Engineering Mathematics

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Content of this course

1. Fourier Series and Fourier Integral.

2. Partial Differential Equation and Its Solutions.

3. Complex Analysis. (The theory of functions of a complex variable)

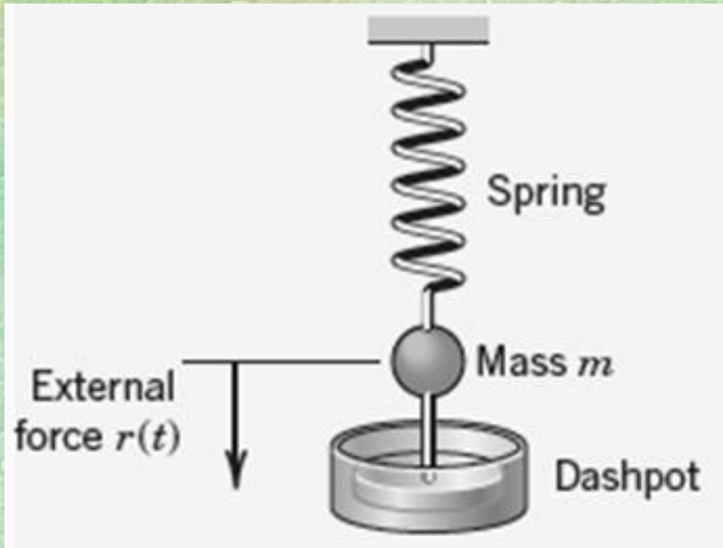
Partial Differential Equation and Its Solutions

- Introduction to Partial Differential Equations
- Derivation of Partial Differential Equations
- D'Alembert Solution for Wave Equations
- Classification of Partial Differential Equations
- Solving Partial Differential Equations by Separation of Variables

Introduction to Partial Differential Equations

- An ordinary differential equation (ODE) is a differential equation that involves functions of a single independent variable and its derivatives.
- The order of a differential equation is the order of the highest derivative present in the equation.

Example 1: Mass-Spring System



$$my'' = r(t) - cy' - ky$$

$$my'' + cy' + ky = r(t)$$

$y(t)$?

t is the independent variable

Introduction to Partial Differential Equations

An example of an ordinary differential equation:

$$y'' + 5y' + 5y = sint$$

$y(t)$? $y(t) = f(t, k_1, k_2)$

Initial conditions are required to solve an ordinary differential equation.

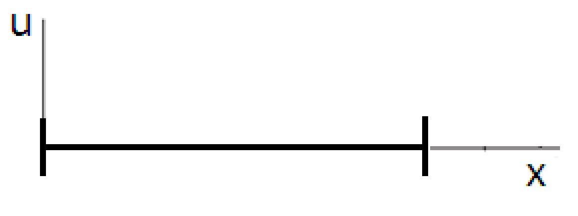
$y(0) = 2$ $y'(0) = 1$

Introduction to Partial Differential Equations

- A partial differential equation (PDE) is a differential equation that involves partial derivatives of one or more dependent variables with respect to multiple independent variables.
- The order of a partial differential equation is the highest order of partial derivative that appears in the equation.

Example 2: The following equation, known as the wave equation, is an example of a partial differential equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$



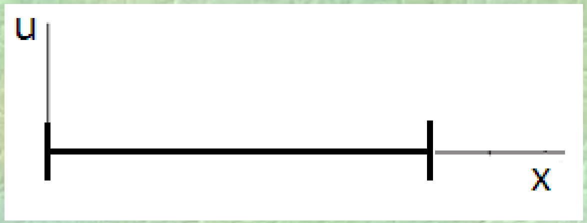
$\mathrm{u}(x,t)$?

t and x are independent variables

Introduction to Partial Differential Equations

Wave equation as an example of a partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$



$u(x,t) = ?$ $\quad$ $u(x,t) = f(x, t, k_1, k_2, \ldots\ldots)$

To find the solution to a partial differential equation, boundary conditions and initial conditions are necessary.

Examples of boundary conditions

$u(0,t) = 0 \qquad u(L,t) = 0 \qquad t \geq 0$

Examples of initial conditions

$u(x,0) = \varphi(x) \qquad \dfrac{\partial u}{\partial t}(x,0) = \theta(x)$

Partial Differential Equation and Its Solutions

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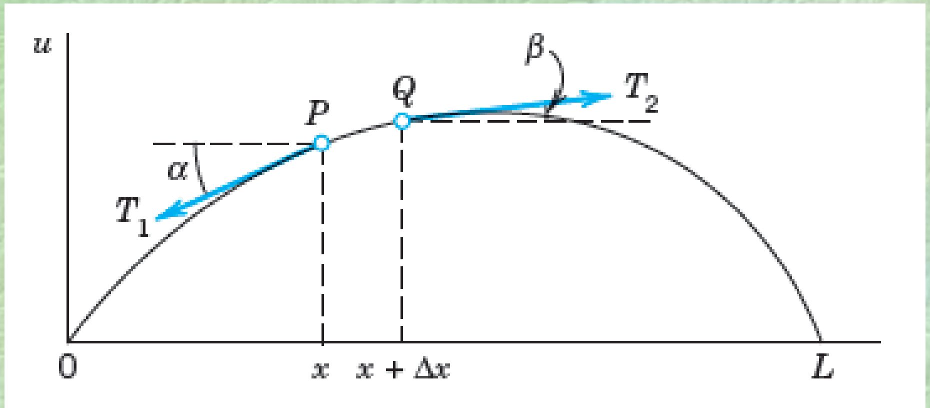
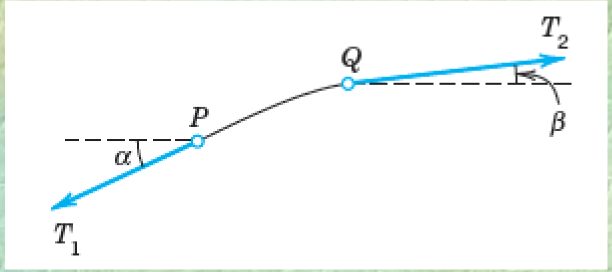
Derivation of Partial Differential Equations

Vibration Study of a Stretched Flexible String





$$T_2 sin\beta - T_1 sin\alpha = ma \qquad T_1 cos\alpha = T_2 cos\beta = T = const.$$

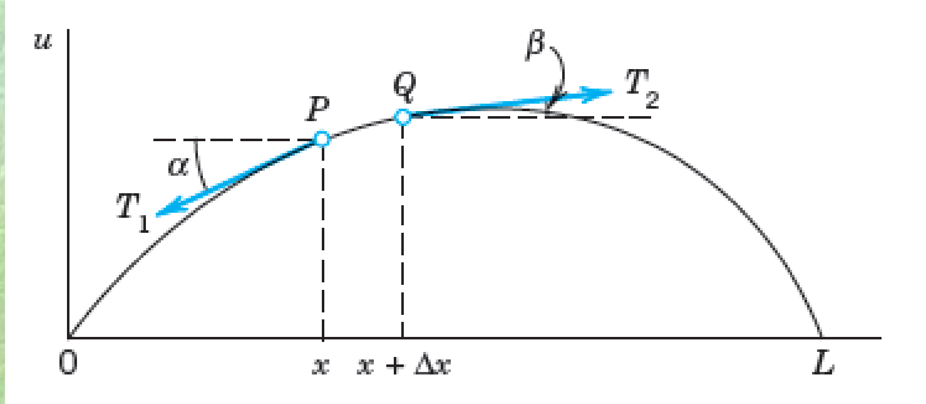
$$T_2 sin\beta - T_1 sin\alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2}$$

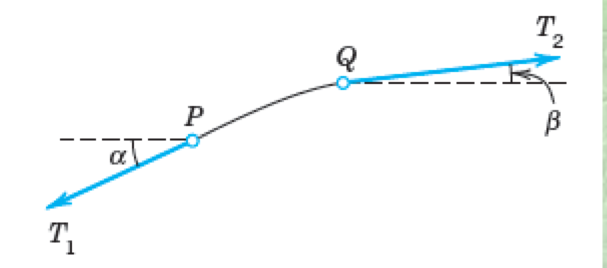
$$\frac{T_2 sin\beta}{T_2 cos\beta} - \frac{T_1 sin\alpha}{T_1 cos\alpha} = \frac{\rho \Delta x}{T}\frac{\partial^2 u}{\partial t^2} = tan\beta - tan\alpha$$

$$\left[\left(\frac{\partial u}{\partial x}\right)|_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)|_{x}\right] = \frac{\rho \Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

Derivation of Partial Differential Equations

Vibration Study of a Stretched Flexible String





$$\left[\left(\frac{\partial u}{\partial x}\right)|_{x+\Delta x}-\left(\frac{\partial u}{\partial x}\right)|_{x}\right]=\frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)|_{x+\Delta x}-\left(\frac{\partial u}{\partial x}\right)|_{x}\right]=\frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

$$c^2=\frac{T}{\rho}$$

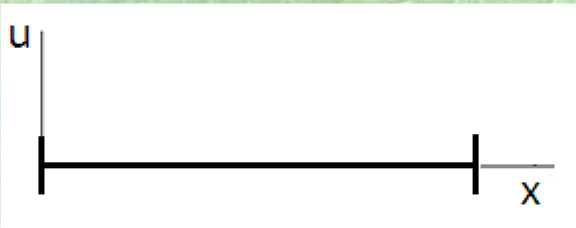
$$\boxed{\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}}$$

One-Dimensional Wave Equation

Derivation of Partial Differential Equations

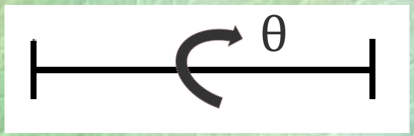
One-Dimensional Wave Equation

Flexible String $c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$



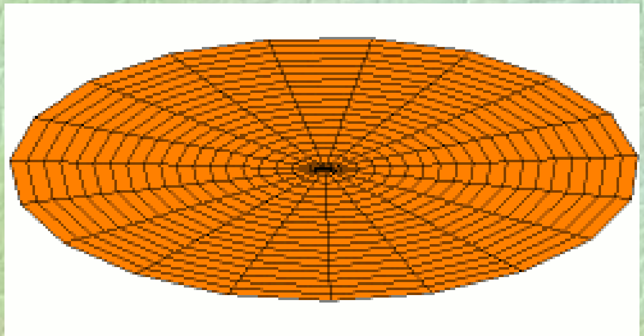
A rod with one end fixed $c^2 \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}$

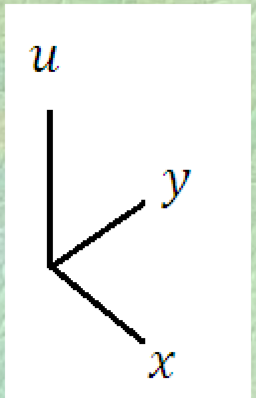
A rod with both ends fixed $c^2 \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}$



Derivation of Partial Differential Equations

Two-Dimensional Wave Equation





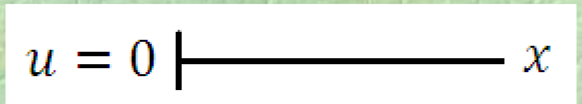
$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \qquad \alpha^2 = \frac{Tg}{w}$$

$$u\,(x, y, t) \quad ?$$

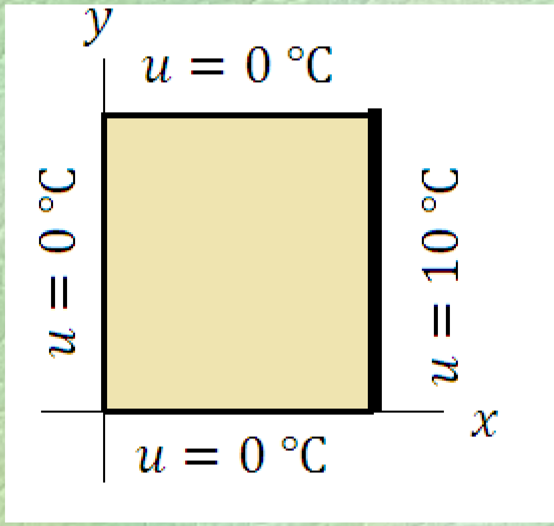
Derivation of Partial Differential Equations

One-Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad u(x,t)$$



Two-Dimensional Heat Equation



$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad u(x,y,t)$$

Three-Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad u(x,y,z,t)$$

Derivation of Partial Differential Equations

Two-Dimensional Laplace Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad u(x,y)$$

Three-Dimensional Laplace Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \qquad u(x,y,z)$$

Two-Dimensional Poisson Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y) \qquad u(x,y)$$

Three-Dimensional Poisson Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=f(x,y,z) \qquad u(x,y,z)$$

Exercises

Exercise 1: Check whether the following functions satisfy the wave equation or not.

1) $u = x^3 + 3xt^2$
2) $u = \sin(wct)\sin(wx)$

Exercise 2: Prove that the vibration of an elastic string under the influence of an external force $p(x,t)$ per unit length, applied perpendicular to the string, satisfies the following equation.

$$u_{tt} = c^2 u_{xx} + \frac{p(x,t)}{\rho}$$

Partial Differential Equation and Its Solutions

- ❑ Introduction to Partial Differential Equations
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- ❑ D'Alembert Solution for One-Dimensional Wave Equations
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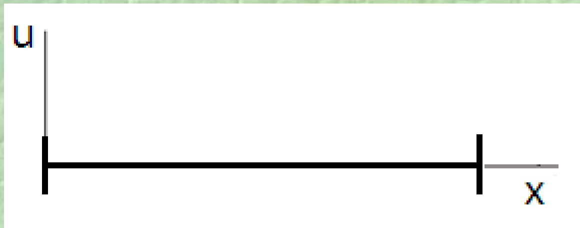
D'Alembert Solution for One-Dimensional Wave Equations

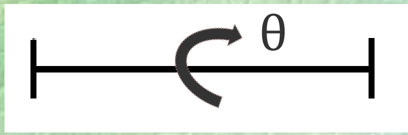
One-Dimensional Wave Equation

Flexible String $c^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$



A rod with one end fixed $c^2 \dfrac{\partial^2 \theta}{\partial x^2} = \dfrac{\partial^2 \theta}{\partial t^2}$

A rod with both ends fixed $c^2 \dfrac{\partial^2 \theta}{\partial x^2} = \dfrac{\partial^2 \theta}{\partial t^2}$



D'Alembert Solution for One-Dimensional Wave Equations

Suppose $f(x-at)$ is a function with a second derivative. Then, according to the chain rule:

$$\frac{\partial f(x-at)}{\partial t} = -af'(x-at) \qquad \frac{\partial f(x-at)}{\partial x} = f'(x-at)$$

$$\frac{\partial^2 f(x-at)}{\partial t^2} = a^2 f''(x-at) \qquad \frac{\partial^2 f(x-at)}{\partial x^2} = f''(x-at)$$

Given these rules, it is clear that the function $u=f(x-at)$ satisfies the wave equation.

Similarly, the function $u=g(x+at)$ will also satisfy the wave equation.

Therefore, the general solution of the wave equation can be written as follows:

$$u = f(x-at) + g(x+at)$$

D'Alembert Solution for One-Dimensional Wave Equations

Therefore, in the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

The general solution of the wave equation for any *f* and *g* that have a second derivative can be written as follows:

$$u = f(x - at) + g(x + at)$$

For example, all of the following functions are solutions to the wave equation.

$$u = sin(x - at) + 10(x + at) \qquad u = \sin(cos(x - at)) + \cos(e^{x+at})$$

$$u = cos(x - at) + e^{x+at} \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

D'Alembert Solution for One-Dimensional Wave Equations

Therefore, in the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

The general solution of the wave equation for any f and g that have a second derivative can be written as follows:

$$u = f(x - at) + g(x + at)$$

A valid solution must satisfy the initial conditions. Therefore, suppose:

$$u(x\,, 0) = \varphi(x) \qquad\qquad \frac{\partial u}{\partial t}\Big|_{x,0} = \theta(x)$$

D'Alembert Solution for One-Dimensional Wave Equations

Therefore, in the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad u = f(x - at) + g(x + at)$$

The initial conditions are:

$$u(x\,,0) = \varphi(x) \longrightarrow u(x,0) = \boxed{\varphi(x) = f(x) + g(x)}$$

$$\frac{\partial u}{\partial t}|_{x,0} = \theta(x) \longrightarrow \frac{\partial u}{\partial t}|_{x,0} = \theta(x) = -af'(x) + ag'(x)$$

$$\boxed{\int_{x_0}^{x} \theta(s)ds = -a(f(x) - f(x_0)) + a(g(x) - g(x_0))}$$

D'Alembert Solution for One-Dimensional Wave Equations

Therefore, in the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad\qquad u = f(x - at) + g(x + at)$$

To satisfy the initial conditions, it is necessary to:

$$\boxed{f(x) + g(x) = \varphi(x)} \qquad \boxed{-f(x) + g(x) = \frac{1}{a}\int_{x_0}^{x} \theta(s)ds - f(x_0) + g(x_0)}$$

$$g(x) = \frac{1}{2}\left[\varphi(x) + \frac{1}{a}\int_{x_0}^{x} \theta(s)ds - f(x_0) + g(x_0)\right] \longrightarrow u = f(x - at) + g(x + at)$$

$$f(x) = \frac{1}{2}\left[\varphi(x) - \frac{1}{a}\int_{x_0}^{x} \theta(s)ds + f(x_0) - g(x_0)\right]$$

$$\boxed{u(x,t) = \frac{\varphi(x - at) + \varphi(x + at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \theta(s)ds}$$

D'Alembert Solution for One-Dimensional Wave Equations

Example 3: Consider a string that extends infinitely in both directions. The initial displacement of the points on the string is given by the following function, and then the string is released from rest. Find the equation of motion for the string.

$$\varphi(x) = \frac{1}{1+8x^2}$$

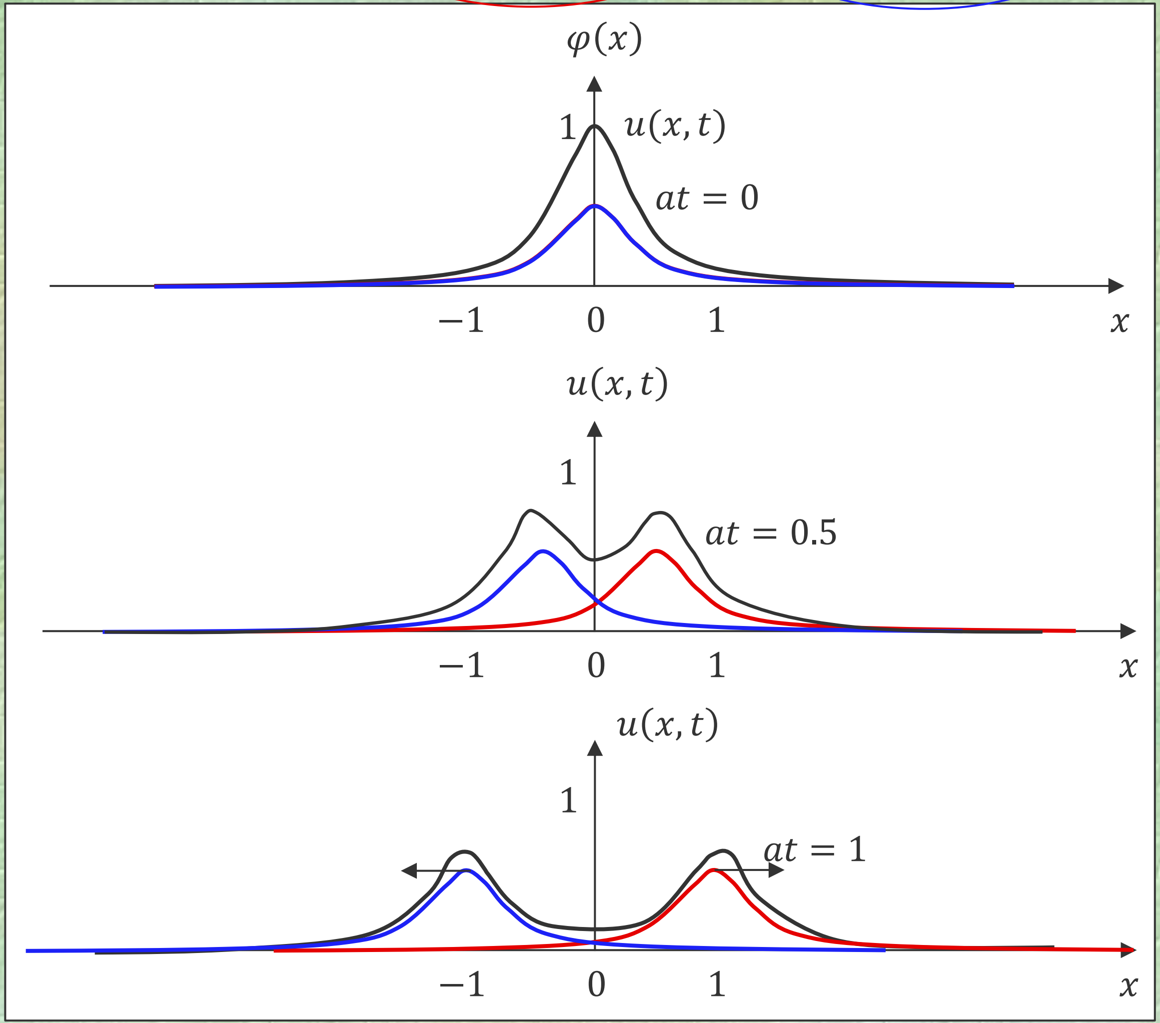
Solution: Since the string has no initial velocity, $\theta(x)=0$, and therefore, the equation of motion for the string is:

$$u(x,t) = \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\cancel{\int_{x-at}^{x+at}\theta(s)ds}$$

$$u(x,t) = \frac{\varphi(x-at)+\varphi(x+at)}{2} = \frac{1}{2}\left[\frac{1}{1+8(x-at)^2} + \frac{1}{1+8(x+at)^2}\right]$$

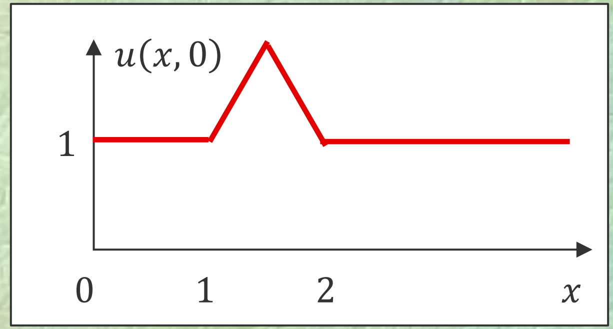
D'Alembert Solution for One-Dimensional Wave Equations

$$\varphi(x) = \frac{1}{1+8x^2} \qquad u(x,t) = \frac{0.5}{1+8(x-at)^2} + \frac{0.5}{1+8(x+at)^2}$$



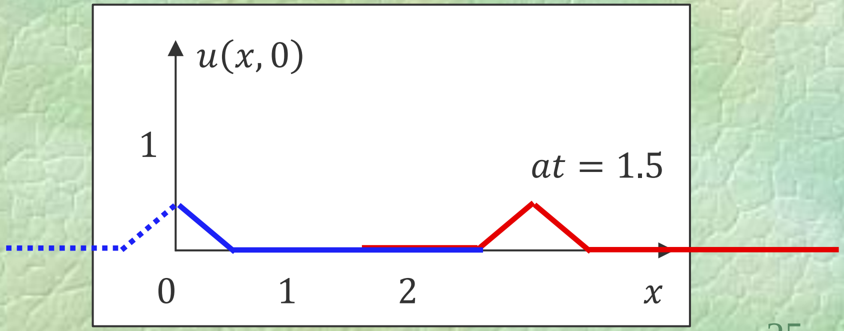
D'Alembert Solution for One-Dimensional Wave Equations

Example 4: A semi-infinite string is released from rest as shown in the figure below. Determine the state of the string as a function of time and position.



Solution: Since the string has no initial velocity, θ(x)=0, and the equation of motion for the string is:

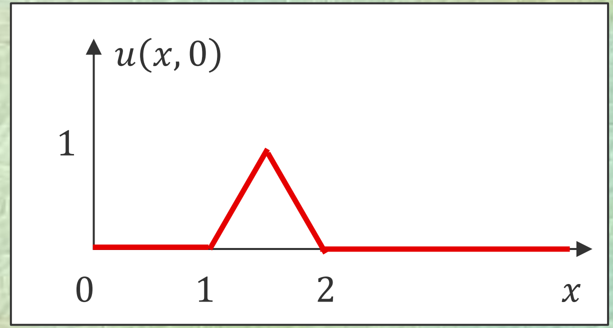
$$u(x,t) = \frac{\varphi(x-at)+\varphi(x+at)}{2}$$



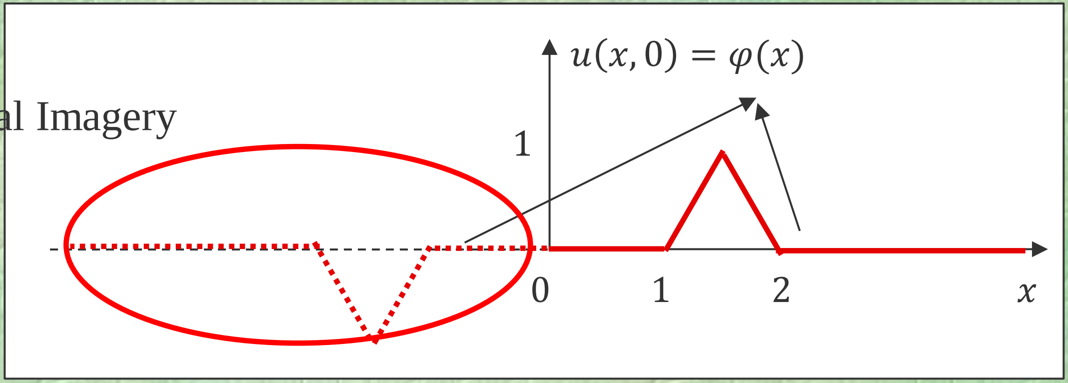
Boundary conditions are not satisfied.

D'Alembert Solution for One-Dimensional Wave Equations

Example 4: A semi-infinite string is released from rest as shown in the figure below. Determine the state of the string as a function of time and position.



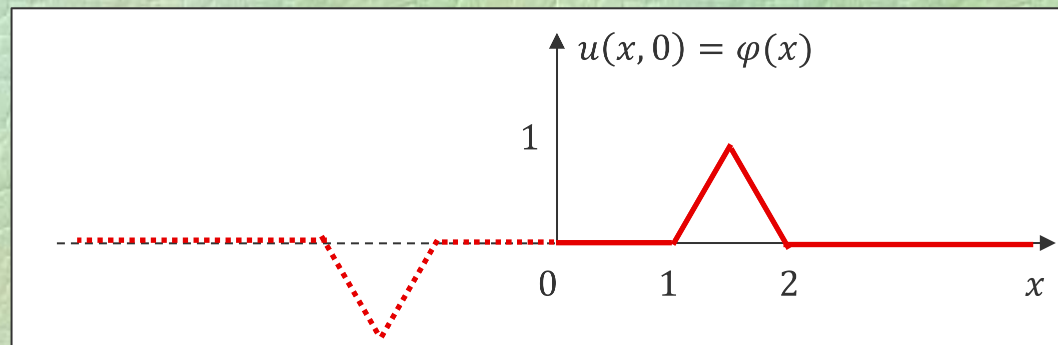
Virtual Imagery



Boundary conditions are satisfied.

