LINEAR CONTROL SYSTEMS

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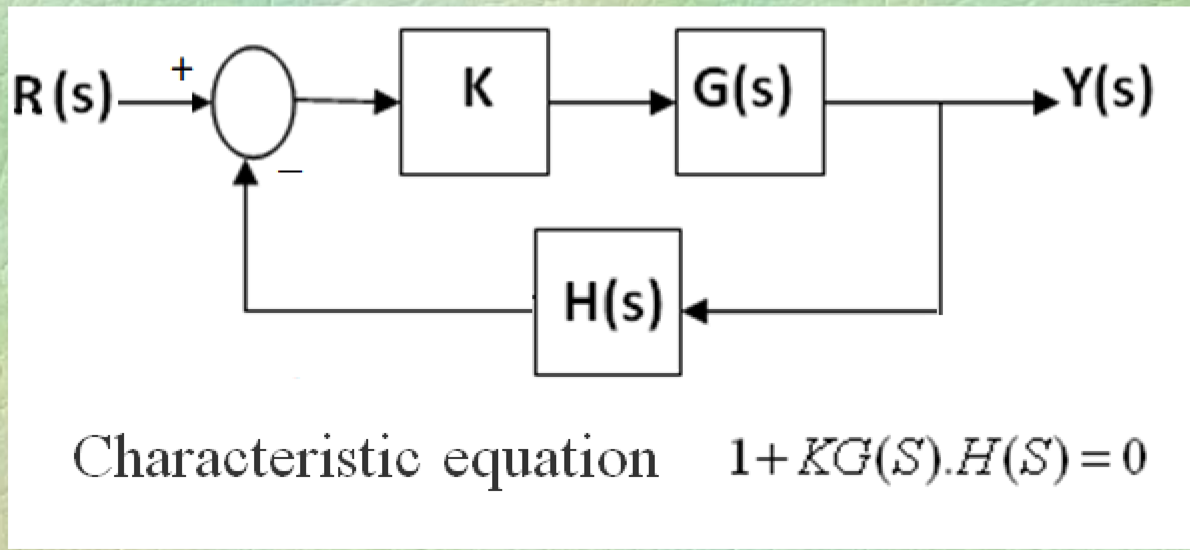
Lecture 6

Root Locus Criteria

Topics to be covered include:

- Root locus criterion.
  - Root loci (RL).
  - Complement root loci (CRL).
  - Complete root loci.

Root locus



$$KG(S).H(S) = -1 \Rightarrow \left|KG(S).H(S)\right| \angle KG(S).H(S) = 1\angle \pm 180(2h+1)$$

Magnitude and angle conditions for drawing the root locus

$$\begin{cases} \left|KG(S).H(S)\right| = 1 \\ \angle KG(S).H(S) = \pm 180(2h+1) \end{cases}$$

Root locus

Root locus, shows the position of roots of the following equation for different values of *k*

$$1+kf(s)=0$$

The highest degree of the numerator and denominator must have the same sign.

Root loci (RL) $k \in R^{+}$

Complement root loci (CRL) $k \in R^{-}$

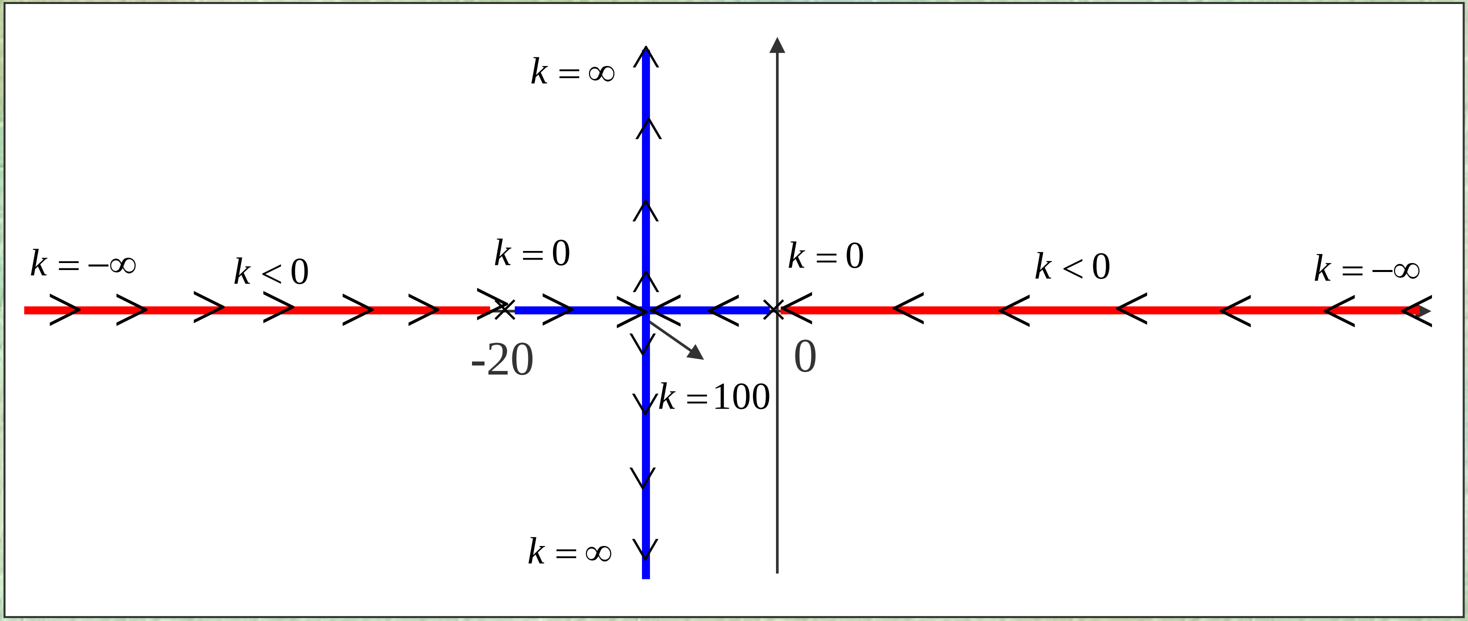
Complete root loci $k \in R$

Root locus

Example 1: Find the complete root locus of the following system.

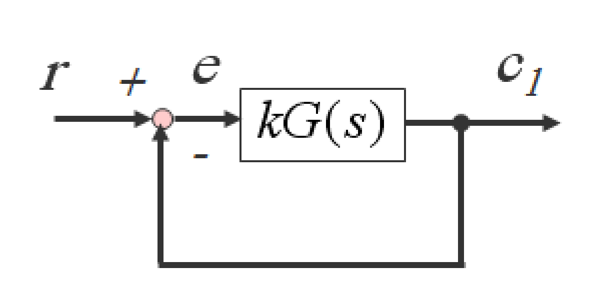
$$1 + k\frac{1}{s(s+20)} = 0$$

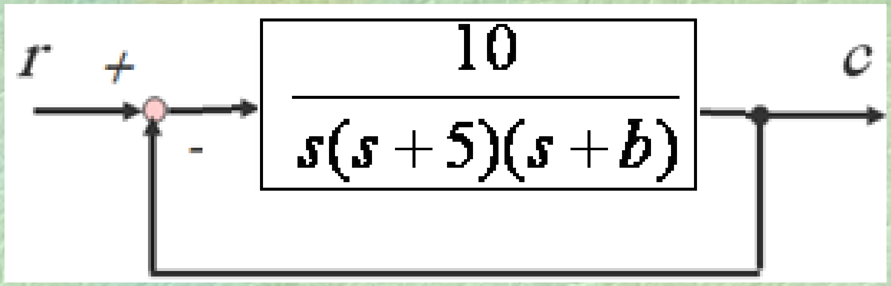
$$s^2 + 20s + k = 0 \quad \Rightarrow \quad s = -10 \pm \sqrt{100 - k}$$

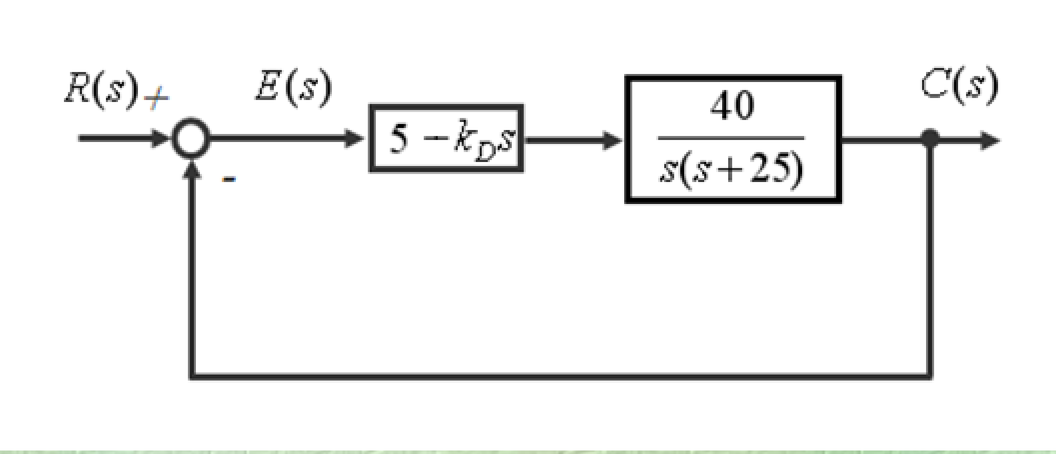


Root locus

Why is plotting the root locus important to us?





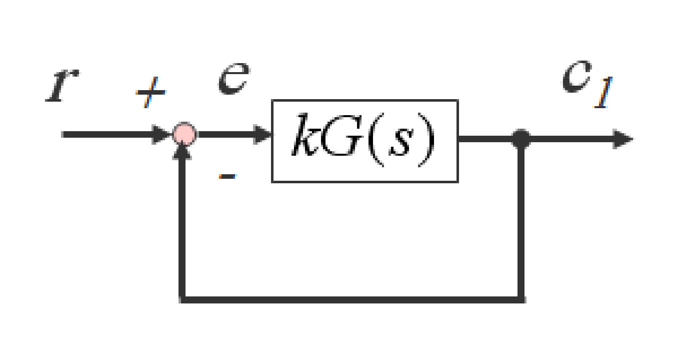


How to plot the root locus $1+kf(s)=0$

1- Standardization: Express the system's characteristic equation exactly in the following form(The highest degrees of the numerator and denominator should have the same sign.):

$$1+kf(s)=0$$

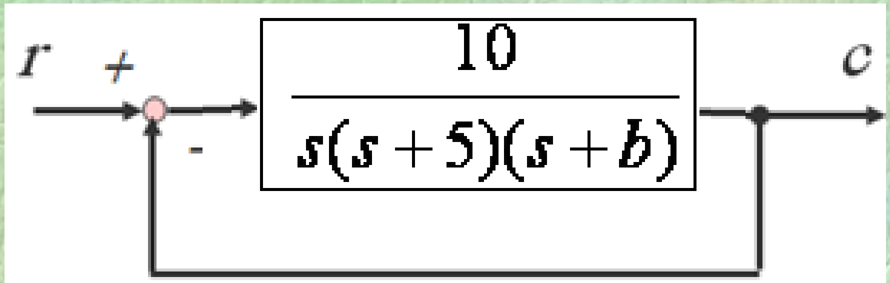
Example 2: Standardize the characteristic equation of the following system:



$$T(s)=\frac{kG(s)}{1+kG(s)} \Rightarrow \Delta(s)=1+kG(s)=0$$

How to plot the root locus $1+kf(s)=0$

Example 3: Standardize the characteristic equation of the following system:



$$T(s)=\frac{\frac{10}{s(s+5)(s+b)}}{1+\frac{10}{s(s+5)(s+b)}}$$

$$\Delta(s)=1+\frac{10}{s(s+5)(s+b)}=0$$

$$s(s+5)(s+b)+10=0$$

$$s^2(s+5)+bs(s+5)+10=0$$

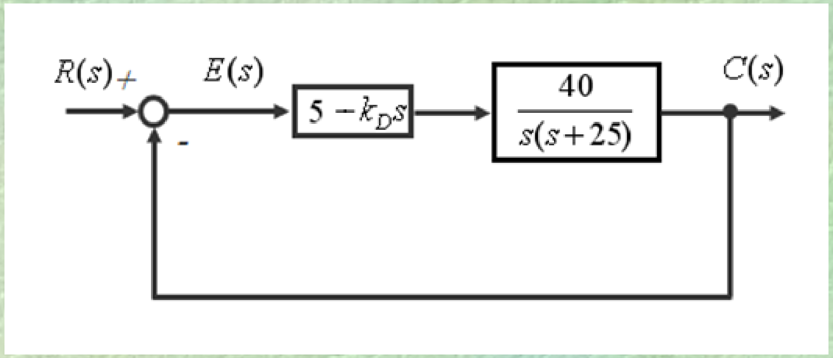
$$s^2(s+5)+10+bs(s+5)=0$$

$$\frac{s^2(s+5)+10}{s^2(s+5)+10}+\frac{bs(s+5)}{s^2(s+5)+10}=0$$

$$1+b\frac{s(s+5)}{s^2(s+5)+10}=0$$

How to plot the root locus $1+kf(s)=0$

Exercise 1: Standardize the characteristic equation of the following system:

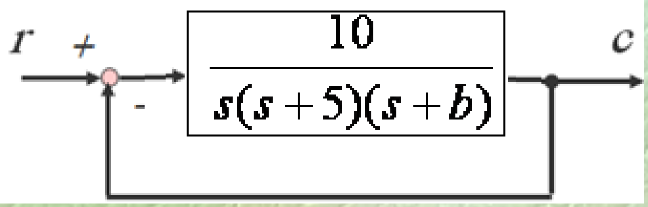


$$1 + k_D \frac{-40s}{s(s+25)+200} = 0 \qquad 1 + (-k_D)\frac{40s}{s(s+25)+200} = 0$$

How to plot the root locus $1+kf(s)=0$

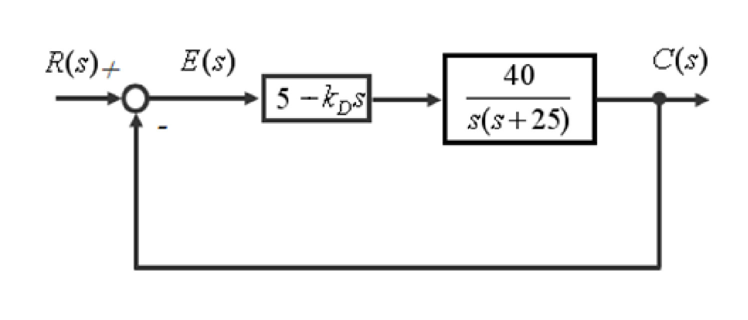
2- Determine the number of branches and the locations of poles and zeros:

Identify the poles and zeros of *f*(*s*). The number of branches is equal to the degree of the characteristic equation.



$$1+b\frac{s(s+5)}{s^2(s+5)+10}=0$$

Poles, zeros, number of branches?



$$1+(-k_D)\frac{40s}{s(s+25)+200}=0$$

Poles, zeros, number of branches?

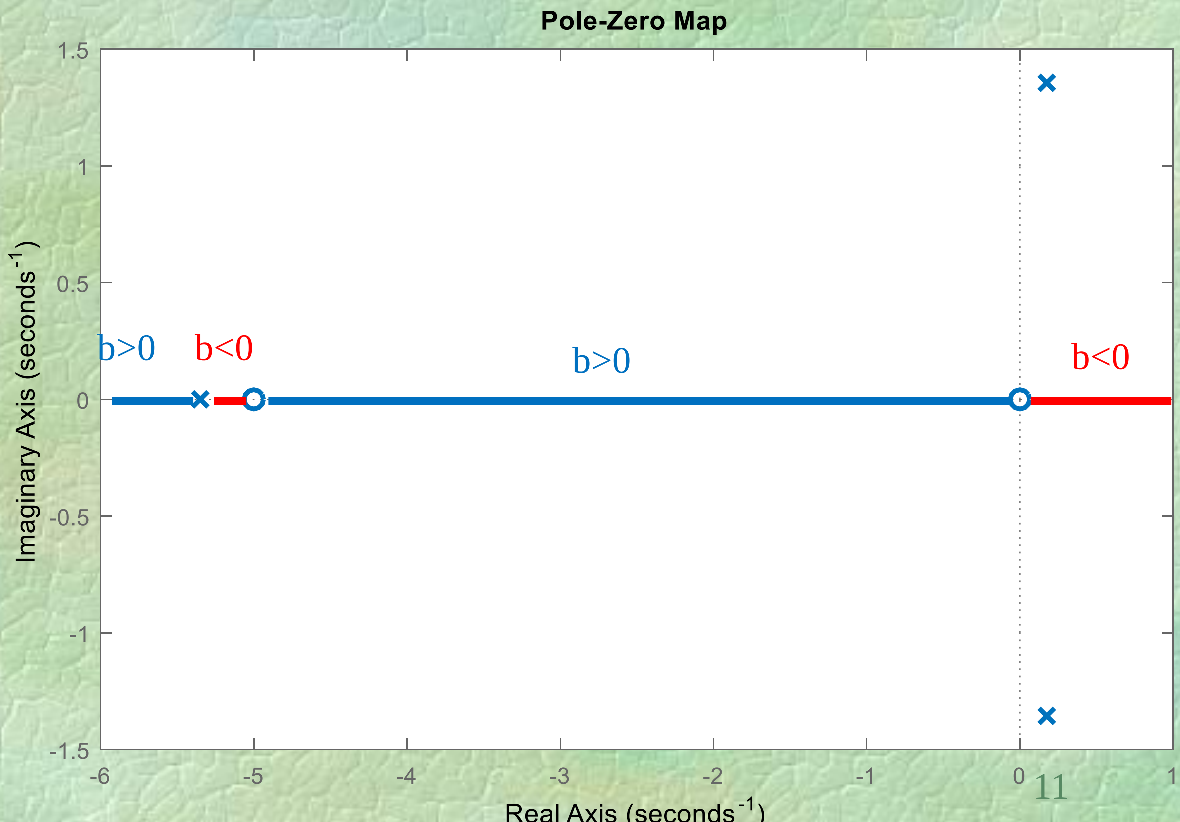
$$1+k\frac{s^2}{s+200}=0$$

Poles, zeros, number of branches?

How to plot the root locus $1+kf(s)=0$

3- Geometric locus on the real axis: Move from the right on the real axis with $k<0$, and after passing each pole or zero on the real axis, change the sign of k. The entire real axis is part of the locus (for $k>0$) or its complement (for $k<0$).

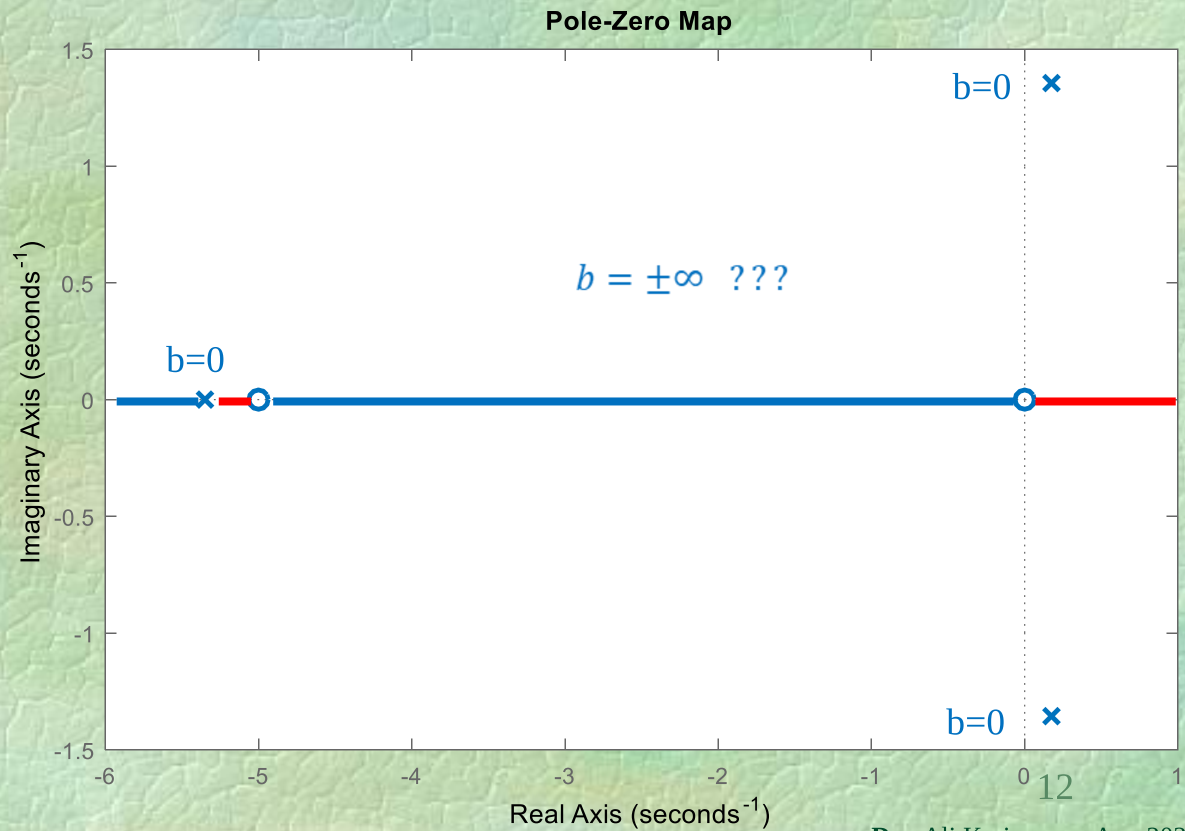
$$1+b\frac{s(s+5)}{s^2(s+5)+10}=0$$



How to plot the root locus $1+kf(s)=0$

4- Determine the starting and ending points of the root locus branches: The branches of the root locus start at the open-loop poles and end at the open-loop zeros.

$$1+b\frac{s(s+5)}{s^2(s+5)+10}=0$$



How to plot the root locus $1+kf(s)=0$

5- Number of asymptotes, angles of asymptotes, and intersection points of asymptotes with the real axis

Number of asymptotes: n_p-n_z

$$\begin{cases} k>0 \quad \theta = \dfrac{(2m+1)\pi}{|n_p-n_z|} \qquad m=0,1,2,\ldots \\ k<0 \quad \theta = \dfrac{2m\pi}{|n_p-n_z|} \qquad m=0,1,2,\ldots \end{cases}$$

The angle of asymptotes with respect to the positive real axis

Asymptotes center

$$\delta = \frac{\sum_{i=1}^{n_p} p_i - \sum_{i=1}^{n_z} z_i}{n_p-n_z}$$

The intersection point of asymptotes with the real axis

How to plot the root locus $1 + kf(s) = 0$

Example 4: Find the complete root locus for the following system.

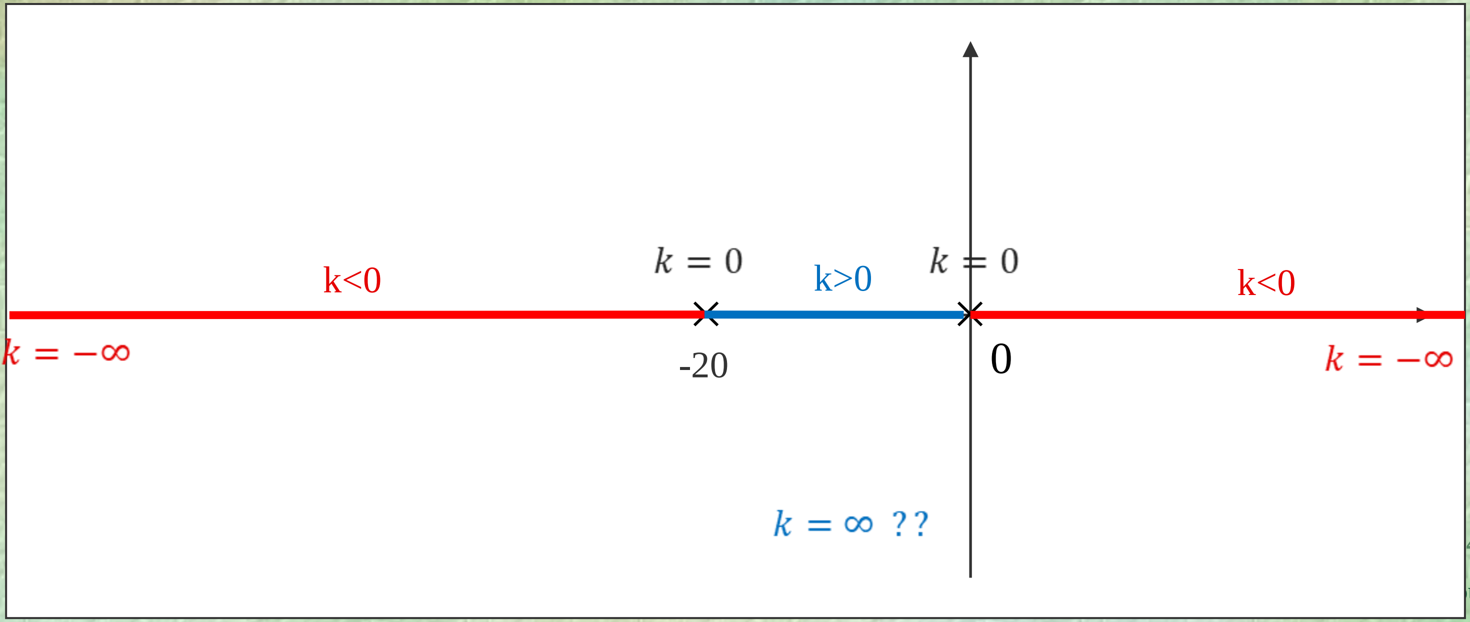
$$1 + k\frac{1}{s(s+20)} = 0$$

1- Standardization:

2- Determine the number of branches, poles and zeros:

3- Geometric locus on the real axis:

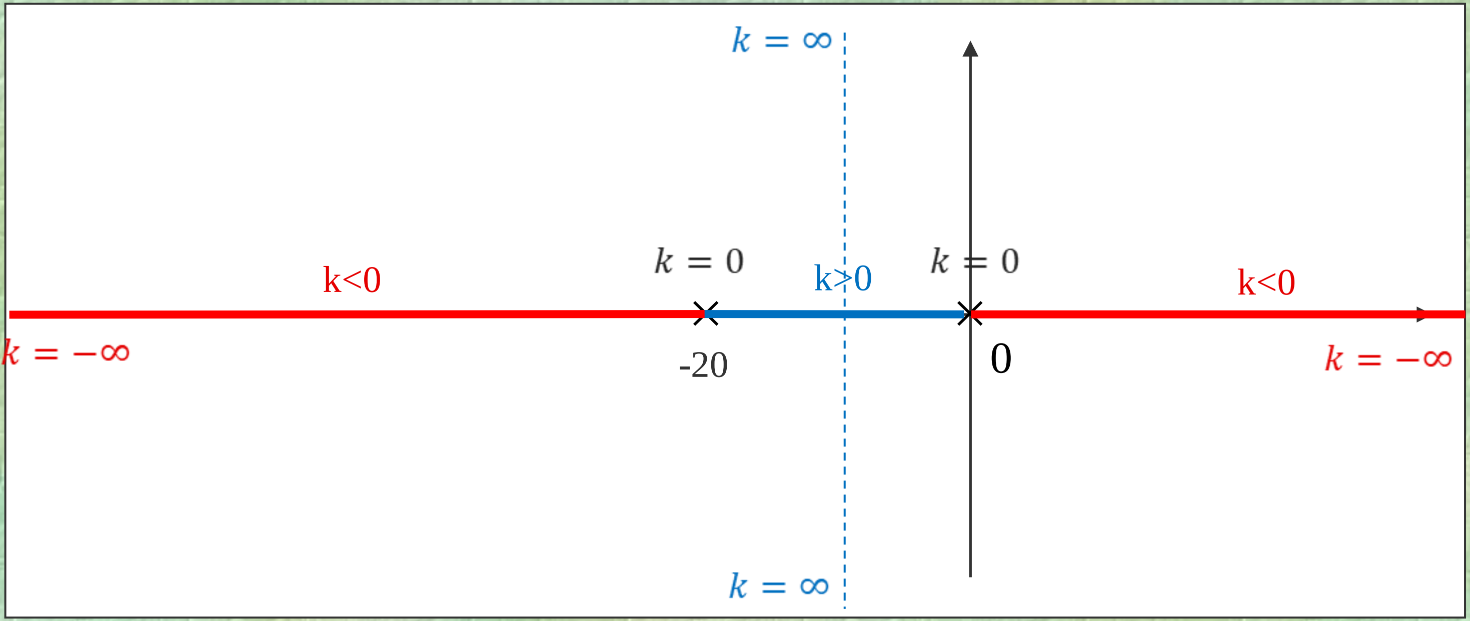
4- Determine the starting and ending points of the root locus branches:



How to plot the root locus $1+kf(s)=0$

Example 4: Find the complete root locus for the following system.

$$1+k\frac{1}{s(s+20)}=0$$



5- Number, angles, and intersection points of asymptotes with the real axis:

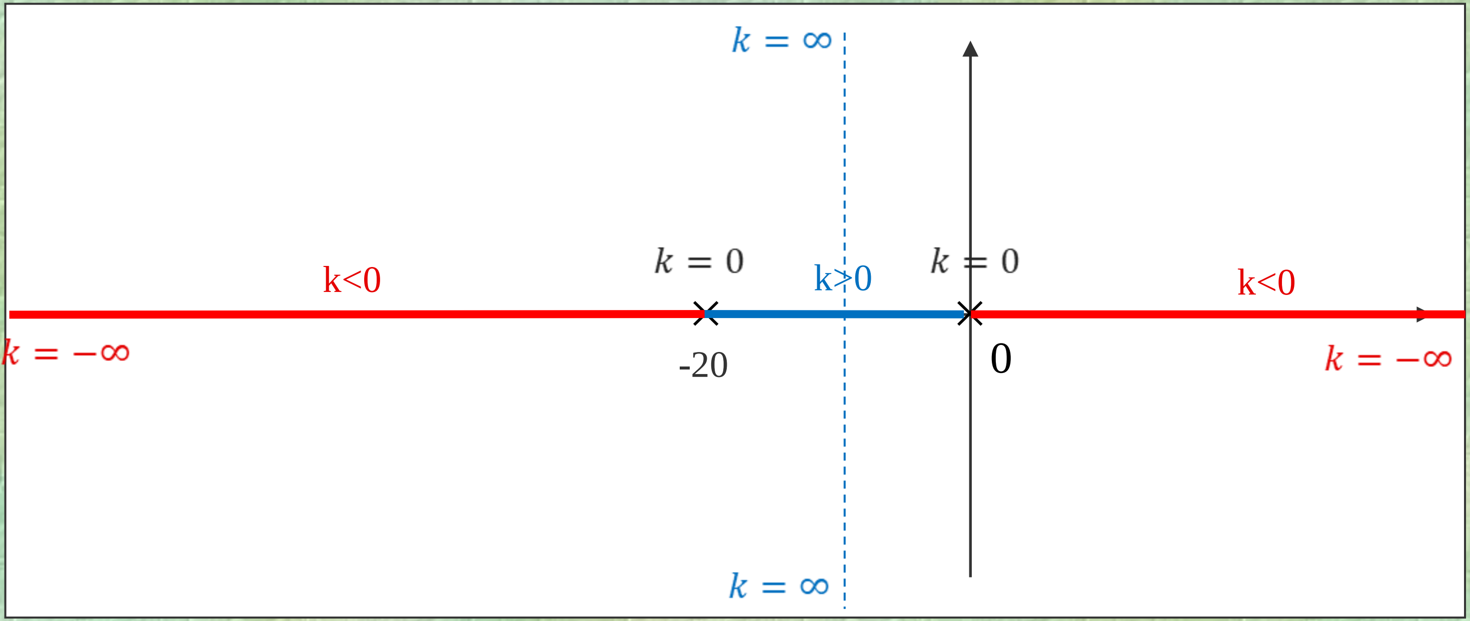
Asymptotes center

$$\delta=\frac{\sum_{i=1}^{n_p}p_i-\sum_{i=1}^{n_z}z_i}{n_p-n_z}=\frac{-20-0}{2}=-10$$

$$\begin{cases} k>0 \quad \theta=\dfrac{(2m+1)\pi}{|n_p-n_z|}=\dfrac{\pi}{2},-\dfrac{\pi}{2} \\ k<0 \quad \theta=\dfrac{2m\pi}{|n_p-n_z|}=0,\pi \end{cases}$$

How to plot the root locus $1+kf(s)=0$

Example 4: Find the complete root locus for the following system. $1+k\frac{1}{s(s+20)}=0$



We need a new criterion.

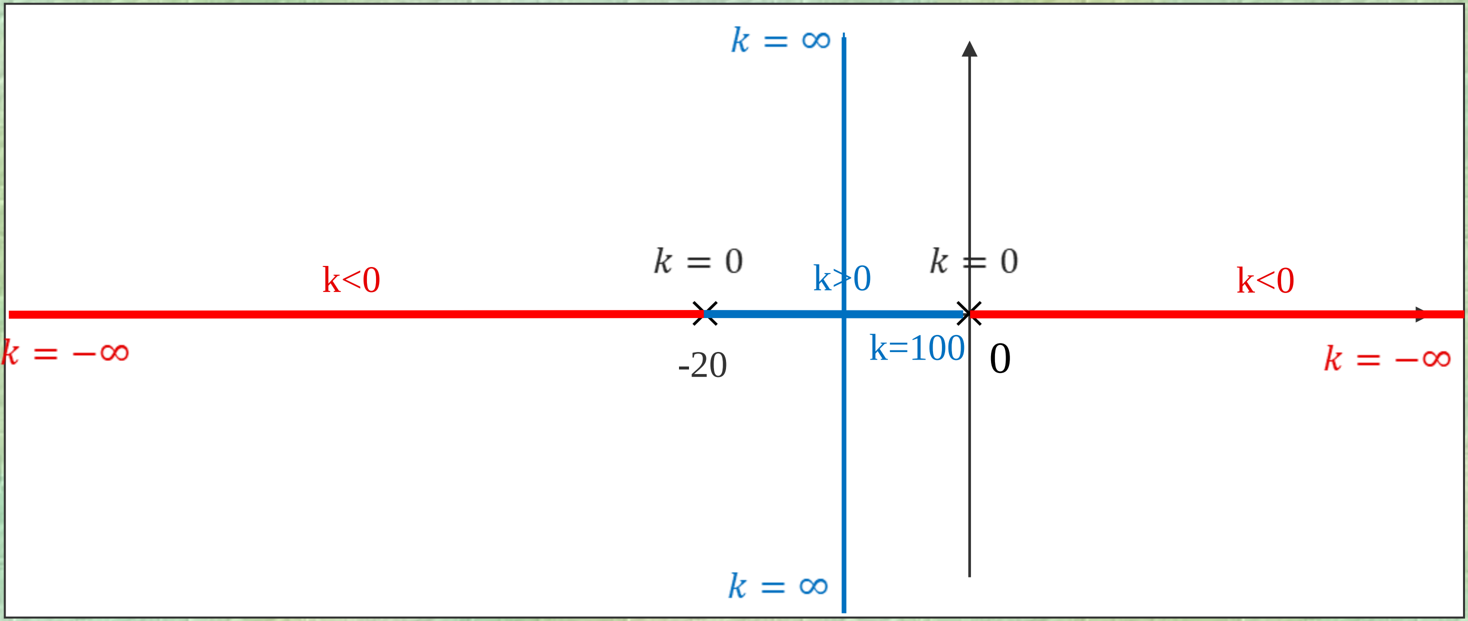
6- Calculation of breakaway or break-in points:

$$k=-\frac{1}{f(s)} \qquad \frac{\partial k}{\partial s}=0 \qquad s=\text{break point}$$

How to plot the root locus $1 + kf(s) = 0$

Example 4: Find the complete root locus for the following system.

$$1 + k\frac{1}{s(s+20)} = 0$$



6- Calculation of breakaway or break-in points:

$$k = -\frac{1}{f(s)} \qquad \frac{\partial k}{\partial s} = 0 \qquad s = \text{break point}$$

$$k = -s(s+2) \qquad \frac{\partial k}{\partial s} = -2s - 20 = 0 \qquad s = -10 \qquad k = 100$$

How to plot the root locus $1 + kf(s) = 0$

Example 5: Find the complete root locus for the given system.

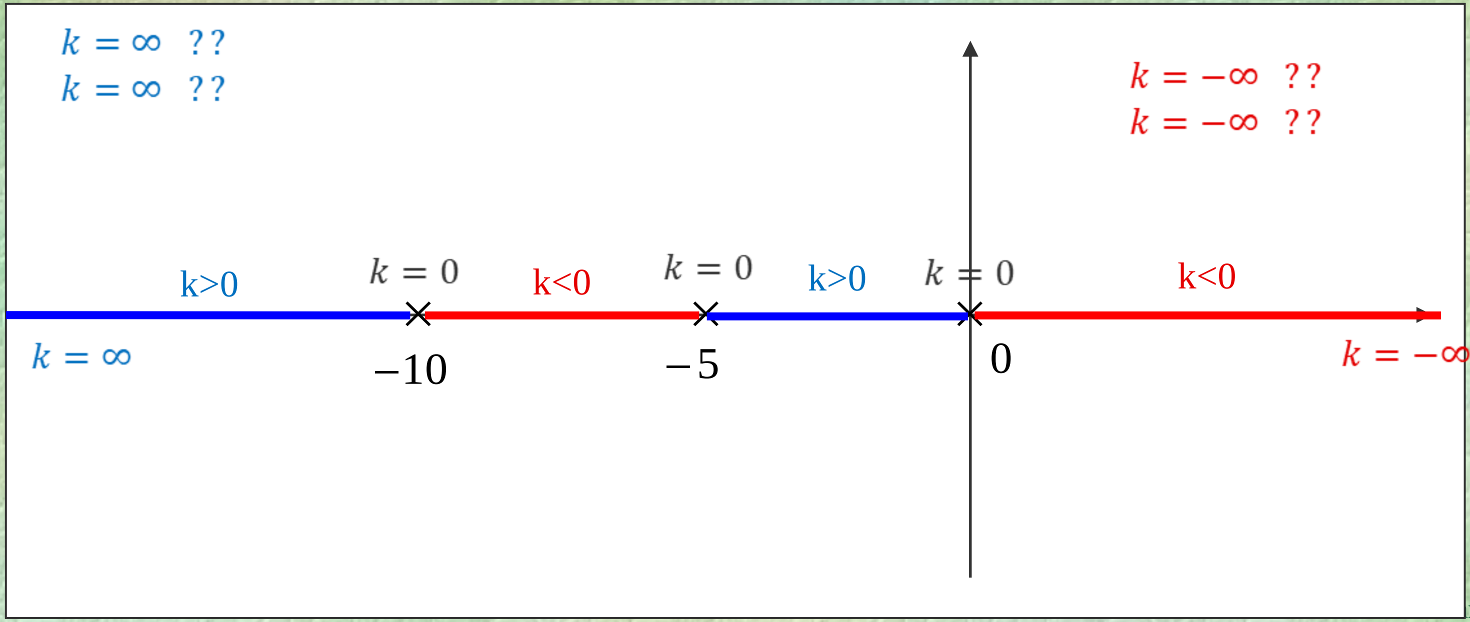
$$1 + k\frac{1}{s(s+5)(s+10)} = 0$$

1- Standardization:

2- Determine the number of branches, poles and zeros:

3- Geometric locus on the real axis:

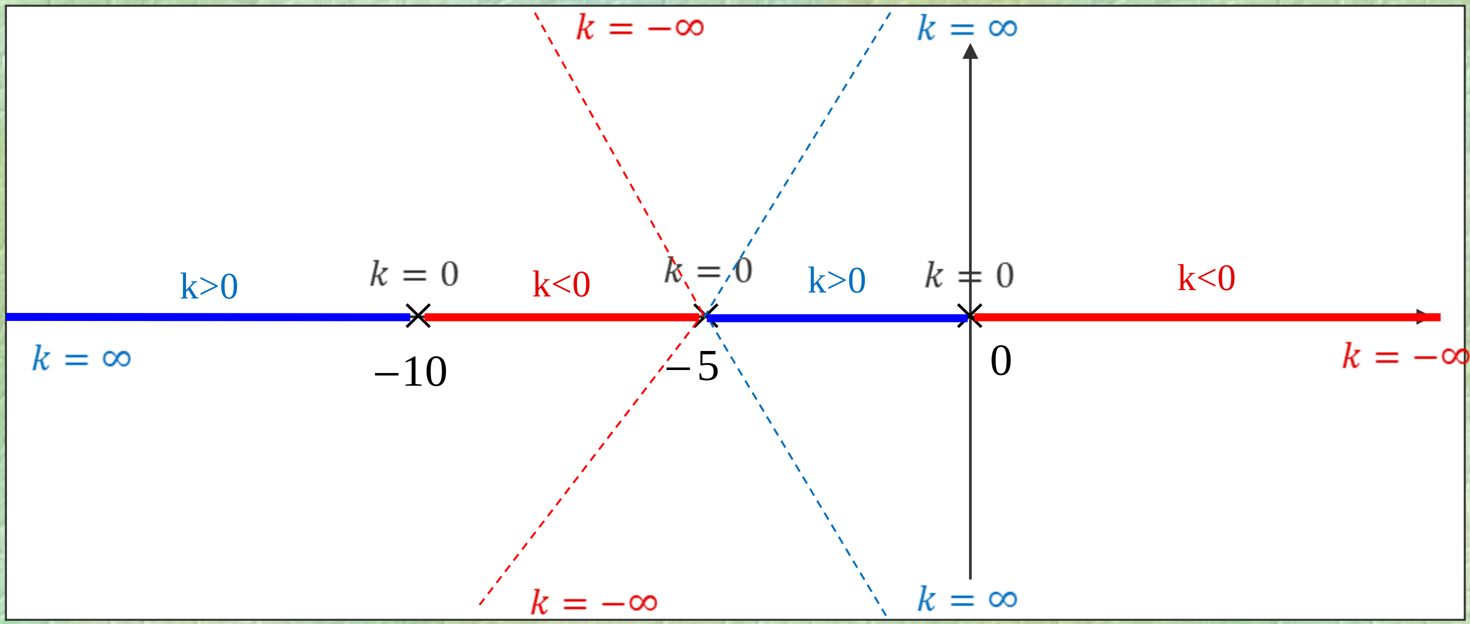
4- Determine the starting and ending points of the root locus branches:



How to plot the root locus $1+kf(s)=0$

Example 5: Find the complete root locus for the given system.

$$1+k\frac{1}{s(s+5)(s+10)}=0$$



5- Number, angles, and intersection points of asymptotes with the real axis:

Asymptotes center

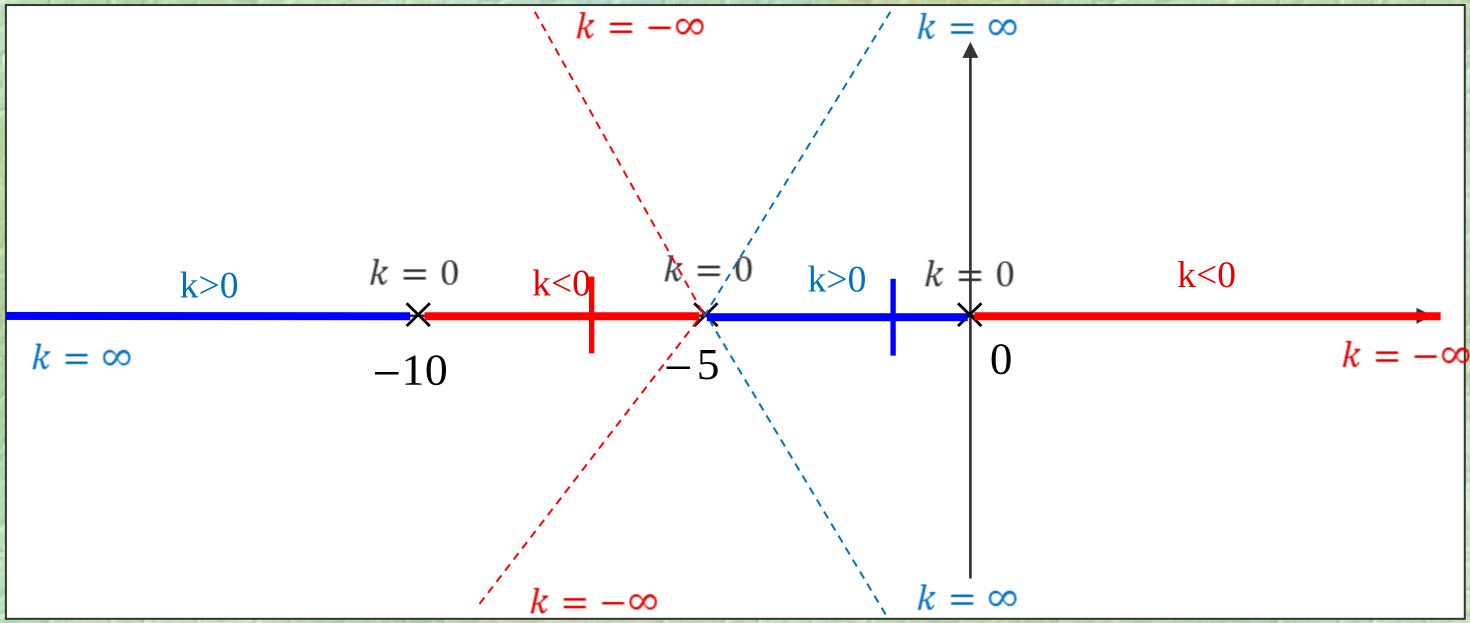
$$\delta=\frac{\sum_{i=1}^{n_p} p_i-\sum_{i=1}^{n_z} z_i}{n_p-n_z}=\frac{-10-5}{3}=-5$$

$$\begin{cases} \text{k}>0 \quad \theta=\dfrac{(2m+1)\pi}{|n_p-n_z|}=\dfrac{\pi}{3},\dfrac{3\pi}{3},\dfrac{5\pi}{3} \\ \text{k}<0 \quad \theta=\dfrac{2m\pi}{|n_p-n_z|}=0,\dfrac{2\pi}{3},\dfrac{4\pi}{3} \end{cases}$$

How to plot the root locus $1+kf(s)=0$

Example 5: Find the complete root locus for the given system.

$$1+k\frac{1}{s(s+5)(s+10)}=0$$



6- Calculation of breakaway or break-in points:

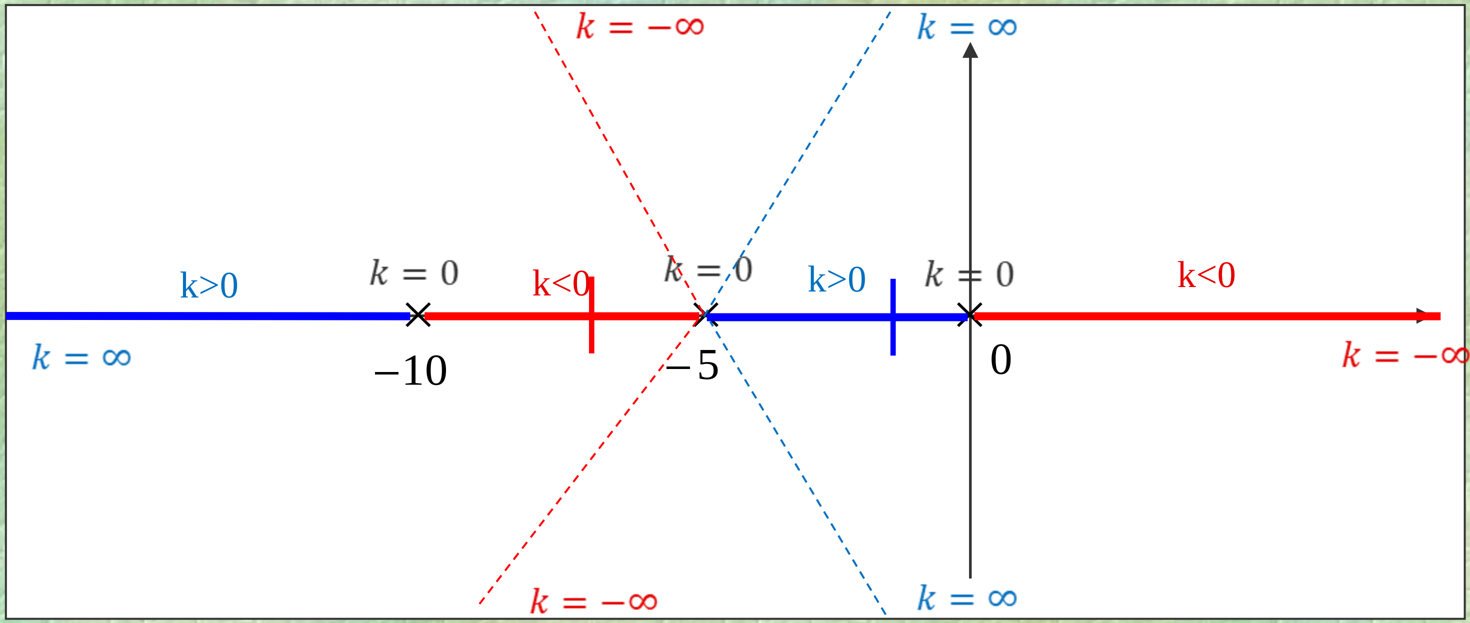
$$k=-\frac{1}{f(s)} \qquad \frac{\partial k}{\partial s}=0 \qquad s=\text{break point}$$

We need a new criterion.

How to plot the root locus $1 + kf(s) = 0$

Example 5: Find the complete root locus for the given system.

$$1 + k\frac{1}{s(s+5)(s+10)} = 0$$

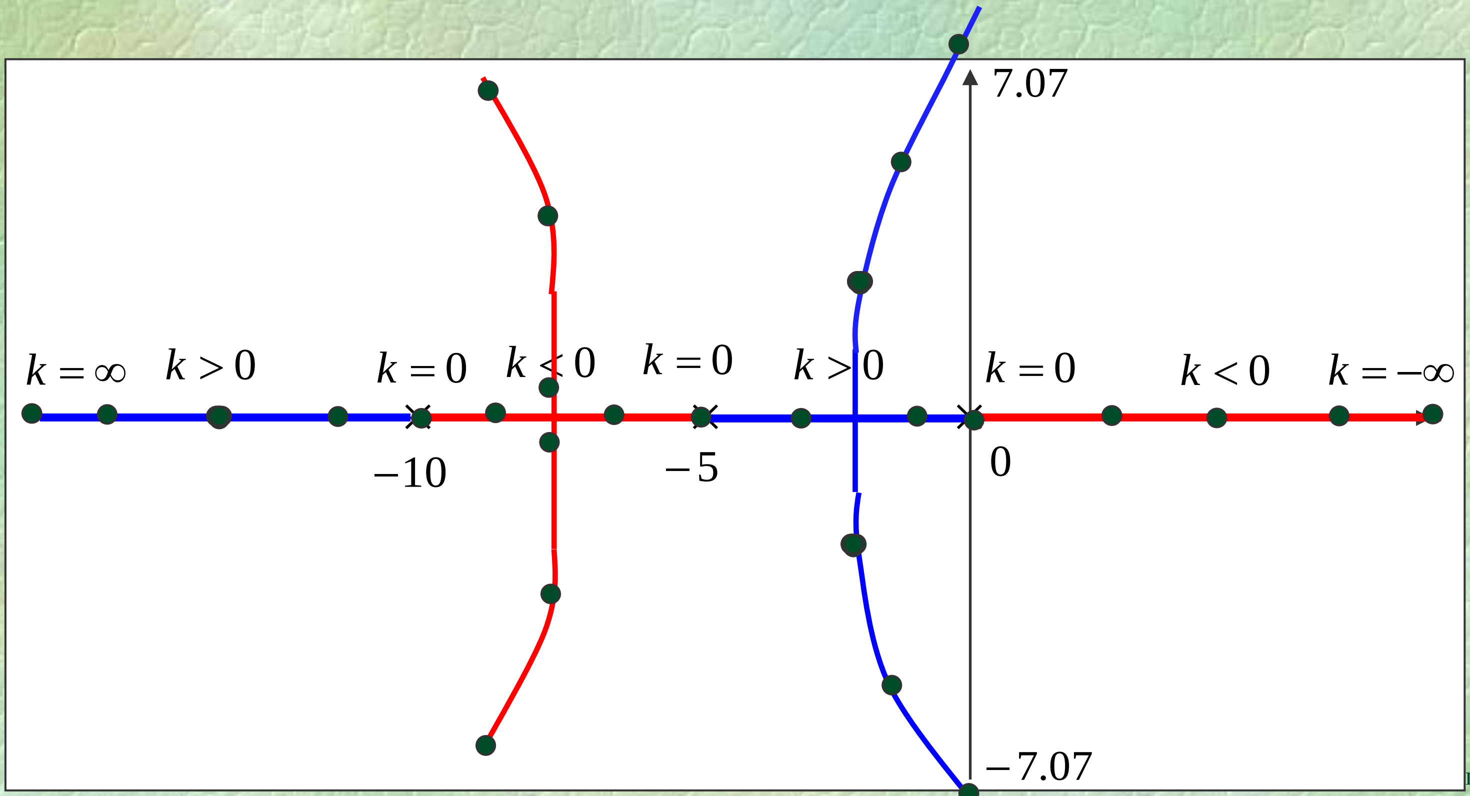


7- Intersection points of branches with the imaginary axis: The intersection points with the imaginary axis are found using the Routh-Hurwitz method.

How to plot the root locus $1 + kf(s) = 0$

Example 5: Find the complete root locus for the given system.

$$1 + k\frac{1}{s(s+5)(s+10)} = 0$$

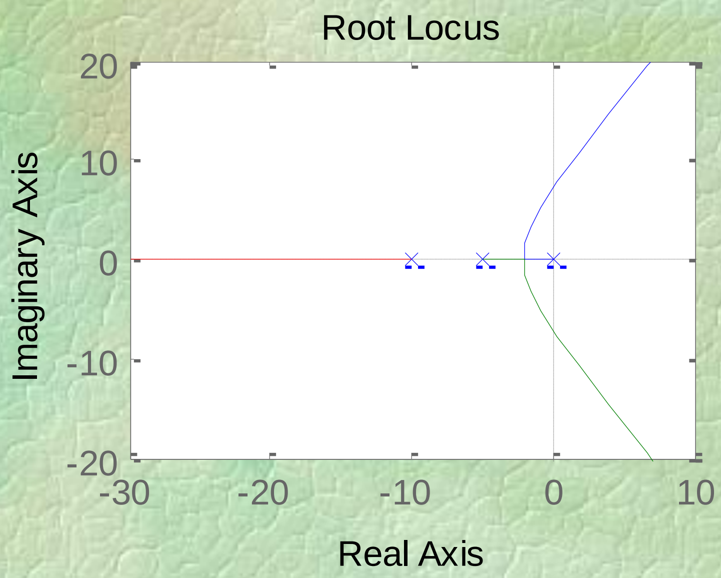


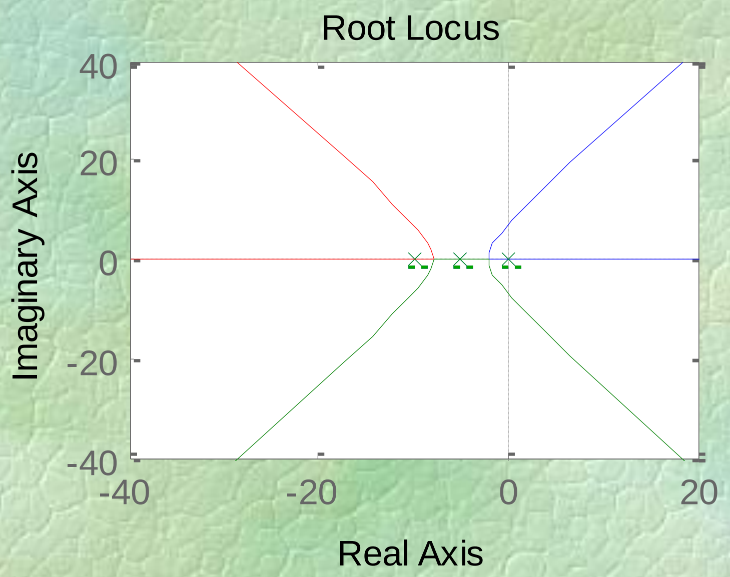
How to plot the root locus $1 + kf(s) = 0$

MATLAB

$$1 + k\frac{1}{s(s+5)(s+10)} = 0$$

rlocus(1,[1 15 50 0]);   hold on; rlocus(-1,[1 15 50 0])





How to plot the root locus $1+kf(s)=0$

Example 6: Find the complete root locus for the given system.

$$1+10\frac{(s+k)(s+3)}{s(s^2-1)}=0$$

. . . .

We need a new criterion in this example.

8- Find the arrival angles and departure angles.

Departure angles.

$$\sum\angle \text{Zeros}-\sum\angle \text{poles}=\pi$$

Arrival angles.

$$\sum\angle \text{Zeros}-\sum\angle \text{poles}=0$$

How to plot the root locus $1+kf(s)=0$

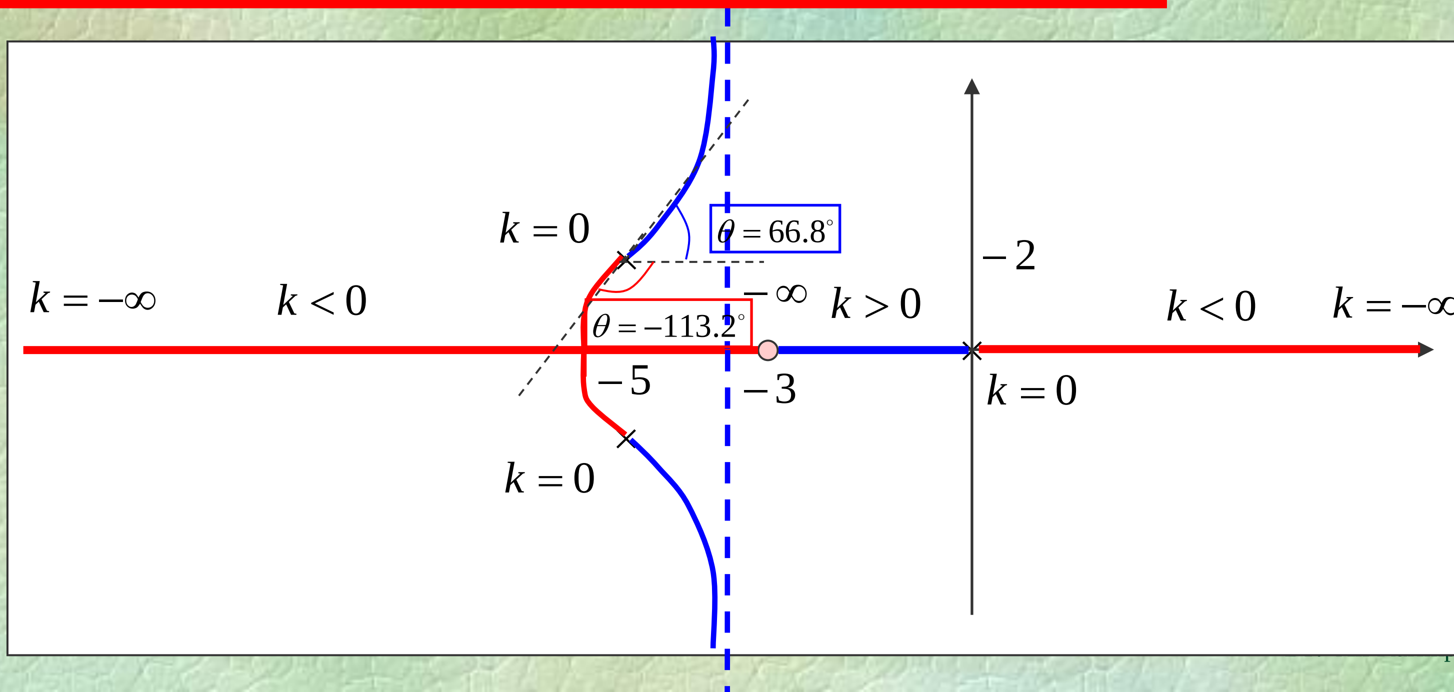
8: Find the arrival angles and departure angles.

$$135-\theta-90^\circ-(180^\circ-\tan^{-1}\frac{2}{5})=-\theta-113.2^\circ=\pm180^\circ$$

$$\theta=66.8^\circ$$

$$135-\theta-90^\circ-(180^\circ-\tan^{-1}\frac{2}{5})=-\theta-113.2^\circ=0^\circ$$

$$\theta=-113.2^\circ$$



Calculation of k on the Root Loci

$$1 + kf(s) = 0 \quad \xrightarrow[\text{of magnitude}]{\text{Condition}} \quad |k| = \frac{1}{|f(s)|}$$

Let $$f(s) = C\frac{\prod_{i=1}^{m}|s+z_i|}{\prod_{j=1}^{n}|s+p_j|}$$

$$|k|_{s_1} = \frac{\prod_{j=1}^{n}|s_1+p_j|}{C\prod_{i=1}^{m}|s_1+z_i|}$$

Summary

The Root Locus procedure

Specify the equation **exactly** in the following form.

$$1 + kf(s) = 0$$

1) How many branches in root loci?

2) Poles and zeros of f(s)?

3) Real axis?

4) Imaginary axis?

5) Asymptotes and centered of asymptotes?

6) Break points?

7) Intersection with the imaginary axis?

8) Arival and departure angle?

Exercises

Exercise 2: A unity feedback (negative sign) control system has an open loop transfer function

$$G(s)=\frac{k}{s(1+0.02s)(1+0.05s)}$$

Sketch the complete root loci, and find the corresponding k when the root loci crosses jω axis.

Exercise 3: The transfer function of a single-loop control system are given as:

$$G(s)=\frac{10}{s^2(s+1)(s+3)} \qquad H(s)=1+T_d s$$

Construct the root loci of the **Zeros** of $1+G(s)H(s)=0$ for $-\infty<T_d<\infty$

Exercise 4: The open loop transfer function of a unity-feedback (negative system is:

$$G_p(s)=\frac{K}{(s+5)^n}$$

Construct the complete root loci of the characteristic equation for Let n=1, n=2 and n=3.

Exercises

Exercise 5: The open loop transfer function of a unity-feedback (negative sign)
system is:

$$G(s) = \frac{K(s+\alpha)(s+3)}{s(s^2-1)}$$

a) Construct the root loci for $-\infty<K<\infty$, with $\alpha=5$.
b) Construct the root loci for $-\infty< \alpha <\infty$, with $K=5$.

