Engineering Mathematics

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Content of this course

1. Fourier Series and Fourier Integral.

2. Partial Differential Equation and Its Solutions.

3. Complex Analysis. (The theory of functions of a complex variable)

Part One: Fourier Series and Fourier Integral

- Introduction to Fourier Series
- Determining Fourier Series Coefficients and Related Theorems
- Half-Range Expansions
- Different Representations of Fourier Series
- Applications of Fourier Series in Engineering
- Fourier Integral
- Applications of Fourier Integral in Engineering

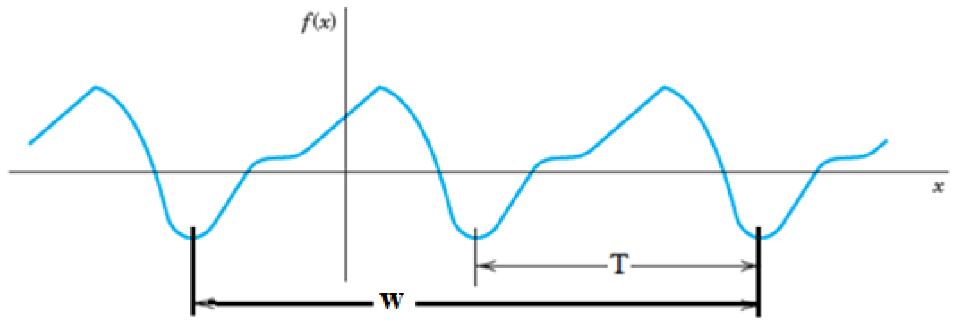
Introduction to Fourier Series

Periodic Function

A function f(x) is periodic if there exists a positive constant T such that

$$f(x + T) = f(x) \qquad \forall x$$

An example of an oscillatory function is:

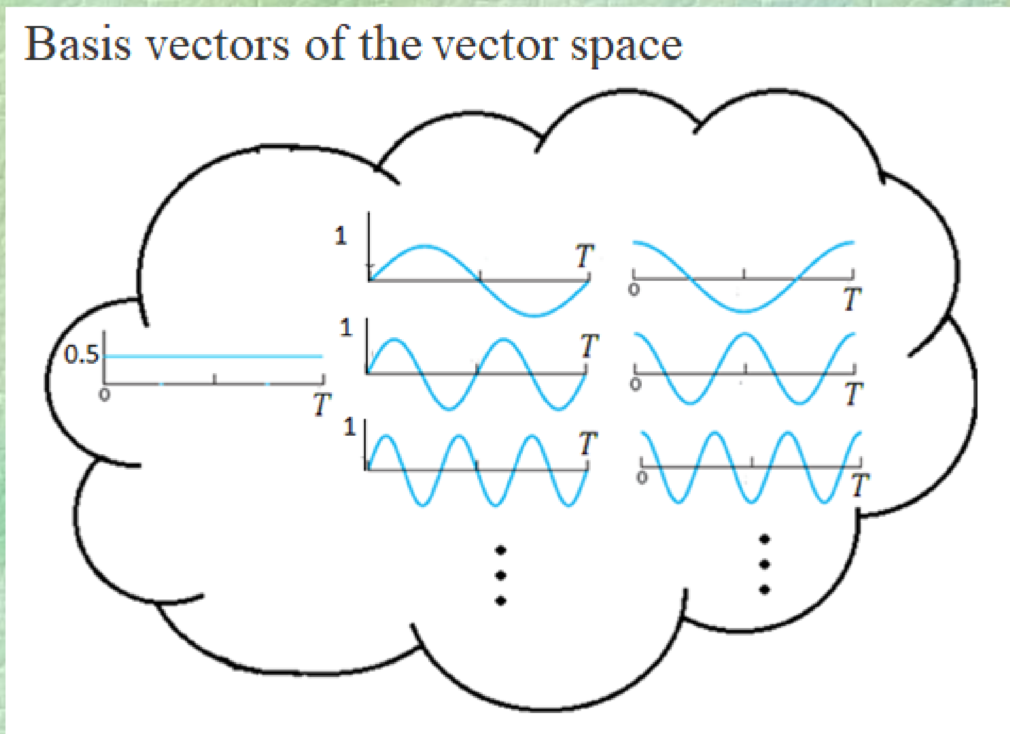


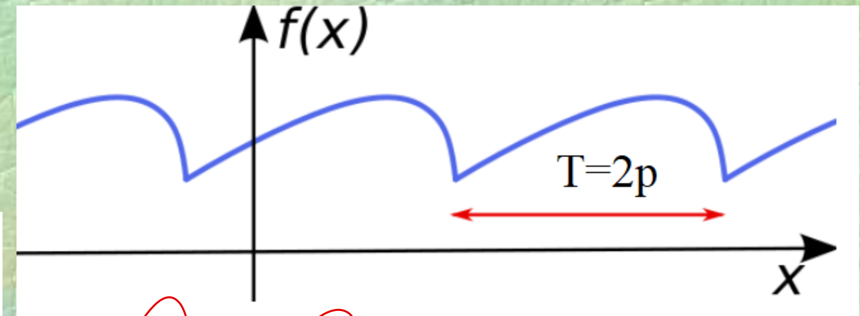
Periodic function with fundamental period T
Periodic function with period 2T, 3T, …

Periodic function decomposition (Fourier series)

Periodic function:

What is the basis for this space?