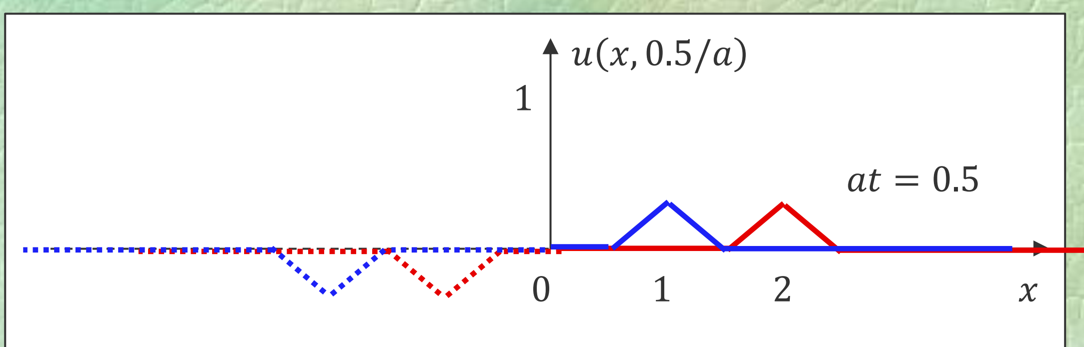
D'Alembert Solution for One-Dimensional Wave Equations

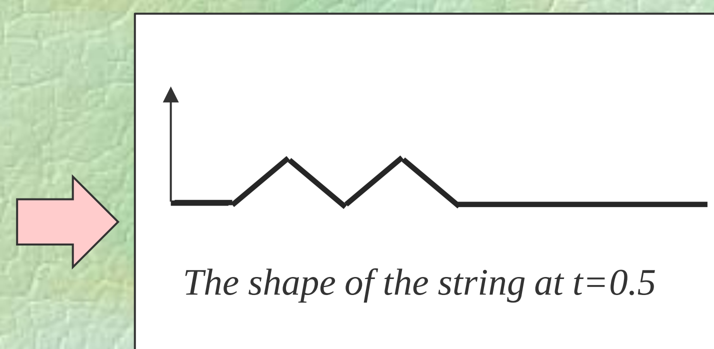
Example 4: A semi-infinite string is released from rest as shown in the figure below. Determine the state of the string as a function of time and position.



Solution: Since the string has no initial velocity, θ(x)=0, and the equation of motion for the string is:

$$u(x,t) = \frac{\varphi(x-at) + \varphi(x+at)}{2}$$

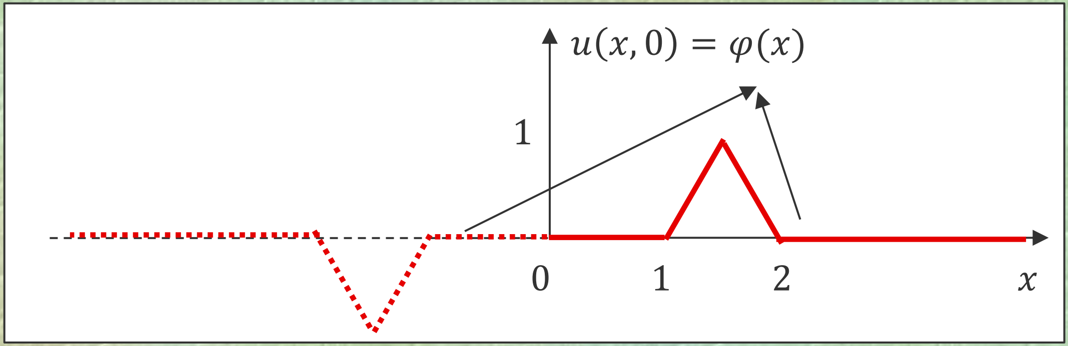


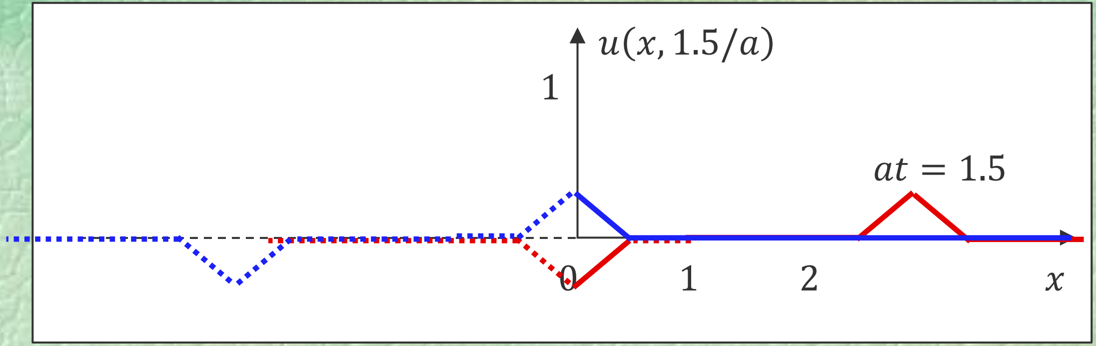


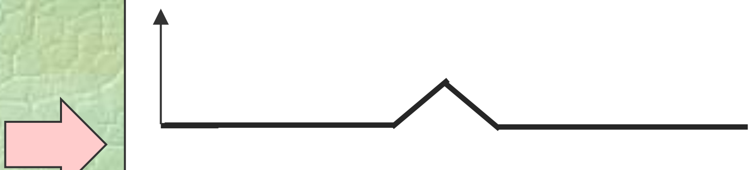
The shape of the string at t=0.5

D'Alembert Solution for One-Dimensional Wave Equations

Example 4: A semi-infinite string is released from rest as shown in the figure below. Determine the state of the string as a function of time and position.



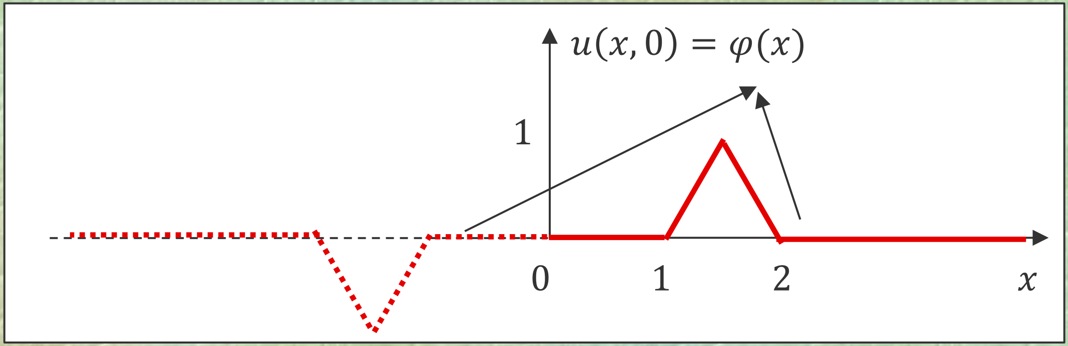


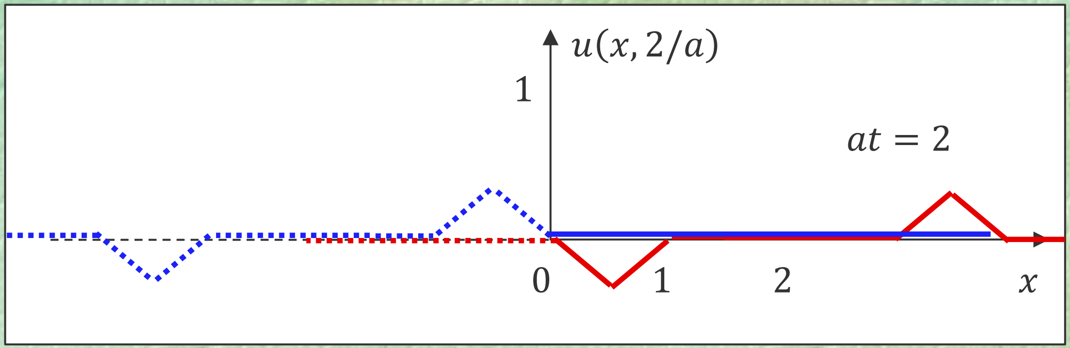


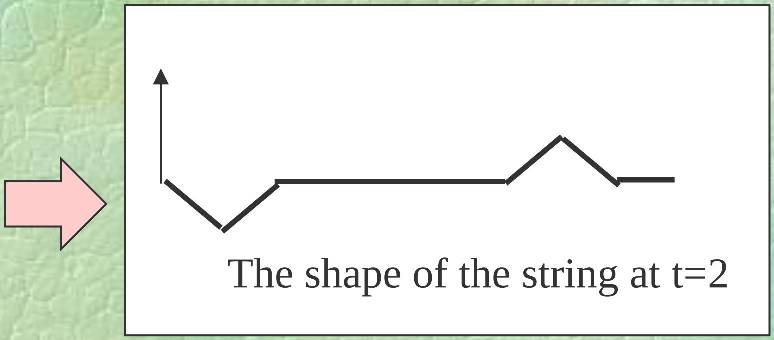
The shape of the string at t=1.5

D'Alembert Solution for One-Dimensional Wave Equations

Example 4: A semi-infinite string is released from rest as shown in the figure below. Determine the state of the string as a function of time and position.

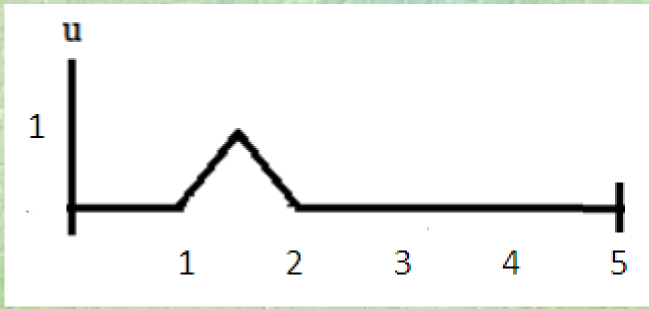






D'Alembert Solution for One-Dimensional Wave Equations

Exercise 3: A finite string is released from rest as shown in the figure below. Determine the subsequent motion of the string.



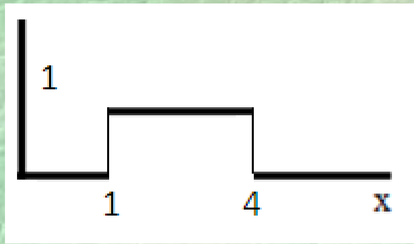
D'Alembert Solution for One-Dimensional Wave Equations

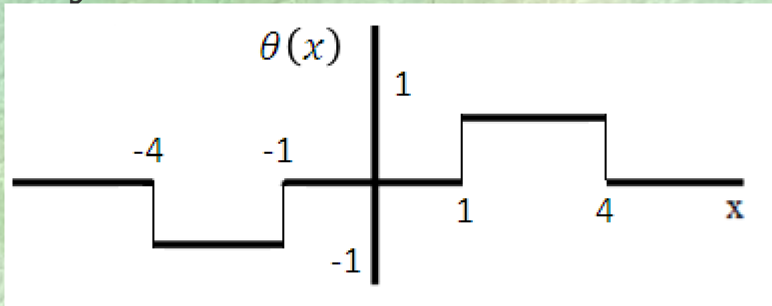
Example 5: A uniform string stretched along the x-axis from $x = 0$ to $x = \infty$ is struck at t=0 such that the part of the string between x=1 and x=4 obtains an initial velocity of one. Determine the subsequent displacement of the string and plot the displacement at the point $x = 1$ as a function of t.

