Graph Theory

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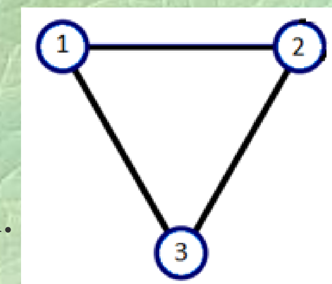
Introduction to Graph Theory

A **graph** G is a combination of vertices (nodes), and edges (*link* or *line*).

A graph G consists of two sets. 1- Set of vertices, V. and 2- Set of unordered edges, *E*.

e.g. V={1, 2, 3}, E={{1, 2}, {2, 3}, {3, 1}})

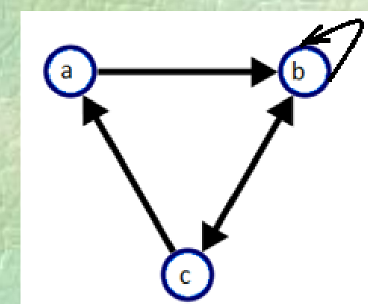


The degree of a vertex u, denoted deg(u), is the number of vertices adjacent to u.

A graph is regular if all vertices have the same degree.

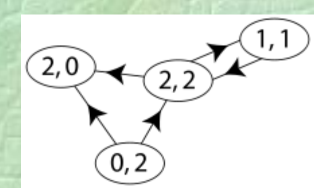
A 2-regular graph

A **directed graph** or **digraph** is a graph in which edges have orientations.
A digraph is a pair G = (V, E), with 1- Set of vertices, V. and 2- Set of ordered pairs of edges, *E*.



e.g. *G* = (V, E) = ({a, b, c}, {(a, b), (b, c), (c, b), (c, a), (b, b)})

E ⊆ V×V order/size of graph is the cardinality of V/E.



The indegree/outdegree of a vertex u is in/out-neighbors.

A digraph is topologically balanced if each vertex has the same indegree and outdegree.

$$\sum_{v\in V} \deg^-(v) = \sum_{v\in V} \deg^+(v) = |E|.$$

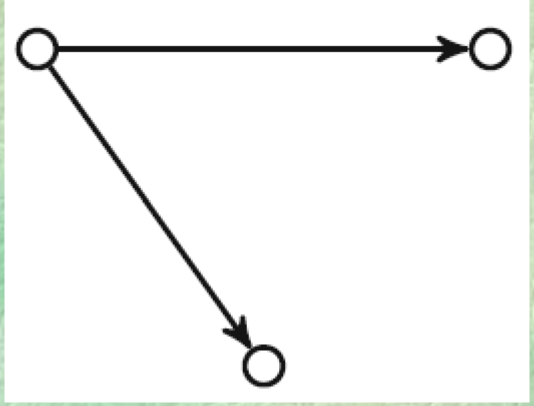
Introduction to Graph Theory

A digraph G is strongly connected if there is a directed path from any vertex to every other vertex. A digraph G is weakly connected if its undirected version is connected.

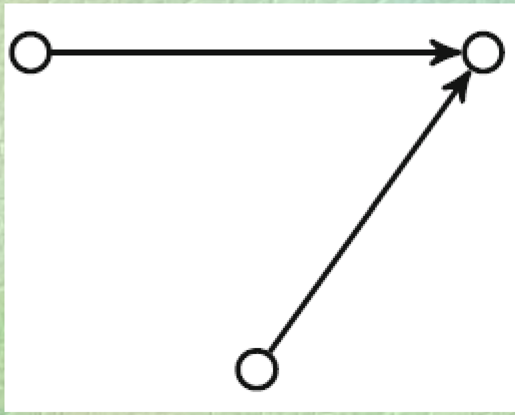
A cycle in a digraph is a simple directed path that its start and end vertices are the same.

A forest or acyclic digraph is a digraph with no cycle.

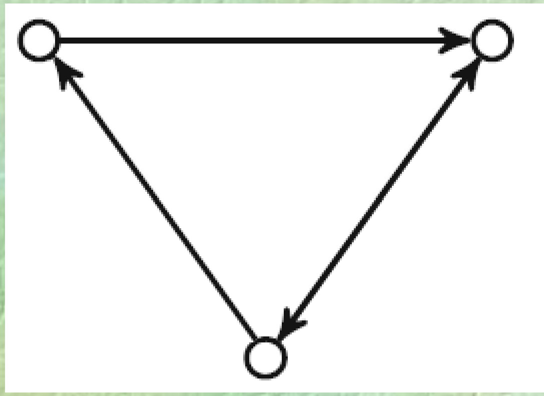
A vertex is globally reachable vertex if it can be reached from any other vertex in a directed way.



Acyclic weakly connected graph without a globally reachable node.



Acyclic weakly connected graph with a globally reachable node.



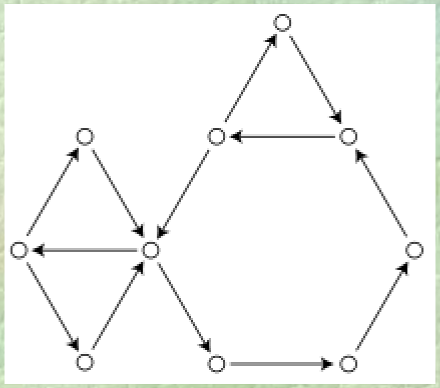
Strongly connected graph with cycle and globally reachable node.

Remark: Every acyclic digraph(cycle-free digraph) has at least one source and at least one sink.

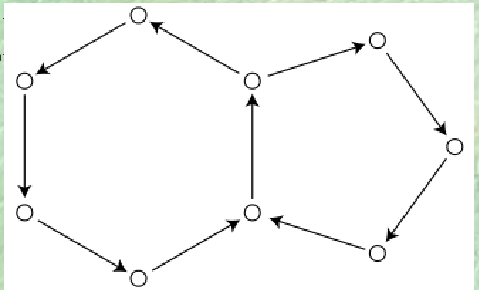
Introduction to Graph Theory

A directed graph is said to be aperiodic if the greatest common divisor of the lengths of its cycles (period of G) is one. Otherwise it is periodic.

A periodic strongly connected digraph with period 3.



An aperiodic strong connected digraph. Since, gcd(5,6)=1.

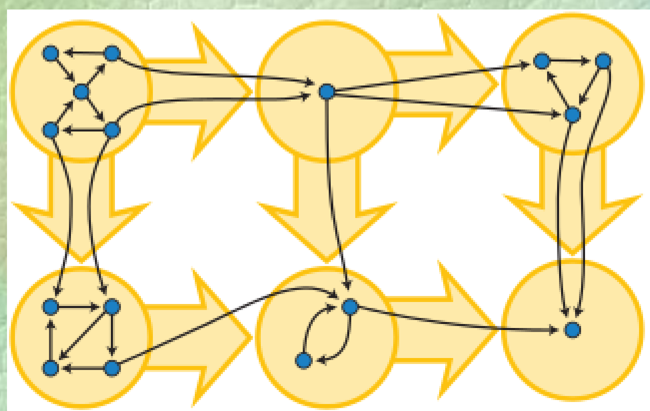


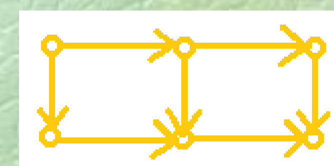
Remark: Periodic and aperiodic graphs has a very interesting property in averaging systems.

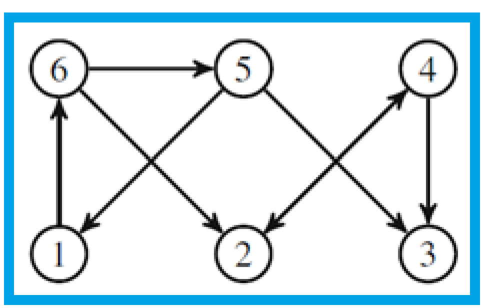
Introduction to Graph Theory

If each strongly connected component is contracted to a single vertex, the resulting graph is the **condensation** of G.

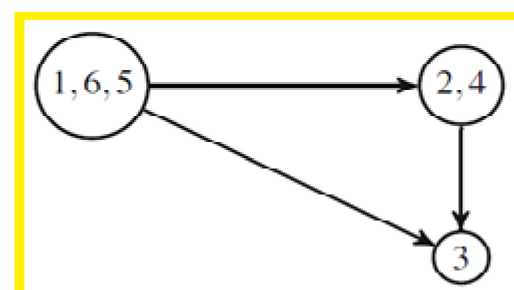
Blue one is a directed graph and yellow one is its condensation.



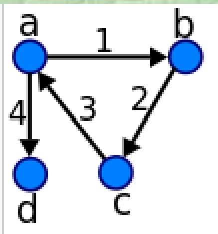




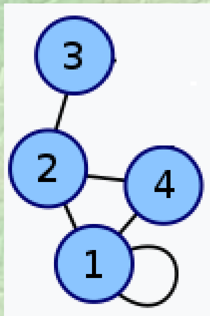
condensation



Weighted directed graphs (directed networks) are directed graphs with weights assigned to their edges, similarly, weighted graphs (weighted networks).



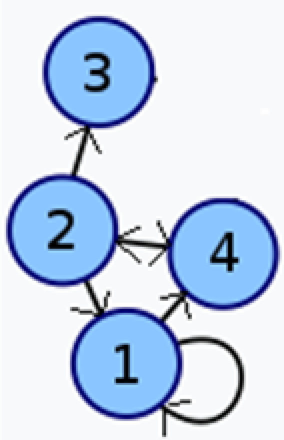
Degree matrix for a graph G=(V, E) is a diagonal matrix, its diagonal elements are $d_{ii}=\deg(v_i)$



$$D=\begin{bmatrix}3&0&0&0\\0&3&0&0\\0&0&1&0\\0&0&0&2\end{bmatrix}$$

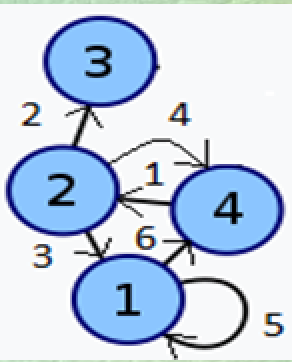
Degree, indegree, outdegree matrices and weighted versions

Indegree/outdegree matrix for a digraph G=(V, E) is a diagonal matrix, its diagonal elements are the number of incoming/outcoming edges at each vertex.



$$D_{out} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad D_{in} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Weighted outdegree/indegree matrix for a graph G=(V, E) is a diagonal matrix, its diagonal elements are sum of out/in weight connected to that edge.

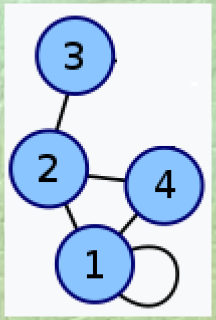


$$D_{out} = \begin{bmatrix} 11 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad D_{in} = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

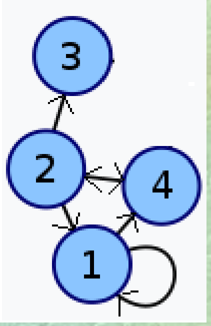
Weighted-Balanced digraph: A digraph is weighted-balanced(topologically balanced) if $D_{out}=D_{in}$

Binary adjacency and weighted adjacency matrices

Binary Adjacency Matrix is a square matrix used to represent a graph. The elements of the matrix indicate whether pairs of vertices are adjacent or not in the graph.



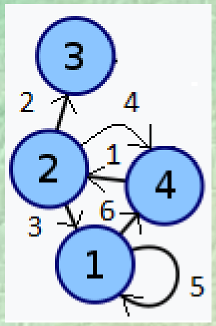
$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$



$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Remark: Adjacency matrix of an undirected graph is symmetric.

Weighted Adjacency Matrix is a square matrix used to represent a graph. The elements of the matrix indicate whether pairs of vertices are adjacent or not in the graph and also the weight of it.



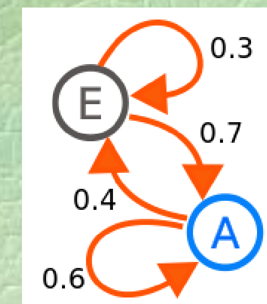
$$A = \begin{bmatrix} 5 & 0 & 0 & 6 \\ 3 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Remark: Sum of i^{th} column is equal to D_{inii} and sum of i^{th} row is equal to D_{outii}. Or similarly,

$$D_{out} = diag(A\mathbb{1}_n\ ), \qquad Din = diag(A^T\mathbb{1}_n)$$

Stochastic matrix

Stochastic Matrix: A Markov chain is a stochastic model describing a sequence of possible events in which the probability of each event depends only on the state attained in the previous event.

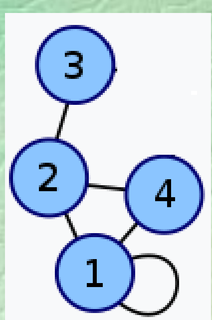


If the Markov process is in state A, then the probability it changes to state E is 0.4, while the probability it remains in state A is 0.6.

Weighted Adjacency Matrix of this system is: $A = \begin{bmatrix} 0.6 & 0.4 \\ 0.7 & 0.3 \end{bmatrix}$

This is a row stochastic matrix since sum of every row is 1.

How to change an adjacency matrix to row stochastic matrix(in strongly connected graphs)?



$$D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$A_{row\ \_\ stochastic} = D^{-1}A = \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}$$

Row stochastic matrix property

Lemma on row stochastic matrices.

Consider A is a row stochastic matrix.
1- A is non-negative.
2- One of eigenvalues of A, and its spectral radius is one.

Proof: By Gersgorin theorem.

Remark: Clearly every row stochastic matrix is not convergent, but it may be semi-convergent.

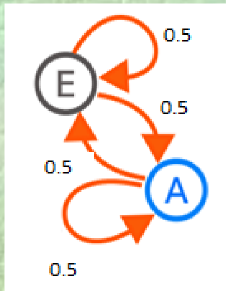
Proof: By Jordan representation.

Stochastic matrix

Stochastic Matrix: A Markov chain is a stochastic model describing a sequence of possible events in which the probability of each event depends only on the state attained in the previous event.

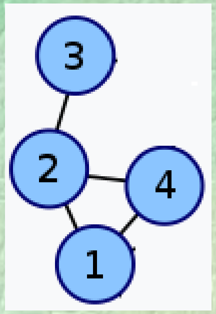
If both rows sum and column sum is equal to one we have double stochastic matrix(weight-balanced).

Weighted Adjacency Matrix of this system is: $A = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$



This is a double stochastic matrix since sum of every row and every column is 1.

How to change an adjacency matrix to double stochastic matrix(in strongly connected graphs)?



$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

1- For each <u>edge</u> try to find a_{ij} by: $a_{ij}=1/(1+\max(d_i,d_j))$

2- Other $a_{ij}=0$

3- Other a_{ii} such that it be double stochastic.

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$A_{double\_stochastic} = \begin{bmatrix} 5/12 & 1/4 & 0 & 1/3 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/4 & 3/4 & 0 \\ 1/3 & 1/4 & 0 & 5/12 \end{bmatrix}$$

Non-negative, irreducible, primitive and positive matrices

Non-negative matrices, $a_{ij} \geq 0$

$$A_1 = \begin{bmatrix} 0 & 0 \\ 5 & 0 \end{bmatrix}$$

irreducible matrices, $\sum_{k=0}^{n-1} A^k > 0$

$$A_2 = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$$

Primitive matrices, $A^k > 0$

$$A_3 = \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}$$

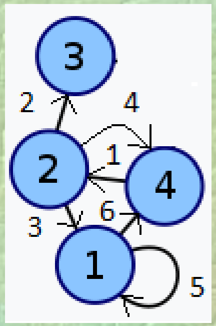
Positive matrices, $a_{ij} > 0$

$$A_4 = \begin{bmatrix} 2 & 2 \\ 3 & 2 \end{bmatrix}$$

Adjacency matrices and reducibility

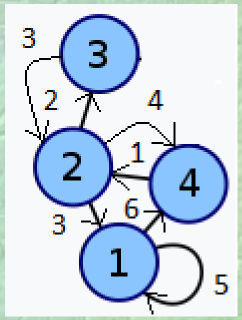
Theorem: A weighted digraph is strongly connected if and only if its adjacency matrix A is irreducible(irreducible means $\sum_{k=0}^{n-1} A^k > 0$).

Example: Following graph is not strongly connected so A is reducible.



$$A = \begin{bmatrix} 5 & 0 & 0 & 6 \\ 3 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Example: Following graph is strongly connected so A is irreducible.



$$A = \begin{bmatrix} 5 & 0 & 0 & 6 \\ 3 & 0 & 2 & 4 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Adjacency matrices and reducibility

Theorem: (Perron–FrobeniusTheorem). Let A is non-negative(n≥2), then

1- One of the eigenvalues of A is its spectral radius($\lambda = \rho(A)$), and its right and left eigenvector can be selected non-negative.

2- If A is irreducible then $\lambda = \rho(A)$ is strictly positive and simple and the right and left eigenvectors of λ can be selected positive.

$$A_2 = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$$

3- If A is primitive then $\lambda = \rho(A)$ is strictly positive and simple and larger than other eigenvalues.

$$A_4 = \begin{bmatrix} 2 & 2 \\ 3 & 2 \end{bmatrix}$$

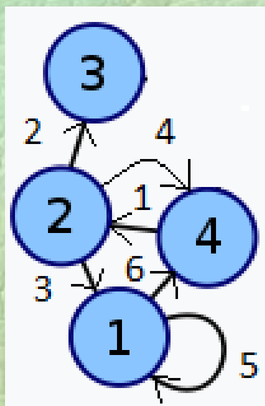
Remark: If A is row stochastic then in case 3 it is semi convergent.

Remark: If λ is simple and strictly larger, in magnitude, than all other eigenvalues, then A/λ is semi-convergent and

$$\lim_{k\to\infty}\left(\frac{A^k}{\lambda^k}\right) = vw^T$$

Laplacian matrix

Laplacian matrix is the difference of outdegree matrix and adjacency matrix



$$D_{out}=\begin{bmatrix}11&0&0&0\\0&9&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\quad A=\begin{bmatrix}5&0&0&6\\3&0&2&4\\0&0&0&0\\0&1&0&0\end{bmatrix}\quad L=\begin{bmatrix}6&0&0&-6\\-3&9&-2&-4\\0&0&0&0\\0&-1&0&1\end{bmatrix}$$

Remark: Laplacian matrix of an undirected graph is symmetric.

Remark: Laplacian matrix is said to be irreducible if G is strongly connected.

Remark: Row sum of Laplacian matrix is equal to zero. So, $L.\mathbf{1}_n=0$. (0 is eigenvalue and $\mathbf{1}_n$ is eigenvector)

Remark: If G is weight-balanced then column sum of Laplacian matrix is equal to zero.($\mathbf{1}_n^T.L=0$)

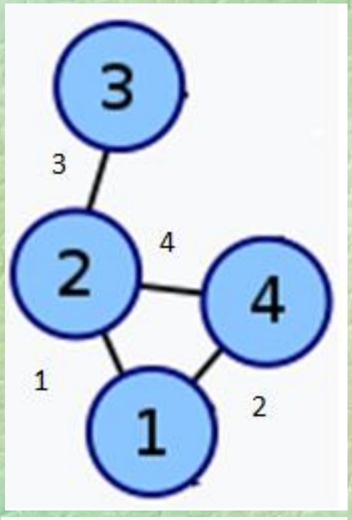
Remark: Laplacian matrix has zero eigenvalue and other eigenvalues are strictly-positive real part. (Gersgorin property)

Remark: Following statements are equivalent for a weighted graph with *n* vertex.

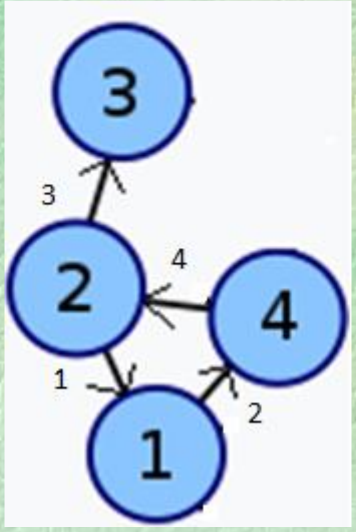
G contains globally reachable node ≈ the 0 eigenvalue of L is simple ≈ Rank(L) = n-1

Incidence matrix

For a graph/digraph with n vertex and m edge the incidence matrix is an $n \times m$ matrices with 0 and 1 (for graphs) and 0, 1, and -1 (for digraph)



$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$



$$B = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

References

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[2] "Introduction to Averaging Dynamics over Networks" Fagnani, Fabio ; Frasca, Paolo, Springer, 2018

[3] "Graph theory" V.K.Balakrishnan, Mc Graw-Hill 1997