0.5 $\quad cos1\frac{2\pi}{T}x$ $\quad cos2\frac{2\pi}{T}x$

$sin1\frac{2\pi}{T}x$ $\quad \ldots$

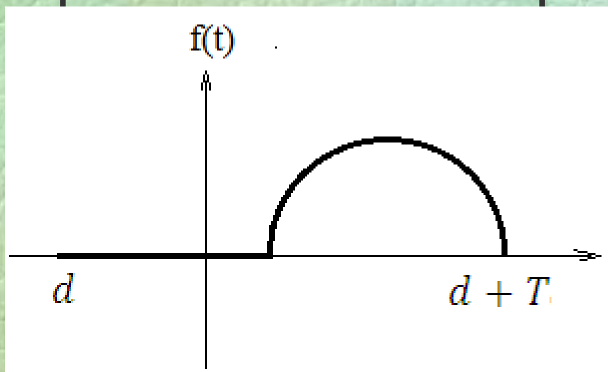
$sin2\frac{2\pi}{T}x$ $\quad \ldots$

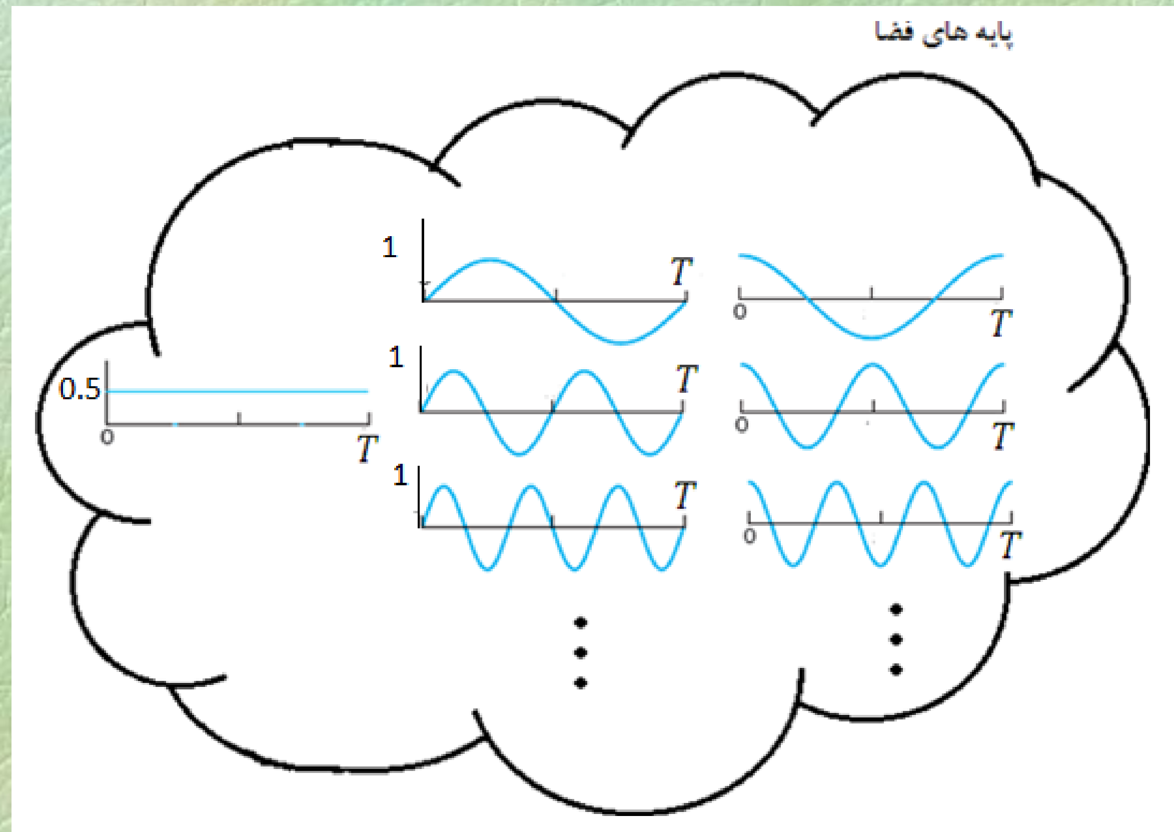
Note1: It is not finite dimensional.

Note2: What is the representation of f(x)?

Introduction to Fourier Series

Representation of a periodic function in terms of basis functions:





$$f(t) = a_0 \frac{1}{2} + a_1 \cos\left(\frac{2\pi}{T} t\right) +$$

$$\ldots\ldots a_n \cos\left(\frac{2nn}{T} t\right) + \cdots +$$

$$b_1 \sin\left(\frac{2\pi}{T} t\right) + \ldots\ldots$$

$$b_n \sin\left(\frac{2n\pi}{T} t\right) + \ldots\ldots\ldots$$

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi k}{T} t\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2\pi k}{T} t\right)$$

Part One: Fourier Series and Fourier Integral

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- Determining Fourier Series Coefficients and Related Theorems
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Determining Fourier Series Coefficients for Functions and Related Theorems

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2k\pi}{T}t\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2k\pi}{T}t\right)$$

By integrating both sides of the equation over one period, we have:

$$\int_d^{d+T} f(t)dt = \int_d^{d+T} \frac{a_0}{2} dt + \cancelto{0}{\int_d^{d+T} \sum_{k=1}^{\infty} a_k \cos\left(\frac{2k\pi}{T}t\right) dt} + \cancelto{0}{\int_d^{d+T} \sum_{k=1}^{\infty} b_k \sin\left(\frac{2k\pi}{T}t\right) dt}$$

$$\boxed{a_0 = \frac{2}{T}\int_d^{d+T} f(t)dt}$$

What is the meaning of this coefficient?

Determining Fourier Series Coefficients for Functions and Related Theorems

$$f(t)=\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos\left(\frac{2k\pi}{T}t\right)+\sum_{k=1}^{\infty}b_k\sin\left(\frac{2k\pi}{T}t\right)$$

To determine the coefficients a_n, it is sufficient to multiply both sides of the equation by $\cos\left(\frac{2n\pi}{T}t\right)$ and then integrate over one period.

$$\int_d^{d+T} f(t)\cos\left(\frac{2n\pi}{T}t\right)dt=$$

$$\underbrace{\cancel{\int_d^{d+T}\frac{a_0}{2}\cos\left(\frac{2n\pi}{T}t\right)dt}}_{0}+\int_d^{d+T}\sum_{k=1}^{\infty}a_k\cos\left(\frac{2k\pi}{T}t\right)\cos\left(\frac{2n\pi}{T}t\right)dt+\underbrace{\cancel{\int_d^{d+T}\sum_{k=1}^{\infty}b_k\sin\left(\frac{2k\pi}{T}t\right)\cos\left(\frac{2n\pi}{T}t\right)dt}}_{0}$$

$$\boxed{a_n=\frac{2}{T}\int_d^{d+T} f(t)\cos\left(\frac{2n\pi}{T}t\right)dt}$$

Determining Fourier Series Coefficients for Functions and Related Theorems

$$f(t)=\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos\left(\frac{2k\pi}{T}t\right)+\sum_{k=1}^{\infty}b_k\sin\left(\frac{2k\pi}{T}t\right)$$

To determine the coefficients b_n, it is sufficient to multiply both sides of the equation by $\sin\left(\frac{2n\pi}{T}t\right)$ and then integrate over one period.

$$\int_d^{d+T} f(t)\sin\left(\frac{2n\pi}{T}t\right)dt=$$

$$\underbrace{\int_d^{d+T}\frac{a_0}{2}\sin\left(\frac{2n\pi}{T}t\right)dt}_{0}+\underbrace{\int_d^{d+T}\sum_{k=1}^{\infty}a_k\cos\left(\frac{2k\pi}{T}t\right)\sin\left(\frac{2n\pi}{T}t\right)dt}_{0}+\int_d^{d+T}\sum_{k=1}^{\infty}b_k\sin\left(\frac{2k\pi}{T}t\right)\sin\left(\frac{2n\pi}{T}t\right)dt$$

$$\boxed{b_n=\frac{2}{T}\int_d^{d+T} f(t)\sin\left(\frac{2n\pi}{T}t\right)dt}$$

Determining Fourier Series Coefficients for Functions and Related Theorems

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2k\pi}{T}t\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2k\pi}{T}t\right)$$

$$\boxed{a_0 = \frac{2}{T}\int_{d}^{d+T} f(t)dt}$$

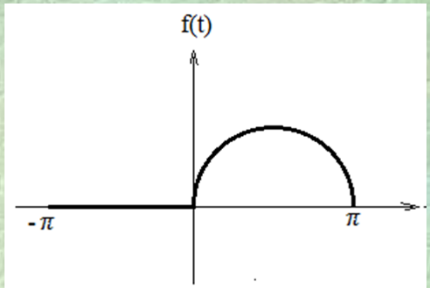
$$\boxed{a_n = \frac{2}{T}\int_{d}^{d+T} f(t)\cos\left(\frac{2n\pi}{T}t\right)dt}$$

$$\boxed{b_n = \frac{2}{T}\int_{d}^{d+T} f(t)\sin\left(\frac{2n\pi}{T}t\right)dt}$$

Determining Fourier Series Coefficients for Functions and Related Theorems

Example 1: Calculate the Fourier series of the periodic function *f*(*t*), which is defined over one period as follows:

$$f(t) = \begin{cases} 0 & -\pi < t < 0 \\ sint & 0 < t < \pi \end{cases}$$



$$T = 2\pi$$

$$a_n = \frac{2}{T}\int_d^{d+T} f(t)\cos\left(\frac{2n\pi}{T}t\right)dt = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\cos\left(\frac{2n\pi}{T}t\right)dt = \frac{1}{\pi}\int_0^{\pi} sint\cos(nt)\,dt$$

$$a_n = \frac{1}{\pi}[-\frac{1}{2}\frac{\cos(n+1)\,t}{n+1} - \frac{1}{2}\frac{\cos(1-n)\,t}{1-n}]_0^{\pi} \qquad a_n = \begin{cases} \dfrac{2}{\pi(1-n^2)} & n = 2k \\ 0 & n = 2k+1 \end{cases}$$

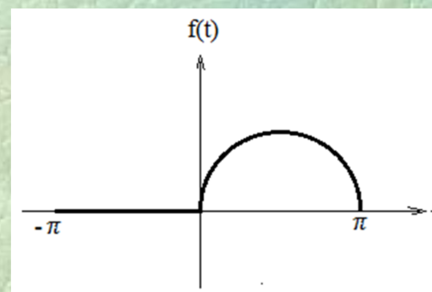
$$n \neq 1$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin(t)\cos(t)\,dt = \frac{1}{2\pi}\int_0^{\pi} \sin(2t)\,dt = 0$$

Determining Fourier Series Coefficients for Functions and Related Theorems

Example 1: Calculate the Fourier series of the periodic function f(t)f(t)f(t), which is defined over one period as follows:

$$f(t) = \begin{cases} 0 & -\pi < t < 0 \\ sint & 0 < t < \pi \end{cases}$$



$$T = 2\pi$$

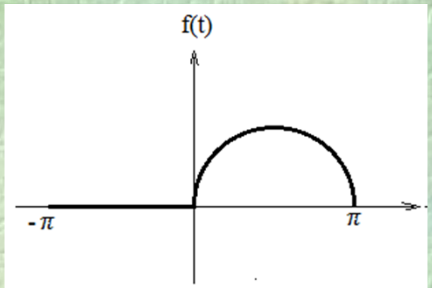
$$b_n = \frac{2}{T}\int_d^{d+T} f(t)\sin\left(\frac{2n\pi}{T}t\right)dt \;=\frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\sin\left(\frac{2n\pi}{T}t\right)dt$$

$$b_n = \frac{1}{\pi}\left(\int_0^{\pi} sin(t)\,\mathrm{s} = \mathrm{in}(nt)\,dt\right)\;[\frac{1}{2}\left\{\frac{\sin(1-n)\,t}{1-n} - \frac{\sin(1+n)t}{1+n}\right\}]_0^{\pi} \;=0 \qquad n\neq 1$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi}\sin^2(t)\,dt = \frac{1}{2} \qquad n=1$$

Determining Fourier Series Coefficients for Functions and Related Theorems

Example 1: Calculate the Fourier series of the periodic function f(t)f(t)f(t), which is defined over one period as follows:



$$f(t) = \begin{cases} 0 & -\pi < t < 0 \\ sint & 0 < t < \pi \end{cases}$$

$$T = 2\pi$$

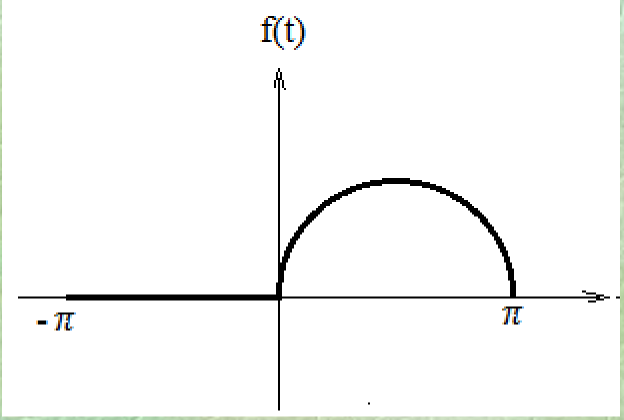
$$a_n = \begin{cases} \dfrac{2}{\pi(1-n^2)} & n = 2k \\ 0 & n = 2k+1 \end{cases} \qquad b_1 = \frac{1}{2} \qquad b_n = 0 \qquad n \neq 1$$

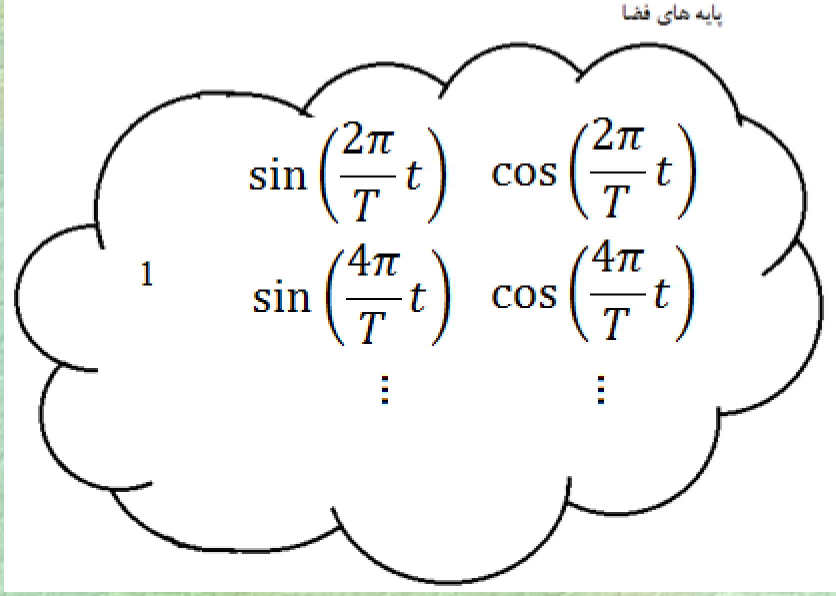
$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2k\pi}{T} t\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2k\pi}{T} t\right)$$

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\sin(t) - \frac{2}{3\pi}\cos(2t) - \frac{2}{15\pi}\cos(4t) - \cdots$$

Determining Fourier Series Coefficients for Functions and Related Theorems

Representation of a periodic function in terms of its basis functions:



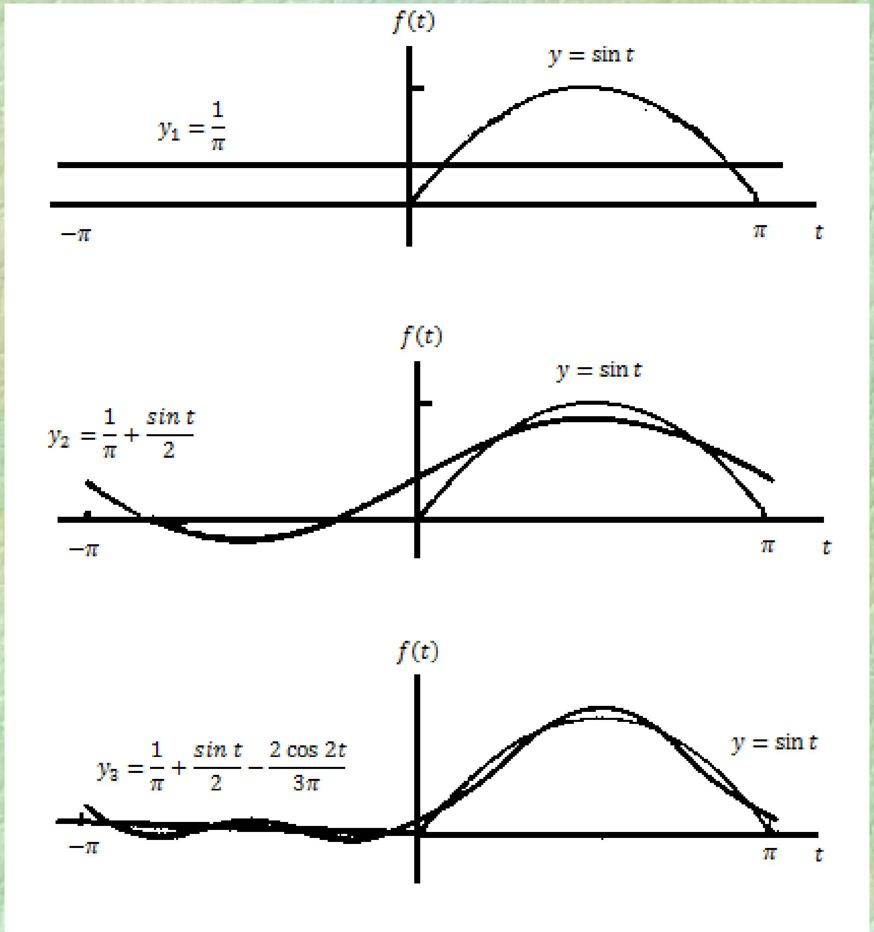


$$f(t) = \begin{cases} 0 & -\pi < t < 0 \\ sint & 0 < t < \pi \end{cases}$$

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\sin(t) - \frac{2}{3\pi}\cos(2t) - \frac{2}{15\pi}\cos(4t) - \cdots$$

Determining Fourier Series Coefficients for Functions and Related Theorems

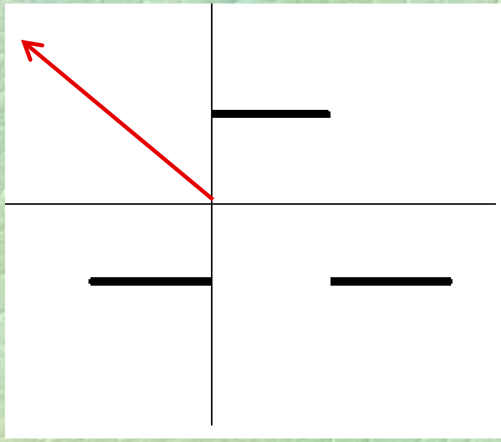
Approximation of a function using the first few terms of its Fourier series



Determining Fourier Series Coefficients for Functions and Related Theorems

Dirichlet's Theorem: If *f(t)* is a bounded periodic function with a finite number of maxima and minima in each period, and it has a finite number of discontinuities in each period, then the Fourier series of *f(t)* will converge to *f(t)* at the points of continuity of *f(t)*, and it will converge to the average of the left-hand and right-hand limits at the points of discontinuity.

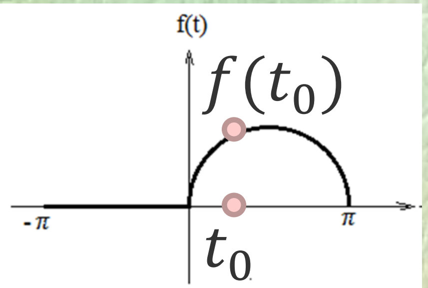
Point to which the series converges



Determining Fourier Series Coefficients for Functions and Related Theorems

Example 2: Verify Dirichlet's Theorem

$$f(t)=\begin{cases}0 & -\pi<t<0\\ sint & 0<t<\pi\end{cases}$$

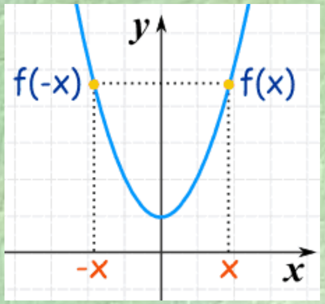


$$f(t)=\frac{1}{\pi}+\frac{1}{2}\sin(t)-\frac{2}{3\pi}\cos(2t)-\frac{2}{15\pi}\cos(4t)-\cdots$$

$$\frac{1}{\pi}+\frac{1}{2}\sin(t_0)-\frac{2}{3\pi}\cos(2t_0)-\frac{2}{15\pi}\cos(4t_0)-\cdots \quad\rightarrow\quad f(t_0)$$

Determining Fourier Series Coefficients for Functions and Related Theorems

Theorem: If *f(t)* is an even function, then:



$$b_n = \frac{2}{T}\int_d^{d+T} f(t)\sin\left(\frac{2n\pi}{T}t\right)dt$$

.
.
.
.
.

$$b_n = 0$$

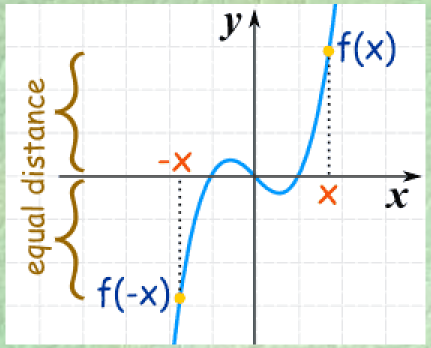
$$a_n = \frac{2}{T}\int_d^{d+T} f(t)\cos\left(\frac{2n\pi}{T}t\right)dt$$

.
.
.
.
.

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos\left(\frac{2n\pi}{T}t\right)dt$$

Determining Fourier Series Coefficients for Functions and Related Theorems

Theorem: If *f(t)* is an odd function, then:



$$a_n = \frac{2}{T}\int_d^{d+T} f(t)\cos\left(\frac{2n\pi}{T}t\right)dt$$

.

.

.

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$$a_n = 0$$

$$b_n = \frac{2}{T}\int_d^{d+T} f(t)\sin\left(\frac{2n\pi}{T}t\right)dt$$

.

.

.

.

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin\left(\frac{2n\pi}{T}t\right)dt$$

Determining Fourier Series Coefficients for Functions and Related Theorems

Example 3: Calculate the Fourier series coefficients of the following function:

$$f(t) = 4 - t^2 \qquad -2<t<2$$

Solution: Since the function is even,

$$b_n = 0$$

$$a_n = \frac{4}{4}\int_0^2 (4 - t^2)\cos\left(\frac{n\pi t}{2}\right)dt = \frac{16(-1)^{n+1}}{n^2\pi^2} \qquad n \neq 0$$

$$a_0 = \frac{4}{4}\int_0^2 (4 - t^2)dt = \frac{16}{3}$$

$$f(t) = \frac{8}{3} + \frac{16}{\pi^2}\cos\left(\frac{\pi t}{2}\right) - \frac{16}{2^2\pi^2}\cos\left(\frac{2\pi t}{2}\right) + \frac{16}{3^2\pi^2}\cos\left(\frac{3\pi t}{2}\right) + \cdots$$

Determining Fourier Series Coefficients for Functions and Related Theorems

$$f(t) = \frac{8}{3} + \frac{16}{\pi^2}\cos\left(\frac{\pi t}{2}\right) - \frac{16}{2^2\pi^2}\cos\left(\frac{2\pi t}{2}\right) + \frac{16}{3^2\pi^2}\cos\left(\frac{3\pi t}{2}\right) + \cdots$$

$$\frac{8}{3} + \frac{16}{\pi^2}\cos\left(\frac{\pi.0}{2}\right) - \frac{16}{2^2\pi^2}\cos\left(\frac{2\pi.0}{2}\right) + \frac{16}{3^2\pi^2}\cos\left(\frac{3\pi.0}{2}\right) + \cdots$$

$$= \frac{8}{3} + \frac{16}{\pi^2}\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right) \quad \rightarrow f(0) = 4$$

Calculation of the series sum using the periodic function:

$$4 = \frac{8}{3} + \frac{16}{\pi^2}\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right) \quad \rightarrow \quad 1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$$

Determining Fourier Series Coefficients for Functions and Related Theorems

Exercise 1: The periodic function below is given over one period:

$$f(t) = \begin{cases} 0 & -1 < t < 0 \\ 1 & 0 < t < 1 \end{cases}$$

a) Write the Fourier series of the function.

b) Sum the first 5 terms of the series and compare it with the actual function.

Exercise 2: Calculate the Fourier series coefficients of the periodic function below.

$$f(t) = t \qquad -1 < t < 1 \qquad T = 2$$

Sum the first 5 terms of the series and compare it with the actual function.

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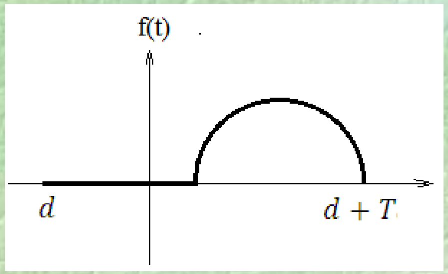
Reminder



$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2k\pi}{T}t\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2k\pi}{T}t\right)$$

Euler coefficients:

$$\boxed{a_n = \frac{2}{T}\int_{d}^{d+T} f(t)\cos\left(\frac{2n\pi}{T}t\right)dt} \qquad \boxed{b_n = \frac{2}{T}\int_{d}^{d+T} f(t)\sin\left(\frac{2n\pi}{T}t\right)dt}$$

Theorem: If f(t) is an even function:

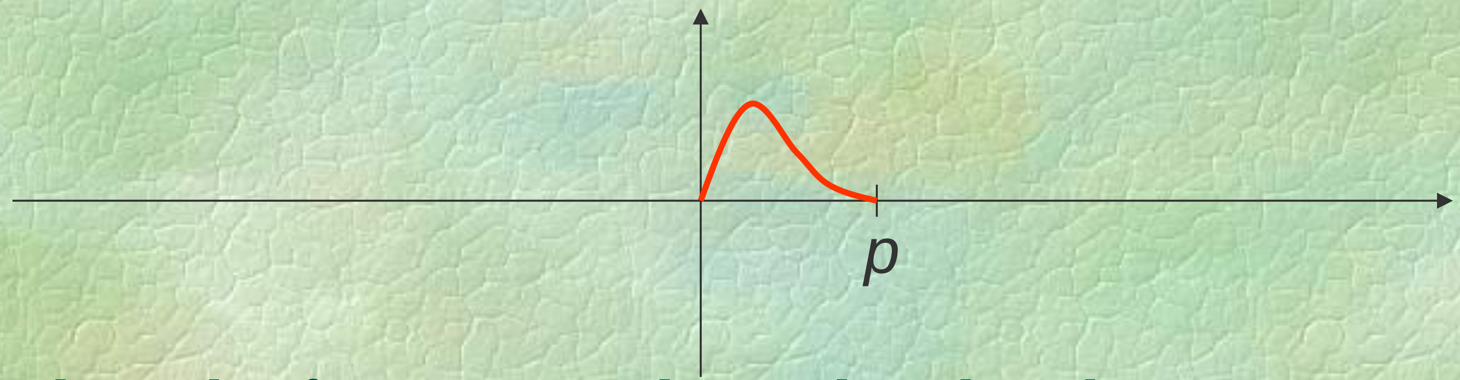
$$\boxed{b_n = 0} \qquad \boxed{a_n = \frac{4}{T}\int_{0}^{\frac{T}{2}} f(t)\cos\left(\frac{2n\pi}{T}t\right)dt}$$

Theorem: If f(t) is an odd function:

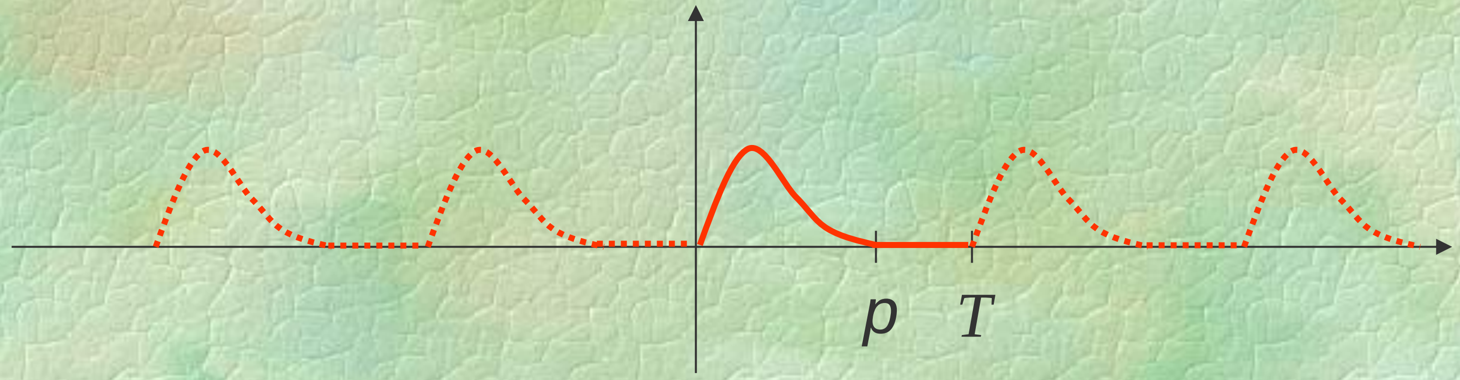
$$\boxed{a_n = 0} \qquad \boxed{b_n = \frac{4}{T}\int_{0}^{\frac{T}{2}} f(t)\sin\left(\frac{2n\pi}{T}t\right)dt}$$

Half-Range Expansions

Consider the following function:



Expanding the function as desired and making it periodic

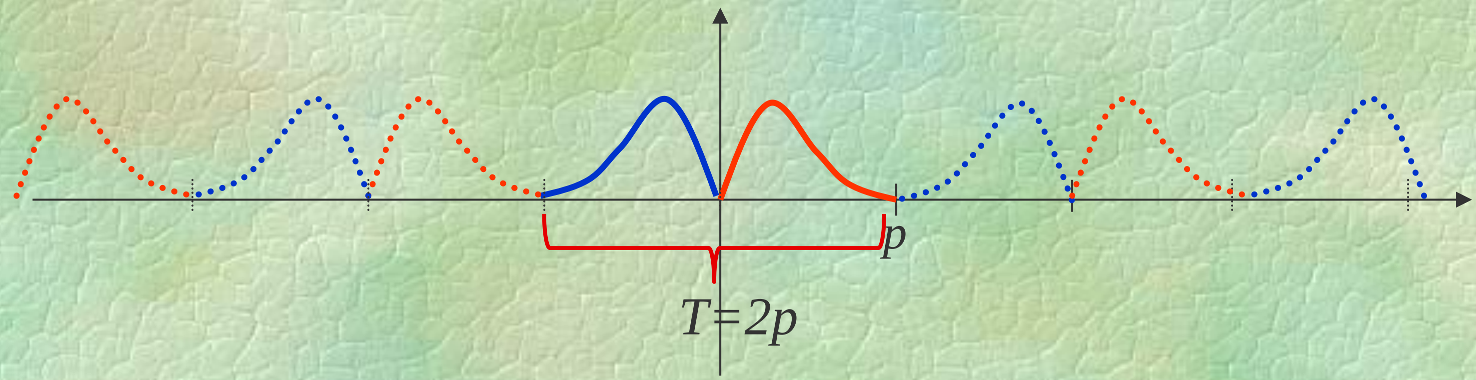


Determine the Fourier series coefficients

Half-Range Expansions

Consider the following function:

Expand the function as an even function and determine its Fourier series (cosine half-range expansion).



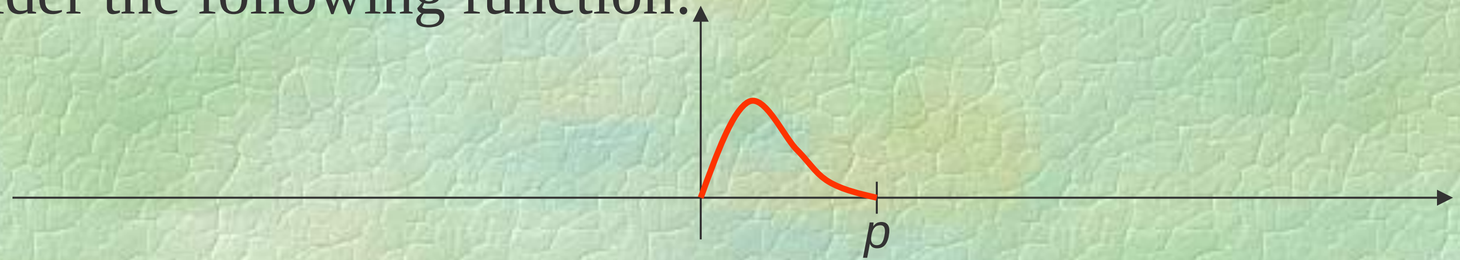
$$\boxed{b_n = 0}$$

$$\boxed{a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos\left(\frac{2n\pi}{T}t\right)dt}$$

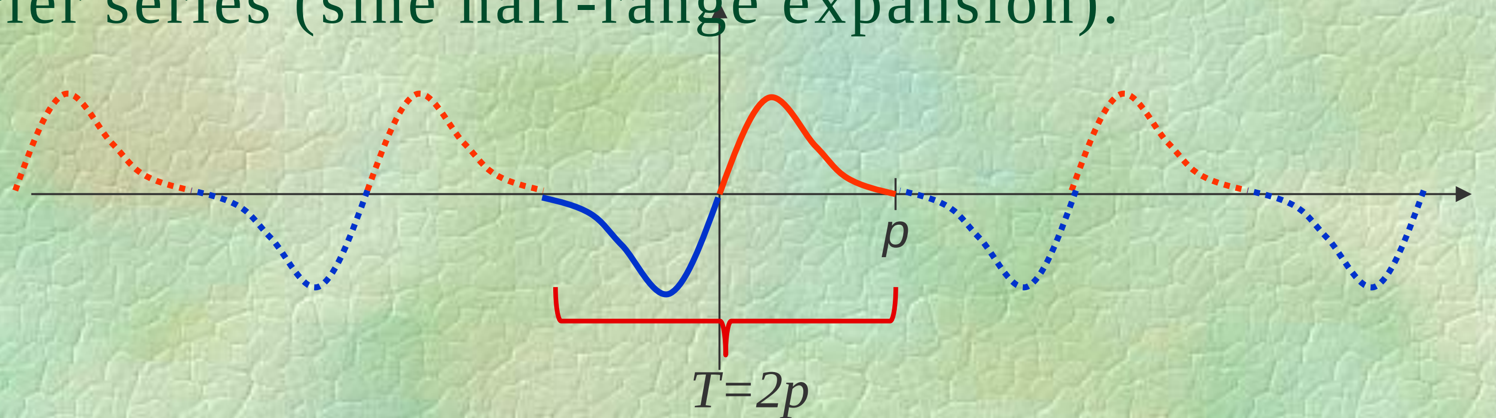
$$\boxed{a_n = \frac{2}{p}\int_0^{p} f(t)\cos\left(\frac{n\pi}{p}t\right)dt}$$

Half-Range Expansions

Consider the following function:



Expand the function as an odd function and determine its Fourier series (sine half-range expansion).



$$\boxed{a_n = 0}$$

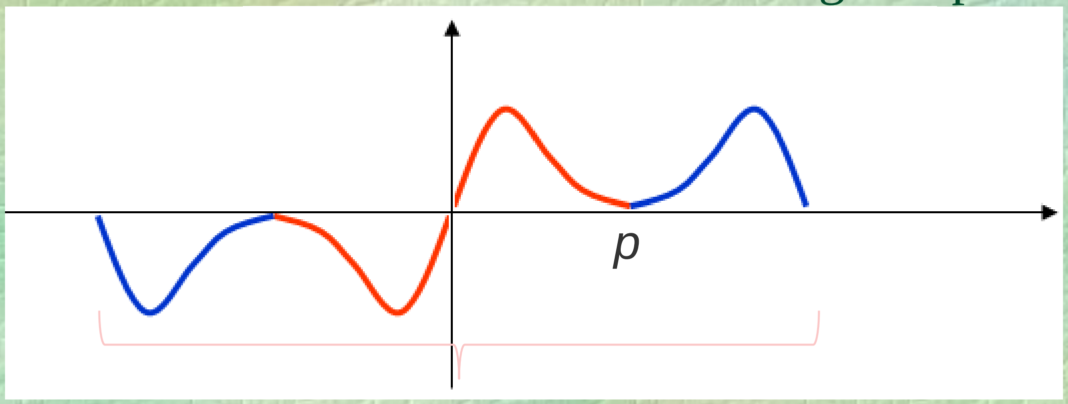
$$\boxed{b_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin\left(\frac{2n\pi}{T}t\right)dt}$$

$$\boxed{b_n = \frac{2}{p}\int_0^{p} f(t)\sin\left(\frac{n\pi}{p}t\right)dt}$$

Another Fourier Series

Consider the following function:

Expand the function symmetrically with respect to the line t=p and then determine the sine half-range expansion.



T=4p

$$\boxed{a_n = 0}$$

$$\boxed{b_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin\left(\frac{2n\pi}{T}t\right)dt}$$

$$\boxed{b_n = \frac{1}{p}\int_0^{2p} f(t)\sin\left(\frac{n\pi}{2p}t\right)dt}$$

$$\boxed{b_n = \begin{cases} \dfrac{2}{p}\displaystyle\int_0^{p} f(t)\sin\left(\frac{n\pi}{2p}t\right)dt & \mathrm{n} = 2k+1 \\ 0 & \mathrm{n=2k} \end{cases}}$$

Half-Range Expansions

Example 4: The following function is defined over the given interval:

$$f(t) = t - t^2 \qquad 0 < t < 1$$

The goal is to find the cosine half-range expansion.

Solution: To calculate the sine half-range expansion, we first need to expand the function as an even function. Therefore, the period is taken as 2.

$$a_n = \frac{2}{p}\int_0^p f(t)\cos\left(\frac{n\pi}{p}t\right)dt \qquad p = 1$$

$$a_n = 2\int_0^1 (t - t^2)\cos(n\pi t)\,dt = \begin{cases} \dfrac{-4}{n^2\pi^2} & n = 2k, n \neq 0 \\ 0 & n = 2k+1 \end{cases}$$

$$a_0 = 2\int_0^1 (t - t^2)\,dt = \frac{1}{3} \qquad f(t) = \frac{1}{6} - \frac{4}{\pi^2}\left[\frac{\cos(2\pi t)}{4} + \frac{\cos(4\pi t)}{16} + \frac{\cos(6\pi t)}{36} + \cdots\right]$$

Half-Range Expansions

Example 5: The following function is defined over the given interval:

$$f(t) = t - t^2 \qquad 0 < t < 1$$

The goal is to find the sine half-range expansion.

Solution: To calculate the sine half-range expansion, we first need to expand the function as an odd function. Therefore, the period is taken as 2.

$$b_n = \frac{2}{p}\int_0^p f(t)\sin\left(\frac{n\pi}{p}t\right)dt \qquad p = 1$$

$$b_n = 2\int_0^1 (t - t^2)\sin(n\pi t)\,dt = \begin{cases} \dfrac{8}{n^3\pi^3} & n = 2k+1 \\ 0 & n = 2k \end{cases}$$

$$f(t) = \frac{8}{\pi^3}\left[\sin(\pi t) + \frac{\sin(3\pi t)}{27} + \frac{\sin(5\pi t)}{125} + \cdots\right]$$

Half-Range Expansions

Exercise 3: The following function is defined over the given interval:

$$f(t) = t - t^2 \qquad 0 < t < 1$$

Which is more suitable: a sine half-range expansion or a cosine half-range expansion?

The need for and application of the desired series

- Use in partial differential equations
- Rate at which coefficients approach zero

Half-Range Expansions

Example 6: The following function is defined over the given interval:

$$f(t) = t - t^2 \qquad 0 < t < 1$$

Cosine half-range expansion.

$$a_n = 2\int_0^1 (t - t^2)\cos(n\pi t)\,dt = \begin{cases} \dfrac{-4}{n^2\pi^2} & n = 2k, n \neq 0 \\ 0 & n = 2k+1 \end{cases}$$

$$a_0 = 2\int_0^1 (t - t^2)\,dt = \frac{1}{3} \qquad f(t) = \frac{1}{6} - \frac{4}{\pi^2}\left[\frac{\cos(2\pi t)}{4} + \frac{\cos(4\pi t)}{16} + \frac{\cos(6\pi t)}{36} + \cdots\right]$$

Sine half-range expansion.

$$b_n = 2\int_0^1 (t - t^2)\sin(n\pi t)\,dt = \begin{cases} \dfrac{8}{n^3\pi^3} & n = 2k+1 \\ 0 & n = 2k \end{cases}$$

$$f(t) = \frac{8}{\pi^3}\left[\sin(\pi t) + \frac{\sin(3\pi t)}{27} + \frac{\sin(5\pi t)}{125} + \cdots\right]$$

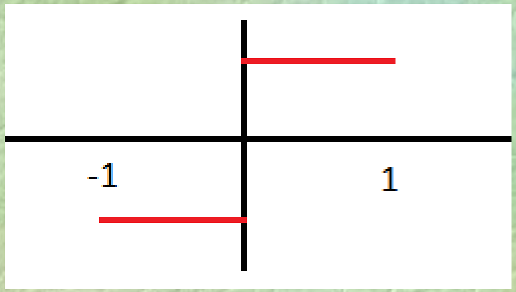
Half-Range Expansions

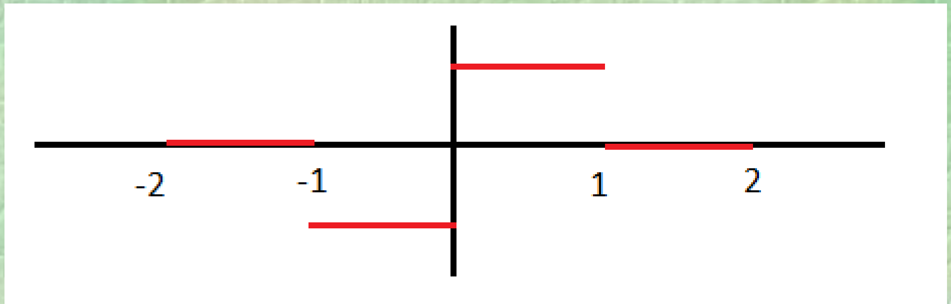
Exercise 4: Given the function defined as follows, write the Fourier series that includes only sine terms and converges to the function over the specified interval.

$$f(t) = 1 \qquad 0 < t < 1$$

Is the requested series unique?

Hint: See following figures:



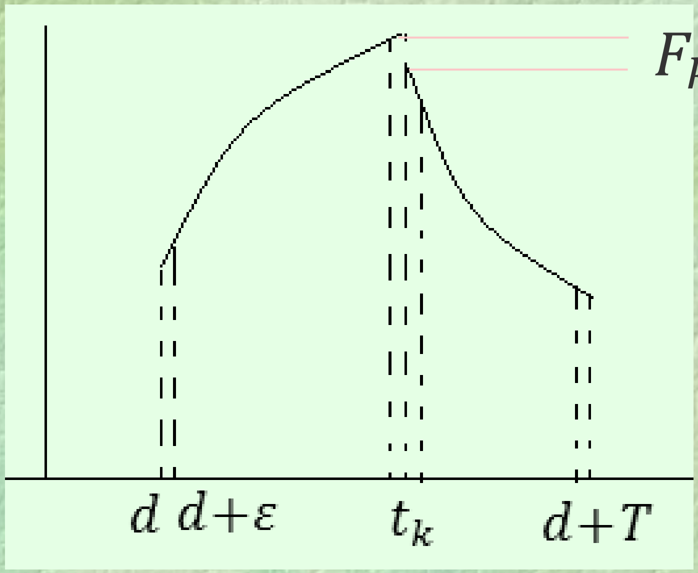


Part One: Fourier Series and Fourier Integral

- ❑ Introduction to Fourier Series
- ❑ Determining Fourier Series Coefficients and Related Theorems
- ❑ Half-Range Expansions
- ❑ Different Representations of Fourier Series
- ❑ Applications of Fourier Series in Engineering
- ❑ Fourier Integral
- ❑ Applications of Fourier Integral in Engineering

Calculate the Fourier series coefficients without integration

Determining the Fourier series coefficients without integration using the concept of jump discontinuities



$F_k < 0$

$$F_k = f(t_k^+) - f(t_k^-)$$

Calculate the Fourier series coefficients without integration

Theorem: If a periodic function *f* satisfies the Dirichlet conditions and has jump discontinuities $F_1, F_2, \ldots, F_m$ at points $t_1, t_2, \ldots, t_m$, where t_1 can be d but t_m cannot be d+T, then the Fourier series coefficients are given by:

$$a_n = -\frac{T}{2n\pi} b'_n - \frac{1}{n\pi} \sum_{k=1}^{m} F_k \sin \frac{2n\pi}{T} t_k \qquad n \neq 0$$

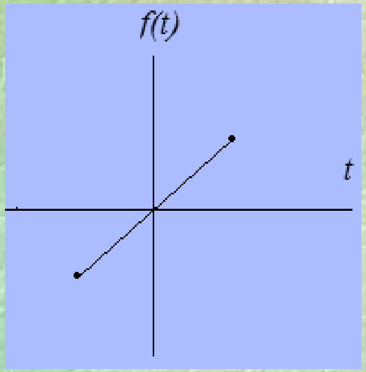
$$b_n = \frac{T}{2n\pi} a'_n + \frac{1}{n\pi} \sum_{k=1}^{m} F_k \cos \frac{2n\pi}{T} t_k$$

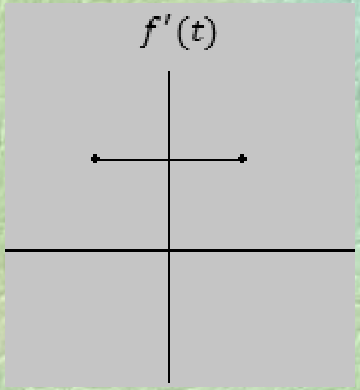
Calculate the Fourier series coefficients without integration

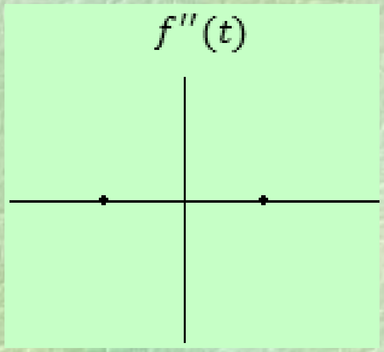
Example 7: Compute the Fourier series coefficients of the following periodic function.

$$f(t) = t \qquad -1 < t < 1 \qquad T = 2$$

Solution: The function and their derivatives are as follows:



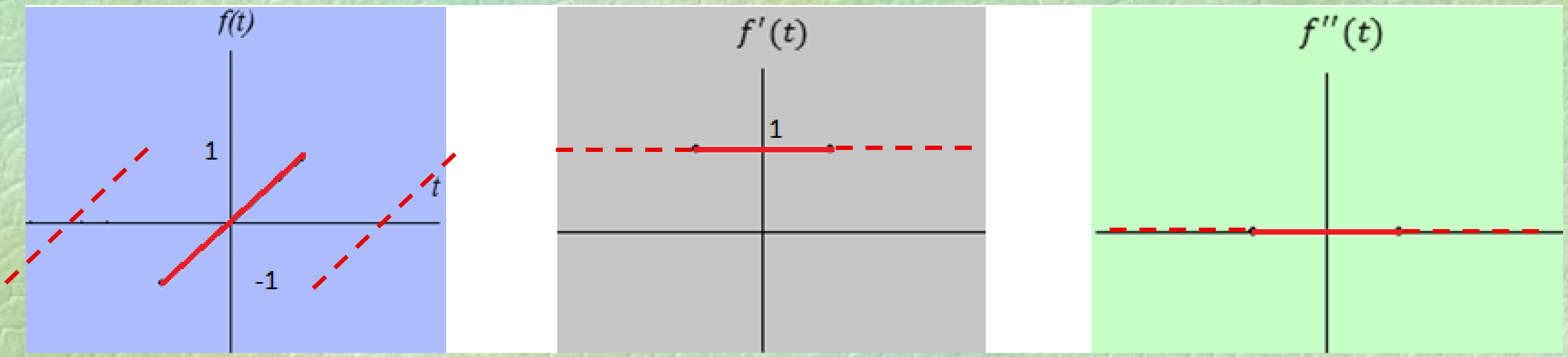




Since the function is odd, therefore:

$$a_n = 0$$

Calculate the Fourier series coefficients without integration



$$b_n = \frac{T}{2n\pi} a'_n + \frac{1}{n\pi} \sum_{k=1}^{m} F_k \, cos \frac{2n\pi}{T} t_k$$

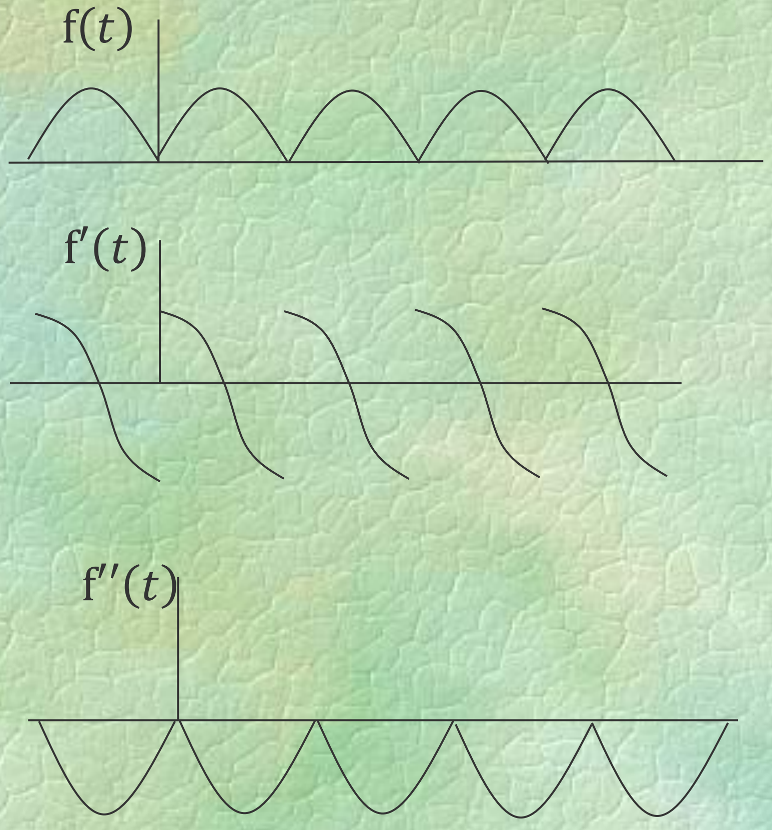
$$b_n = \frac{1}{n\pi} a'_n + \frac{1}{n\pi} F_1 \cos(-n\pi t) = \frac{1}{n\pi} 0 - \frac{1}{n\pi} 2\cos(n\pi) \; = -\frac{2}{n\pi} (-1)^n$$

$$\boxed{b_n = \frac{2(-1)^{n+1}}{n\pi}}$$

Calculate the Fourier series coefficients without integration

Exercise 5: A periodic function *f*(*t*) is defined over one period as follows. