
Averaging Systems

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Introduction to Averaging Systems

Consensus(Rendezvous) problem. Assume that member of a system have dynamics of type

$$x_v(t + 1) = x_v(t) + u_v(t)$$

$x_v(t)$ and $u_v(t)$ are states and inputs of every vertex $v \in V$. The control goal is to make all units converge their state to the same point.

There are many variants of this problem, for example wireless sensor networks or social influence networks.

Important issues in this problem:

- Given a graph G , in which condition there is a consensus(rendezvous) solution or

$$\lim_{t \rightarrow +\infty} x_v(t) = x^*, \quad \forall v \in V.$$

- What is the velocity of convergence to the consensus(rendezvous) point?
- Given a graph G , in which condition there is a disagreement, and what happen in this situation?

Introduction to Averaging Systems

Consensus(Rendezvous) problem.

Now if

$$x_v(t + 1) = x_v(t) + u_v(t)$$

$$u_v(t) = \sum_{w \in V} K_{vw} x_w(t)$$

Now we have

$$X(t + 1) = PX(t)$$

Where $X(t) = [x_1^T(t) \ x_2^T(t) \ \dots \ x_m^T(t)]^T$ and m is number of vertices and $x_i(t) \in \mathbb{R}^n$.

As a simplified situation consider every vertex has one state so:

$$x(t + 1) = Px(t)$$

Now we need a convergence s.t.

$$\lim_{t \rightarrow +\infty} x_v(t) = x^*, \quad \forall v \in V.$$

Necessary condition for convergent to x^* ?

1 as largest eigenvalue and $\mathbf{1}_n$ as eigenvector (i.e. $P\mathbf{1}_n = \mathbf{1}_n$) and so P must be(why?)

Now according to graph structure there is some different situations.

Introduction to Averaging Systems

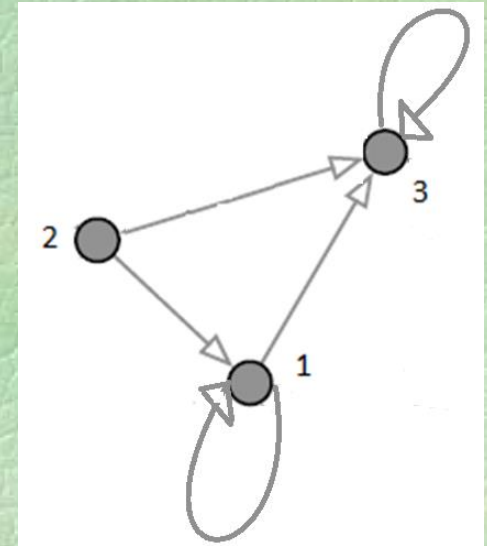
Example 1: A graph with just one globally reachable vertex.

$$P = \begin{bmatrix} 0.1 & 0 & 0.9 \\ 0.4 & 0 & 0.6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\lambda = 1 \quad v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad w^T = [0 \quad 0 \quad 1]$$

$$\lim_{k \rightarrow \infty} x(k) = v w^T x(0) \quad (\text{why?})$$

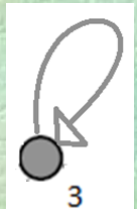
$$\lim_{k \rightarrow \infty} x_1(k) = \lim_{k \rightarrow \infty} x_2(k) = \lim_{k \rightarrow \infty} x_3(k) = x_3(0)$$



Remark: The global reachable vertex(no. 3) has a high social influence and others nothing.

Remark: P is reducible so the graph is not strongly connected.

Remark: The global vertex 3 is a sink.



Introduction to Averaging Systems

Example 2: A graph with two globally reachable vertex.

$$P = \begin{bmatrix} 0.1 & 0 & 0.9 \\ 0.4 & 0 & 0.6 \\ 0.15 & 0 & 0.85 \end{bmatrix}$$

$$\lambda = 1 \quad v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad w^T = [0.143 \quad 0 \quad 0.857]$$

$$\lim_{k \rightarrow \infty} x(k) = v w^T x(0) \quad (\text{why?})$$

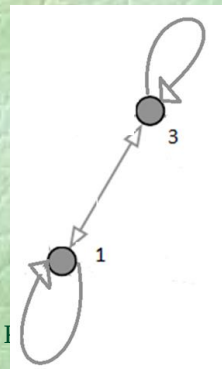
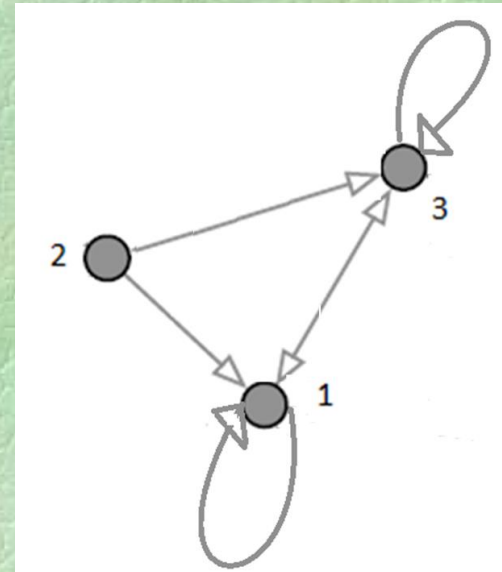
$$\lim_{k \rightarrow \infty} x_1(k) = \lim_{k \rightarrow \infty} x_2(k) = \lim_{k \rightarrow \infty} x_3(k) = 0.143x_1(0) + 0.857x_3(0)$$

Remark: The global reachable vertices (no. 3 and no. 1) contribute in final value.

Remark: Compare social influence of vertex 1 and 3.

Remark: P is reducible so the graph is not strongly connected.

Remark: Vertices 1 and 3 are a sink.



Introduction to Averaging Systems

Example 3: A strongly connected graph.

$$P = \begin{bmatrix} 0.1 & 0 & 0.9 \\ 1 & 0 & 0 \\ 0 & 0.6 & 0.4 \end{bmatrix}$$

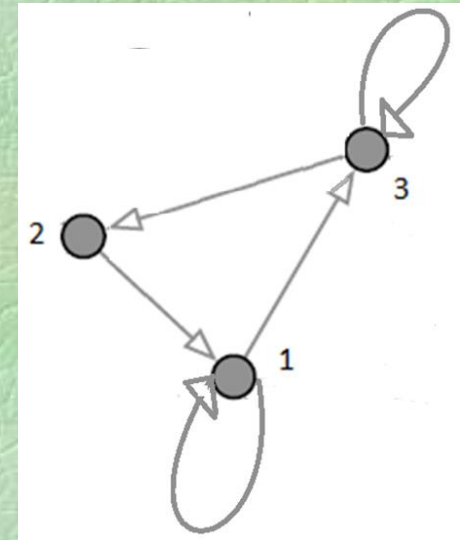
$$\lambda = 1 \quad v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad w^T = [0.295 \quad 0.264 \quad 0.441]$$

$$\lim_{k \rightarrow \infty} x(k) = v w^T x(0) \quad (\text{why?})$$

$$\lim_{k \rightarrow \infty} x_1(k) = \lim_{k \rightarrow \infty} x_2(k) = \lim_{k \rightarrow \infty} x_3(k) = 0.295x_1(0) + 0.264x_2(0) + 0.441x_3(0)$$

Remark: All vertices are global reachable so they contribute in the final value.

Remark: P is irreducible so the graph is strongly connected.



Introduction to Averaging Systems

Example 4: A weight balanced strongly connected graph.

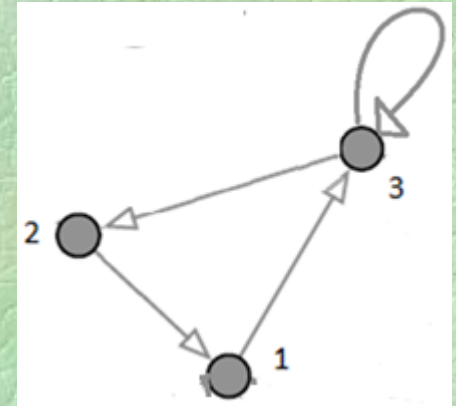
$$P = \begin{bmatrix} 0.417 & 0.333 & 0.25 \\ 0.333 & 0.417 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

$$\lambda = 1 \quad v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad w^T = [0.333 \quad 0.333 \quad 0.333]$$

$$\lim_{k \rightarrow \infty} x(k) = v w^T x(0)$$

$$\lim_{k \rightarrow \infty} x_1(k) = \lim_{k \rightarrow \infty} x_2(k) = \lim_{k \rightarrow \infty} x_3(k) = 0.333x_1(0) + 0.333x_2(0) + 0.333x_3(0)$$

Remark: All vertices are global reachable and weight balanced so they contribute in the final value with same social influence.



Introduction to Averaging Systems

Consensus(Rendezvous) problem.

Different type of
Averaging system

Discrete-time Averaging Systems

$$x(t + 1) = Px(t)$$

Continuous-time Averaging Systems

$$\dot{x}(t) = -L x(t)$$

Time-varying Averaging Systems(different row stochastic matrix
or graph change)

$$x(t + 1) = P(t)x(t) \quad \text{or} \quad \dot{x}(t) = -L(t) x(t)$$

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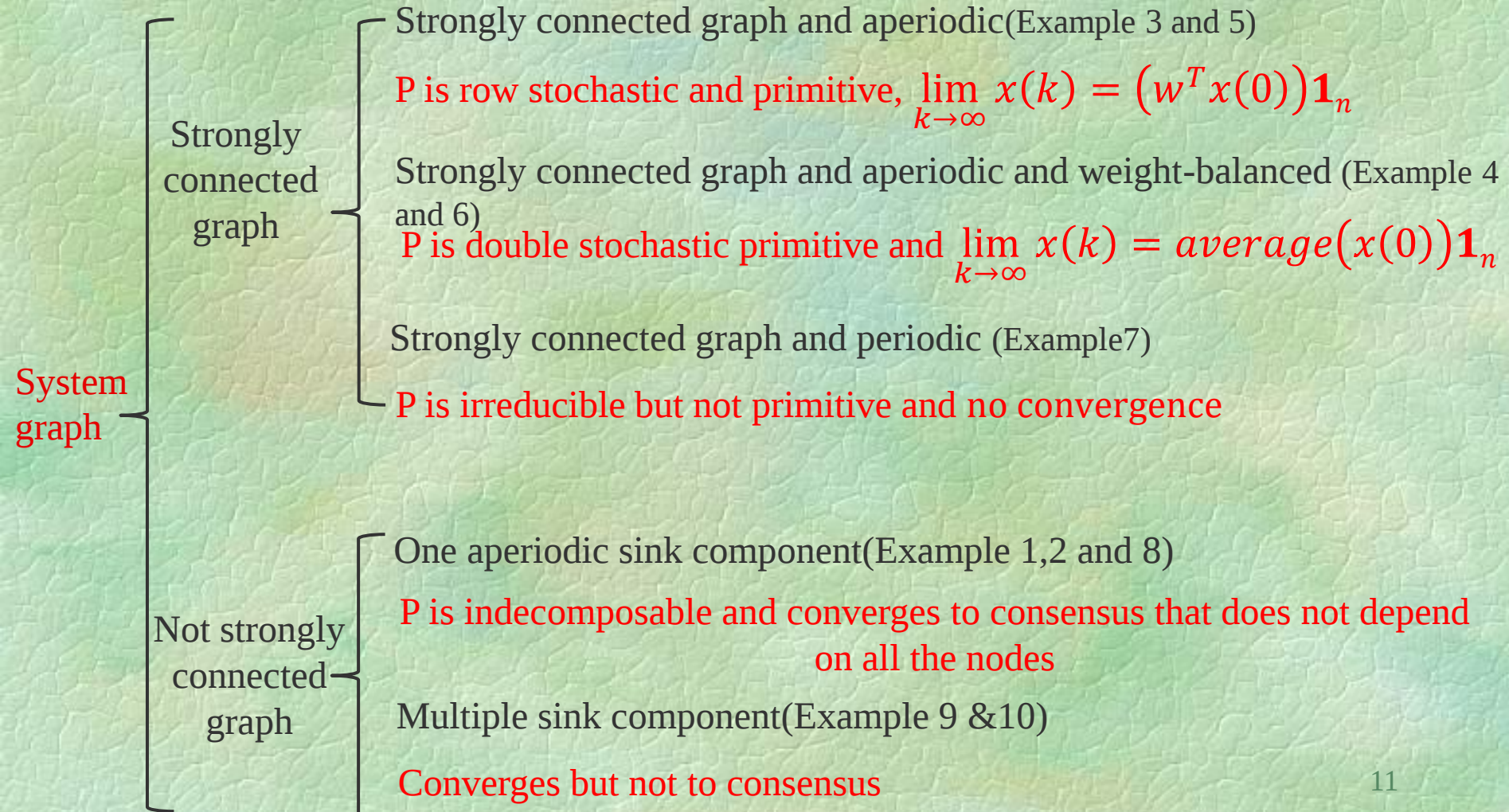
Positive Network Control Systems

Convergence Rates, Scalability and Optimization of Averaging Systems

Discrete-time Averaging Systems

Consensus(Rendezvous) problem in discrete time is:

$$x(t + 1) = Px(t)$$



Discrete-time Averaging Systems

Example 5: Strongly connected graph and aperiodic

$$P = \begin{bmatrix} 0 & 0.7 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 0.9 & 0.1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

P is row stochastic and primitive, eigenvalues of P are:

Left eigenvalues corresponding to eigenvalue 1 is:

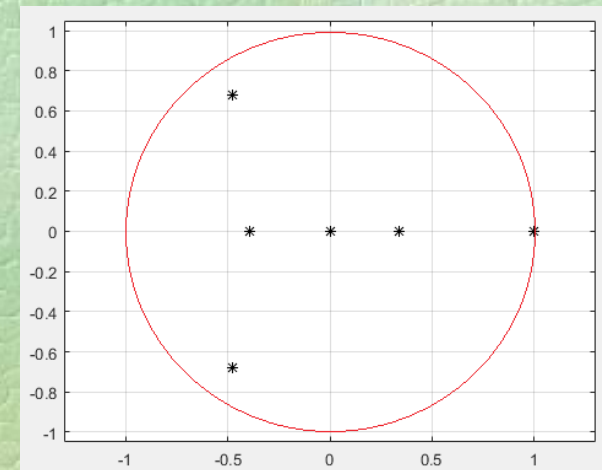
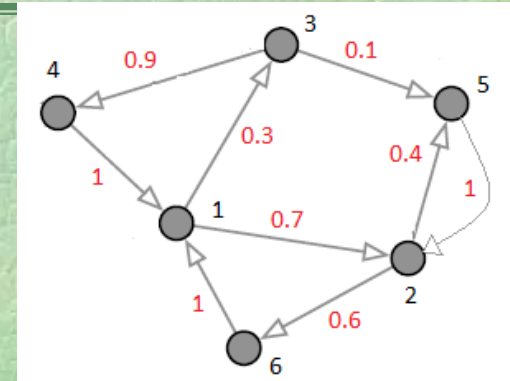
$$w = [0.2479 \quad 0.3017 \quad 0.0744 \quad 0.0669 \quad 0.1281 \quad 0.1810]^T$$

All states converges to:

$$x_i(k \rightarrow \infty) = 0.2479x_1(0) + 0.3017x_2(0) + 0.0744x_3(0) + 0.0669x_4(0) + 0.1281x_5(0) + 0.1810x_6(0) \quad i=1, 2, \dots, 6$$

Remark 1: Vertex 1 and 2 are more important (more social influence).

