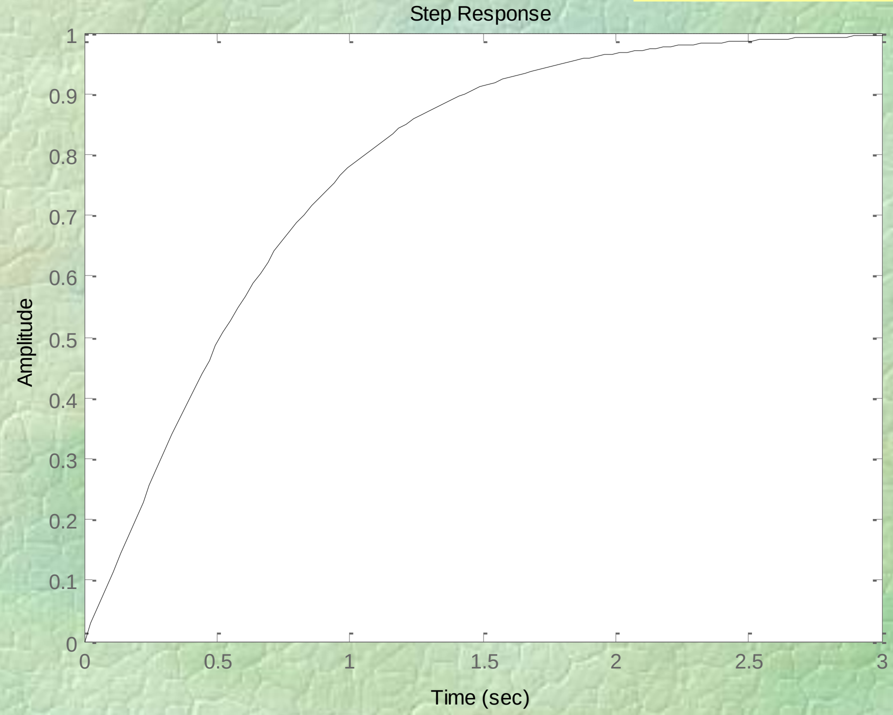
$p_1= -2,\ p_2=-3$

# Poles of Transfer function

**Example 2:** Poles and their physical meaning.

Transfer function

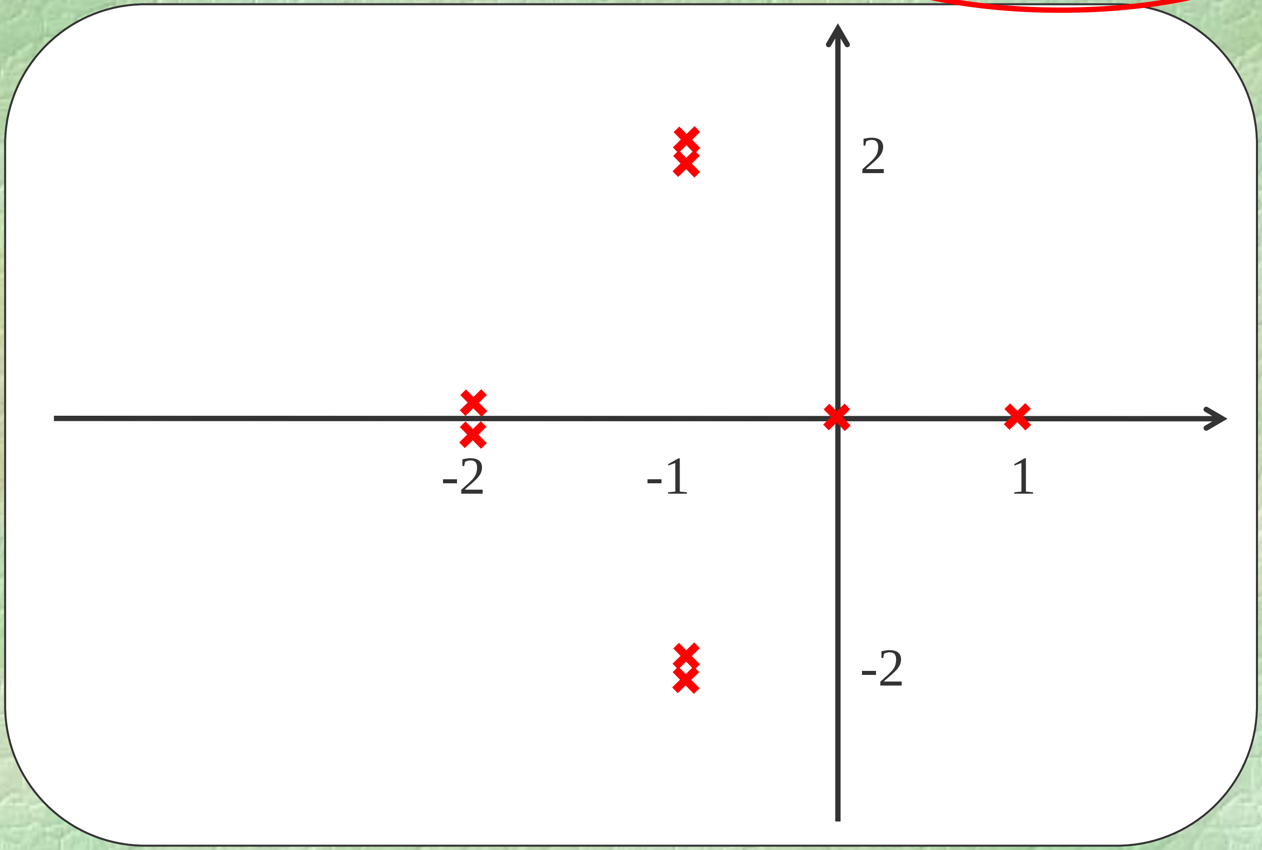
$$G(s) = \frac{s+6}{s^2+5s+6}$$




MATLAB step([1 6],[1 5 6])

# Example 3: Derive the poles of Y(s).

$$y(t) = \left(1 + e^{t} + e^{-2t} + te^{-2t} + te^{-t}\sin(2t)\right)u(t)$$




Remark1: Which terms go to infinity?

Remark 2: Poles on the imaginary axis?

# Zeros of Transfer function

Transfer function $$G(s)=\frac{n(s)}{d(s)}$$

Zero: *s=z* is a value of s that the absolute value of TF is zero.

Who to find it? Let *n(s)=0* and find the roots

Why is it important? It also shows the behavior of system.

Why?????

# Zeros of Transfer function

**Example 4:** Zeros and their physical meaning.

Transfer functions

$$G_1(s)=\frac{s+6}{s^2+5s+6} \qquad G_2(s)=\frac{10s+6}{s^2+5s+6}$$

$z_1= -6$ $\qquad$ $z_2=-0.6$

$$c_1(s)=\frac{s+6}{s(s^2+5s+6)}=\frac{1}{s}+\frac{-2}{s+2}+\frac{1}{s+3} \qquad c_1(t)=\left(1-2e^{-2t}+1e^{-3t}\right)u(t)$$

# Zeros of Transfer function

**Example 4:** Zeros and their physical meaning.

Transfer functions

$$G_1(s)=\frac{s+6}{s^2+5s+6} \qquad G_2(s)=\frac{10s+6}{s^2+5s+6}$$

$z_1= -6$ $\qquad$ $z_2=-0.6$

$$c_2(s)=\frac{10s+6}{s(s^2+5s+6)}=\frac{1}{s}+\frac{7}{s+2}+\frac{-8}{s+3} \qquad c_2(t)=\left(1+7e^{-2t}-8e^{-3t}\right)u(t)$$

# Zeros of Transfer function

**Example 4:** Zeros and their physical meaning.

Transfer functions

$$G_1(s)=\frac{s+6}{s^2+5s+6} \qquad G_2(s)=\frac{10s+6}{s^2+5s+6}$$

$z_1= -6$ $\qquad$ $z_2=-0.6$

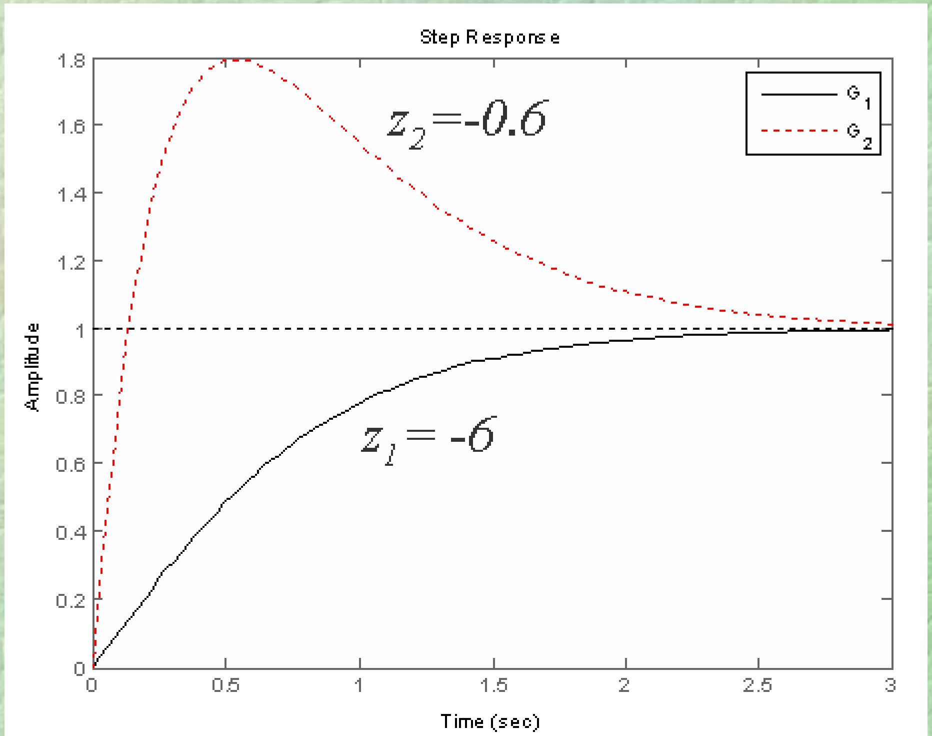
$$G_1(s)=\frac{s+6}{s^2+5s+6} \qquad \begin{cases} p_1=-2 & p_2=-3 \\ z_1=-6 & z_2=\infty \end{cases}$$

Number of poles and zeros?

# Zeros of Transfer function

**Example 4:** Zeros and their physical meaning.

Transfer function $G_1(s) = \dfrac{s+6}{s^2+5s+6}$ $\quad G_2(s) = \dfrac{10s+6}{s^2+5s+6}$




MATLAB step([1 6],[1 5 6]) ;hold on;step([10 6],[1 5 6])

# Zeros and their physical meaning

**Example 5:** Zeros and their physical meaning.

$$\ddot{y} + 5\dot{y} + 6y = \dot{u} + u$$

Transfer function of the system is:

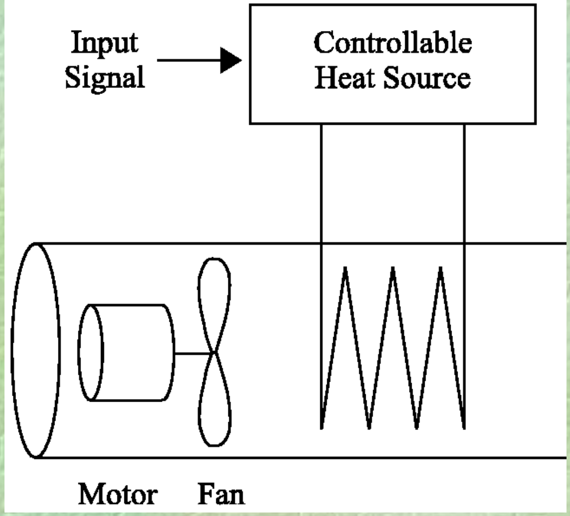
$$\frac{y(s)}{u(s)} = \frac{s+1}{s^2+5s+6}$$

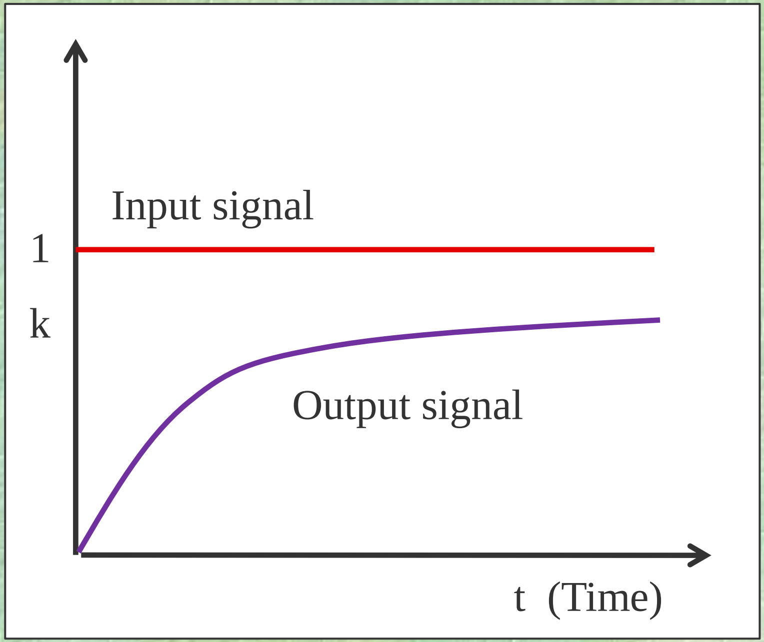
It has a zero at $z=-1$

What does it mean?

$u=ae^{-1t}$ and suitable initial condition leads to $y(t)=\underline{0}$

# **Example 6:** A thermal system




Input signal is an step function so:

$$u(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$$

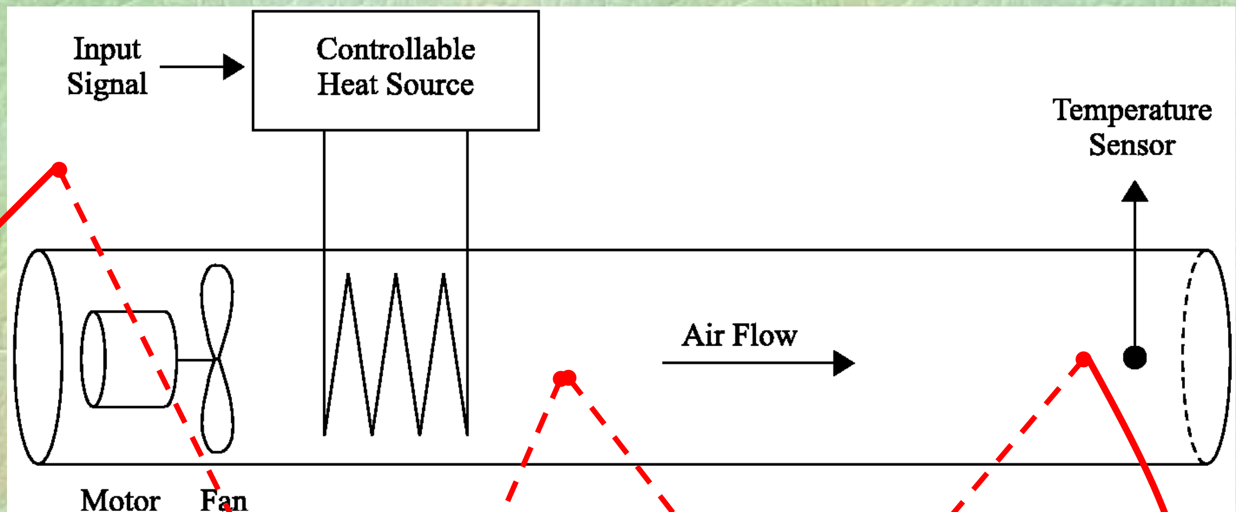
$$u(s)=\frac{1}{s}$$

$$y(t)=\begin{cases}k-ke^{\frac{-t}{\tau}} & t\geq 0\\ 0 & t<0\end{cases}$$

$$y(s)=\frac{k}{s}-\frac{k}{s+1/\tau}$$

$$\Rightarrow \quad g(s)=\frac{y(s)}{u(s)}=\frac{k}{\tau s+1}$$

# Example 7: A system with pure time delay




$$g(s)=\frac{k}{\tau s+1}$$

$$h(s)=e^{-sT_d}$$

$$h(s)g(s)=\frac{ke^{-sT_d}}{\tau s+1}$$

# External and Internal Description Model. (SS and TF Description)

*Topics to be covered include:*

- External Description.(TF Model)
- Internal Model Description. (SS Model)

# Internal Description Model. (SS Description)

- State space solution.

- Eigenvalues of the matrix $A$ and poles of transfer function.

# State Space Models

LTI systems

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

If initial condition and input are defined, then x(t), y(t) ?

$$\dot{x} = Ax + Bu \qquad x(t_0) = x_0$$

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

State transition equation

$$e^{At} = L^{-1}((sI - A)^{-1})$$ State transition matrix

# State transition equation

**Example 8:** Find $x_1(t)$ and $x_2(t)$ for unit step.

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad e^{At} = \begin{bmatrix} e^{-2t} & e^{-2t} - e^{-3t} \\ 0 & e^{-3t} \end{bmatrix}$$

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} b u(\tau) d\tau$$

$$x(t) = \begin{bmatrix} e^{-2t} & e^{-2t} - e^{-3t} \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-2(t-\tau)} & e^{-2(t-\tau)} - e^{-3(t-\tau)} \\ 0 & e^{-3(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(\tau) d\tau$$

$$x(t) = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} \\ 2e^{-3t} \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-2(t-\tau)} - e^{-3(t-\tau)} \\ e^{-3(t-\tau)} \end{bmatrix} d\tau = \begin{bmatrix} \cdots\cdots \\ \cdots\cdots \end{bmatrix}$$

These are eigenvalues of A. $\quad |sI - A| = 0$

# State transition equation

**Example 8:** Find $x_1(t)$ and $x_2(t)$ for unit step.

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{-2t} & e^{-2t} - e^{-3t} \\ 0 & e^{-3t} \end{bmatrix}$$

The eigenvalues of A?

# State transition equation

**Example 9:** Find $x_1(s)$ and $x_2(s)$

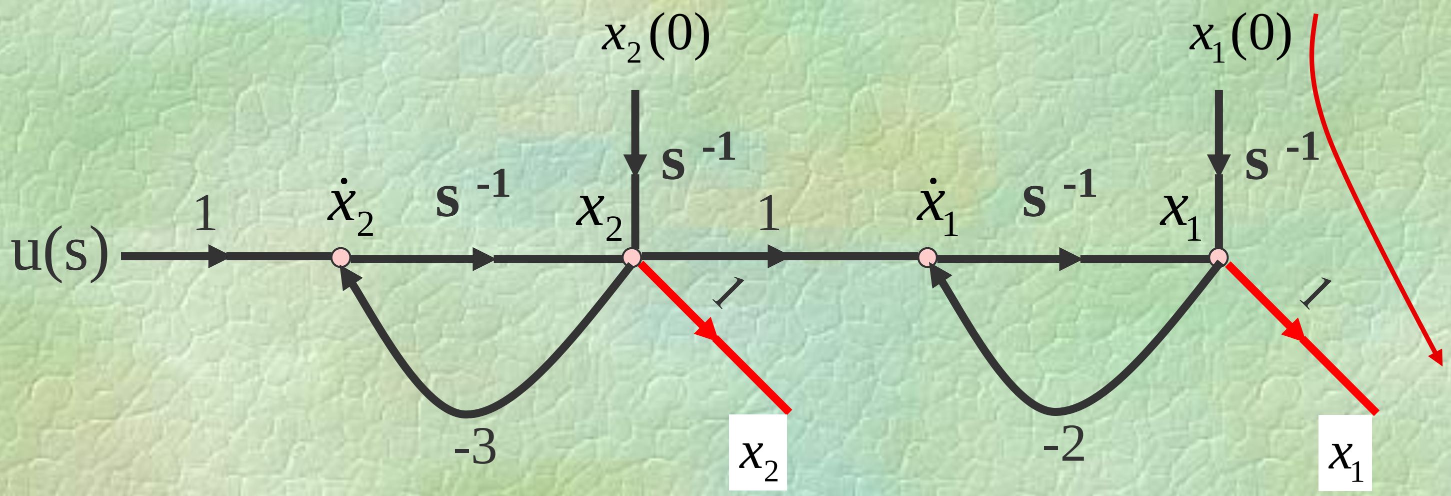
$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} Bu(s)$$

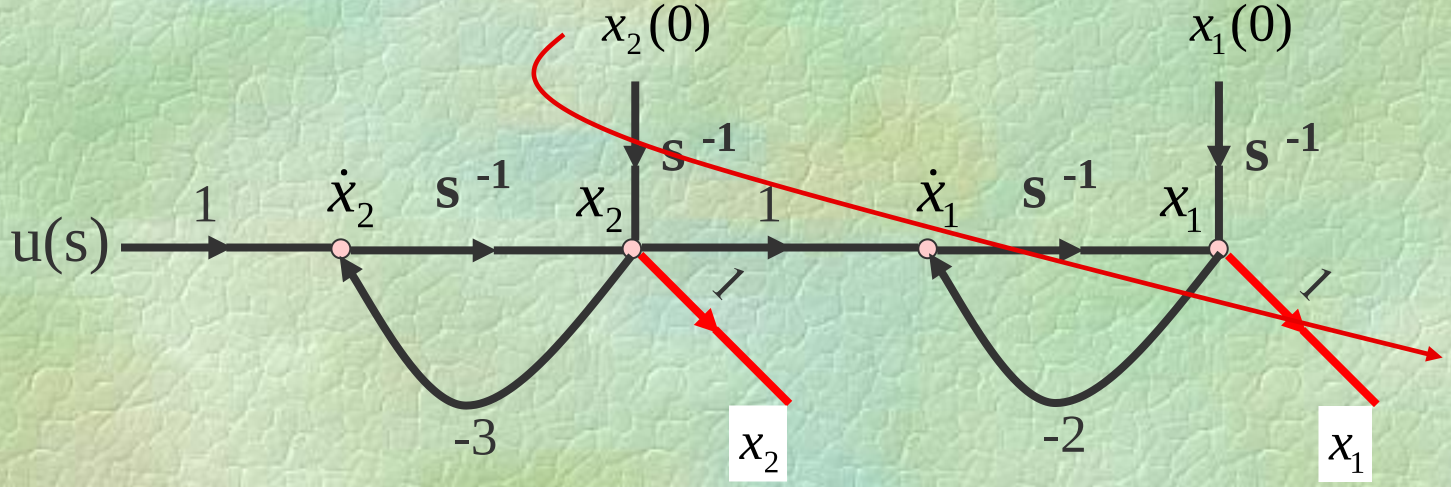
$$x(s) = \begin{bmatrix} s+2 & -1 \\ 0 & s+3 \end{bmatrix}^{-1} x(0) + \begin{bmatrix} s+2 & -1 \\ 0 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(s)$$

$$\begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{s+2} & \frac{1}{(s+2)(s+3)} \\ 0 & \frac{1}{s+3} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{(s+2)(s+3)} \\ \frac{1}{s+3} \end{bmatrix} u(s) = \begin{bmatrix} \frac{1}{s+2} x_1(0) + \frac{1}{(s+2)(s+3)} x_2(0) + \frac{1}{(s+2)(s+3)} u(s) \\ \frac{1}{s+3} x_2(0) + \frac{1}{s+3} u(s) \end{bmatrix}$$

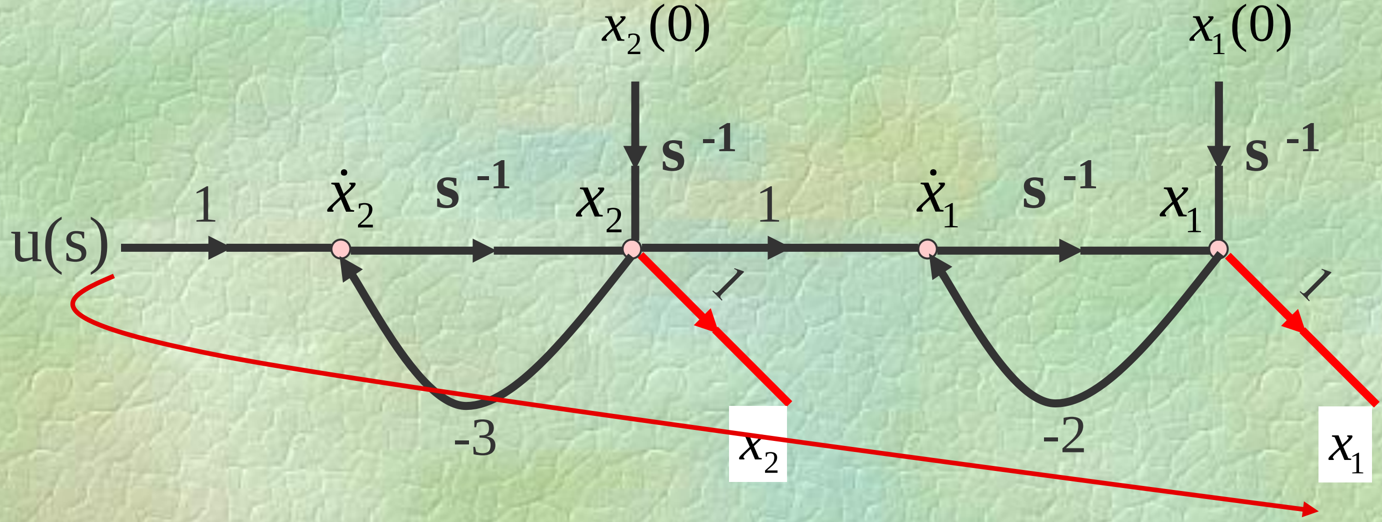
# State transition equation



# State transition equation



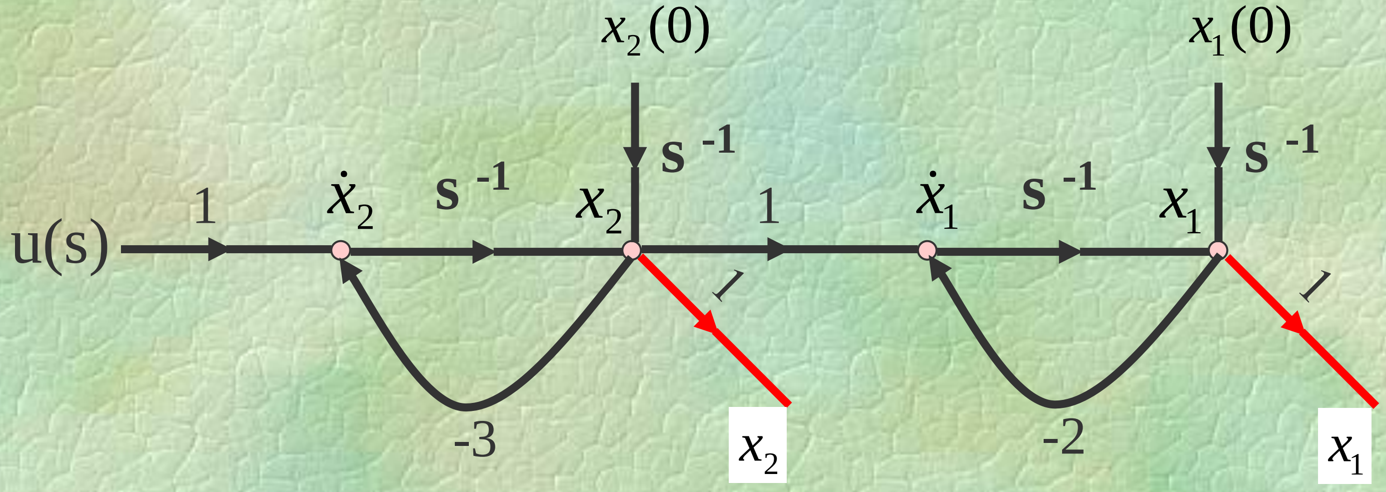
# State transition equation



# State transition equation

**Example 10:** Derive $x_1(s)$ by using state diagram.

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$




$$x_1(s) = \frac{1}{s+2} x_1(0) + \frac{1}{(s+2)(s+3)} x_2(0) + \frac{1}{(s+2)(s+3)} u(s)$$

# Eigenvalues of matrix A and poles of transfer function

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 1 \quad 0] x$$

$$g(s) = \frac{1}{(s+2)(s+3)}$$

# Eigenvalues of matrix A and poles of transfer function

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 1 \quad 0]x$$

$$g(s) = \frac{1}{(s+2)(s+3)}$$

MATLAB

[num,den]=ss2tf(A,b,c,0), g=tf(num,den) , g=minreal(g)

Eigenvalues of *A*

مقادیر ویژه *A*

$\lambda_1 = -3$

$\lambda_2 = -2$

$\lambda_3 = -1$

Poles of *system*

قطبهای سیستم

$p_1 = -3$

$p_2 = -2$

What happened to -1 ?

# Exercises

**Exercise 1:** Find the poles and zeros of following system.

$$G(s)=\frac{s^2+4}{s^3+3s^2+3s+3}$$

**Exercise 2:** Is it possible to apply a nonzero input to the following system for t>0, but the output be zero for t>0? Sh

$$y''+5y'+6y=2u'+u$$

**Exercise 3:** Find the step response of following syst

$$G(s)=\frac{s^2+4}{s^3+3s^2+3s+3}$$

**Exercise 4:** Find the step response of following system for a=1,3,6 and 9.

$$G(s)=\frac{as+2}{s^2+2s+2}$$

**Exercise 5:** Find the poles of Y(s)

$$y(t)=\left(1+t+e^{-2t}+t\sin t\right)u(t)$$

**Exercise 6:** Check F.V.T and I.V.T for following system.

$$Y(s)=\frac{1}{s(s-1)}$$

# Exercises

**Exercise 7:** Find the transfer function of the following system.

a) By use of the $g(s)=d+c(sI-A)^{-1}b$.

b) Through use of state diagram

$$\dot{x}=\begin{bmatrix}-2 & 1 & 3\\ 0 & -3 & 2\\ 1 & 4 & -1\end{bmatrix}x+\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}u$$
$$y=\begin{bmatrix}1 & 2 & 0\end{bmatrix}x$$

**Exercise 8:** Find *y(t)* for initial condition $[1\ 3\ \ -1]^T$ and the unit step as input.

$$\dot{x}=\begin{bmatrix}-2 & 1 & 3\\ 0 & -3 & 2\\ 1 & 4 & -1\end{bmatrix}x+\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}u$$
$$y=\begin{bmatrix}1 & 2 & 0\end{bmatrix}x$$

**Exercise 9:** a)Find the eigenvalues of following system.

b) Find the pole of transfer function.

c) Are the system controllable and observable?

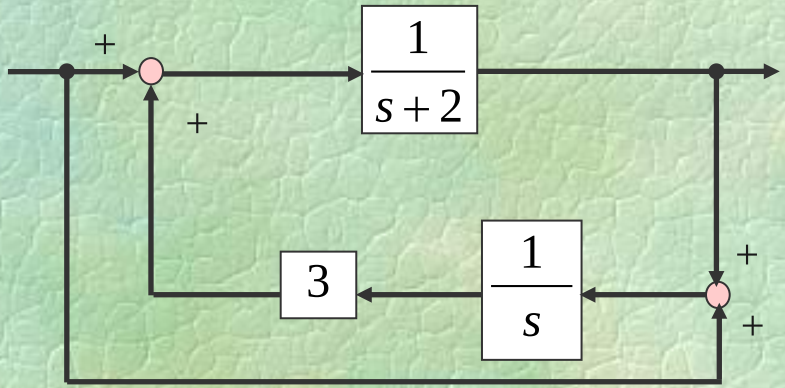
$$\dot{x}=\begin{bmatrix}-2 & 1 & 3\\ 0 & -3 & 2\\ 1 & 4 & -1\end{bmatrix}x+\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}u$$
$$y=\begin{bmatrix}1 & 2 & 0\end{bmatrix}x$$

# Exercises

**Exercise 11:** Suppose $x_1(0)=1$, $x_2(0)=3$ and $x_3(0)=2$ and $r(t)=u(t)$ . Find $x(t)$ and $y(t)$ for $t\geq 0$

$$\dot{x}(t)=\begin{bmatrix}-1 & 4 & 0\\ 2 & -2 & 1\\ 0 & 0 & -3\end{bmatrix}x(t)+\begin{bmatrix}1\\1\\0\end{bmatrix}r(t) \qquad y(t)=[1 \quad 0 \quad 0]x(t)$$

**Exercise 12:** Check the controllability and observability of the following system by transfer function method.




**Exercise 13:** Find a state space realization for following system such that it is not observable and controllable.(Final 1391)

$$g(s)=\frac{s+2}{s+1}$$



Dr. Ali Karimpour Oct 2020