Solution: It is clear that the initial position is given by:

$$\varphi(x) = 0$$

On the other hand, the given initial velocity is:





$$\theta(x) = u(x+1) - u(x+4) + u(x-1) - u(x-4)$$

D'Alembert Solution for One-Dimensional Wave Equations

Now, given the initial velocity and position, the solution is:

$$\varphi(x) = 0$$

$$\theta(x) = u(x+1) - u(x+4) + u(x-1) - u(x-4)$$

$$u(x,t) = \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\theta(s)ds$$

$$u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at}[u(s+1) - u(s+4) + u(s-1) - u(s-4)]ds$$

D'Alembert Solution for One-Dimensional Wave Equations

$$u(x,t)$$

$$= \frac{1}{2a}\left[(s+1)u(s+1) - (s+4)u(s+4) + (s-1)u(s-1) - (s-4)u(s-4)\right]_{x-at}^{x+at}$$

$$u(x,t) = \frac{1}{2a}[(x+at+1)u(x+at+1) - (x+at+4)u(x+at+4)$$

$$+(x+at-1)u(x+at-1) - (x+at-4)u(x+at-4)$$

$$-(x-at+1)u(x-at+1) + (x-at+4)u(x-at+4)$$

$$-(x-at-1)u(x-at-1) + (x-at-4)u(x-at-4)]$$

D'Alembert Solution for One-Dimensional Wave Equations

At x=1

$$u(1,t)=\frac{1}{2a}[(1+at+1)u(1+at+1)-(1+at+4)u(1+at+4)$$

$$+(1+at-1)u(1+at-1)-(1+at-4)u(1+at-4)$$

$$-(1-at+1)u(1-at+1)+(1-at+4)u(1-at+4)$$

$$-(1-at-1)u(1-at-1)+(1-at-4)u(1-at-4)]$$

D'Alembert Solution for One-Dimensional Wave Equations

At x=1

$$u(1,t) = \frac{1}{2a}[(at+2)u(at+2) - (at+5)u(at+5)$$

$$+(at)u(at) - (at-3)u(at-3)$$

$$-(-at+2)u(-at+2) + (-at+5)u(-at+5)$$

$$-(-at)u(-at) + (-at-3)u(-at-3)]$$

D'Alembert Solution for One-Dimensional Wave Equations

$$u(1,t) = \frac{1}{2a}[(at+2)u(at+2) - (at+5)u(at+5) + (at)u(at) - (at-3)u(at-3)$$

$$-(-at+2)u(-at+2) + (-at+5)u(-at+5) - (-at)u(-at) + (-at-3)u(-at-3)]$$

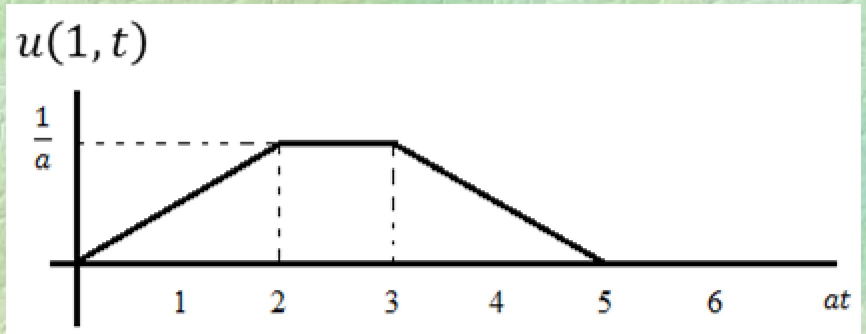
$$u(1,t) = \frac{1}{2a}[(at+2) - (at+5) + at - (-at+2) + (-at+5)] = \frac{t}{2} \qquad 0 < at < 2$$

$$u(1,t) = \frac{1}{2a}[(at+2) - (at+5) + at + (-at+5)] = \frac{1}{a} \qquad 2 < at < 3$$

$$u(1,t) = \frac{1}{2a}[(at+2) - (at+5) + at - (at-3) + (-at+5)] = \frac{-at+5}{2a} \qquad 3 < at < 5$$

$$u(1,t) = \frac{1}{2a}[(at+2) - (at+5) + at - (at-3)] = 0 \qquad 5 < at$$

The position of the string at x=1 as a function of time is:



Exercises

Exercise 4: Determine the displacement $u(x,t)$ of an oscillating string of length L with fixed endpoints and $c=1$. The initial velocity is zero and the initial displacement is $f(x)$.

1) $f(x) = kx(1-x)$

2) $f(x) = k\sin^2(\pi x)$

Exercise 5: Determine the solution to the following wave equation.

$u_{xx} - u_{tt} = 0$, $0<x<L$, $t>0$

$u(x,0) = \sin\frac{\pi}{L}x$, $u_t(x,0) = -\sin\frac{\pi}{L}x$, $0<x<L$

$u(0,t) = u(L,t) = 0$ for all t.

Partial Differential Equation and Its Solutions

- ❑ Introduction to Partial Differential Equations
- ❑ Derivation of Partial Differential Equations
- ❑ D'Alembert Solution for One-Dimensional Wave Equations
- ❑ Classification of Partial Differential Equations
- ❑ Solving Partial Differential Equations by Separation of Variables

Classification of Partial Differential Equations

Can other partial differential equations be solved using the d'Alembert method?

Answer: Consider the following equation:

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = 0$$

First, the possibility of finding solutions in the form $u=f(x+\lambda y)$ for the equation is examined.

$$Af''(x+\lambda y) + 2B\lambda f''(x+\lambda y) + C\lambda^2 f''(x+\lambda y) = 0$$

$$C\lambda^2 + 2B\lambda + A = 0$$

Classification of Partial Differential Equations

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = 0$$

$$u = f(x + \lambda y)$$

$$C\lambda^2 + 2B\lambda + A = 0$$

$$\begin{cases} \text{Hyperbolic} & B^2 - AC > 0 \\ \text{Parabolic} & B^2 - AC = 0 \\ \text{Elliptical} & B^2 - AC < 0 \end{cases}$$

For example, in the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$$

$$C\lambda^2 + 2B\lambda + A = \lambda^2 - a^2 = 0$$

$$\lambda = +a, -a$$

$$u = f(x + at) + g(x - at)$$

Classification of Partial Differential Equations

In general, the equation:

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} = g(u, u_x, u_y, x, y)$$

is called hyperbolic, parabolic, or elliptical depending on the sign of the following expression:

$$B^2(x,y) - A(x,y)C(x,y)$$

$$\begin{cases} \text{Hyperbolic} & B^2(x,y) - A(x,y)C(x,y) > 0 \\ \text{Parabolic} & B^2(x,y) - A(x,y)C(x,y) = 0 \\ \text{Elliptical} & B^2(x,y) - A(x,y)C(x,y) < 0 \end{cases}$$

Classification of Partial Differential Equations

In general, the equation:

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} = g(u, u_x, u_y, x, y)$$

is called hyperbolic, parabolic, or elliptical depending on the sign of the following expression:

$$B^2(x,y) - A(x,y)C(x,y)$$

The wave equation

$$c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \qquad B^2 - AC = 0 - c^2(-1) = c^2 > 0$$ Hyperbolic

The heat equation (heat transfer)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad B^2 - AC = 0 - 0c^2 = 0$$ Parabolic

The Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad B^2 - AC = 0 - 1 = -1 < 0$$ Elliptical

Classification of Partial Differential Equations

Theorem: Consider the following equation:

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} = g\big(u, u_x, u_y, x, y\big) \qquad I$$

Form the following auxiliary equation from the above equation:

$$A(x,y)(y')^2 - 2B(x,y)y' + C(x,y) = 0$$

Independent solutions of the auxiliary equation are the characteristics of the equation I.

$$\psi(x,y) = C_2 \qquad \varphi(x,y) = C_1$$

In this case:

Classification of Partial Differential Equations

a) If the given equation is hyperbolic, with a change of variables:

$$s = \varphi(x, y) \qquad\qquad r = \psi(x, y)$$

the original equation (Eq. I) is transformed into the following equation:

$$u_{rs} = G(u, u_r, u_s, r, s)$$

b) If the given equation is parabolic, with a change of variables:

$$r = x \qquad\qquad s = \psi(x, y)$$

the original equation (Eq. I) is transformed into the following equation:

$$u_{rr} = G(u, u_r, u_s, r, s)$$

c) If the given equation is elliptical, with a change of variables:

$$r = \frac{\varphi(x, y) + \psi(x, y)}{2} \qquad\qquad s = \frac{\varphi(x, y) - \psi(x, y)}{2i}$$

the original equation (Eq. I) is transformed into the following equation:

$$u_{rr} + u_{ss} = G(u, u_r, u_s, r, s)$$

Classification of Partial Differential Equations

Example 6: In the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} = g(u,u_x,u_y,x,y)$$

$$A(x,y)(y')^2 - 2B(x,y)y' + C(x,y) = 0$$

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad 1(x')^2 - a^2 = 0 \qquad x' = -a \ , \qquad a$$

$$x' = -a \quad \rightarrow \quad x = -at + c_1 \quad \rightarrow c_1 = x + at$$

$$x' = a \quad \rightarrow \quad x = at + c_2 \quad \rightarrow c_2 = x - at$$

hyperbolic $\rightarrow$ $s = x + at$

$r = x - at$

Classification of Partial Differential Equations

Example (continued): Using substitutions from the theorem

$$r = x - at \qquad\qquad s = x + at$$

And according to the chain rule for partial derivatives, the following relationships are obtained.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \qquad \frac{\partial^2 u}{\partial x^2} = \cdots = \frac{\partial^2 u}{\partial r^2} + 2\frac{\partial^2 u}{\partial r \partial s} + \frac{\partial^2 u}{\partial s^2}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} = -a\frac{\partial u}{\partial r} + a\frac{\partial u}{\partial s} \qquad \frac{\partial^2 u}{\partial t^2} = \cdots = a^2\frac{\partial^2 u}{\partial r^2} - 2a^2\frac{\partial^2 u}{\partial r \partial s} + a^2\frac{\partial^2 u}{\partial s^2}$$

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} \quad \rightarrow \quad \frac{\partial^2 u}{\partial r \partial s} = 0 \quad \rightarrow \quad \frac{\partial u}{\partial s} = k(s) \quad \rightarrow \quad u = f(s) + g(r)$$

$$\rightarrow \quad u(x,t) = f(x + at) + g(x - at)$$

Classification of Partial Differential Equations

Example 7: Determine the characteristics of the following equation, assuming that y is not zero. Then, if possible, transform the equation into its canonical form and solve it.

$$x\frac{\partial^2 u}{\partial x^2} - y\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} = 0$$

Solution: The characteristics are obtained by solving the following auxiliary equation.

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} = g(u, u_x, u_y, x, y)$$

$$A(x,y)(y')^2 - 2B(x,y)y' + C(x,y) = 0$$

$$x\,(y')^2 + yy' + 0 = 0$$

$$y'(xy' + y) = 0$$

$$y' = 0$$

$$\boxed{y = c_1}$$

$$x\frac{dy}{dx} + y = 0$$

$$\boxed{xy = c_2}$$

Classification of Partial Differential Equations

Example (continued): Since $B^2 - AC > 0$, it is hyperbolic. Therefore, using substitutions from the theorem and according to the chain rule for partial derivatives, the following relationships are obtained.

$$s = xy \qquad r = y$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = r\frac{\partial u}{\partial s} \qquad \frac{\partial^2 u}{\partial x^2} = \cdots = r^2\frac{\partial^2 u}{\partial s^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial u}{\partial s} + y\left[\frac{\partial^2 u}{\partial r \partial s}\frac{\partial r}{\partial y} + \frac{\partial^2 u}{\partial s^2}\frac{\partial s}{\partial y}\right] = \frac{\partial u}{\partial s} + r\frac{\partial^2 u}{\partial r \partial s} + s\frac{\partial^2 u}{\partial s^2}$$

$$x\frac{\partial^2 u}{\partial x^2} - y\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} = 0 \quad \rightarrow \quad -r^2\frac{\partial^2 u}{\partial r \partial s} = 0 \quad \rightarrow \quad \frac{\partial^2 u}{\partial r \partial s} = 0$$

$$\rightarrow \quad \frac{\partial u}{\partial s} = k(s) \quad \rightarrow \quad u = f(s) + g(r) \quad \rightarrow \quad u = f(xy) + g(y)$$

Partial Differential Equation and Its Solutions

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Separation of Variables

To solve the following equation: $$\frac{\partial^2\theta}{\partial t^2} = a^2 \frac{\partial^2\theta}{\partial x^2}$$

Assuming that the angle of twist θ is expressed as the product of two functions, one depending only on x and the other depending only on t, we have:

$$\theta(x,t) = X(x)T(t)$$

By substituting into the wave equation and dividing both sides by XT, we get:

$$\frac{\partial^2\theta}{\partial t^2} = \mathrm{X}T'' \quad , \quad \frac{\partial^2\theta}{\partial x^2} = X''T$$

$$\frac{\partial^2\theta}{\partial t^2} = a^2 \frac{\partial^2\theta}{\partial x^2} \longrightarrow \mathrm{X}T'' = a^2 X''T$$

$$\longrightarrow \frac{T''}{T} = a^2 \frac{X''}{X} = \mu$$

Separation of Variables

$$\frac{\partial^2\theta}{\partial t^2} = a^2\frac{\partial^2\theta}{\partial x^2} \longrightarrow \frac{T''}{T} = a^2\frac{X''}{X} = \mu$$

$$\begin{cases} X'' = \dfrac{\mu}{a^2}X \\ T'' = \mu T \end{cases}$$

$\mu > 0 \quad \mu = 0 \quad \mu < 0$

Case 1: Assuming that $\mu > 0$ $\quad \mu = \lambda^2$

$$\frac{T''}{T} = a^2\frac{X''}{X} = \mu$$

$$X'' = \frac{\lambda^2}{a^2}X \qquad X = Ce^{\frac{\lambda}{a}x} + De^{-\frac{\lambda}{a}x}$$

$$T'' = \lambda^2 T \qquad T = Ae^{\lambda t} + Be^{-\lambda t}$$

$$\theta(x,t) = X(x)T(t) = (Ce^{\frac{\lambda}{a}x} + De^{-\frac{\lambda}{a}x})(Ae^{\lambda t} + Be^{-\lambda t})$$

Separation of Variables

Case 2: Assuming that $\mu = 0$

$$\frac{T''}{T} = a^2 \frac{X''}{X} = \mu$$

$$X'' = 0 \qquad X = Cx + D$$

$$T'' = 0 \qquad T = At + B$$

$$\theta(x,t) = X(x)T(t) = (Cx + D)(At + B)$$

Case 3: Assuming that $\mu < 0$ $\qquad \mu = -\lambda^2$

$$\frac{T''}{T} = a^2 \frac{X''}{X} = -\lambda^2$$

$$X'' = -\frac{\lambda^2}{a^2} X \qquad X = C \cos \frac{\lambda}{a} x + D \sin \frac{\lambda}{a} x$$

$$T'' = -\lambda^2 T \qquad T = A\cos \lambda t + B\sin \lambda t$$

$$\theta(x,t) = X(x)T(t) = (C \cos \frac{\lambda}{a} x + D \sin \frac{\lambda}{a} x)(A\cos \lambda t + B\sin \lambda t)$$

Separation of Variables

Case 1: Assuming that $\mu > 0, \quad \mu = \lambda^2$

$$X'' = \frac{\lambda^2}{a^2} X$$

$$\theta(x,t) = X(x)T(t) = (Ce^{\lambda\frac{x}{a}} + De^{-\lambda\frac{x}{a}})(Ae^{\lambda t} + Be^{-\lambda t})$$

$$T'' = \lambda^2 T$$

Case 2: Assuming that $\mu = 0$

$$X'' = 0$$

$$\theta(x,t) = X(x)T(t) = (Cx + D)(At + B)$$

$$T'' = 0$$

Case 3: Assuming that $\mu < 0, \quad \mu = -\lambda^2$

$$X'' = -\frac{\lambda^2}{a^2} X$$

$$\theta(x,t) = X(x)T(t) = (C\cos\frac{\lambda}{a}x + D\sin\frac{\lambda}{a}x)(A\cos\lambda t + B\sin\lambda t)$$

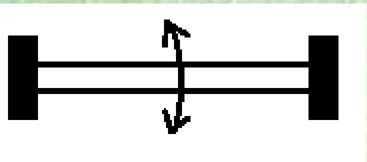
$$T'' = -\lambda^2 T$$

Separation of Variables

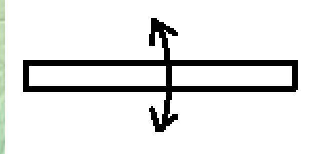
The remaining part is to calculate the values of the unknowns $A\ , B, C, D, and\ \lambda$

Consider following situations:

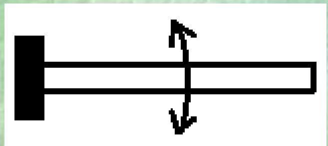
- a. Two-ended fixed shaft:



- b. Two-ended free shaft:

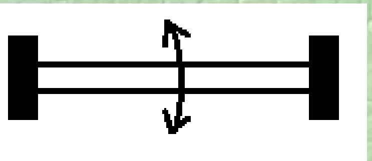


- c. One-ended free and one-ended fixed shaft:



Separation of Variables

Two-ended fixed shaft:



$$\theta(0,t) = 0 \qquad \forall\, t \qquad B1 \qquad\qquad \theta(l,t) = 0 \qquad \forall\, t \qquad B2$$

$$\theta(x,0) = f(x) \qquad I1 \qquad\qquad \frac{\partial\theta}{\partial t}(x,0) = g(x) \qquad I2$$

Assuming that $\mu < 0, \quad \mu = -\lambda^2$

$$\theta(x,t) = \left(C\cos\frac{\lambda}{a}x + D\sin\frac{\lambda}{a}x\right)(A\cos\lambda t + B\sin\lambda t)$$

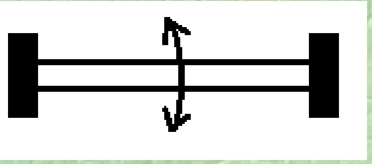
Applying boundary condition 1 (B1)

$$\theta(0,t) = 0 = C(A\cos\lambda t + B\sin\lambda t) \longrightarrow \cancel{A = B} = 0 \;\; or \;\; C = 0$$

$$\theta(x,t) = \left(\sin\frac{\lambda}{a}x\right)(A\cos\lambda t + B\sin\lambda t) \qquad Note\ to\ D\ \ ??$$

Separation of Variables

Two-ended fixed shaft:



$$\cancel{\theta(0,t) = 0 \qquad \forall\, t \qquad B1}$$

$$\theta(l,t) = 0 \qquad \forall\, t \qquad B2$$

$$\theta(x,0) = f(x) \qquad I1$$

$$\frac{\partial\theta}{\partial t}(x,0) = g(x) \qquad I2$$

Assuming that $\mu < 0, \quad \mu = -\lambda^2$

$$\theta(x,t) = \left(\sin\frac{\lambda}{a}x\right)(A\cos\lambda t + B\sin\lambda t)$$

Applying boundary condition 2 (B2)

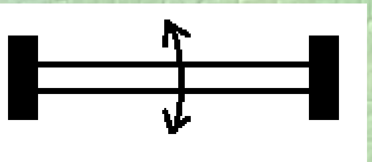
$$\theta(l,t) = 0 = \sin\frac{\lambda}{a}l(A\cos\lambda t + B\sin\lambda t) \longrightarrow \cancel{A = B = 0} \;\; or \;\; \sin\frac{\lambda}{a}l = 0$$

$$\frac{\lambda}{a}l = n\pi \qquad \lambda_n = \frac{n\pi a}{l}$$

$$\theta_n(x,t) = \left(\sin\frac{n\pi}{l}x\right)\left(A_n\cos\frac{n\pi a}{l}t + B_n\sin\frac{n\pi a}{l}t\right)$$

Separation of Variables

Two-ended fixed shaft:



$\cancel{\theta(0,t) = 0 \quad \forall\, t \quad B1}$ $\qquad$ $\cancel{\theta(l,t) = 0 \quad \forall\, t \quad B2}$

$\theta(x,0) = f(x) \qquad I1$ $\qquad$ $\dfrac{\partial\theta}{\partial t}(x,0) = g(x) \qquad I2$

Assuming that $\mu < 0, \quad \mu = -\lambda^2$

$$\theta_n(x,t) = \left(\sin\frac{n\pi}{l}x\right)\left(A_n \cos\frac{n\pi a}{l}t + B_n \sin\frac{n\pi a}{l}t\right)$$

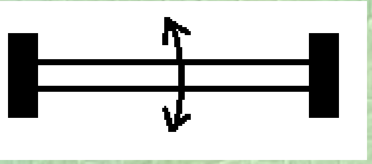
Applying initial condition 1 (I1)

$$\theta(x,t) = \sum_{n=1}^{\infty}\theta_n(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi}{l}x\left(A_n\cos\frac{n\pi a}{l}\mathrm{t} + B_n\sin\frac{n\pi a}{l}t\right)$$

$\theta(x,0) = f(x) \qquad \theta(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{l}x = f(x) \qquad A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x\,dx$

Separation of Variables

Two-ended fixed shaft:



$$\cancel{\theta(0,t) = 0 \qquad \forall\, t \qquad B1} \qquad\qquad \cancel{\theta(l,t) = 0 \qquad \forall\, t \qquad B2}$$

$$\cancel{\theta(x,0) = f(x) \qquad\qquad I1} \qquad\qquad \frac{\partial\theta}{\partial t}(x,0) = g(x) \qquad\qquad I2$$

Assuming that $\mu < 0, \quad \mu = -\lambda^2$

$$\theta(x,t) = \sum_{n=1}^{\infty} sin\frac{n\pi}{l}x(A_n\cos\frac{n\pi a}{l}\mathrm{t} + B_n\, sin\frac{n\pi a}{l}t) \qquad A_n = \frac{2}{l}\int_0^l f(x)\, sin\frac{n\pi}{l}xdx$$

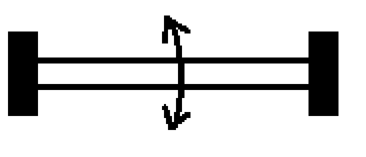
Applying initial condition 2 (I2)

$$\frac{\partial\theta}{\partial t} = \sum_{n=1}^{\infty} sin\frac{n\pi}{l}x(-A_n sin\frac{n\pi a}{l}t + B_n cos\frac{n\pi a}{l}t)\frac{n\pi a}{l} \qquad g(x) = \sum_{n=1}^{\infty}(\frac{n\pi a}{l}B_n)\, sin\frac{n\pi}{l}x$$

$$\frac{n\pi a}{l}B_n = \frac{2}{l}\int_0^l g(x)\, sin\frac{n\pi}{l}xdx \qquad\qquad B_n = \frac{2}{n\pi a}\int_0^l g(x)\, sin\frac{n\pi}{l}xdx$$

Separation of Variables

Two-ended fixed shaft:



$$\cancel{\theta(0,t) = 0 \qquad \forall\, t \qquad B1}$$

$$\cancel{\theta(l,t) = 0 \qquad \forall\, t \qquad B2}$$

$$\cancel{\theta(x,0) = f(x) \qquad I1}$$

$$\cancel{\frac{\partial\theta}{\partial t}(x,0) = g(x) \qquad I2}$$

Assuming that $\mu < 0, \quad \mu = -\lambda^2$

$$\boxed{\theta(x,t) = \sum_{n=1}^{\infty} sin\frac{n\pi}{l}x(A_n\cos\frac{n\pi a}{l}\mathrm{t} + B_n\, sin\frac{n\pi a}{l}t)}$$

$$\boxed{A_n = \frac{2}{l}\int_0^l f(x)\, sin\frac{n\pi}{l}\, xdx}$$

$$\boxed{B_n = \frac{2}{n\pi a}\int_0^l g(x)\, sin\frac{n\pi}{l}\, xdx}$$

Exercises

Exercise 6: Solve the following equation.

$u_{xx} - 4u_{xy}+3u_{yy}=0$

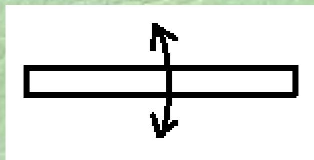
(Hint: v=x+y , z=3x+y)

Exercise 7: We know that the heat equation is a parabolic equation. Explain why?

Exercise 8: If $\mathrm{A}u_{xx} + 2Bu_{xy}+\mathrm{C}u_{yy}= \mathrm{f}(\mathrm{x},\ \mathrm{y},\ u_x,\ u_y)$ be a hyperbolic equation, then by changing of variable to v=φ(x, y) and z=ψ(x, y), it can be transform to $u_{vz} = F$ (v, z, u, u_v, u_z). Show that for the wave equation ψ =x-ct and φ =x+ct.

Exercise 9: If $\mathrm{A}u_{xx} + 2Bu_{xy}+\mathrm{C}u_{yy}= \mathrm{f}(\mathrm{x},\ \mathrm{y},\ u_x,\ u_y)$ be a parabolic equation, then by changing of variable to v=x and z=ψ(x, y), it can be transform to $u_{vv} = F$ (v, z, u, u_v, u_z). Investigate the validity of this for the equation $u_{xx} + 2u_{xy}+u_{yy}=0$.

Separation of Variables



Two-ended free shaft:

$$\frac{\partial\theta}{\partial x}(0,t)=0 \qquad \forall\, t \qquad B1 \qquad\qquad \frac{\partial\theta}{\partial x}(l,t)=0 \qquad \forall\, t \qquad B2$$

$$\theta(x,0)=f(x) \qquad I1 \qquad\qquad \frac{\partial\theta}{\partial t}(x,0)=g(x) \qquad I2$$

Assuming that $\mu < 0, \quad \mu = -\lambda^2$

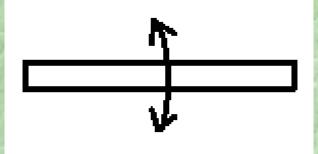
$$\theta(x,t) = \left(C\cos\frac{\lambda}{a}x + D\sin\frac{\lambda}{a}x\right)(A\cos\lambda t + B\sin\lambda t)$$

Applying boundary condition 1 (B1)

$$\frac{\partial\theta}{\partial x}(0,t)=0=D\frac{\lambda}{a}(A\cos\lambda t + B\sin\lambda t) \longrightarrow \cancel{A=B=0} \;\; or \;\; D=0 \;\; or \;\; \cancel{\lambda=0}$$

$$\theta(x,t)=\left(\cos\frac{\lambda}{a}x\right)(A\cos\lambda t + B\sin\lambda t) \qquad Note\ to\ C\ \ ??$$

Separation of Variables



Two-ended free shaft:

$$\cancel{\frac{\partial\theta}{\partial x}(0,t) = 0 \qquad \forall\, t \qquad B1} \qquad\qquad \frac{\partial\theta}{\partial x}(l,t) = 0 \qquad \forall\, t \qquad B2$$

$$\theta(x,0) = f(x) \qquad I1 \qquad\qquad \frac{\partial\theta}{\partial t}(x,0) = g(x) \qquad I2$$

Assuming that $\mu < 0, \quad \mu = -\lambda^2$

$$\theta(x,t) = (cos\frac{\lambda}{a}x)(Acos\,\lambda t + Bsin\,\lambda t)$$

Applying boundary condition 2 (B2)

$$\frac{\partial\theta}{\partial x}(l,t) = 0 = -\frac{\lambda}{a}sin\frac{\lambda}{a}l(Acos\,\lambda t + Bsin\,\lambda t) \longrightarrow \cancel{A = B = 0}\ \ or\ \ sin\frac{\lambda}{a}l = 0$$

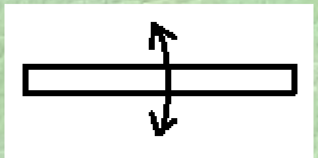
$$or\ \ \cancel{\lambda = 0}$$

$$\frac{\lambda}{a}l = n\pi \qquad\qquad \lambda_n = \frac{n\pi a}{l}$$

$$\theta_n(x,t) = (\,cos\frac{n\pi}{l}x)(A_n cos\frac{n\pi a}{l}t + B_n sin\frac{n\pi a}{l}t)$$

Separation of Variables

Two-ended free shaft:



$$\cancel{\frac{\partial\theta}{\partial x}(0,t) = 0 \qquad \forall\, t \qquad B1} \qquad\qquad \cancel{\frac{\partial\theta}{\partial x}(l,t) = 0 \qquad \forall\, t \qquad B2}$$

$$\theta(x,0) = f(x) \qquad I1 \qquad\qquad \frac{\partial\theta}{\partial t}(x,0) = g(x) \qquad I2$$

Assuming that $\mu < 0, \quad \mu = -\lambda^2$

$$\theta_n(x,t) = \left(cos\frac{n\pi}{l}x\right)\left(A_n cos\frac{n\pi a}{l}t + B_n sin\frac{n\pi a}{l}t\right)$$

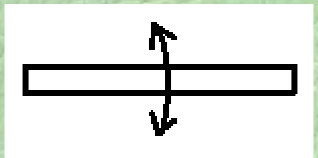
Applying initial condition 1 (I1)

$$\theta(x,t) = \sum_{n=1}^{\infty}\theta_n(x,t) = \sum_{n=1}^{\infty} cos\frac{n\pi}{l}x\left(A_n \cos\frac{n\pi a}{l}\mathrm{t} + B_n\, sin\frac{n\pi a}{l}t\right)$$

$$\theta(x,0) = f(x) \qquad \theta(x,0) = \sum_{n=1}^{\infty} A_n cos\frac{n\pi}{l}x = f(x) \qquad A_n = \frac{2}{l}\int_0^l f(x) cos\frac{n\pi}{l}x dx$$

Separation of Variables

Two-ended free shaft:



$$\cancel{\frac{\partial\theta}{\partial x}(0,t)=0 \qquad \forall\, t \qquad B1} \qquad\qquad \cancel{\frac{\partial\theta}{\partial x}(l,t)=0 \qquad \forall\, t \qquad B2}$$

$$\cancel{\theta(x,0)=f(x) \qquad I1} \qquad\qquad \frac{\partial\theta}{\partial t}(x,0)=g(x) \qquad I2$$

Assuming that $\mu<0, \quad \mu=-\lambda^2$

$$\theta(x,t)=\sum_{n=1}^{\infty} cos\frac{n\pi}{l}x(A_n\cos\frac{n\pi a}{l}\mathrm{t}+B_n\, sin\frac{n\pi a}{l}t) \qquad A_n=\frac{2}{l}\int_0^l f(x)\, cos\frac{n\pi}{l}xdx$$

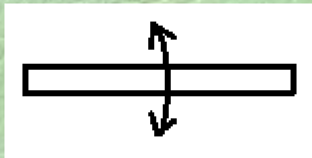
Applying initial condition 2 (I2)

$$\frac{\partial\theta}{\partial t}=\sum_{n=1}^{\infty} cos\frac{n\pi}{l}x(-A_n sin\frac{n\pi a}{l}t+B_n cos\frac{n\pi a}{l}t)\frac{n\pi a}{l} \qquad g(x)=\sum_{n=1}^{\infty}(\frac{n\pi a}{l}B_n)\, cos\frac{n\pi}{l}x$$

$$\frac{n\pi a}{l}B_n=\frac{2}{l}\int_0^l g(x)\, cos\frac{n\pi}{l}xdx \qquad\qquad B_n=\frac{2}{n\pi a}\int_0^l g(x)\, cos\frac{n\pi}{l}xdx$$

Separation of Variables

Two-ended free shaft:



$$\cancel{\frac{\partial\theta}{\partial x}(0,t)=0 \qquad \forall\, t \qquad B1} \qquad\qquad \cancel{\frac{\partial\theta}{\partial x}(l,t)=0 \qquad \forall\, t \qquad B2}$$

$$\cancel{\theta(x,0)=f(x) \qquad I1} \qquad\qquad \cancel{\frac{\partial\theta}{\partial t}(x,0)=g(x) \qquad I2}$$

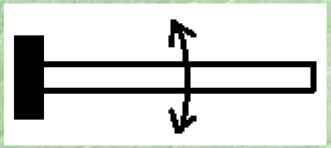
Assuming that $\mu<0, \quad \mu=-\lambda^2$

$$\boxed{\theta(x,t)=\sum_{n=1}^{\infty} cos\frac{n\pi}{l}x(A_n\cos\frac{n\pi a}{l}\mathrm{t}+B_n\, sin\frac{n\pi a}{l}t)}$$

$$\boxed{A_n=\frac{2}{l}\int_0^l f(x)\, cos\frac{n\pi}{l}xdx} \qquad \boxed{B_n=\frac{2}{n\pi a}\int_0^l g(x)\, cos\frac{n\pi}{l}xdx}$$

Separation of Variables

One-ended free and one-ended fixed shaft:



$$\theta(0,t)=0 \quad \forall\, t \quad B1 \qquad \frac{\partial\theta}{\partial x}(l,t)=0 \quad \forall\, t \quad B2$$

$$\theta(x,0)=f(x) \quad I1 \qquad \frac{\partial\theta}{\partial t}(x,0)=g(x) \quad I2$$

Assuming that $\mu < 0, \quad \mu=-\lambda^2$

$$\theta(x,t)= \left(C\cos\frac{\lambda}{a}x + D\sin\frac{\lambda}{a}x\right)(A\cos\lambda t + B\sin\lambda t)$$

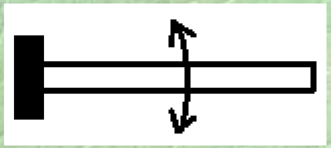
Applying boundary condition 1 (B1)

$$\theta(0,t)=0=C(A\cos\lambda t + B\sin\lambda t) \longrightarrow \cancel{A=B}=0 \;\; or \;\; C=0$$

$$\theta(x,t)=\left(\sin\frac{\lambda}{a}x\right)(A\cos\lambda t + B\sin\lambda t) \quad Note\ to\ D\ ??$$

Separation of Variables

One-ended free and one-ended fixed shaft:



$$\cancel{\theta(0,t) = 0 \quad \forall\, t \quad B1} \qquad \frac{\partial\theta}{\partial x}(l,t) = 0 \quad \forall\, t \quad B2$$

$$\theta(x,0) = f(x) \quad I1 \qquad \frac{\partial\theta}{\partial t}(x,0) = g(x) \quad I2$$

Assuming that $\mu < 0, \quad \mu = -\lambda^2$

$$\theta(x,t) = \left(\sin\frac{\lambda}{a}x\right)(A\cos\lambda t + B\sin\lambda t)$$

Applying boundary condition 2 (B2)

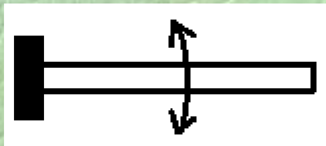
$$\frac{\partial\theta}{\partial x}(l,t) = 0 = \frac{\lambda}{a}\cos\frac{\lambda}{a}l(A\cos\lambda t + B\sin\lambda t) \longrightarrow \cancel{A = B = 0} \;\; or \;\; \cos\frac{\lambda}{a}l = 0$$

$$or \quad \cancel{\lambda = 0}$$

$$\frac{\lambda}{a}l = \frac{(2n-1)\pi}{2} \qquad \lambda_n = \frac{(2n-1)\pi a}{2l}$$

Separation of Variables

One-ended free and one-ended fixed shaft:



$$\cancel{\theta(0,t) = 0 \quad \forall\, t \qquad B1} \qquad \cancel{\frac{\partial\theta}{\partial x}(l,t) = 0 \quad \forall\, t \qquad B2}$$

$$\theta(x,0) = f(x) \qquad I1 \qquad \frac{\partial\theta}{\partial t}(x,0) = g(x) \qquad I2$$

Assuming that $\mu < 0, \quad \mu = -\lambda^2$

$$\theta_n(x,t) = \ sin\frac{(2n-1)\pi}{2l}x(A_n\cos\frac{(2n-1)\pi a}{2l}\mathrm{t} + B_n\ sin\frac{(2n-1)\pi a}{2l}t)$$

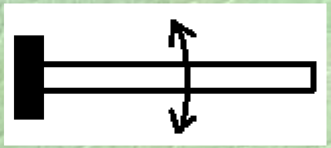
Applying initial condition 1 (I1)

$$\theta(x,t) = \sum_{n=1}^{\infty} sin\frac{(2n-1)\pi}{2l}x(A_n\cos\frac{(2n-1)\pi a}{2l}\mathrm{t} + B_n\ sin\frac{(2n-1)\pi a}{2l}t)$$

$$\theta(x,0) = \sum_{n=1}^{\infty} A_n sin\frac{(2n-1)\pi}{2l}x = f(x) \qquad A_n = \frac{2}{l}\int_0^l f(x)\, sin\frac{(2n-1)\pi}{2l}xdx$$

Separation of Variables

One-ended free and one-ended fixed shaft:



$$\cancel{\theta(0,t)=0 \qquad \forall\, t \qquad B1} \qquad \cancel{\frac{\partial\theta}{\partial x}(l,t)=0 \qquad \forall\, t \qquad B2}$$

$$\cancel{\theta(x,0)=f(x) \qquad I1} \qquad \frac{\partial\theta}{\partial t}(x,0)=g(x) \qquad I2$$

Assuming that $\mu<0, \quad \mu=-\lambda^2$

$$\theta(x,t)=\sum_{n=1}^{\infty} sin\frac{(2n-1)\pi}{2l}x(A_n\cos\frac{(2n-1)\pi a}{2l}\text{t}+B_n\, sin\frac{(2n-1)\pi a}{2l}t)$$

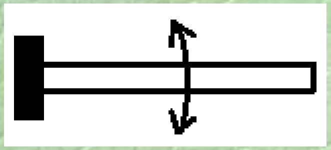
$$A_n=\frac{2}{l}\int_0^l f(x)\, sin\frac{(2n-1)\pi}{2l}xdx$$

Applying initial condition 2 (I2)

$$g(x)=\sum_{n=1}^{\infty}\left[\frac{(2n-1)\pi a}{2l}B_n\right] sin\frac{(2n-1)\pi x}{2l} \qquad B_n=\frac{4}{(2n-1)\pi a}\int_0^l g(x)\, sin\frac{(2n-1)\pi}{2l}xdx$$

Separation of Variables

One-ended free and one-ended fixed shaft:



$$\cancel{\theta(0,t) = 0 \quad \forall\, t \qquad B1} \qquad \cancel{\frac{\partial\theta}{\partial x}(l,t) = 0 \quad \forall\, t \qquad B2}$$

$$\cancel{\theta(x,0) = f(x) \qquad I1} \qquad \cancel{\frac{\partial\theta}{\partial t}(x,0) = g(x) \qquad I2}$$

Assuming that $\mu < 0, \quad \mu = -\lambda^2$

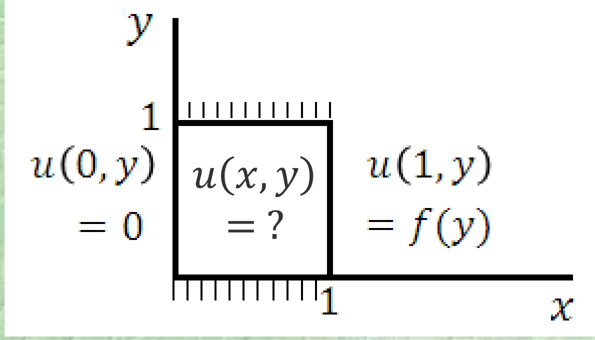
$$\boxed{\theta(x,t) = \sum_{n=1}^{\infty} sin\frac{(2n-1)\pi}{2l}x\left(A_n\cos\frac{(2n-1)\pi a}{2l}\mathrm{t} + B_n\, sin\frac{(2n-1)\pi a}{2l}t\right)}$$

$$\boxed{A_n = \frac{2}{l}\int_0^l f(x)\, sin\frac{(2n-1)\pi}{2l}xdx}$$

$$\boxed{B_n = \frac{4}{(2n-1)\pi a}\int_0^l g(x)\, sin\frac{(2n-1)\pi}{2l}xdx}$$

Separation of Variables

Example 8: Consider a metal sheet in steady-state as shown.



The heat flow in the sheet can be considered two-dimensional.

The two-dimensional heat equation is:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \xrightarrow{\frac{\partial u}{\partial t} = 0} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(x, y) = X(x)Y(y)$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \mu$$

Separation of Variables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \mu$$

$$X'' = \mu X \qquad\qquad Y'' = -\mu Y$$

$$\mu > 0 \quad \mu = 0 \quad \mu < 0$$

Separation of Variables

Case 1: Assuming that $\mu < 0, \;\; \mu = -\lambda^2$

$X'' = -\lambda^2 X$

$$u(x,y) = X(x)Y(y) = (A\cos\lambda x + B\sin\lambda x)(C\cosh\lambda y + D\sinh\lambda y)$$

$Y'' = \lambda^2 Y$

Case 2: Assuming that $\mu = 0$

$X'' = 0$

$$u(x,y) = X(x)Y(y) = (Ax + B)(Cy + D)$$

$Y'' = 0$

Case 3: Assuming that $\mu > 0, \;\; \mu = \lambda^2$

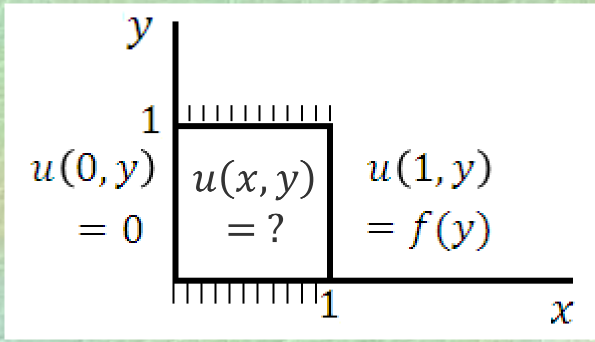
$X'' = \lambda^2 X$

$$u(x,y) = X(x)Y(y) = (A\cosh\lambda x + B\sinh\lambda x)(C\cos\lambda y + D\sin\lambda y)$$

$Y'' = -\lambda^2 Y$

Separation of Variables

Problem Statement



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(0,y) = 0 \qquad B1$$

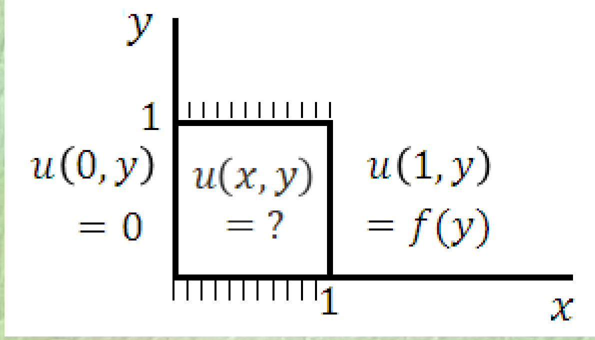
$$\frac{\partial u}{\partial y}(x,0) = 0 \qquad B2$$

$$\frac{\partial u}{\partial y}(x,1) = 0 \qquad B3$$

$$u(1,y) = f(y) \qquad B4$$

Separation of Variables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



$$u(0, y) = 0 \qquad B1$$

$$\frac{\partial u}{\partial y}(x, 0) = 0 \qquad B2$$

$$\frac{\partial u}{\partial y}(x, 1) = 0 \qquad B3$$

$$u(1, y) = f(y) \qquad B4$$

Assume that $\mu < 0$, $\mu = -\lambda^2$

$$u(x, y) = X(x)Y(y) = (A\cos\lambda x + B\sin\lambda x)(C\cosh\lambda y + D\sinh\lambda y)$$

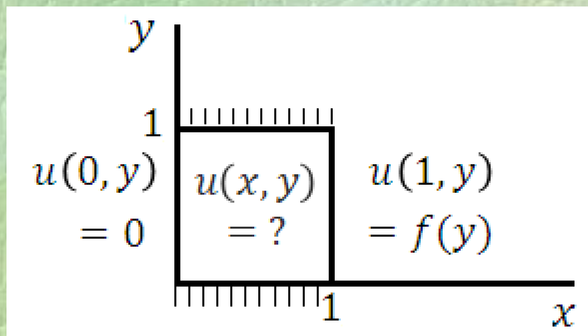
Applying boundary condition 1 (B1)

$$u(0, y) = (A)(C\cosh\lambda y + D\sinh\lambda y) = 0 \longrightarrow \cancel{C = D} = 0 \ \ or\ A = 0$$

$$u(x, y) = \sin\lambda x(C\cosh\lambda y + D\sinh\lambda y) \qquad \text{B} =?$$

Separation of Variables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



$$\cancel{u(0, y) = 0} \qquad B1$$

$$\frac{\partial u}{\partial y}(x, 0) = 0 \qquad B2$$

$$\frac{\partial u}{\partial y}(x, 1) = 0 \qquad B3$$

$$u(1, y) = f(y) \qquad B4$$

Assume that $\mu < 0, \ \ \mu = -\lambda^2$

$$u(x, y) = \sin \lambda x(C\cosh \lambda y + D\sinh \lambda \mathrm{y})$$

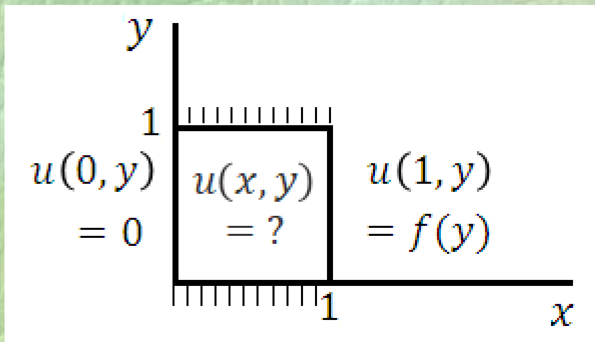
Applying boundary condition 2 (B2)

$$\frac{\partial u}{\partial y}(x, 0) = \sin\lambda x(D\lambda) = 0 \longrightarrow \cancel{\lambda = 0} \ \ or \ \ D = 0$$

$$u(x, y) = C\sin \lambda \mathrm{x} \, . \cosh \lambda y$$

Separation of Variables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



$\cancel{u(0, y) = 0}$ $\quad B1$

$\cancel{\frac{\partial u}{\partial y}(x, 0) = 0}$ $\quad B2$

$\frac{\partial u}{\partial y}(x, 1) = 0$ $\quad B3$

$u(1, y) = f(y)$ $\quad B4$

Assume that $\mu < 0, \ \ \mu = -\lambda^2$

$$u(x, y) = C \sin \lambda x \, . \cosh \lambda y$$

Applying boundary condition 3 (B3)

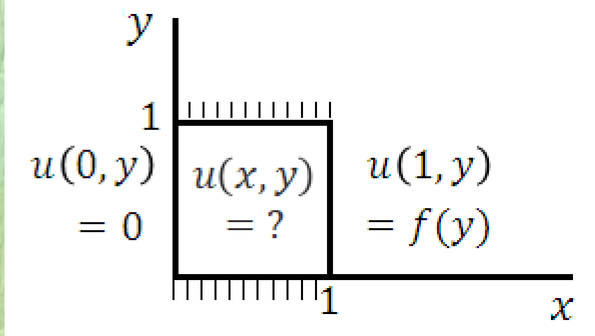
$$\frac{\partial u}{\partial y}(x, 1) = C\lambda \sin \lambda x . \sinh \lambda = 0 \longrightarrow \quad \lambda = 0 \ \ or \ \ C = 0$$

$$u(x, y) = 0 \longrightarrow \quad \cancel{\mu < 0}$$

Assume that $\mu = 0$

Separation of Variables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



$$u(0, y) = 0 \qquad B1$$

$$\frac{\partial u}{\partial y}(x, 0) = 0 \qquad B2$$

$$\frac{\partial u}{\partial y}(x, 1) = 0 \qquad B3$$

$$u(1, y) = f(y) \qquad B4$$

Assume that $\mu = 0$

$$u(x, y) = X(x)Y(y) = (Ax + B)(Cy + D)$$

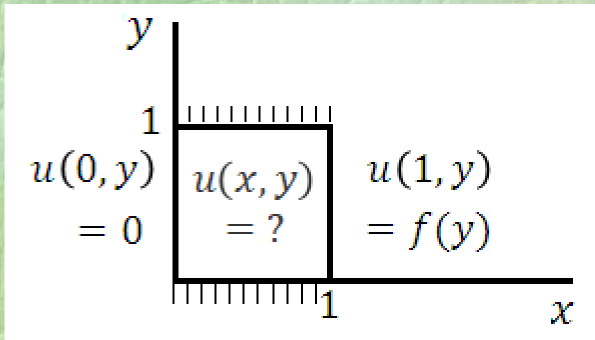
Applying boundary condition 1 (B1)

$$u(0, y) = B(Cy + D) = 0 \longrightarrow \cancel{C = D} = 0 \;\; or\; B = 0$$

$$u(x, y) = x(Cy + D) \qquad \text{A} =?$$

Separation of Variables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



$$\cancel{u(0, y) = 0} \qquad B1$$

$$\frac{\partial u}{\partial y}(x, 0) = 0 \qquad B2$$

$$\frac{\partial u}{\partial y}(x, 1) = 0 \qquad B3$$

$$u(1, y) = f(y) \qquad B4$$

Assume that $\mu = 0$

$$u(x, y) = x(Cy + D)$$

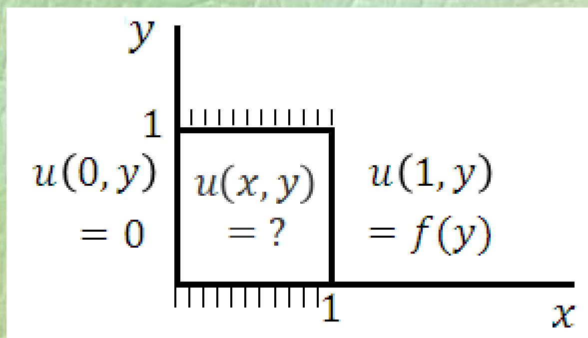
Applying boundary condition 2 (B2)

$$\frac{\partial u}{\partial y}(x, 0) = Cx = 0 \longrightarrow C = 0$$

$$\boxed{u(x, y) = Dx}$$

Separation of Variables

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$



$$\cancel{u(0,y)=0} \qquad B1$$

$$\cancel{\frac{\partial u}{\partial y}(x,0)=0} \qquad B2$$

$$\cancel{\frac{\partial u}{\partial y}(x,1)=0} \qquad B3$$

$$u(1,y)=f(y) \qquad B4$$

Assume that $\mu = 0$

$$u(x,y)=Dx$$

Applying boundary condition 3 (B3) $\quad \frac{\partial u}{\partial y}(x,1)=0$

Applying boundary condition 4 (B4)

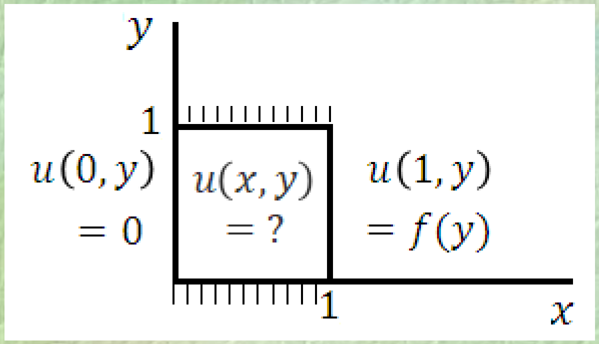
$$u(1,y)=D=f(y)$$

This is valid for constant f

Assume that $\mu > 0, \quad \mu = \lambda^2$

Separation of Variables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



$$u(0, y) = 0 \qquad B1$$

$$\frac{\partial u}{\partial y}(x, 0) = 0 \qquad B2$$

$$\frac{\partial u}{\partial y}(x, 1) = 0 \qquad B3$$

$$u(1, y) = f(y) \qquad B4$$

Assume that $\mu > 0, \;\; \mu = \lambda^2$

$$u(x, y) = X(x)Y(y) = (A\cosh \lambda x + B\sinh \lambda x)(C\cos \lambda y + D\sin \lambda y)$$

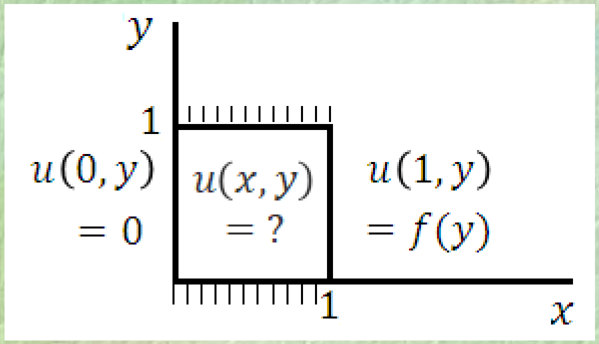
Applying boundary condition 1 (B1)

$$u(0, y) = (A)(C\cos \lambda y + D\sin \lambda y) = 0 \longrightarrow \cancel{C = D} = 0 \;\; or \;\; A = 0$$

$$u(x, y) = \sinh \lambda x(C\cos \lambda y + D\sin \lambda y) \qquad B = ?$$

Separation of Variables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



$$\cancel{u(0, y) = 0} \qquad B1$$

$$\frac{\partial u}{\partial y}(x, 0) = 0 \qquad B2$$

$$\frac{\partial u}{\partial y}(x, 1) = 0 \qquad B3$$

$$u(1, y) = f(y) \qquad B4$$

Assume that $\mu > 0,\ \ \mu = \lambda^2$

$$u(x, y) = sinh\ \lambda x(Ccos\ \lambda y + Dsin\ \lambda y)$$

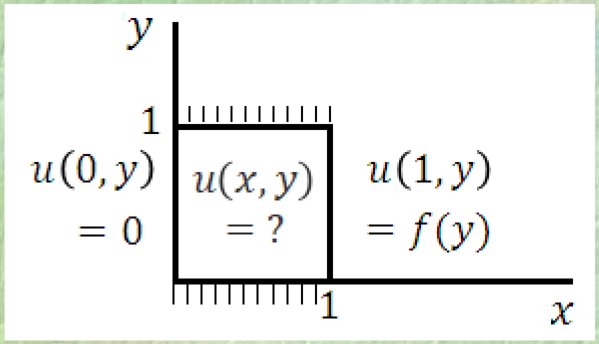
Applying boundary condition 2 (B2)

$$\frac{\partial u}{\partial y}(x, 0) = sinh\lambda x(D\lambda) = 0 \longrightarrow D = 0\ \ or\ \ \cancel{\lambda = 0}$$

$$u(x, y) = Csinh\ \lambda x.\ cos\ \lambda y$$

Separation of Variables

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$



$$\cancel{u(0,y)=0} \qquad B1$$

$$\cancel{\frac{\partial u}{\partial y}(x,0)=0} \qquad B2$$

$$\frac{\partial u}{\partial y}(x,1)=0 \qquad B3$$

$$u(1,y)=f(y) \qquad B4$$

Assume that $\mu > 0,\ \ \mu=\lambda^2$

$$u(x,y)=C\sinh\lambda x.\cos\lambda y$$

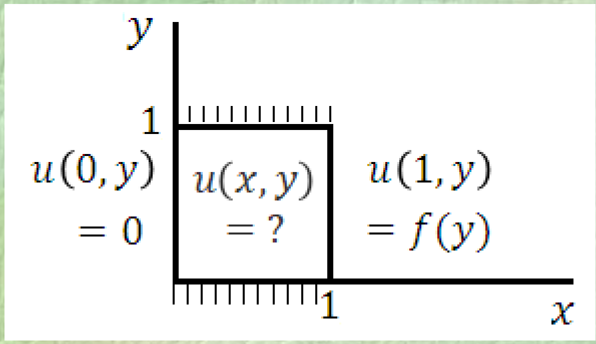
Applying boundary condition 3 (B3)

$$\frac{\partial u}{\partial y}(x,1)=-C\lambda\sinh\lambda x.\sin\lambda=0 \longrightarrow \cancel{C=0}\ or\ \lambda=\mathrm{n}\pi\ \ or\ \cancel{\lambda=0}$$

$$u_n(x,y)=c_n\sinh n\pi x.\cos n\pi y$$

Separation of Variables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



$$\cancel{u(0, y) = 0} \qquad B1$$

$$\cancel{\frac{\partial u}{\partial y}(x, 0) = 0} \qquad B2$$

$$\cancel{\frac{\partial u}{\partial y}(x, 1) = 0} \qquad B3$$

$$u(1, y) = f(y) \qquad B4$$

Assume that $\mu > 0, \ \ \mu = \lambda^2$

$$u_n(x, y) = C_n \sinh n\pi x . \cos n\pi y$$

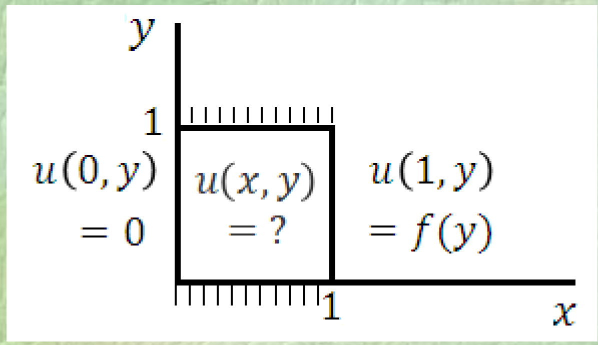
Applying boundary condition 4 (B4)

$$u(x, y) = Dx + \sum_{n=1}^{\infty} C_n \sinh n\pi x \cos n\pi y$$

$$u(1, y) = f(y) = D + \sum_{n=1}^{\infty} C_n \sinh n\pi \cos n\pi y$$

Separation of Variables

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$



$$\cancel{u(0,y)=0} \qquad B1$$

$$\cancel{\frac{\partial u}{\partial y}(x,0)=0} \qquad B2$$

$$\cancel{\frac{\partial u}{\partial y}(x,1)=0} \qquad B3$$

$$u(1,y)=f(y) \qquad B4$$

Assume that $\mu > 0$, $\mu = \lambda^2$

$$u_n(x,y)=c_n sinh\, n\pi x.\, cos\, n\pi y$$

Applying boundary condition 4 (B4)

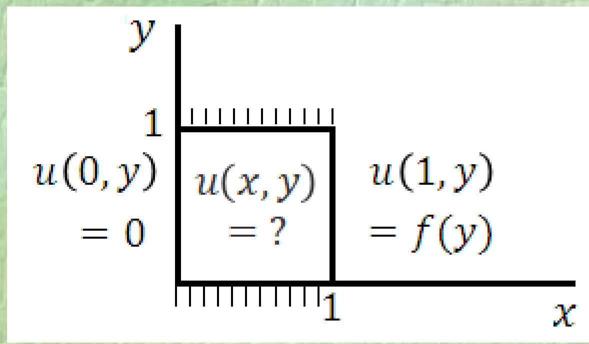
$$u(1,y)=f(y)=D+\sum_{n=1}^{\infty} C_n sinhn\pi\, cosn\pi y$$

$$D=\int_0^1 f(y)dy \qquad C_n \sinh n\pi = 2\int_0^1 f(y)\cos n\pi y\, dy$$

$$D=\int_0^1 f(y)dy \qquad C_n=\frac{2}{\sinh n\pi}\int_0^1 f(y)\cos n\pi y\, dy$$

Separation of Variables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



$$\cancel{u(0, y) = 0} \qquad B1$$

$$\cancel{\frac{\partial u}{\partial y}(x, 0) = 0} \qquad B2$$

$$\cancel{\frac{\partial u}{\partial y}(x, 1) = 0} \qquad B3$$

$$\cancel{u(1, y) = f(y)} \qquad B4$$

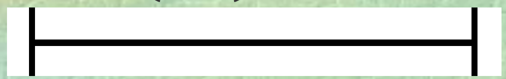
Assume that $\mu > 0, \quad \mu = \lambda^2$

$$u(x, y) = Dx + \sum_{n=1}^{\infty} C_n sinhn\pi x \; cosn\pi y$$

$$D = \int_0^1 f(y)dy \qquad\qquad C_n = \frac{2}{\sinh n\pi} \int_0^1 f(y) \cos n\pi y \, dy$$

Separation of Variables

Example 9: A rod with a length of l is completely insulated on its lateral surface, and the rod is so thin that the heat flow in it can be considered one-dimensional. Determine the temperature at any point of the rod at any given time.



Solution: The one-dimensional heat equation must be used.

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}$$

$u(0,t) = 50 \qquad B1 \qquad u(l,t) = 100 \qquad B2 \qquad u(x,0) = 100 \qquad I1$

$$u(x,t) = X(x)T(t) \qquad\qquad \frac{X''}{X} = a^2 \frac{T'}{T} = \mu$$

Separation of Variables

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}$$

$$\downarrow$$

$$\frac{X''}{X} = a^2 \frac{T'}{T} = \mu$$

$$X'' = \mu X \qquad\qquad T' = \frac{\mu}{a^2} T$$

$$\mu > 0 \quad \mu = 0 \quad \mu < 0$$

Separation of Variables

Case 1: Assuming that $\mu > 0, \quad \mu = \lambda^2$

$X'' = \lambda^2 X$

$T' = \dfrac{\lambda^2}{a^2} T$

$$u(x,t) = X(x)T(t) = (A\cosh \lambda x + B\sinh \lambda x)(C\exp \frac{\lambda^2}{a^2} t)$$

Case 2: Assuming that $\mu = 0$

$X'' = 0$

$T' = 0$

$$u(x,t) = X(x)T(t) = (Ax + B)C$$

Case 3: Assuming that $\mu < 0, \quad \mu = -\lambda^2$

$X'' = -\lambda^2 X$

$T' = -\dfrac{\lambda^2}{a^2} T$

$$u(x,t) = X(x)T(t) = (A\cos \lambda x + B\sin \lambda x)\, C\exp(-\frac{\lambda^2}{a^2} t)$$

Separation of Variables

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}$$

$\cancel{u(0,t) = 50}$ $\quad B1$

$u(l,t) = 100 \quad B2$

$u(x,0) = 100 \quad I1$

Assume that $\mu = 0$

$$u(x,t) = (Ax + B)$$

Applying boundary condition 1 (B1)

$u(0,t) = B = 50$ $\qquad$ $u(x,t) = (Ax + 50)$

Applying boundary condition 2 (B2)

$u(l,t) = Al + 50 = 100$ $\qquad$ $A = 50/l$ $\qquad$ $u(x,t) = \dfrac{50}{l}x + 50$

Separation of Variables

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}$$

$$\cancel{u(0,t) = 50} \qquad B1$$

$$\cancel{u(l,t) = 100} \qquad B2$$

$$u(x,0) = 100 \qquad I1$$

Assume that $\mu = 0$

$$u(x,t) = \frac{50}{l}x + 50$$

Applying initial condition 1 (I1)

$$u(x,0) = \frac{50}{l}x + 50 = 100 \qquad ??$$

Assume that $\mu < 0$, $\mu = -\lambda^2$

Separation of Variables

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}$$

$u(0,t) = 50 \qquad B1$

$u(l,t) = 100 \qquad B2$

$u(x,0) = 100 \qquad I1$

Assume that $\mu < 0, \ \mu = -\lambda^2$

$$u(x,t) = (A\cos \lambda x + B\sin \lambda \mathrm{x}) \exp(-\frac{\lambda^2}{a^2} t)$$

Applying boundary condition 1 (B1)

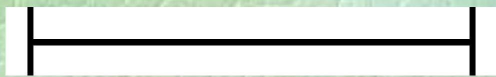
$$u(0,t) = A\exp(-\frac{\lambda^2}{a^2} t) = 50$$

Unacceptable

Assume combination of $\mu < 0, \ \mu = 0$

$$u(x,t) = \frac{50}{l} x + 50 + (A\cos \lambda x + B\sin \lambda \mathrm{x}) \exp(-\frac{\lambda^2}{a^2} t)$$

Separation of Variables



$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}$$

$u(0,t) = 50 \qquad B1$

$u(l,t) = 100 \qquad B2$

$u(x,0) = 100 \qquad I1$

Assume combination of μ < 0, μ = 0

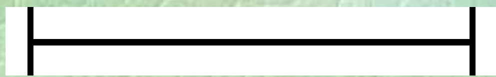
$$u(x,t) = \frac{50}{l}x + 50 + (A\cos\lambda x + B\sin\lambda \mathrm{x})\ \exp(-\frac{\lambda^2}{a^2}t)$$

Applying boundary condition 1 (B1)

$$u(0,t) = 50 + A\exp(-\frac{\lambda^2}{a^2}t) = 50 \qquad\qquad A = 0$$

$$u(x,t) = \frac{50}{l}x + 50 + B\sin\lambda \mathrm{x}\ \exp(-\frac{\lambda^2}{a^2}t)$$

Separation of Variables



$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}$$

$\cancel{u(0,t) = 50}$ $\quad B1$

$u(l,t) = 100 \quad B2$

$u(x,0) = 100 \quad I1$

Assume combination of $\mu < 0$, $\mu = 0$

$$u(x,t) = \frac{50}{l}x + 50 + B\sin\lambda x \exp(-\frac{\lambda^2}{a^2}t)$$

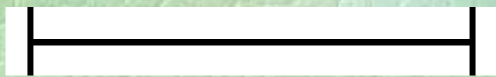
Applying boundary condition 2 (B2)

$$u(l,t) = 50 + 50 + B\sin\lambda l \exp(-\frac{\lambda^2}{a^2}t) = 100$$

$\cancel{B = 0}$ $or\ \lambda l = n\pi$

$$u_n(x,t) = \frac{50}{l}x + 50 + B_n \sin\frac{n\pi}{l}x \exp\left(-\frac{n^2\pi^2}{a^2 l^2}t\right)$$

Separation of Variables



$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}$$

$\cancel{u(0,t) = 50}$ $B1$

$\cancel{u(l,t) = 100}$ $B2$

$u(x,0) = 100$ $I1$

Assume combination of $\mu < 0$, $\mu = 0$

$$u_n(x,t) = \frac{50}{l}x + 50 + B_n sin\frac{n\pi}{l}x\exp\left(-\frac{n^2\pi^2}{a^2l^2}t\right)$$

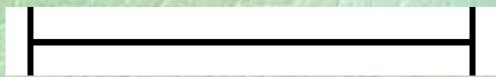
Applying initial condition 1 (I1)

$$u(x,t) = 50 + \frac{50}{l}x + \sum_{n=1}^{\infty} B_n\, sin\frac{n\pi}{l}x\exp\left(-\frac{n^2\pi^2}{a^2l^2}t\right)$$

$$u(x,0) = 100 = 50 + \frac{50}{l}x + \sum_{n=1}^{\infty} B_n\, sin\frac{n\pi}{l}x \qquad B_n = \frac{2}{l}\int_0^l (50 - \frac{50}{l}x)\sin\frac{n\pi}{l}x\,dx$$

$$B_n = \frac{100}{n\pi}$$

Separation of Variables



$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}$$

$\cancel{u(0,t) = 50}$ $\quad B1$

$\cancel{u(l,t) = 100}$ $\quad B2$

$\cancel{u(x,0) = 100}$ $\quad I1$

Assume combination of $\mu < 0$, $\mu = 0$

$$u(x,t) = 50 + \frac{50}{l} x + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{l} x \exp\left(-\frac{n^2\pi^2}{a^2 l^2} t\right)$$

$$B_n = \frac{100}{n\pi}$$

$$\boxed{u(x,t) = 50 + \frac{50}{l} x + \frac{100}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{l} x \exp\left(-\frac{n^2\pi^2}{a^2 l^2} t\right)}$$

Separation of Variables

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \qquad u(x,y,z)=X(x)Y(y)Z(z)$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \longrightarrow \frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}=0$$

$$\longrightarrow \quad \frac{X''}{X}=-\frac{Y''}{Y}-\frac{Z''}{Z}$$

$$\frac{X''}{X}=\mathrm{a} \qquad \frac{Y''}{Y}=\mathrm{b} \qquad \frac{Z''}{Z}=-\mathrm{a}-\mathrm{b}$$

Exercises

Exercise 10: Obtain the solution to the one-dimensional wave differential equation for the following conditions.

$$L=\pi ، c^2=1 ، g(x)= 0 ، f(x) =k\sin(2x)$$

Exercise 11: Determine the temperature distribution in a rod with a length of 80 cm, assuming the initial temperature is $100\sin(\pi x/80)$ and the temperature at both ends is zero.

Exercise 12: Determine the temperature of a rod with length L if both ends are at zero degrees and the initial temperature of the rod is $f(x)$.

$$f(x) = \begin{cases} x & 0 < x < L/2 \\ L - x & \dfrac{L}{2} < x < L \end{cases}$$

References

- Advanced Engineering Mathematics , E. Kreyszig

- Advanced Engineering Mathematics, C. R. Wylie