Remark 2: Final values depending on all nodes (every element of w is strictly positive).



Discrete-time Averaging Systems

Example 6: Strongly connected graph and aperiodic and weight-balanced

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0 & 0.5 \end{bmatrix}$$

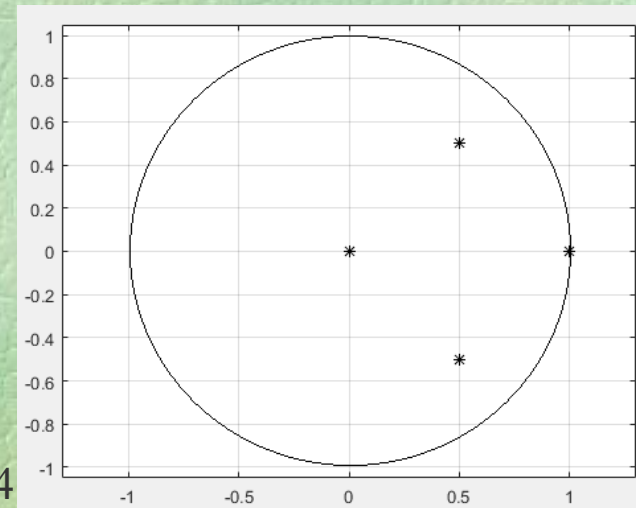
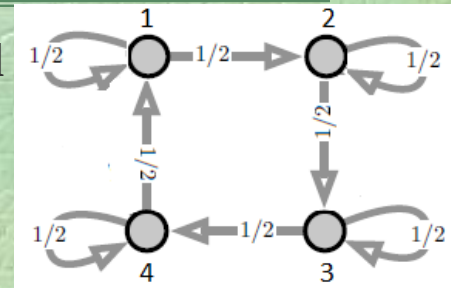
P is double stochastic and primitive, eigenvalues of P are:

Left eigenvalues corresponding to eigenvalue 1 is:

$$w = [0.25 \ 0.25 \ 0.25 \ 0.25]^T$$

All states converges to:

$$x_i(k \rightarrow \infty) = 0.25x_1(0) + 0.25x_2(0) + 0.25x_3(0) + 0.25x_4(0) \quad i=1, 2, \dots, 4$$



Remark 1: All vertices have similar effect. (similar social influence).

Remark 2: Final values depending on all nodes (every element of w is strictly positive).

Remark 3: The averaging system leads to average consensus.

Discrete-time Averaging Systems

Example 7: Strongly connected graph and periodic

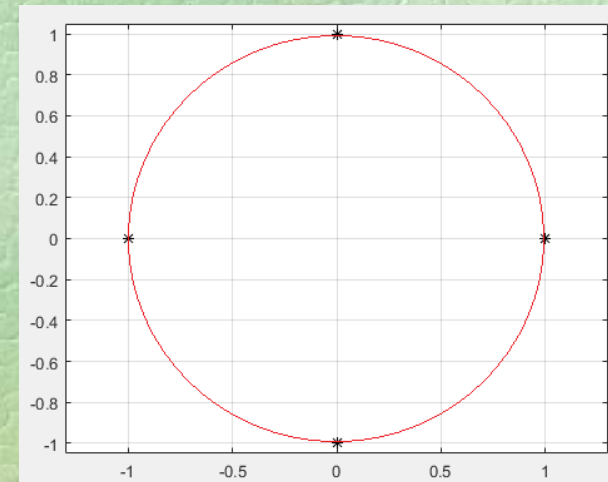
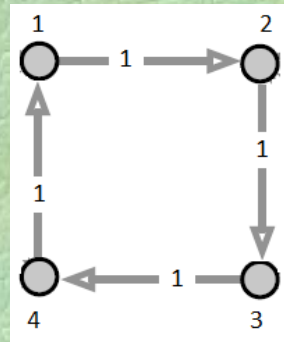
$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

P is double stochastic with period of 4 but not primitive, eigenvalues of P are:

There is no convergence and the final value has a periodic value with period of 4:

$$\begin{array}{cccc} x(k) = & \xrightarrow{\quad} & x(k+1) = & \xrightarrow{\quad} & x(k+2) = & \xrightarrow{\quad} & x(k+3) = \\ \begin{bmatrix} x_3(0) \\ x_4(0) \\ x_1(0) \\ x_2(0) \end{bmatrix} & & \begin{bmatrix} x_4(0) \\ x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} & & \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} & & \begin{bmatrix} x_2(0) \\ x_3(0) \\ x_4(0) \\ x_1(0) \end{bmatrix} \end{array}$$

$\xleftarrow{\quad}$



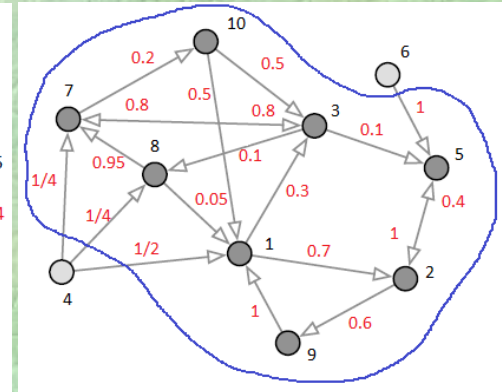
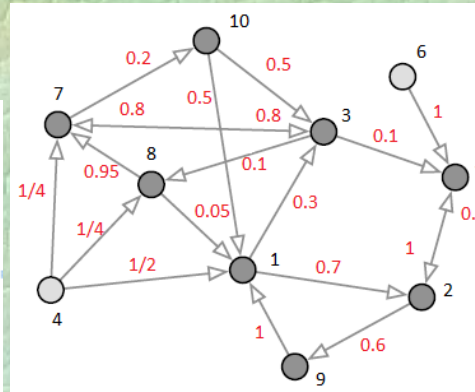
Remark 1: No convergence.

Remark 2: Periodic response.

Discrete-time Averaging Systems

Example 8: One aperiodic sink component

$$P = \begin{bmatrix} 0 & 0.7 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & 0 & 0.80 & 0.10 & 0 & 0 & 0 \\ 0.50 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0.25 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 \\ 0.05 & 0 & 0 & 0 & 0 & 0 & 0.95 & 0 & 0 & 0 & 0 \\ 1.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.50 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



P is row stochastic but not primitive (1 sink 2 sources),
eigenvalues of P are:

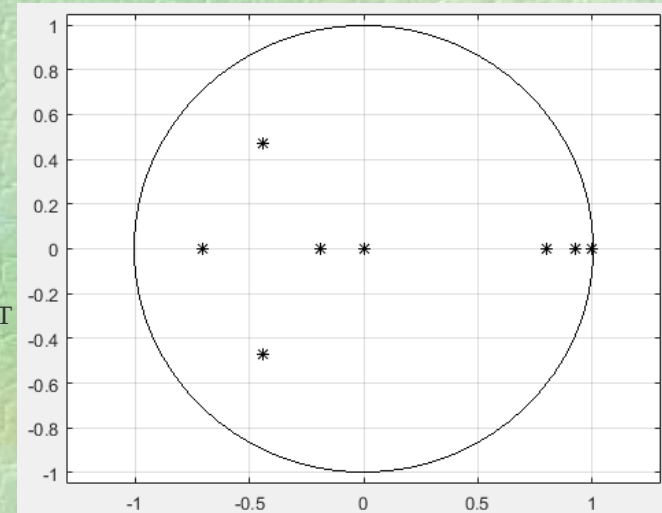
Left eigenvalues corresponding to eigenvalue 1 is:

$$w = [0.1360 \ 0.1936 \ 0.2097 \ 0 \ 0.0989 \ 0 \ 0.1877 \ 0.0210 \ 0.1161 \ 0.0375]^T$$

All states converges to:

$$x_i(k \rightarrow \infty) = 0.1244x_1(0) + 0.1673x_2(0) + 0.1333x_3(0) + 0.0803x_5(0) + 0.1066x_7(0) + 0.2663x_8(0) + 0.1004x_9(0) + 0.0213x_{10}(0)$$

$$i = 1, 2, \dots, 10$$



Remark 1: Vertex 4 and 6 (sources) has no effect on the final value. (no Social influence).

Remark 2: Final values depending on all vertices in the sink.

Discrete-time Averaging Systems

Example 9: Multiple sink component

Multiple sink component,
one aperiodic sink and one periodic.

$$P = \begin{bmatrix} 0 & 0 & 1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.00 & 0 & 0 \\ 0.50 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0.25 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 \\ 0.05 & 0 & 0 & 0 & 0 & 0 & 0.95 & 0 & 0 & 0 \\ 1.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.50 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

P is row stochastic and not primitive, eigenvalues of P are:

Left eigenvalues corresponding to eigenvalue 1 is:

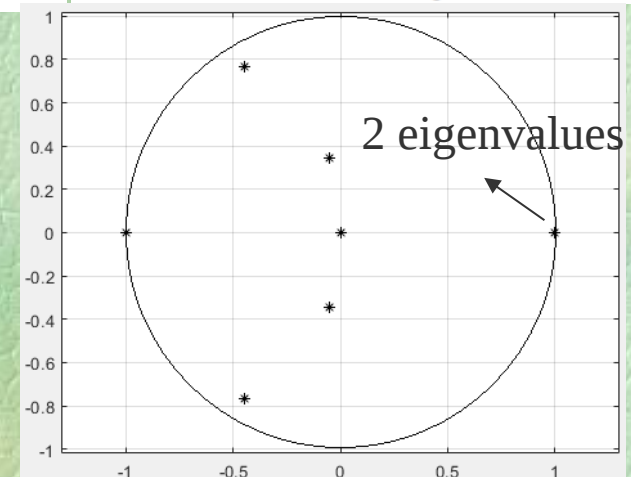
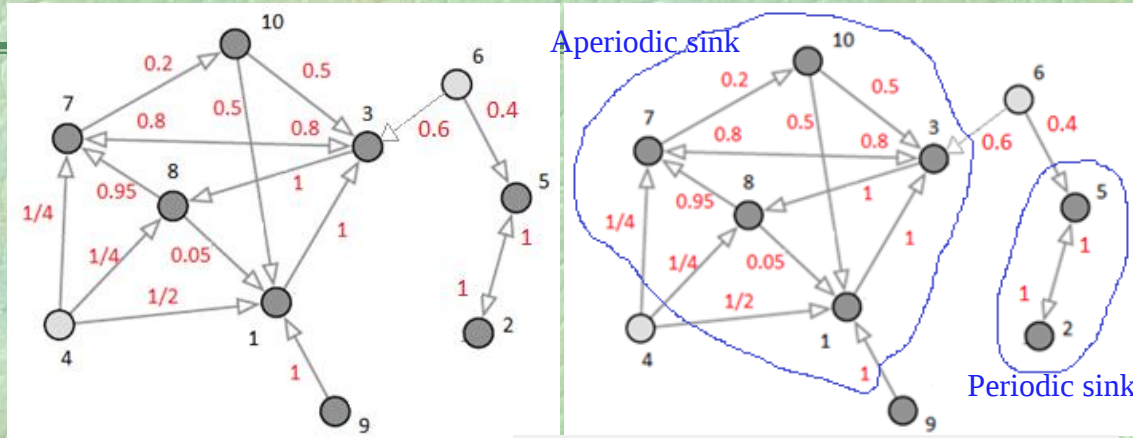
$$w = [0.0441 \ 0 \ 0.3044 \ 0 \ 0 \ 0 \ 0.2892 \ 0.3044 \ 0 \ 0.0578]^T$$

All states of aperiodic sink converges to:

$$x_i(k \rightarrow \infty) = 0.0441x_1(0) + 0.3044x_3(0) + 0.2892x_7(0) + 0.3044x_8(0) + 0.0578x_{10}(0) \quad i=1, 3, 7, 8, 10$$

All states of periodic sink jump to the following parts (period is 2):

$$x_2(k) = x_5(0), x_5(k) = x_2(0) \text{ and } x_2(k+1) = x_2(0), x_5(k+1) = x_5(0)$$



Discrete-time Averaging Systems

Theorem: (Consensus for row-stochastic matrices with a globally-reachable aperiodic strongly-connected component) Let A be a row-stochastic matrix and let G be its associated digraph. The following statements are equivalent:

- (A1) The eigenvalue 1 is simple and all other eigenvalues μ satisfy $|\mu| < 1$,
- (A2) A is semi-convergent and $\lim_{k \rightarrow \infty} A^k = \mathbf{1}_n \mathbf{w}^T$, $\mathbf{w} \geq 0$, and $\mathbf{1}_n^T \mathbf{w} = 1$,
- (A3) G contains a globally reachable node and the subgraph of globally reachable nodes is aperiodic.

If any, and therefore all, of the previous conditions are satisfied, then the matrix A is said to be indecomposable and the following properties hold:

- (i) $\mathbf{w} \geq 0$ is the left dominant eigenvector of A and $w_i > 0$ if and only if node i is globally reachable;
- (ii) the solution to the averaging model $\mathbf{x}(k+1) = A\mathbf{x}(k)$ is $\lim_{k \rightarrow \infty} \mathbf{x}(k) = \mathbf{w}^T \mathbf{x}(0) \mathbf{1}_n$,
- (iii) if additionally A is doubly-stochastic, then $\mathbf{w} = \frac{1}{n} \mathbf{1}_n$, $\lim_{k \rightarrow \infty} \mathbf{x}(k) = \text{average}(\mathbf{x}(0)) \mathbf{1}_n$

Discrete-time Averaging Systems

Theorem: (Convergence for row-stochastic matrices with multiple aperiodic sinks).[1]

Let A be a row-stochastic matrix, let G be its associated digraph, and let $M \geq 2$ be the number of sinks in the condensation digraph $C(G)$. If each of the M sinks, is aperiodic, then

- (i) the semi-simple eigenvalue $\rho(A)=1$ has multiplicity equal M and is strictly larger than the magnitude of all other eigenvalues, hence A is semi-convergent.
- (ii) there exist M different left eigenvectors of A (corresponding to 1 eigenvalues), denoted by $\omega^m \in \mathbb{R}^n$ for $m \in \{1, 2, \dots, M\}$, with property that $\omega^m \geq 0$, $\omega^m \mathbf{1}_n = 1$, ω_i^m is positive if and only if node i belongs to m -th sink.
- (iii) the solution to the averaging model $x(k+1)=Ax(k)$ with initial condition $x(0)$ satisfy

$$\lim_{k \rightarrow \infty} x_i(k) = \begin{cases} (\omega^m)^\top x(0), & \text{if node } i \text{ belongs to the } m\text{-th sink,} \\ (\omega^m)^\top x(0), & \text{if node } i \text{ is connected with the } m\text{-th sink and no other} \\ & \text{sink,} \\ \sum_{m=1}^M z_{i,m} ((\omega^m)^\top x(0)), & \text{if node } i \text{ is connected to more than one sink,} \end{cases}$$

where, for each node i connected to more than one sink, the coefficients $z_{i,m}$ is used.

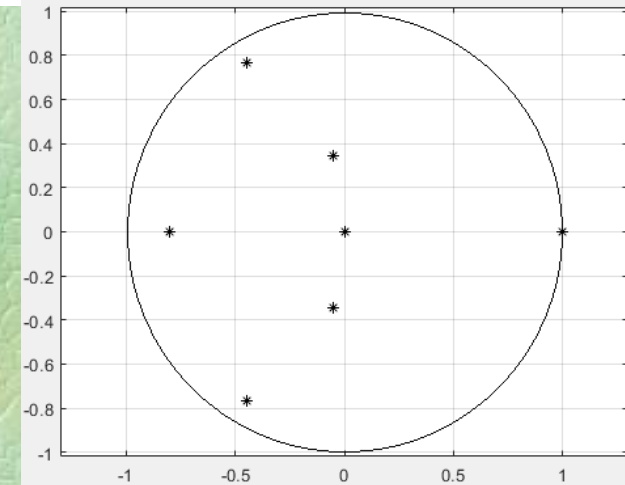
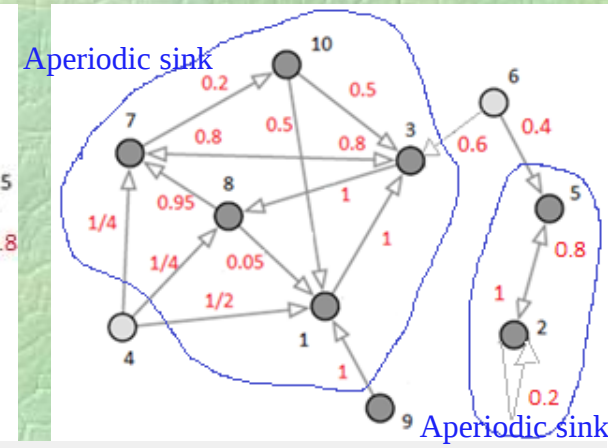
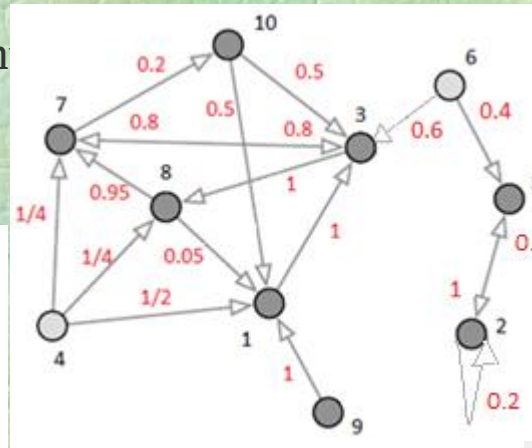
See example 10.

Discrete-time Averaging Systems

Example 10: Multiple sink components

Two aperiodic sink.

$$P = \begin{bmatrix} 0 & 0 & 1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.00 & 0 & 0 \\ 0.50 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0.25 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 \\ 0.05 & 0 & 0 & 0 & 0 & 0 & 0.95 & 0 & 0 & 0 \\ 1.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.50 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



P is row stochastic and not primitive, eigenvalues of P are:

Left eigenvalues corresponding to eigenvalue 1 are:

$$w_1 = [0.0441 \ 0 \ 0.3044 \ 0 \ 0 \ 0 \ 0.2892 \ 0.3044 \ 0 \ 0.0578]^T$$

$$w_2 = [0 \ 0.5556 \ 0 \ 0 \ 0.4444 \ 0 \ 0 \ 0 \ 0 \ 0]^T$$

All states of aperiodic sink 1 converges to:

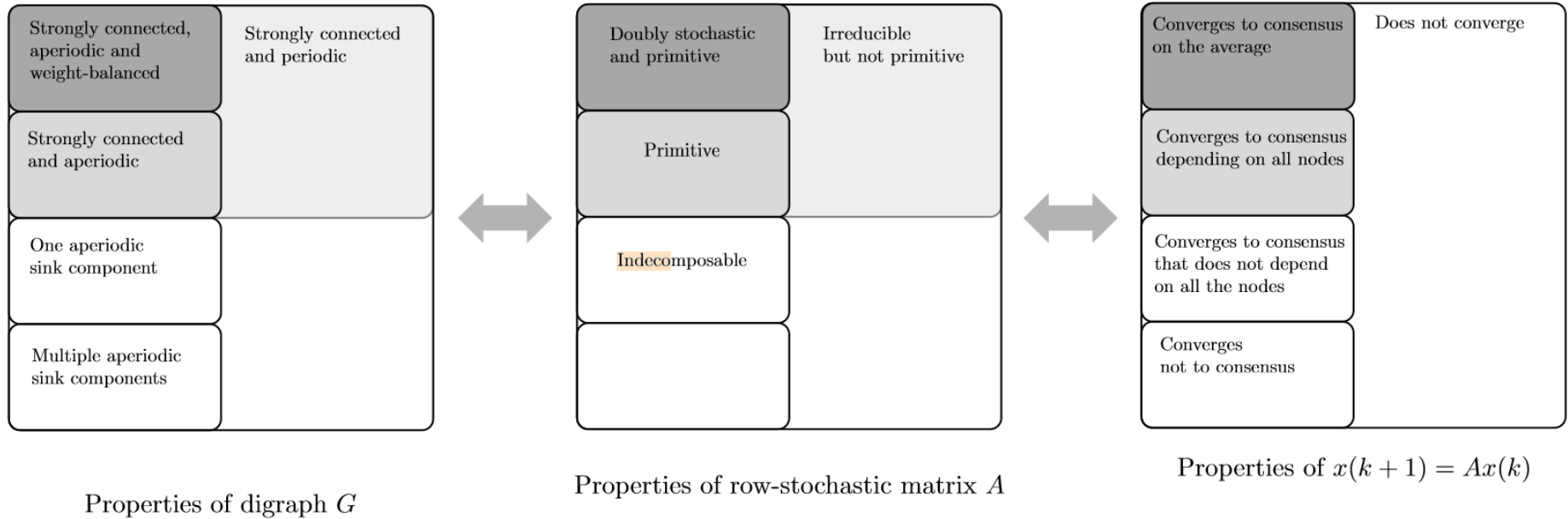
$$x_i(k \rightarrow \infty) = 0.0441x_1(0) + 0.3044x_3(0) + 0.2892x_7(0) + 0.3044x_8(0) + 0.0578x_{10}(0) \quad i=1, 3, 7, 8, 10$$

All states of aperiodic sink 2 converges to:

$$x_i(k \rightarrow \infty) = 0.5556x_2(0) + 0.4444x_5(0) \quad i=2, 5$$

Remark 1: No consenses. Ali Karimpour Aug 2024

Summary of the asymptotic behavior of discrete-time averaging systems and its relationships with properties of matrices and graphs



Centrality measure of each node

It is of interest to determine the relative importance of a node in a network (social influence).

Centrality
measures
of a graph

1- **Degree centrality:** Degree centrality is number of indegree of each edge or column sum of that edge in adjacency matrix. $c_{\text{degree}} = A^T \mathbb{1}_n$

2- **Eigenvector centrality:** Eigenvector centrality is the dominant eigenvector of the adjacency matrix associated with the dominant eigenvalue. $\rho(A)c_{\text{ev}} = A^T c_{\text{ev}}$

3- **Katz centrality:** Katz centrality is the combination of degree and eigenvector centrality. $c_K = \alpha A^T (c_K + \mathbb{1}_n)$ where $\alpha < 1/\rho(A)$

4- **PageRank centrality:** PageRank centrality is for row stochastic matrices and is:

$$c_{\text{pr}} = \alpha A^T c_{\text{pr}} h + \frac{1-\alpha}{n} \mathbb{1}_n \quad 0 < \alpha < 1$$

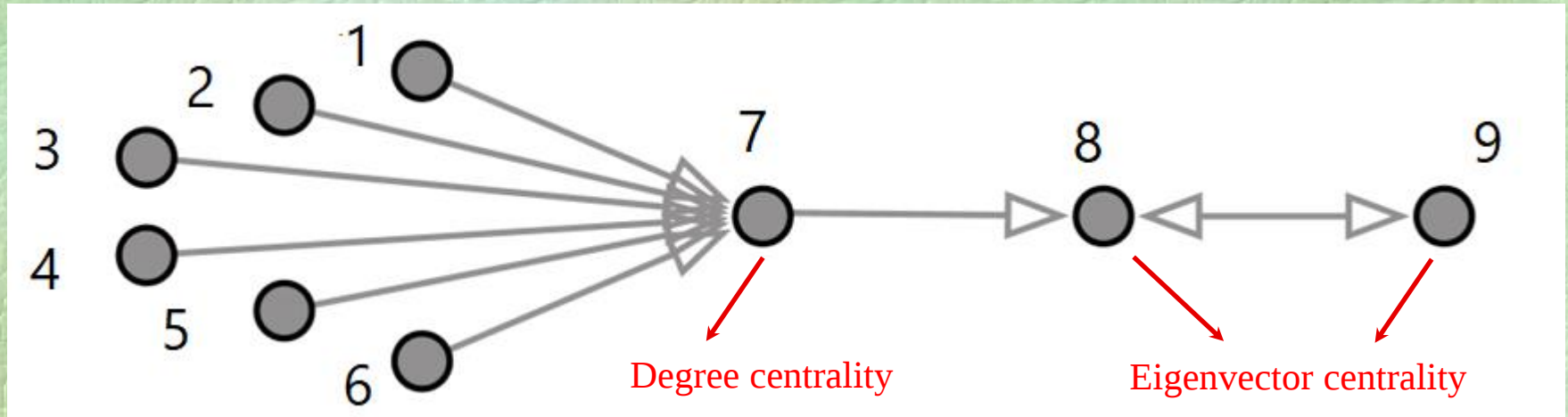
5- **Closeness centrality:** Closeness centrality is for strongly connected graph and is:

$$c_{\text{closeness}}(i) = \frac{1}{\sum_{j=1}^n d_{i \rightarrow j}}$$

6- **Betweenness centrality:** Betweenness centrality is for strongly connected graph and is:

$$c_{\text{betweenness}}(i) = \frac{\sum_{j,k=1}^n d_{k \rightarrow i \rightarrow j}}{\sum_{h=1}^n \sum_{j,k=1}^n d_{k \rightarrow h \rightarrow j}}$$

Centrality measure of each node



$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$x(999) = \begin{bmatrix} x_9(0) \\ x_9(0) \\ x_9(0) \\ x_9(0) \\ x_9(0) \\ x_9(0) \\ x_9(0) \\ x_9(0) \\ x_8(0) \end{bmatrix}$$

$$x(1000) = \begin{bmatrix} x_8(0) \\ x_8(0) \\ x_8(0) \\ x_8(0) \\ x_8(0) \\ x_8(0) \\ x_8(0) \\ x_9(0) \\ x_8(0) \end{bmatrix}$$

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Continuous-time Averaging Systems

This part considers a certain class of differential equations and show when consensus in steady state happen.

$$x(t + 1) = Px(t)$$

Now subtract $x(k)$ from both side

$$x(k + 1) - x(k) = (P - I_n)x(k) = -Lx(k);$$

where L is Laplacian and by dividing both side by $\tau \rightarrow 0^+$ we have

$$\dot{x}(t) = -\bar{L}x(t)$$

where $\tau\bar{L} = L$

A flock of auklets exhibit
swarm behaviour



Continuous-time Averaging Systems

A flock of animals exhibit swarm behaviour.

Every fish tries to fix their position with regards to neighbours as following manner.

$$\dot{\theta}_i = \text{average}\{\theta_j ; \text{for all neighbors of } i\} - \theta_i$$

Equivalently:

$$\dot{\theta} = -L \theta$$

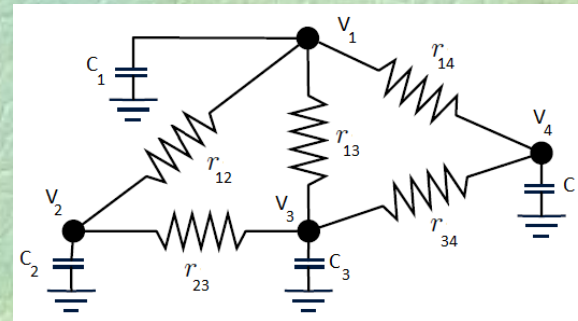
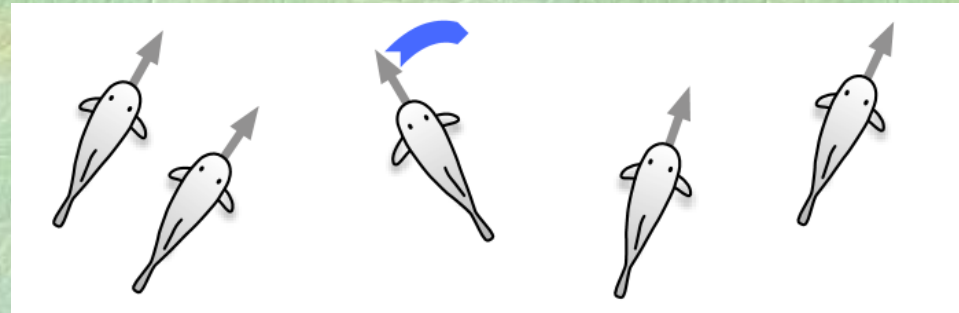
An RC electrical circuit.

$$A = \begin{bmatrix} 0 & 1/r_{12} & 1/r_{13} & 1/r_{14} \\ 1/r_{12} & 0 & 1/r_{23} & 0 \\ 1/r_{13} & 1/r_{23} & 0 & 1/r_{34} \\ 1/r_{14} & 0 & 1/r_{34} & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1/r_{12} + 1/r_{13} + 1/r_{14} & 0 & 0 & 0 \\ 0 & 1/r_{12} + 1/r_{23} & 0 & 0 \\ 0 & 0 & 1/r_{23} + 1/r_{13} + 1/r_{34} & 0 \\ 0 & 0 & 0 & 1/r_{14} + 1/r_{34} \end{bmatrix}$$

$$L = D - A$$

$$\dot{V} = -c^{-1} L V$$



Continuous-time Averaging Systems

Now we want to speak about the solution of following problem.

$$\dot{x} = -L x$$

Theorem: (Consensus for Laplacian matrices with globally reachable nodes). If a Laplacian matrix L has associated digraph G with a globally reachable node, then

- 1- The eigenvalue 0 of $-L$ is simple and all other eigenvalues of $-L$ have negative real part,
- 2- The solution at steady state is:

$$\lim_{t \rightarrow \infty} x(t) = (w^T x(0)) \mathbf{1}_n$$

where w is the left eigenvector of L corresponding to eigenvalue 0 s.t. $w^T \mathbf{1}_n = 1$

- 3- $w_i \geq 0$ for all nodes i and $w_i > 0$ if and only if node i is globally reachable,
- 4- If additionally G is weight-balanced, then G is strongly connected and

$$\lim_{t \rightarrow \infty} x(t) = \text{average}(x(0)) \mathbf{1}_n$$

Remark: There is a method to derive a weight-balanced, for strongly connected graph.

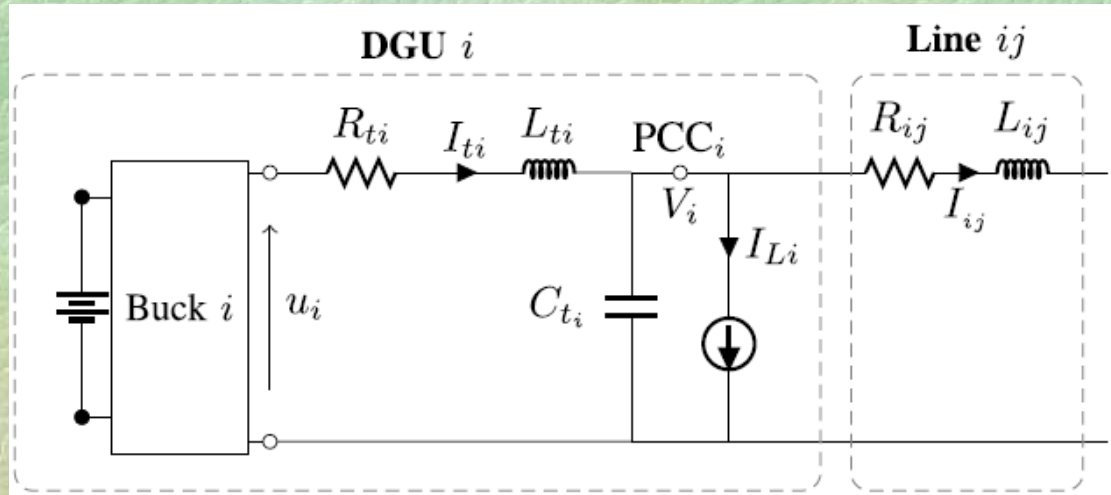
$$\bar{L} = \frac{1}{\ell_{\max}} \text{diag}(w) L,$$

$$\bar{A} = I_n - \bar{L}.$$

Where $\ell_{\max}$ is a maximum of diagonal elements of L and w is left eigenvector of the eigenvalue 0.

Continuous-time Averaging Systems

DC MICROGRID MODEL [3]



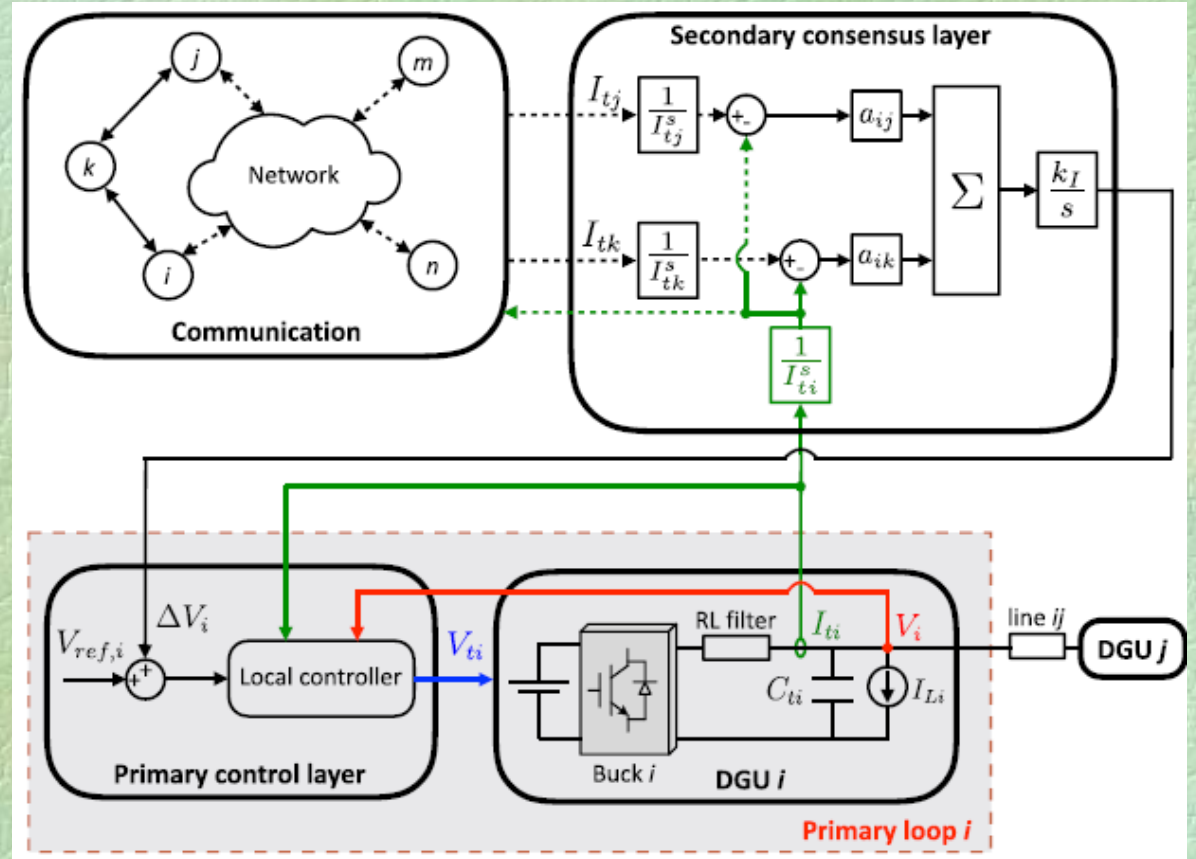
$$L_{ti} \dot{I}_{ti} = -R_{ti} I_{ti} - V_i + u_i$$

$$C_{ti} \dot{V}_i = I_{ti} - I_{Li} - \sum_{j \in \mathcal{N}_i} I_{ij}$$

$$L_{ij} \dot{I}_{ij} = (V_i - V_j) - R_{ij} I_{ij}.$$

Continuous-time Averaging Systems

DC MICROGRID CONSENSUS CONTROL [4]



First, we augment system with additional state variables (distributed integrators) θ_i , for each nodes as:

$$\dot{\theta}_i = - \sum_{j \in \mathcal{N}_i^c} \gamma_{ij} (w_i I_{ti} - w_j I_{tj})$$

Comparing Discrete-time and Continuous-time Averaging Systems

Consensus(Rendezvous) problem in discrete time and continuous time is:

$$x(k+1) = Px(k)$$

$$\dot{x}(t) = -\bar{L}x(t)$$

Consensus

Discrete
time

Strongly connected graph and aperiodic(Example 3 and 5)

P is row stochastic and primitive, $\lim_{k \rightarrow \infty} x(k) = (w^T x(0)) \mathbf{1}_n$

Strongly connected graph and aperiodic and weight-balanced (Example 4 and 6)

P is double stochastic primitive and $\lim_{k \rightarrow \infty} x(k) = \text{average}(x(0)) \mathbf{1}_n$

Continuous
time

If a Laplacian matrix L has associated digraph G with a globally reachable node

$$\lim_{t \rightarrow \infty} x(t) = (w^T x(0)) \mathbf{1}_n$$

If a Laplacian matrix L has associated digraph G with a globally reachable node and additionally G is weight-balanced(so it is strongly connected) then

$$\lim_{t \rightarrow \infty} x(t) = \text{average}(x(0)) \mathbf{1}_n$$

Continuous-time Averaging Systems

Example 11: (Exercise 7-4 part i and ii)

$$A = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & c \\ 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b+c & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$$

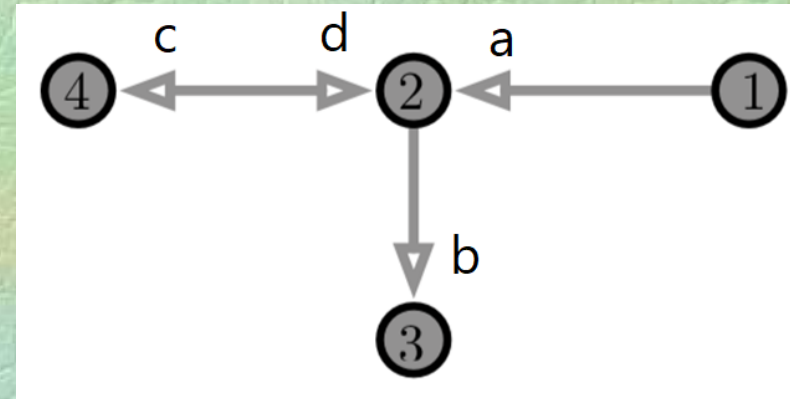
$$L = D - A$$

Let: $a=2, b=3, c=2.5, d=5$

$$\lambda_1 = 0, \quad \lambda_2 = -0.3661, \quad \lambda_3 = -0.8147, \quad \lambda_4 = -27.1196$$

$$\omega_1^T = [0 \quad 0 \quad 1 \quad 0]$$

$$\lim_{t \rightarrow \infty} x(t) = (w^T x(0)) \mathbf{1}_n = \begin{bmatrix} x_3(0) \\ x_3(0) \\ x_3(0) \\ x_3(0) \end{bmatrix}$$



Continuous-time Averaging Systems

Example 11: (Exercise 7-4 part iii **change the graph**)

$$A = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & c \\ e & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b+c & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$$

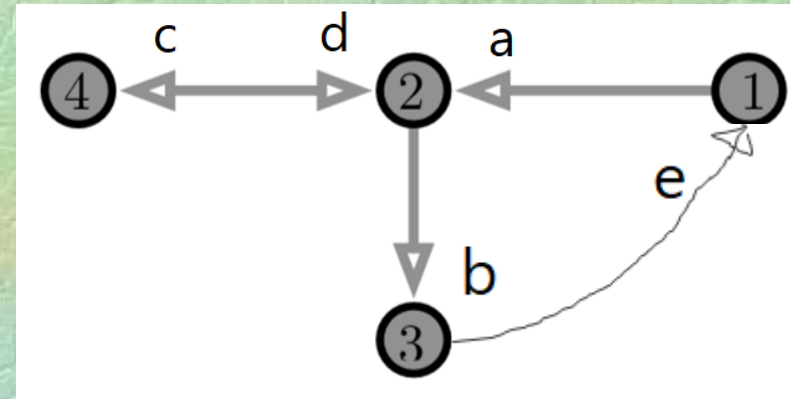
$$L = D - A$$

Let: $a=2, b=3, c=2.5, d=5, e=12$

$$\lambda_1 = 0, \quad \lambda_2 = -3.4282 \quad \lambda_3 = \lambda_4 = -10.5359 \pm 1.6602j$$

$$\omega_1^T = [0.4615 \quad 0.3077 \quad 0.0769 \quad 0.1538]$$

$$\lim_{t \rightarrow \infty} x(t) = (w^T x(0)) \mathbf{1}_n = \begin{bmatrix} 0.4615x_1(0) + 0.3077x_2(0) + 0.0769x_3(0) + 0.1538x_4(0) \\ 0.4615x_1(0) + 0.3077x_2(0) + 0.0769x_3(0) + 0.1538x_4(0) \\ 0.4615x_1(0) + 0.3077x_2(0) + 0.0769x_3(0) + 0.1538x_4(0) \\ 0.4615x_1(0) + 0.3077x_2(0) + 0.0769x_3(0) + 0.1538x_4(0) \end{bmatrix}$$



Continuous-time Averaging Systems

Example 11: (Exercise 7-4 part iii change the graph averaging)

$$L = \begin{bmatrix} 2 & -2 & 0 & 0 \\ 0 & 5.5 & -3 & -2.5 \\ -12 & 0 & 12 & 0 \\ 0 & -5 & 0 & 5 \end{bmatrix}$$

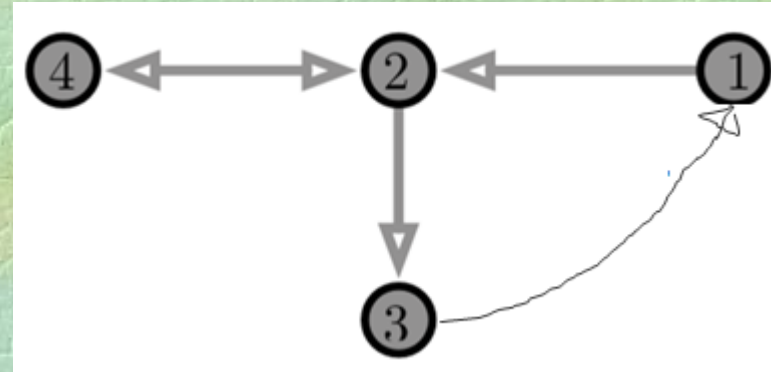
$$\bar{L} = \frac{1}{\ell_{\max}} \text{diag}(w)L,$$

$$\bar{L} = \begin{bmatrix} 0.0769 & -0.0769 & 0 & 0 \\ 0 & 0.1410 & -0.0769 & -0.0641 \\ -0.0769 & 0 & 0.0769 & 0 \\ 0 & -0.0641 & 0 & 0.0641 \end{bmatrix}$$

$$\lambda_1 = 0, \quad \lambda_2 = -0.0653 \quad \lambda_3 = \lambda_4 = -0.1468 \pm 0.0408j$$

$$\omega_1^T = [0.25 \quad 0.25 \quad 0.25 \quad 0.25]$$

$$\lim_{t \rightarrow \infty} x(t) = (w^T x(0)) \mathbf{1}_n = \begin{bmatrix} 0.25x_1(0) + 0.25x_2(0) + 0.25x_3(0) + 0.25x_4(0) \\ 0.25x_1(0) + 0.25x_2(0) + 0.25x_3(0) + 0.25x_4(0) \\ 0.25x_1(0) + 0.25x_2(0) + 0.25x_3(0) + 0.25x_4(0) \\ 0.25x_1(0) + 0.25x_2(0) + 0.25x_3(0) + 0.25x_4(0) \end{bmatrix}$$



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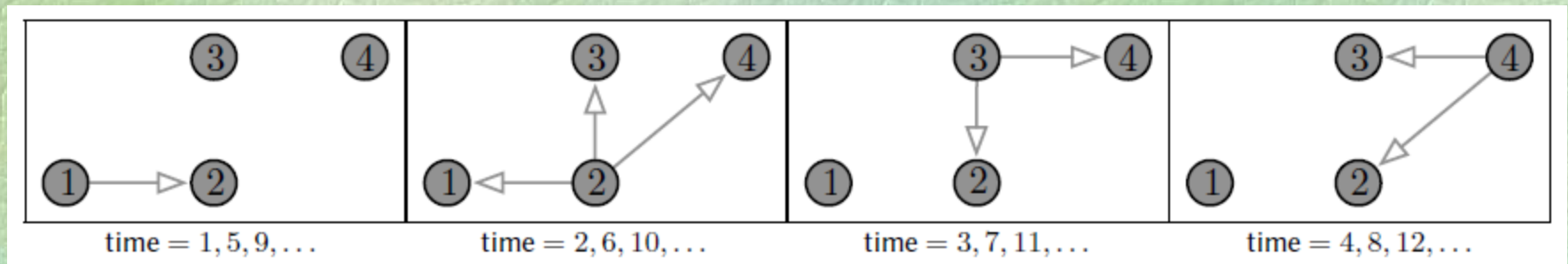
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Time-varying Averaging Systems

In time-varying systems the averaging row-stochastic matrix is not constant over the time, but instead changes values and, possibly, switches among a finite number of values.

For example in **round-robin scheduling**, where the n agents are numbered and, for each i , agent i talks(change its value) only at times $i, n+i, 2n+i, \dots, kn+i$



$$x(n+1) = A_n A_{n-1} \dots A_1 x(1):$$

As an another example in **asynchronous execution**, each time(not in synchrony) an agent wakes up, the available information from its neighbours varies. At an iteration instant for agent i , assuming agent i has new messages/information from agents $i_1, \dots, i_m$, agent i will implement:

$$x_i^+ = \frac{1}{m+1} x_i + \frac{1}{m+1} (x_{i_1} + x_{i_2} + \dots + x_{i_m})$$

Time-varying Averaging Systems

Now we want to speak about the convergence of the following system in different time-varying situation

$$x(k + 1) = A(k)x(k)$$

- Types of time-varying systems
- The graph is connected at all times.
 - Different symmetric and doubly-stochastic adjacency matrix.
 $x(t)$ **can converges exponentially fast** to $average(x(0))\mathbb{1}_n$.
 - Different primitive row-stochastic adjacency matrix.
 $x(t)$ **can converges** to $(w^T x(0))\mathbb{1}_n$.
 - The graph is not connected at all times.
 - The union digraph contains a globally reachable node (row-stochastic adjacency matrix).
 $x(t)$ **can converges exponentially fast** to $(w^T x(0))\mathbb{1}_n$.
 - The corresponding graphs to be connected over time. (symmetric row-stochastic adjacency matrix).
 $x(t)$ **can converges exponentially fast** to $average(x(0))\mathbb{1}_n$.

Time-varying Averaging Systems

The graph is connected at all times, but **different doubly-stochastic adjacency** matrix.

Theorem: (Convergence over digraphs strongly-connected at all times). Let the set of $\{A(k)\}$ be a sequence of symmetric and doubly-stochastic matrices with associated digraphs $\{G(k)\}$ so that:

(AC1) each non-zero edge weight $a_{ij}(k)$, including the self-loops weights $a_{ii}(k)$, is larger than a constant $\varepsilon > 0$ (at all time), and

(AC2) each digraph $G(k)$ is strongly connected and aperiodic.

Then the solution to $x(k+1) = A(k)x(k)$ converges exponentially fast to $\text{average}(x(0))\mathbb{1}_n$.

Proof: Define disagreement vector $\delta(k) = x(k) - \text{average}(x(0))\mathbb{1}_n$ now one must show that disagreement vector converges to zero. (Remark: $\delta(k) \in \mathbb{1}_n^\perp$)

Since $\delta(k+1) = A\delta(k)$

$$\|\delta(k+1)\|_2 = \|A(k)\delta(k)\|_2 \leq \rho_{ess}(A(k))\|\delta(k)\|_2$$

The first inequality follows from the property of $\mathbb{1}_n^\perp$ and symmetric property of $A(k)$ (Double-stochastic). Now let $c = \max\{\rho_{ess}(A(k))\}$ which is less than 1:

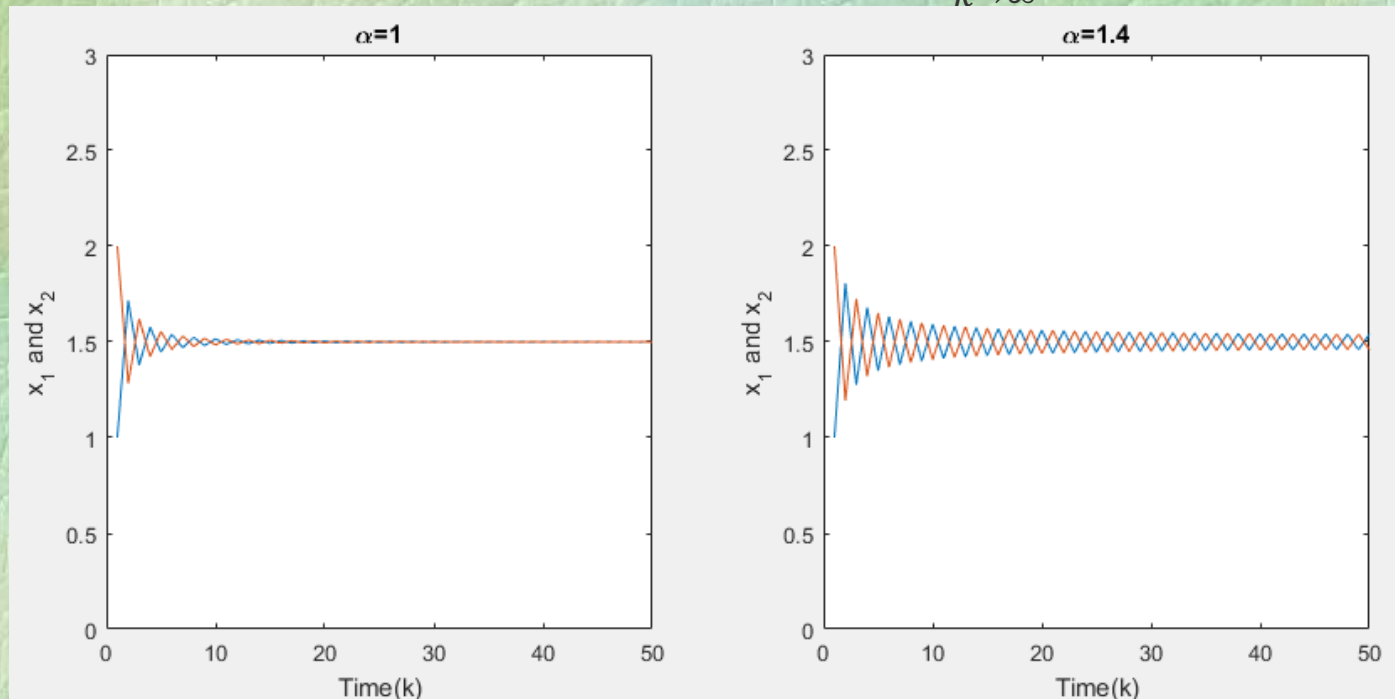
$$\|\delta(k)\|_2 \leq c^k \|\delta(0)\|_2$$

Time-varying Averaging Systems

Example 12(Importance of assumption AC1): Following example violate assumption (AC1).

$$A(k) = \begin{bmatrix} 1 - \exp(-1/(k+1)^\alpha) & \exp(-1/(k+1)^\alpha) \\ \exp(-1/(k+1)^\alpha) & 1 - \exp(-1/(k+1)^\alpha) \end{bmatrix} \quad k \in \mathbb{Z}_{\geq 0} \text{ and } \alpha \geq 1.$$

For any $\alpha \geq 1$ and any k we know that $\rho_{ess}(A(k)) < 1$. But $\lim_{k \rightarrow \infty} \rho_{ess}(A(k)) = 1$



Remark: For $\alpha=1$, $A(k)$ converges to A_∞ very slowly so $x(k)$ converges to $\text{average}(x(0))\mathbb{I}_n$, but for $\alpha>1$ $A(k)$ converges to A_∞ very fast so $x(k)$ oscillates indefinitely.

Time-varying Averaging Systems

The graph is connected at all times, but **different row-stochastic adjacency** matrix(Ex 12.6).

Theorem: (Convergence over digraphs strongly-connected at all times). Consider a sequence of row-stochastic matrices $\{A(k)\}$ with associated digraphs $\{G(k)\}$ so that

(AC1) Each non-zero edge weight $a_{ij}(k)$, including the self-loops weights $a_{ii}(k)$, is larger than a constant $\varepsilon > 0$ (at all time), and

(AC2) Each digraph $G(k)$ is strongly connected and aperiodic.

(AC3) There is a positive vector $w \in \mathbb{R}^n$ satisfying $\mathbb{1}_n^T w = 1$ and $w^T A(k) = w^T$ for all k .

Then the solution to $x(k+1) = A(k)x(k)$ converges to $\lim_{k \rightarrow \infty} x(k) = (w^T x(0)) \mathbb{1}_n$.

Time-varying Averaging Systems

The graph is not connected at all times, but the union digraph contains a globally reachable node and at any time we have **row-stochastic adjacency** matrix.

Theorem: (Consensus for time-varying algorithms) Let the set of $\{A(k)\}$ be a sequence of row-stochastic matrices with associated digraphs $\{G(k)\}$ so that:

(A1) each digraph $G(k)$ has a self-loop at each node;

(A2) Each non-zero edge weight $a_{ij}(k)$, including the self-loops weights $a_{ii}(k)$, is larger than a constant $\varepsilon > 0$ (at all time), and

(A3) There exists a duration $\delta \in \mathbb{N}$ such that, for all times k , the union digraph $G(k) \cup G(k+1) \dots \cup G(k+\delta-1)$ contains a globally reachable node. Then

- (i) There exists a non-negative vector $w \in \mathbb{R}^n$ normalized to $w_1 + \dots + w_n = 1$ such that $\lim_{k \rightarrow \infty} A(k) \cdot A(k-1) \cdot \dots \cdot A(0) = \mathbb{1}_n w^T$;
- (ii) The solution to $x(k+1) = A(k)x(k)$ converges exponentially fast to $(w^T x(0)) \mathbb{1}_n$
- (iii) If additionally each matrix in the sequence is double-stochastic, then the solution to $x(k+1) = A(k)x(k)$ converges to $\text{average}(x(0)) \mathbb{1}_n$

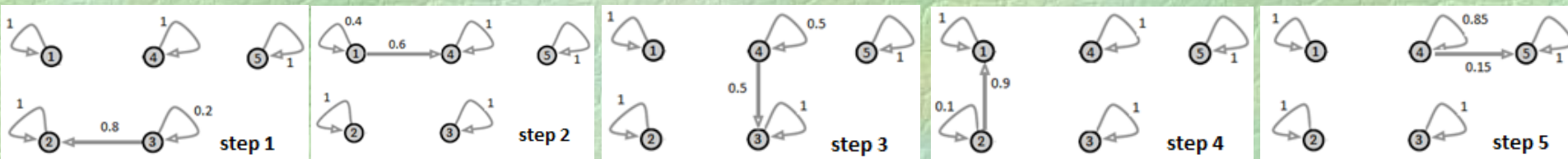
Note: In a sequence with property (A2), edges can appear and disappear, but the weight of each edge (that appears an infinite number of times) does not go to zero as $k \rightarrow \infty$

Time-varying Averaging Systems

Proof of theorem by max-min function (After introducing max-min).

Time-varying Averaging Systems

Example 13: (Consensus for time-varying algorithms) Let the set of $\{A(k)\}$ be a sequence of row-stochastic matrices with associated digraphs $\{G(k)\}$ as follows.



Initial value is:

$$x(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

Left eigenvector corresponding to largest eigenvalue of

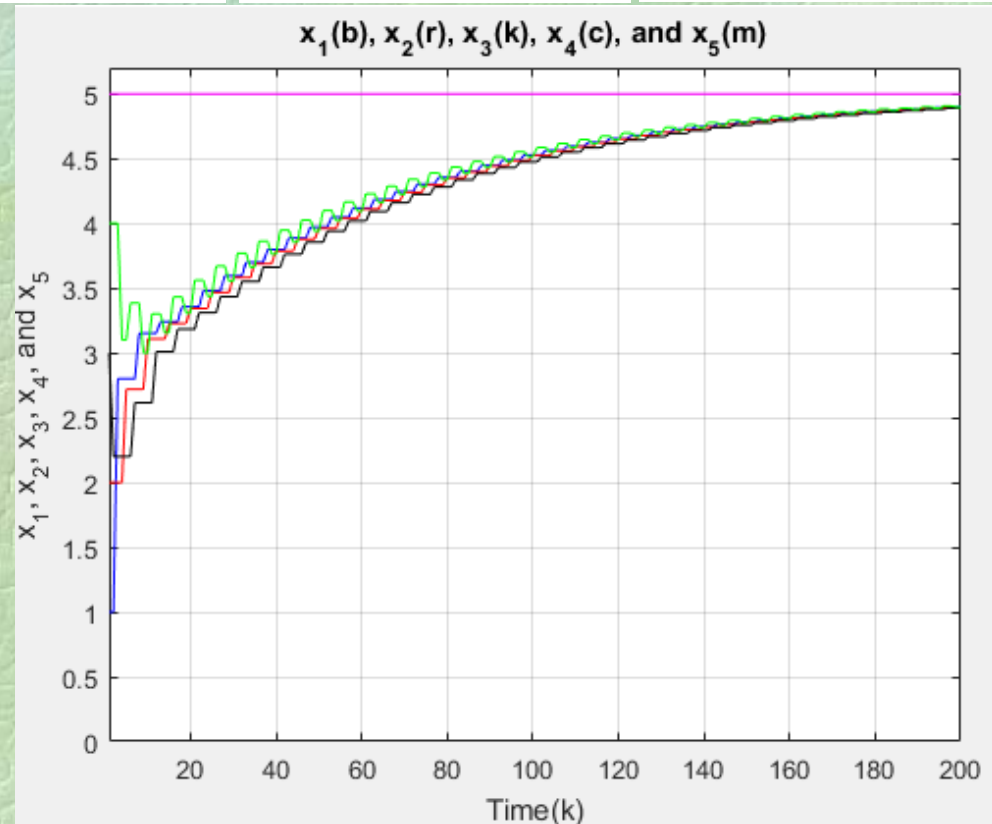
$$A = A_5 A_4 A_3 A_2 A_1$$

is:

$$w = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Consensus happens and converge to

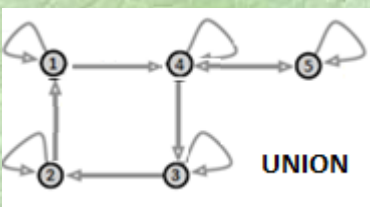
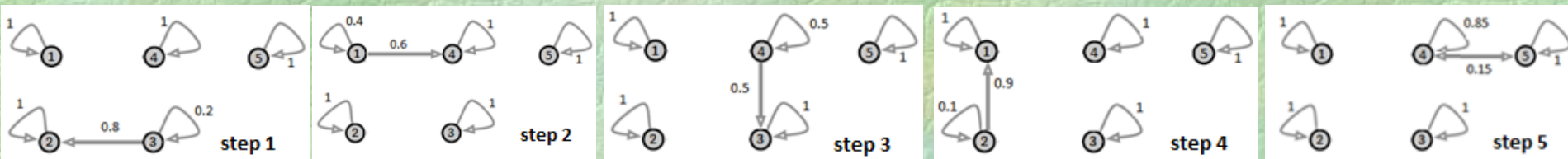
$$(w^T x(0)) \mathbb{1}_5$$



Remark: Vertex 5 is a sink so all converges to vertex 5

Time-varying Averaging Systems

Example 14: (Consensus for time-varying algorithms) Let the set of $\{A(k)\}$ be a sequence of row-stochastic matrices with associated digraphs $\{G(k)\}$ as follows. (V_5 is not sink)



Initial value is:

$$x(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

Left eigenvector corresponding to largest eigenvalue of

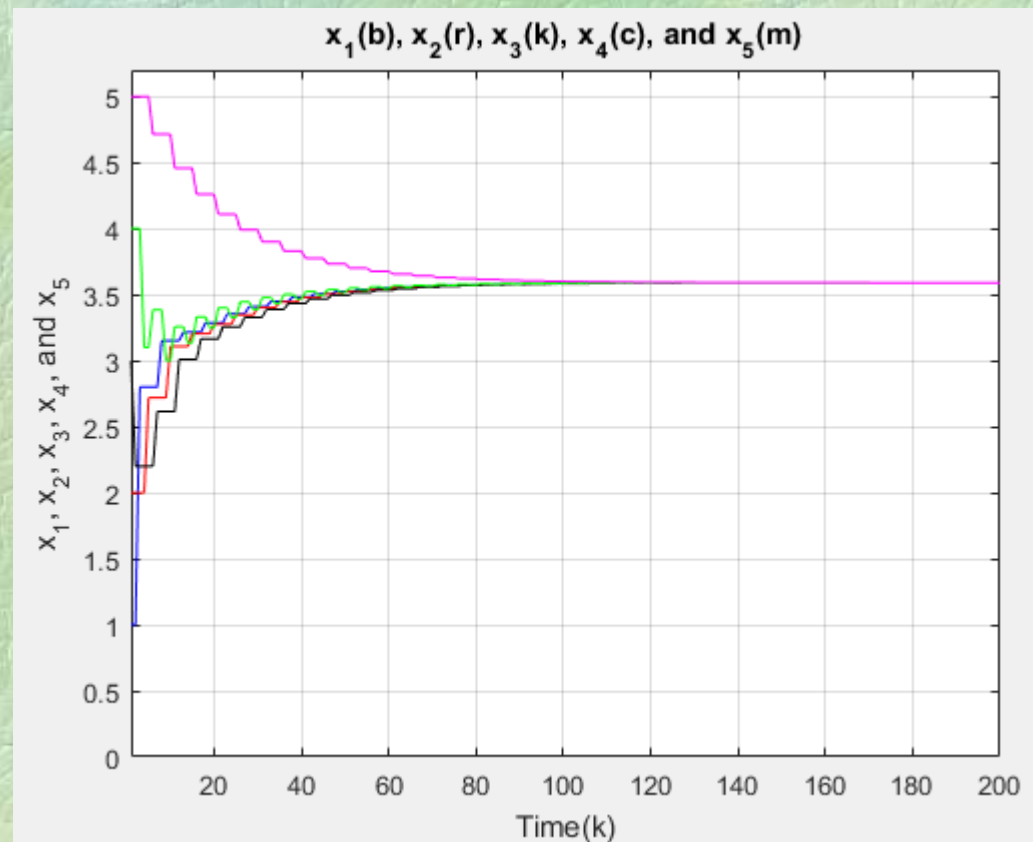
$$A = A_5 A_4 A_3 A_2 A_1$$

is:

$$w = \begin{bmatrix} 0.1106 \\ 0.1843 \\ 0.0415 \\ 0.3318 \\ 0.3318 \end{bmatrix}$$

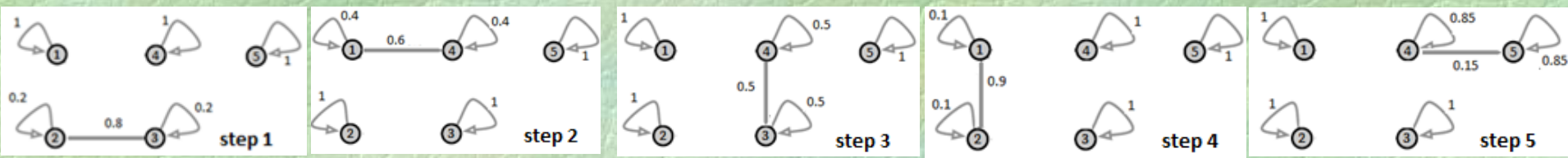
Consensus happens and converge to

$$(w^T x(0)) \mathbb{1}_5$$



Time-varying Averaging Systems

Example 15: (Consensus for time-varying algorithms) Let the set of $\{A(k)\}$ be a sequence of **double-stochastic matrices** with associated digraphs $\{G(k)\}$ as follows.



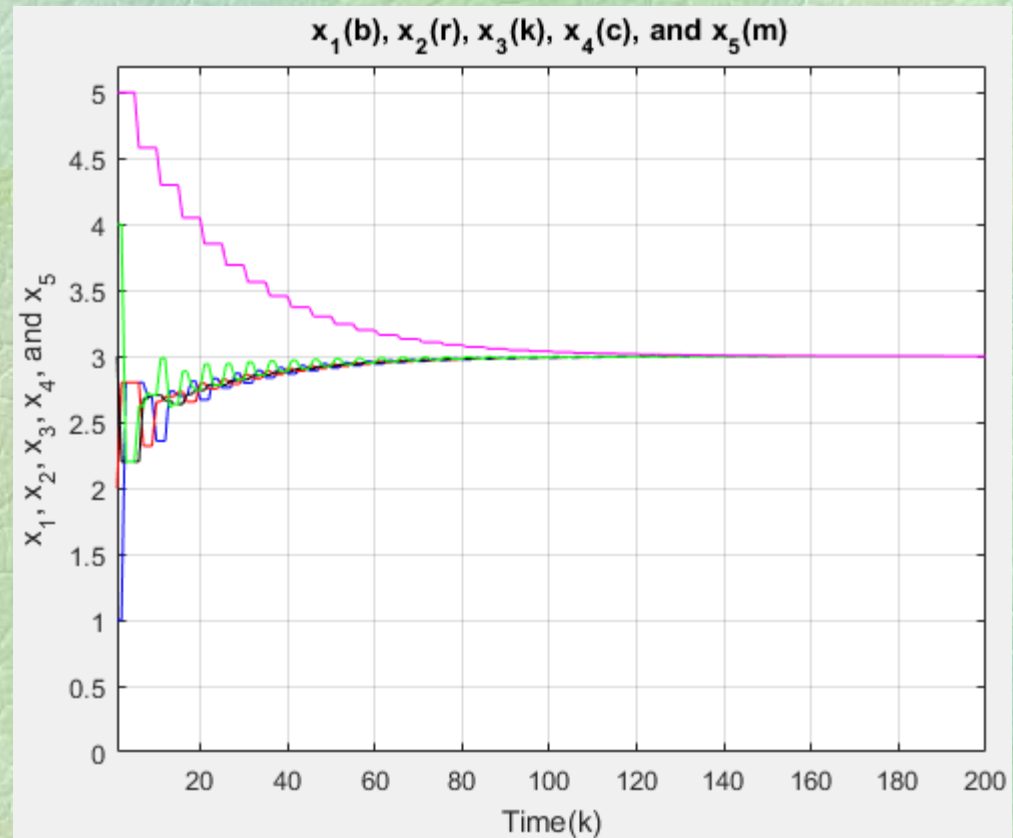
Initial value is: $x(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$

Left eigenvector corresponding to largest eigenvalue of $A = A_5 A_4 A_3 A_2 A_1$ is:

$$w = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}$$

Consensus happens and converge to

$$(w^T x(0)) \cdot \mathbb{1}_5 = \text{average}(x(0)) \cdot \mathbb{1}_5$$



Time-varying Averaging Systems

The graph is not connected at all times, but the adjacency matrix to be symmetric row stochastic and the corresponding graphs to be connected over time.

Theorem: (Consensus for symmetric time-varying algorithms) Let the set of $\{A(k)\}$ be a sequence of symmetric row-stochastic matrices with associated digraphs $\{G(k)\}$ so that:

(A1) each digraph $G(k)$ has a self-loop at each node;

(A2) Each non-zero edge weight $a_{ij}(k)$, including the self-loops weights $a_{ii}(k)$, is larger than a constant $\varepsilon > 0$ (at all time), and

(A4) For all $k \in \mathbb{Z}_{\geq 0}$, the graph $\cup_{\tau > k} G(\tau)$ is connected. Then

$$(i) \lim_{k \rightarrow \infty} A(k)A(k-1) \dots A(0) = \frac{1}{n} \mathbb{1}_n \mathbb{1}_n^T$$

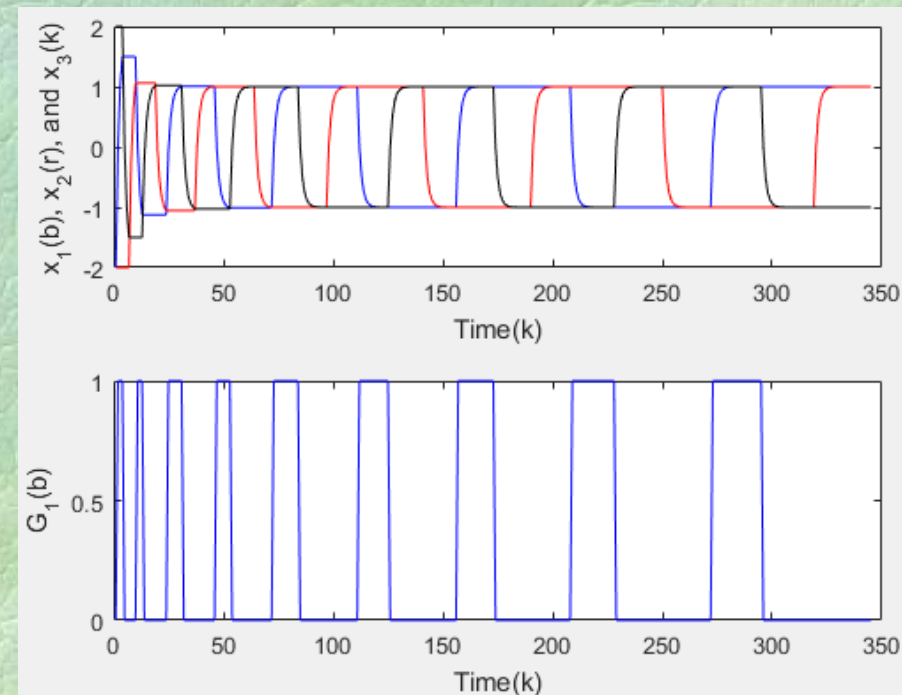
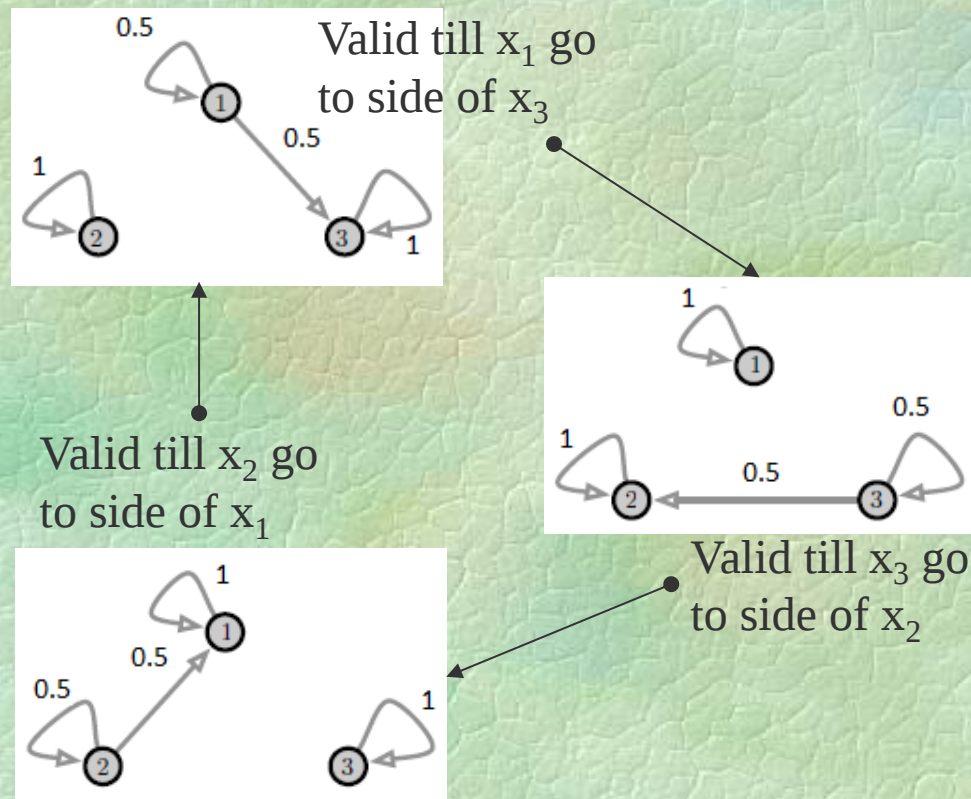
(ii) Then the solution to $x(k+1) = A(k)x(k)$ converges exponentially fast to $\text{average}(x(0)) \mathbb{1}_n$

Note: In this theorem assumption A4 is weaker than assumption A3. A3 requires the existence of a finite time-interval of duration $\delta \in \mathbb{N}$ so that the union graph $G(k) \cup G(k+1) \dots \cup G(k+\delta-1)$ contains a globally reachable node for all times $k \geq 0$.

Time-varying Averaging Systems

Following example shows the important of assumption A3 in the case of row-stochastic adjacency matrices.

Example 16: (Uniform connectivity is required for non-symmetric matrices) Consider following event driven system. Starting point is: $x_1 < -1$, $x_2 < -1$, $x_3 > +1$.



As it is shown there is no consensus(lack of **uniform connectivity**).

The max-min function and row-stochastic matrices

Define **max-min function** $V_{\max-\min} : \mathbb{R}^n \rightarrow \mathbb{R}^+$ by

$$V_{\max-\min}(x) = \max(x_1, x_2, \dots, x_n) - \min(x_1, x_2, \dots, x_n)$$

Remark: For and row-stochastic matrix we have: $V_{\max-\min}(Ax) \leq V_{\max-\min}(x)$

So, $V_{\max-\min}(x(k+1)) \leq V_{\max-\min}(x(k)) \leq \dots \leq V_{\max-\min}(x(0))$

and clearly

$$\min x(0) \leq \min x(k) \leq \min x(k+1) \leq \max x(k+1) \leq \max x(k) \leq \max x(0)$$

Column-maximum row-minimum entry is defined as:

$$\gamma(A) = \max_{j \in \{1, \dots, n\}} \min_{i \in \{1, \dots, n\}} a_{ij} \in [0, 1]$$

Remark: $\gamma(A) > 0$ if and only if A has positive column.

Example 17:

$$A = \begin{bmatrix} 0.4 & 0.6 \\ 1 & 0 \end{bmatrix}, \quad \gamma(A) = 0.4 \quad A = \begin{bmatrix} 0.5 & 0.5 \\ 0.2 & 0.8 \end{bmatrix}, \quad \gamma(A) = 0.5 \quad A=1, \quad \gamma(A) = 1$$

The max-min function and row-stochastic matrices

Theorem: (Alternative convergence analysis for discrete-time averaging)

Given an n -dimensional row-stochastic matrix A with associated digraph G , the following statements hold:

(i) For all $x \in \mathbb{R}^n$, the max-min function satisfies $V_{\max-\min}(Ax) \leq (1 - \gamma(A))V_{\max-\min}(x)$

The following properties of A are equivalent

- a) G contains a globally reachable node and the subgraph of globally reachable node is aperiodic.
- b) There exist a positive integer number h such that A^h has a positive column, and
- c) A is semi-convergent to a rank-one matrix.

Remark: If A satisfies any (and therefore all) of the above property then $V_{\max-\min}(x(k))$ converges exponentially fast to zero such that

$$V_{\max-\min}(x(k)) \leq (1 - \gamma(A^h))^{\lceil \frac{k}{h} \rceil} V_{\max-\min}(x(0))$$

Time-varying algorithms in continuous-time

In continuous system we have:

$$\dot{x}(t) = -Lx(t)$$

Theorem (Consensus for time-varying algorithms in continuous time). Let $A(t)$ be a time-varying adjacency matrix with associated time-varying digraph $G(t)$. Assume

(A1) Each non-zero edge weight $a_{ij}(t)$ is larger than a constant $\varepsilon > 0$,

(A2) There exists a duration $T > 0$ such that, for all $t \in \mathbb{R}_{\geq 0}$, the digraph associated to the adjacency matrix

$$\int_t^{t+T} L(\tau) d\tau$$

Contains a globally reachable node.

Then

a) There exist a non-negative $w \in \mathbb{R}^n$ where $\|w\|_{sum} = 1$ s.t the solution to $\dot{x}(t) = -L(t)x(t)$ converges exponentially fast to $(w^T x(0)) \mathbb{1}_n$

b) If additionally, $\mathbb{1}_n^T L(t) = 0_n^T$ for almost all times t , the solution to $\dot{x}(t) = -L(t)x(t)$ converges to $average(x(0)) \mathbb{1}_n$

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Time-varying Averaging Systems

Positive Network Control Systems

Convergence Rates, Scalability and Optimization of Averaging Systems

Positive network control systems

Positive and compartmental systems, that are, dynamical systems with state variables and output variables that are non-negative for all times given a positive initial state.

Examples of
positive/compartmental
systems

- Transportation network systems.
- Epidemic propagation models.
- Biological networks.
- Water reservoirs.
- Factory storeroom.
- Extinction behaviour.

Positive network control systems

A linear system described by:

$$\begin{aligned}\dot{x} &= Ax + bu \\ y &= cx\end{aligned}$$

or

$$\begin{aligned}x(k+1) &= Ax(k) + bu(k) \\ y(k) &= cx(k)\end{aligned}$$

is said to be a **positive linear system** iff for any nonnegative initial state vector and positive input, the output and the state trajectory are nonnegative.

A matrix is **Metzler** if all non-diagonal elements are non-negative.

Theorem: Consider following continuous linear system and suppose u is non-negative.

$$\begin{aligned}\dot{x} &= Ax + bu \\ y &= cx\end{aligned}$$

The mentioned system is positive continuous linear system iff A is Metzler and all elements of b and c are non-negative.

Theorem: Consider following discrete linear system and suppose u is non-negative.

$$\begin{aligned}x(k+1) &= Ax(k) + bu(k) \\ y(k) &= cx(k)\end{aligned}$$

The mentioned system is positive continuous linear system iff all elements of A , b and c are non-negative.

Proof: Clear.

Positive network control systems

Property of **Metzler** matrix A .

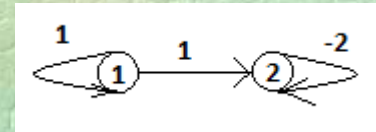
- 1) There exist a real eigenvalue λ_u where $\lambda_u \geq \text{Real}(\lambda_i)$ for $i=1, 2, \dots, n$.
- 2) There exist a nonnegative left and right eigenvector corresponding to λ_u .

If **Metzler** matrix A is irreducible(its induced graph is strongly connected) then.

- 1) There exist a real eigenvalue λ_u where $\lambda_u > \text{Real}(\lambda_i)$ for $i=1, 2, \dots, n$.
- 2) There exist a positive left and right eigenvector corresponding to λ_u .

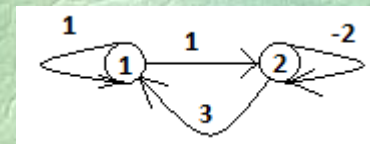
Example 15: A reducible Metzler matrix.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \quad \lambda_u = \lambda_1 = 1, \quad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \lambda_2 = -2$$



Example 16: An irreducible Metzler matrix.

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \quad \lambda_u = \lambda_1 = 1.7913, \quad v_1 = \begin{bmatrix} 0.7842 \\ 0.6205 \end{bmatrix}, \quad \lambda_2 = -2.7913$$



Positive network control systems

Theorem: (Properties of Hurwitz Metzler matrices). Consider following system where u is a constant value. If matrix A is Metzler, the following statements are equivalent:

$$\dot{x} = Ax + bu$$

- 1) A is Hurwitz,
- 2) A is invertible and $-A^{-1} \geq 0$, and
- 3) for all $b \geq 0_n$, there exists a unique $x^* \geq 0_n$ solving $Ax^* + b = 0_n$.

Moreover, if A is Metzler, Hurwitz and irreducible, then $-A^{-1} > 0$.

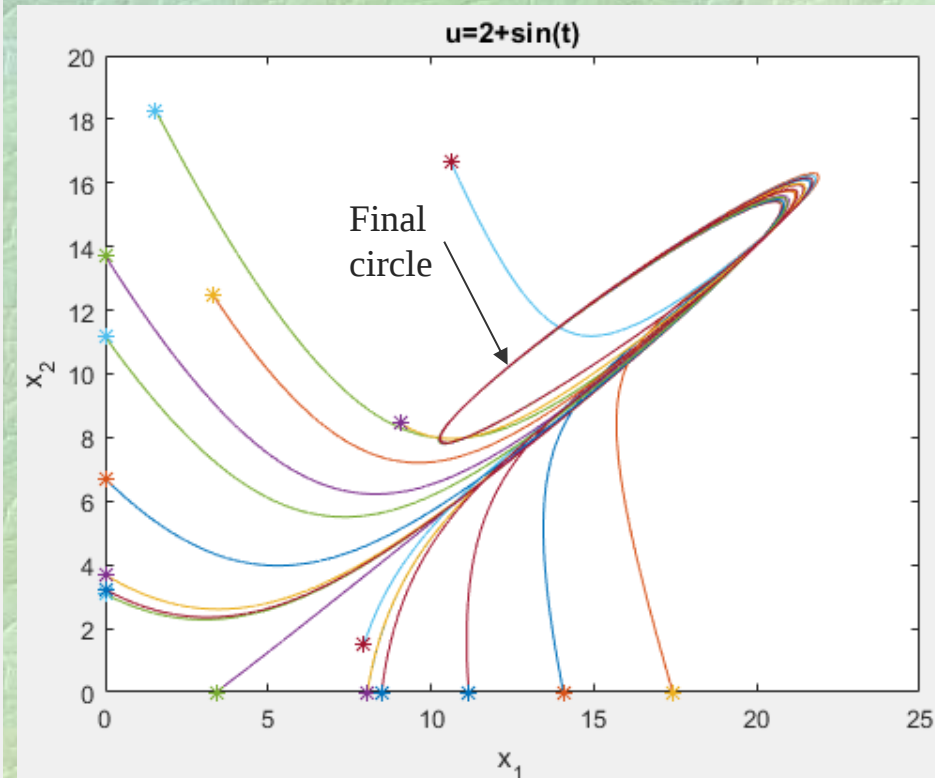
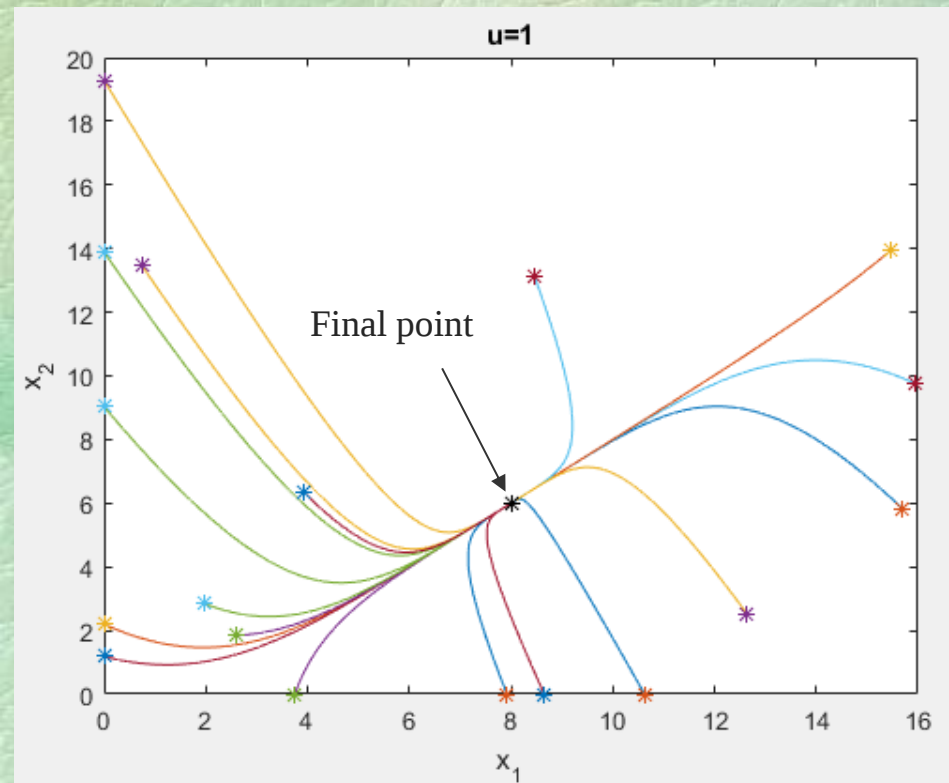
Remark: If u be bounded then there is a bounded x .

Positive network control systems

Example 17: A positive system

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 3 & -4 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \end{bmatrix} u$$

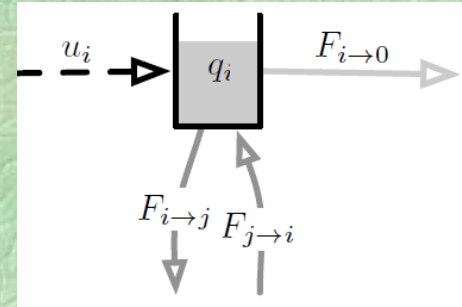
Trajectories of system for 18 different initial condition and $u=1$ and $u=2+\sin(t)$ are shown in the figure.



Compartment systems

In compartmental systems, materials are stored at individual locations and is transferred along each other in special directions. Each compartment contains a time-varying quantity $q_i(t)$.

$$\dot{q}_i(t) = \sum_{j=1, j \neq i}^n (F_{j \rightarrow i} - F_{i \rightarrow j}) - F_{i \rightarrow 0} + u_i$$



If one consider $M(q) = q_1 + q_2 + \dots + q_n$ as the total mass in the system, then

$$\dot{M}(q(t)) = - \sum_{i=1}^n F_{i \rightarrow 0}(q(t)) + \sum_{i=1}^n u_i$$

This equality implies that the total mass is constant in systems without inflows and outflows.

A **compartmental system is linear** if all flows depend linearly upon the mass in the originating compartment, and the inflow from the environment is constant and non-negative. So:

$$F_{i \rightarrow j}(q) = f_{ij}q_i, \quad F_{i \rightarrow 0}(q) = f_{i0}q_i, \quad \text{and} \quad u_i(q) = u_i$$

Compartment systems

Consider following compartmental systems:

$$\dot{q}_i(t) = \sum_{j=1, j \neq i}^n (F_{j \rightarrow i} - F_{i \rightarrow j}) - F_{i \rightarrow 0} + u_i$$

Its vector form is:

$$\dot{q}(t) = Cq(t) + u$$

where C is called the compartmental matrix and is defined by

$$c_{ij} = \begin{cases} f_{ji} & \text{if } i \neq j \\ -f_{i0} - \sum_{h=1, h \neq i}^n f_{ih} & \text{if } i = j \end{cases}$$

Definition (Compartmental matrices). A matrix C is compartmental if
(i) the off-diagonal entries are nonnegative, and
(ii) the column sums are nonpositive (weakly column diagonally dominant).

Remark: A compartmental matrix is Metzler and weakly column diagonally dominant.

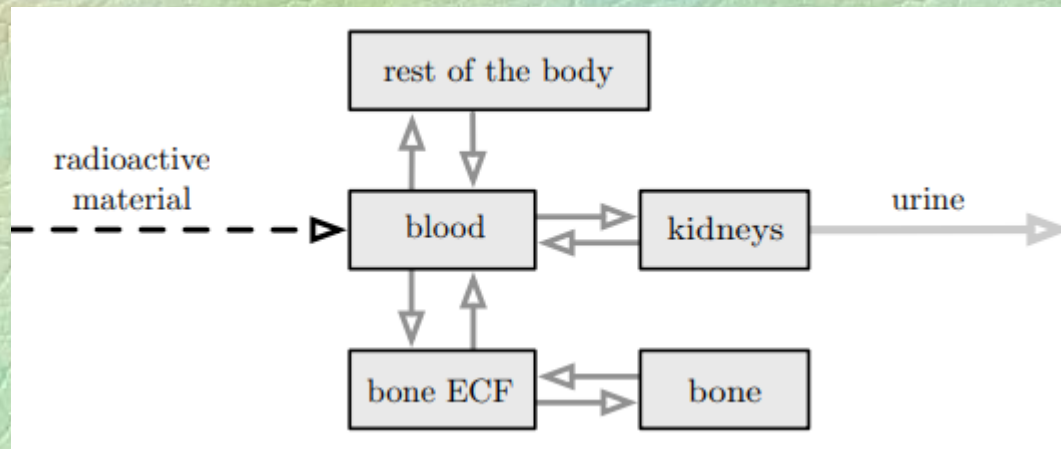
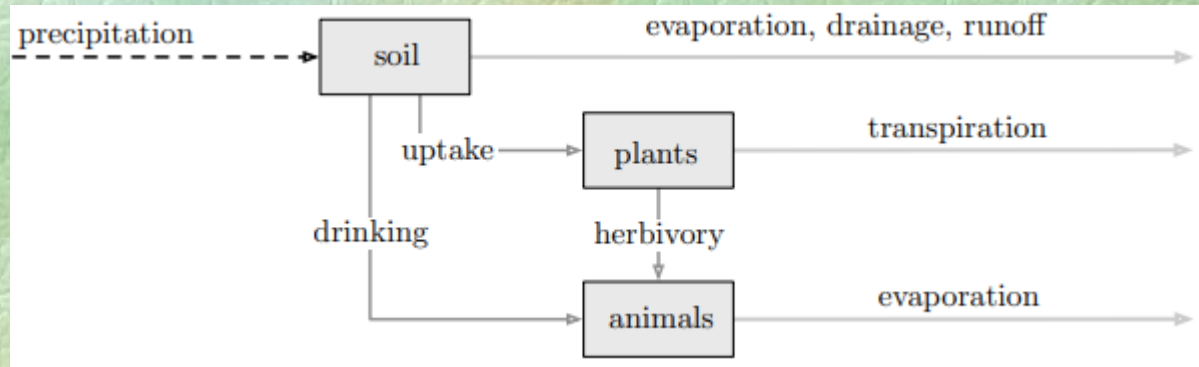
Compartment systems

Lemma: (Spectral properties of compartmental matrices). For a compartmental system with compartmental matrix C ,

- (i) Real part of eigenvalues of C is not positive.
- (ii) C is invertible if and only if C is Hurwitz (i.e., Real part of eigenvalues of C is negative)

Proof: (i) is clear by Gershgorin circle.

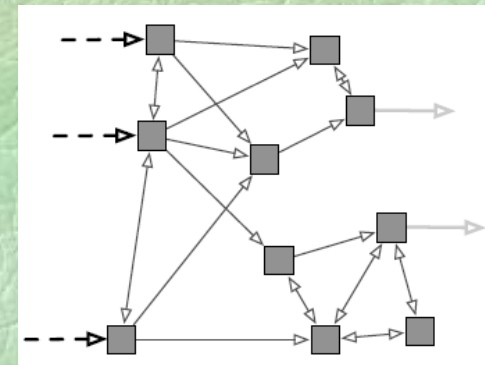
Compartment systems



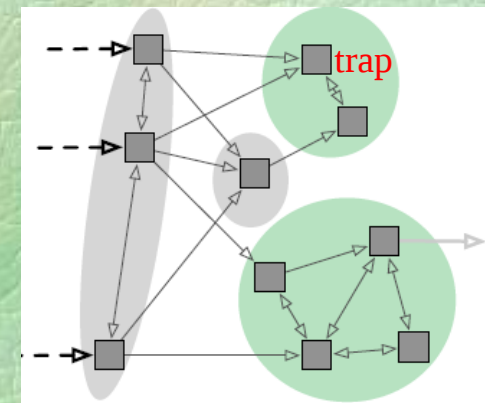
Compartment systems

A compartment system is **outflow-connected** if there exists a directed path from any compartment in S to the environment.

A compartment system is **inflow-connected** if there exists a directed path from the environment to any compartment in S .



A compartment system contains a **trap** if there is no directed path from any of the compartments in S to the environment or to any compartment outside S .



A **simple trap** is a trap that has no traps inside it.

Remark: The system is outflow connected (i.e., all compartments are outflow-connected) if and only if the system contains no trap.

Compartment systems

Theorem (Algebraic graph theory of compartmental systems). Consider the linear compartmental system with dynamics

$$\dot{q}(t) = Cq(t) + u$$

and with compartmental matrix C and compartmental digraph G_F .

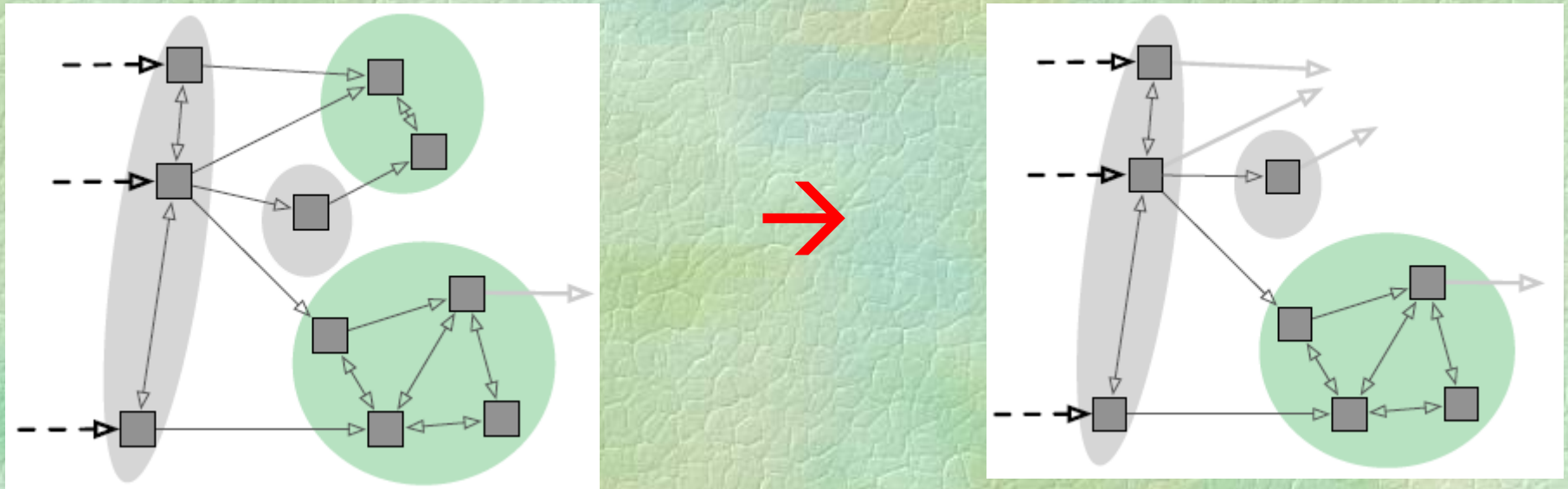
The following statements are equivalent:

- (i) the system is outflow-connected,
- (ii) each sink of the condensation of G_F is outflow-connected, and
- (iii) the compartmental matrix C is Hurwitz.

Moreover, the sinks of the condensation of G_F that are not outflow-connected are precisely the simple traps of the system and their number equals the multiplicity of 0 as a semi simple eigenvalue of C .

Compartment systems

Reduced compartmental system is derived when one remove all simple traps(if exist) from G_F and regard the edges into the removed compartments as outflow edges into the environment.



Compartment systems

Theorem (Asymptotic behavior of compartmental systems). The linear compartmental system

$$\dot{q}(t) = Cq(t) + u$$

with compartmental matrix C and compartmental digraph G_F has the following possible asymptotic behaviors:

(i) if the system is outflow-connected, then the compartmental matrix C is invertible, every solution tends exponentially to the unique equilibrium $q^* = -C^{-1}u \geq 0_n$, and in the i^{th} compartment $q_i^* > 0$ if and only if the i^{th} compartment is inflow-connected to a positive inflow;

(ii) if the system contains one or more simple traps, then:

(a) the reduced compartmental system is outflow-connected and all its solutions converge exponentially fast to the unique non-negative equilibrium $C_{rd}^{-1}u_{rd}$, for $C_{rd} = F_{rd}^T - \text{diag}(F_{rd}\mathbb{1}_n + f_{rd,0})$;

(b) any simple trap H contains non-decreasing mass along time. If H is inflow-connected to a positive inflow, then the mass inside H grows linearly with time. Otherwise, the mass inside H converges to a scalar multiple of the right eigenvector corresponding to the eigenvalue 0 of the compartmental submatrix for H .

Compartment systems

Tables of asymptotic behaviours for averaging and positive discrete-time systems

Dynamics	Assumptions & Asymptotic Behavior	References
averaging system $x(k+1) = Ax(k)$ A row-stochastic	the associated digraph has a globally reachable node $\implies$ $\lim_{k \rightarrow \infty} x(k) = (w^T x(0)) \mathbf{1}_n$ where $w \geq 0$ is the left eigenvector of A with eigenvalue 1 satisfying $\mathbf{1}_n^T w = 1$	Convergence properties: Theorem 5.1. Examples: opinion dynamics & averaging in Chapter 1
affine system $x(k+1) = Ax(k) + b$	A convergent (that is, its spectral radius is less than 1) $\implies \lim_{k \rightarrow \infty} x(k) = (I_n - A)^{-1}b$	Convergence properties: Exercise E2.10. Examples: Friedkin-Johnsen system in Exercise E5.8
positive affine system $x(k+1) = Ax(k) + b$ $A \geq 0, b \geq \mathbf{0}_n$	$x(0) \geq \mathbf{0}_n \implies x(k) \geq \mathbf{0}_n$ for all k , and A convergent (that is, $ \lambda < 1$ for all $\lambda \in \text{spec}(A)$) $\implies \lim_{k \rightarrow \infty} x(k) = (I_n - A)^{-1}b \geq \mathbf{0}_n$	Positivity properties: Exercise E10.9 Examples: Leslie population model in Exercise E4.14

Compartment systems

Tables of asymptotic behaviours for averaging and positive continuous-time systems

Dynamics	Assumptions & Asymptotic Behavior	References
averaging system $\dot{x}(t) = -Lx(t)$ L Laplacian matrix	the associated digraph has a globally reachable node $\implies$ $\lim_{t \rightarrow \infty} x(t) = (w^\top x(0)) \mathbf{1}_n$ where $w \geq 0$ is the left eigenvector of L with eigenvalue 0 satisfying $\mathbf{1}_n^\top w = 1$	Convergence properties: Theorem 7.4. Examples: Flocking system in Section 7
affine system $\dot{x}(t) = Ax(t) + b$	A Hurwitz (that is, its spectral abscissa is negative) $\implies \lim_{t \rightarrow \infty} x(t) = -A^{-1}b$	Convergence properties: Exercise E7.2
positive affine system $\dot{x}(t) = Ax(t) + b$ A Metzler, $b \geq \mathbf{0}_n$	$x(0) \geq \mathbf{0}_n \implies x(t) \geq \mathbf{0}_n$ for all t , and A Hurwitz (that is, $\Re(\lambda) < 0$ for all $\lambda \in \text{spec}(A)$) $\implies \lim_{t \rightarrow \infty} x(t) = -A^{-1}b \geq \mathbf{0}_n$	Positivity properties: Theorem 10.3 and Corollary 10.6. Example: compartmental systems in Section 10.
compartmental system $\dot{q}(t) = Cq(t) + u$ C compartmental, $u \geq \mathbf{0}_n$	$q(0) \geq \mathbf{0}_n \implies q(t) \geq \mathbf{0}_n$ for all t , and system is outflow-connected $\implies \lim_{t \rightarrow \infty} q(t) = -C^{-1}u \geq \mathbf{0}_n$	Algebraic graph theory: Theorem 10.12 Asymptotic behavior: Theorem 10.13

References

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- [3] “A Robust Consensus Algorithm for Current Sharing and Voltage Regulation in DC Microgrids” Cucuzzella, ... ,IEEE TRANSACTIONS ON CONTROL SYSTEMS TECHNOLOGY, 2019
- [4] “Stable current sharing and voltage balancing in DC microgrids: A consensus-based secondary control layer” Tucci, ... , Automatica, 2018.