Multivariable Control Systems

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Lecture 2

References are appeared in the last slide.

Linear Algebra

Topics to be covered include:

- Vector Spaces, Norms
- Singular Value Decomposition
- Unitary, Primitive, Hermitian and positive(negative) definite Matrices
- Relative Gain Array (RGA)
- Matrix Perturbation

Vector Spaces

A set of vectors and a field of scalars with some properties is called vector space.

To see the properties have a look at Linear Algebra written by Hoffman.

$$\forall\ \alpha_1\ and\ \alpha_2 \in Field \quad \& \quad \forall\ v_1\ and\ v_2 \in \text{Vector Space} \quad \Rightarrow \quad \alpha_1 v_1 + \alpha_2 v_2 \in \text{Vector Space}$$

Important vector spaces are:

R^n over the field of real numbers (R)

C^n over the field of complex numbers (C)

Continuous functions on the interval [0,1] over the field of real numbers (R)

Norms

To meter the lengths of vectors in a vector space we need the idea of a norm.

Norm is a function that maps a vector x to a nonnegative real number

$$\| \, . \, \|: \quad F \to R^{+}$$

A Norm must satisfy following properties:

$$1 - \text{Positivity} \quad \|\mathrm{x}\| > 0 \;\; , \forall \; \mathrm{x} \neq 0 \;\text{ and } \|x\| = 0 \,\text{for } \mathrm{x} = 0$$

$$2 - \text{Homogeneity} \quad \|\alpha \mathrm{x}\| = |\alpha| \|x\| \;, \forall \; \mathrm{x} \in \mathrm{F} \text{ and } \forall \; \alpha \in \mathrm{C}$$

$$3 - \text{Triangle inequality} \quad \|\mathrm{x} + \mathrm{y}\| \leq \|x\| + \|y\| \;, \forall \; \mathrm{x}, \mathrm{y} \in \mathrm{F}$$

Norm of vectors

p-norm is: $$\|x\|_p = \left(\sum_i |a_i|^p \right)^{\frac{1}{p}} \qquad p \geq 1$$

For p=1 we have 1-norm or sum norm $$\|x\|_1 = \left(\sum_i |a_i| \right)$$

For p=2 we have 2-norm or euclidian norm $$\|x\|_2 = \left(\sum_i |a_i|^2 \right)^{1/2}$$

For p=∞ we have ∞-norm or max norm $$\|x\|_\infty = \max_i \left\{ |a_i| \right\}$$

Norm of vectors

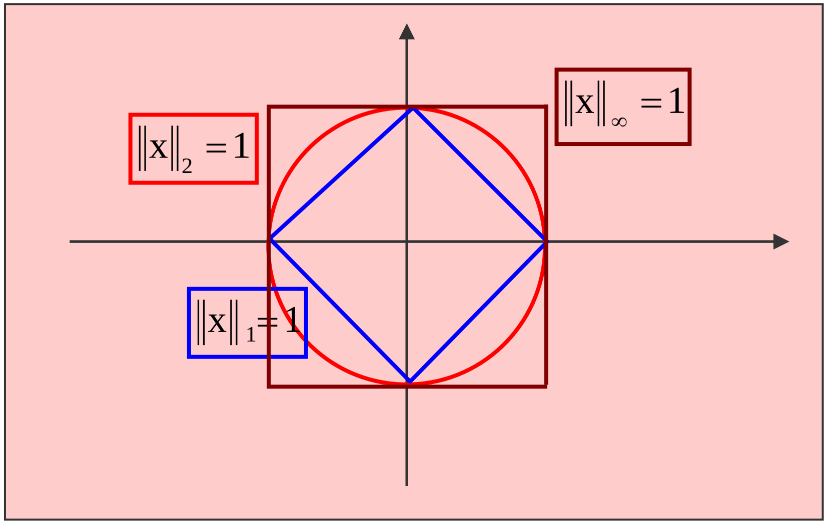
Let $x = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

Then

$$\|x\|_1 = (1+1+2) = 4$$

$$\|x\|_2 = \sqrt{1^2+1^2+2^2} = \sqrt{6}$$

$$\|x\|_\infty = \max(1,1,2) = 2$$



Exercise 2-1: Introduce a non-scalar vector with identical 1, 2 and ∞ norm.

Norm of matrices

We can extend norm of vectors to matrices

Sum matrix norm (extension of 1-norm of vectors) is: $\|A\|_{sum} = \sum_{i,j} |a_{ij}|$

Frobenius norm (extension of 2-norm of vectors) is: $\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2}$

Max element norm (extension of max norm of vectors) is: $\|A\|_{\max} = \max_{i,j} |a_{ij}|$

Matrix norm

A norm of a matrix is called matrix norm if it satisfies

$$\|AB\| \le \|A\| . \|B\|$$

Define the induced-norm of a matrix A as follows:

$$\|A\|_{ip} = \max_{\|x\|_p = 1} \|Ax\|_p$$

Any induced-norm of a matrix, is a matrix norm

Matrix norm for matrices

$$\|A\|_{ip} = \max_{\|x\|_p=1} \|Ax\|_p$$

If we put p=1 so we have

$$\|A\|_{i1} = \max_{\|x\|_1=1} \|Ax\|_1 = \max_j \sum_i |a_{ij}|$$ Maximum column sum

If we put p=inf so we have

$$\|A\|_{i\infty} = \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_i \sum_j |a_{ij}|$$ Maximum row sum

If we put p=2 so we have

$$\|A\|_{i2} = \max_{\|x\|_2=1} \|Ax\|_2 = \max_{\|x\|_2=1} \frac{\|Ax\|_2}{\|x\|_2} = \sigma_1(A) = \sigma_{\max}(A) = \bar{\sigma}(A)$$

Linear Algebra

- Vector Spaces, Norms
- Singular Value Decomposition (SVD)
- Unitary, Primitive, Hermitian and positive(negative) definite Matrices
- Relative Gain Array (RGA)
- Matrix Perturbation

Singular Value Decomposition (SVD)

Theorem 2-1: Let $M \in C^{l\times m}$. Then there exist $\Sigma \in R^{l\times m}$ and unitary matrices $Y \in C^{l\times l}$ and $U \in C^{m\times m}$ such that

$$M = Y\Sigma U^H$$

$$\Sigma = [S \quad 0] \;\; or \;\; \begin{bmatrix} S \\ 0 \end{bmatrix} \qquad S = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & \sigma_r \end{bmatrix} \qquad \sigma_1 \ge \sigma_2 \ge \sigma_3 \ge \dots \ge \sigma_r \ge 0$$

$$r = \min\{l, m\}$$

$$Y = [y_1, y_2, \dots\dots, y_l],\; U = [u_1, u_2, \dots\dots, u_m]$$

Singular Value Decomposition (SVD)

Example 2-1

$$M=\begin{bmatrix}1 & 2 & -1\\ 3 & 4 & 1\\ 4 & 2 & 8\end{bmatrix}$$

$$M=\begin{bmatrix}0.04 & -0.53 & -0.85\\ 0.38 & -0.77 & 0.51\\ 0.92 & 0.34 & -0.17\end{bmatrix}.\begin{bmatrix}9.77 & 0 & 0\\ 0 & 4.53 & 0\\ 0 & 0 & 0\end{bmatrix}.\begin{bmatrix}0.50 & -0.33 & -0.80\\ 0.35 & -0.77 & 0.53\\ 0.79 & 0.55 & 0.27\end{bmatrix}^H$$

$$u_1=\begin{bmatrix}0.50\\ 0.35\\ 0.79\end{bmatrix} \qquad Mu_1=9.77\begin{bmatrix}0.04\\ 0.38\\ 0.92\end{bmatrix}=9.77y_1 \qquad u_2=\begin{bmatrix}-0.33\\ -0.77\\ 0.55\end{bmatrix} \qquad Mu_2=4.53\begin{bmatrix}-0.53\\ -0.77\\ 0.34\end{bmatrix}=4.53y_2$$

$$u_3=\begin{bmatrix}-0.80\\ 0.53\\ 0.27\end{bmatrix}$$

Has no affect on the output or $Mu_3=0$

Singular Value Decomposition (SVD)

Theorem 2-1: Let $M \in C^{l\times m}$. Then there exist $\Sigma \in R^{l\times m}$ and unitary matrices $Y \in C^{l\times l}$ and $U \in C^{m\times m}$ such that

$$M = Y\Sigma U^H$$

Y can be derived from eigenvectors of $M M^H$

U can be derived from eigenvectors of $M^H M$

$\sigma_1, \sigma_2, ..., \sigma_r$ are roots of nonzero eigenvalues of $M^H M$ or $M M^H$

Exercise 2-2: Introduce a non-vector matrix with identical 1, 2 and ∞ norm.

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Norm of real functions(signals)

Consider continuous signals on the interval [0,∞) and p ≥ 1

p-norm is defined as $\|u(t)\|_p = \left(\int_0^\infty |u(t)|^p dt\right)^{1/p}$

∞-norm is defined as $\|u(t)\|_\infty = sup_t|u(t)|$

2-norm is defined as $\|u(t)\|_2 = \left(\int_0^\infty |u(t)|^2 dt\right)^{1/2}$

1-norm is defined as $\|u(t)\|_1 = \int_0^\infty |u(t)| dt$

Norm of real functions(signals)

Exercise 2-3: Derive 1, 2 and ∞ norm of following signals.

$$u_1(t) = \frac{1}{(2t+3)^2}$$

Ans: 1/6, 0.0786 and 1/9

$$u_2(t) = \frac{1}{2t+3}$$

Ans: ∞, 0.4082 and 1/3

$$u_3(t) = sint$$

Ans: ∞, ∞ and 1

Norm of transfer functions(systems)

$$u \rightarrow \boxed{G(s)} \rightarrow y$$

Let G(s) is a stable transfer matrix with impulse response matrix g(t). To evaluate the performance:

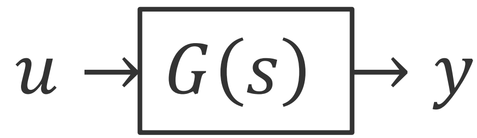
Given u(t), how large can be the output y(t)?

large ?? We consider 2-norm for signals.

$H_2\ norm$: When u(t) is a series of unit impulses.

$H_\infty\ norm$: When u(t) is a any non-zero, finite 2-norm signal.

H_2 *norm* for transfer functions(systems)



Let G(s) is a stable and strictly proper transfer matrix G(s), (D=0 in state space realization). H_2 norm is defined by:

$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} tr\left(G(j\omega)^H G(j\omega)\right) d\omega}$$

$$tr\left(G(j\omega)^H G(j\omega)\right) = \sum_{ij} \left|G_{ij}(j\omega)\right|^2 = \|G(j\omega)\|_F^2$$

By Parseval's theorem, we have

$$\|G(s)\|_2 = \|g(t)\|_2 = \sqrt{\int_0^{\infty} tr\left(g(\tau)^T g(\tau)\right) d\tau}$$

$$tr\left(g(\tau)^T g(\tau)\right) = \sum_{ij} \left|g_{ij}(\tau)\right|^2 = \|g(\tau)\|_F^2$$

H_∞ *norm* for transfer functions(systems)

$$u \rightarrow \boxed{G(s)} \rightarrow y$$

Let G(s) is a stable ~~and strictly proper~~ transfer matrix G(s), (~~D=0 in state space realization~~). H_∞ norm is defined by:

$$\|G(s)\|_\infty = \max_{\substack{u(t)\neq 0 \\ \|u(t)\|_2<\infty}} \frac{\|y(t)\|_2}{\|u(t)\|_2} = \max_{\|u(t)\|_2=1} \|y(t)\|_2$$

It can be shown that:

$$\|G(s)\|_\infty = \max_{\omega} \overline{\sigma}(G(j\omega))$$

Remark: H_2 *norm* is not a matrix norm but H_∞ *norm* is a matrix norm.

Norm of transfer functions(systems)

Exercise 2-4: Derive H_2 of following systems with both formula.

$$g_1(s) = \frac{1}{s+2}$$

Ans: 0.5

$$g_2(s) = \frac{1}{\varepsilon s+1} \quad \varepsilon \to 0$$

Ans: inf

$$g_3(s) = \frac{\varepsilon s}{s^2+\varepsilon s+1} \quad \varepsilon \to 0$$

Ans: 0

$$G_4(s) = \begin{bmatrix} \dfrac{2}{s+10} \\ \dfrac{20}{s+1} \end{bmatrix}$$

Ans: around 20

Norm of transfer functions(systems)

Exercise 2-5: Derive H_∞ of following systems.

$$g_1(s) = \frac{2}{s+1}$$

Ans: 2

$$g_2(s) = \frac{1}{\varepsilon s+1} \quad \varepsilon \to 0$$

Ans: 1

$$g_3(s) = \frac{\varepsilon s}{s^2+\varepsilon s+1} \quad \varepsilon \to 0$$

Ans: 1

$$G_4(s) = \begin{bmatrix} \frac{2}{s+10} \\ \frac{20}{s+1} \end{bmatrix}$$

Ans:
around 14.14

Linear Algebra

- Vector Spaces, Norms
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- Unitary, Primitive, Hermitian and positive(negative) definite Matrices
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- Matrix Perturbation

Unitary and Hermitian Matrices

A matrix $U \in C^{n\times n}$ is unitary if

$$U^H U = I$$

A matrix $Q \in C^{n\times n}$ is Hermitian if

$$Q^H = Q$$

For real matrices Hermitian matrix means symmetric matrix.

Exercise 2-6: Show that for any matrix *V*, $V^H V$ and $V V^H$ are Hermitian matrix and their eigenvalues are real nonnegative.

Primitive Matrices

A matrix $A \in R^{n\times n}$ is nonnegative if its entries are nonnegative numbers.

A matrix $A \in R^{n\times n}$ is positive if all of its entries are strictly positive numbers.

Definition 2.1

A primitive matrix is a square nonnegative matrix where some power (positive integer) of it is positive.

Perron Theorem

Suppose A is a primitive matrix, with spectral radius $\lambda = \rho(A)$. Then λ is a simple root of the characteristic polynomial which is strictly greater than the modulus of any other root, and λ has strictly positive eigenvectors. (Note that Perron theorem is a necessarily condition)

λ is called **Perron-Frobenius eigenvalue** of A.

Primitive Matrices

$$A_1 = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$ with eigenvalues 2 and -1 is primitive.

$$A_2 = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$$ with eigenvalues -2 and 2 is not primitive.

$$A_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$ with eigenvalues 1 and 1 is not primitive.

$$A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$ with eigenvalues 4 and 0 is not primitive.

Positive (Negative) Definite Matrices

A matrix $Q \in C^{n\times n}$ is positive definite if for any $x \in C^n, x \neq 0$

$x^H Qx$ is real and positive

A matrix $Q \in C^{n\times n}$ is negative definite if for any $x \in C^n, x \neq 0$

$x^H Qx$ is real and negative

A matrix $Q \in C^{n\times n}$ is positive semi definite if for any $x \in C^n, x \neq 0$

$x^H Qx$ is real and nonnegative

Negative semi definite is defined similarly

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Relative Gain Array (RGA)

The relative gain array (RGA), was introduced by Bristol (1966).

For a square matrix A

$$RGA(A) = \Lambda(A) = A \times (A^{-1})^T$$

For a non square matrix A

$$RGA(A) = \Lambda(A) = A \times (A^{\dagger})^T$$

Linear Algebra

- Vector Spaces, Norms
- Unitary, Primitive, Hermitian and positive(negative) definite Matrices
- Inner Product
- Singular Value Decomposition (SVD)
- Relative Gain Array (RGA)
- Matrix Perturbation

Matrix Perturbation

1- Additive Perturbation

2- Multiplicative Perturbation

3- Element by Element Perturbation

Additive Perturbation

Theorem 2-2

Suppose $A \in C^{m\times n}$ has full column rank (n). Then

$$\min_{\Delta \in C^{m\times n}} \left\{ \|\Delta\|_2 \mid rank(A+\Delta) < n \right\} = \sigma_n(A) = \underline{\sigma}(A)$$

$$\underline{\sigma}(A) \uparrow$$

Additive Perturbation

Example 2-2

$$A = \begin{bmatrix} 100 & 100 \\ 100.2 & 100 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -5 & 5 \\ 5.01 & -5 \end{bmatrix}$$

$$\Delta A = \begin{bmatrix} 0.04995 & -0.05 \\ -0.05 & 0.04995 \end{bmatrix} \qquad (A + \Delta A)^{-1} = ?$$

$$\underline{\sigma}(A) = 0.1 = \bar{\sigma}(\Delta A)$$

$$\Delta A = \begin{bmatrix} 0 & 0 \\ -0.1 & 0 \end{bmatrix} \qquad (A + \Delta A)^{-1} = \begin{bmatrix} -10 & 10 \\ 10.01 & -10 \end{bmatrix}$$

$$A + \Delta A = \begin{bmatrix} 100 & 100 \\ 100.1 & 100 \end{bmatrix} \qquad \Delta(A^{-1}) = \begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix}$$

Multiplicative Perturbation

Theorem 2-3

Suppose $A \in C^{n\times n}$. Then

$$\min_{\Delta \in C^{n\times n}} \left\{ \|\Delta\|_2 \;|\; rank(I - A\Delta) < n \right\} = \frac{1}{\bar{\sigma}(A)}$$

$\bar{\sigma}(A) \;\downarrow$

Element by element Perturbation

Theorem 2-4: Suppose $A \in C^{n \times n}$ is non-singular and suppose λ_{ij} is the ijth element of the RGA of *A*.

The matrix *A* will be singular if ijth element of *A* perturbed by

$$a_{ijp} = a_{ij}(1 - \frac{1}{\lambda_{ij}})$$

Element by element Perturbation

Example 2-3

$$A = \begin{bmatrix} 100 & 100 \\ 100.2 & 100 \end{bmatrix} \qquad \Lambda(A) = \begin{bmatrix} -500 & 501 \\ 501 & -500 \end{bmatrix}$$

Now according to mentioned theorem if a_{11} multiplied by $(1-\frac{1}{\lambda_{11}}) = 1.002$

then the perturbed A is singular or

$$A_P = \begin{bmatrix} 100*1.002 & 100 \\ 100.2 & 100 \end{bmatrix} = \begin{bmatrix} 100.2 & 100 \\ 100.2 & 100 \end{bmatrix}$$

Exercises

2-1 till 2-6 Mentioned in the lecture.

2-7 The spectral radius of a matrix is: $\rho(A) = \max_i |\lambda_i|$

where λ_i is the eigenvalue of A. Show that the spectral radius is not a norm.

2-8 Suppose A is Hermitian. Find the exact relation between the eigenvalues and singular values of A. Does this hold if A is not Hermitian?

2-9 Verify that if Q is Hermitian then its eigenvalues are real.

2-10 Show that Frobnius norm can be derived by $\sqrt{tr(A^H A)}$

2-11 Show that if rank(A)=1, then, $\|A\|_F = \|A\|_2$

.

Exercises

2-12 Suppose

$$A = \begin{bmatrix} 3 & 4 & 7 \\ -2 & 7 & 5 \end{bmatrix}$$

a) Find SVD of A and then by use of SVD:

b) Find the null space of A.

c) Find the range space of A.

d) If $\|x\|_2 = 2.75$ what is the maximum and minimum of $\|Ax\|_2$

2-13 Find a non primitive matrix such that its spectral radius is a simple root of the characteristic polynomial and its spectral radius is strictly greater than the modulus of any other eigenvalues.

Exercises

2-14 Consider following matrix.(Final)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

One of the SVD matrices of A is:

$$\begin{bmatrix} 0.3620 & -0.8562 & 0.3685 \\ 0.4866 & 0.5107 & 0.7088 \\ 0.7951 & 0.0773 & -0.6015 \end{bmatrix}$$

a) Derive induced norm of A (p=2).

b) Derive least gain of A and corresponding input and output direction.

c) Derive nullity and rank of A.

d) Derive unreachable output direction.

e) Suppose rank of A+B is 2. Derive minimum of $\|B\|_2$.

2-15 Show that any induced norm is a matrix norm(just PhD students).

References

References

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