Exercise 6: The open loop transfer function of a unity-feedback (negative sign)system is:

$$G(s) = \frac{500p}{s(s+10)(s+p)}$$

Construct the root loci for $0<p<\infty$

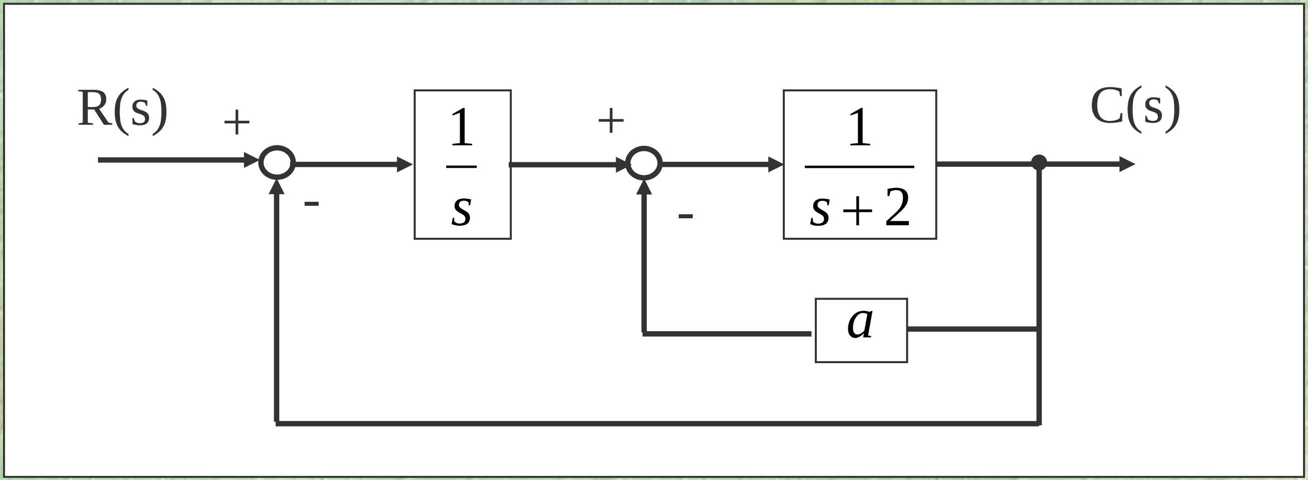
Exercise 7: Consider following system

$$1 + k_1 \frac{1+s}{s^3} = 0$$

Construct the root loci for $0<k_1<\infty$

Exercises

Exercise 8: Construct the root loci of the closed loop poles of the following system for $0<a<\infty$(Midterm spring 2008).



Exercise 9: Consider following system

$$1+k_2\frac{s^2}{s^3+k_1s+k_1}=0$$

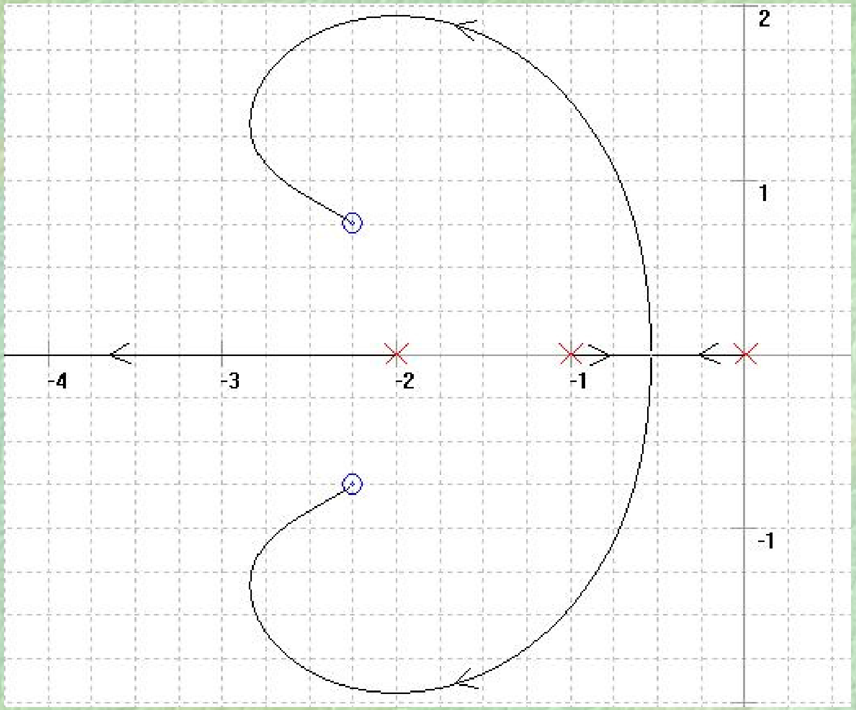
For $k_1=0$, $k_1=1$ and $k_1=10$ construct the root loci for $0<k_2<\infty$.

Exercises

Exercise 10: Find the root-locus graph for the following system.

$$1 + K\frac{s^2 + 4.5s + 5.625}{s(s+1)(s+2)} = 0$$

Answer :



Exercise 11: The open loop transfer function of a unity-feedback (negative sign) system is:

$$G(s) = \frac{10}{s(s-p)}$$

Construct the root loci for $0<p<\infty$(Final 1391)

Exercises

Exercise 12: The open loop transfer function of a unity-feedback (negative sign) system is:

$$G(s) = K\frac{e^{-0.1s}}{s(s+1)(s+2)}$$

Construct the complete root loci of the characteristic equation.

Exercise 13: The open loop transfer function of a unity-feedback (negative sign) system with PD controller is:

$$G(s) = \frac{10(K_p + K_d s)}{s^2}$$

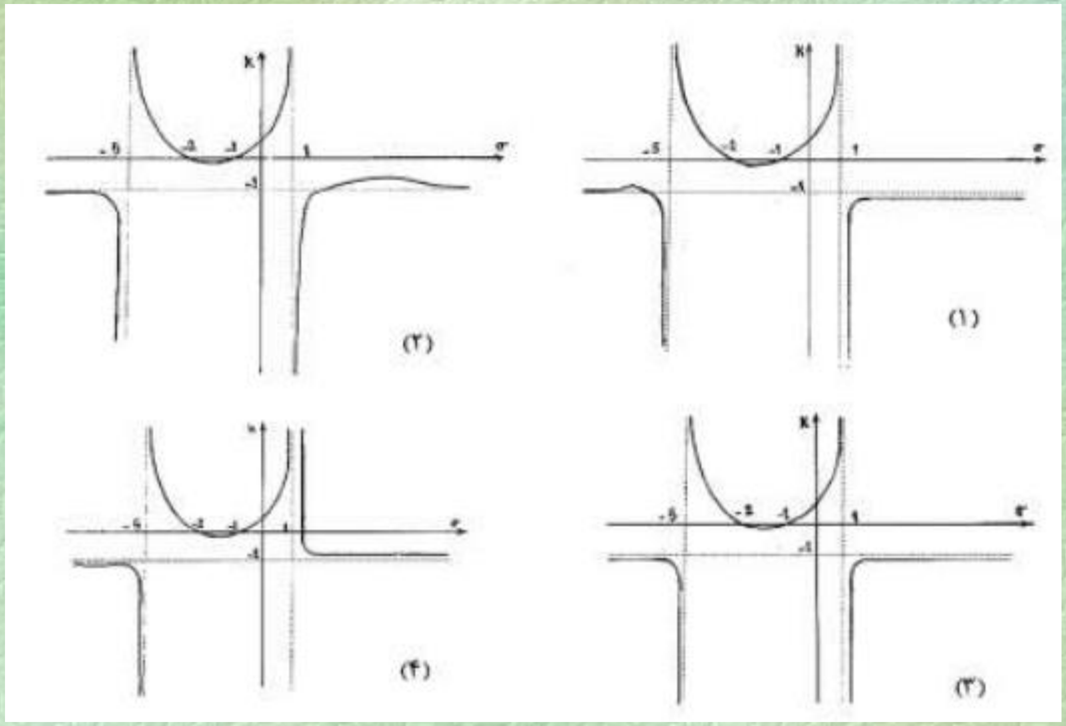
Sketch the root loci for different values of K_p and K_d. (Let Kp=0,1,5,10.)

Exercises

Exercise 14: The open loop transfer function of a unity-feedback (negative sign) system is:

$$G(s)=\frac{k(s-1)(s+5)}{(s+2)(s+1)}$$

Sketch the k versus real part of roots. (University entrance exam 1393)



Answer: 2

Remark: Note that for any k we have two roots.

Exercises

Exercise 15: Consider the open loop transfer function

(University entrance exam 1393)

Which option is correct regarding the poles of the closed-loop transfer function with unit negative feedback?

a) A circle with center -1 and radius 1 is part of the locus.

b) Only at infinity does the geometric locus of the poles of the closed-loop system asymptotically approach the straight lines that intersect at -1.

c) The geometric locus is composed of the straight lines that intersect at -1.

d) A segment of the circle passing through the points -1 and $-2 \pm j\sqrt{3}$ is part of the geometric locus.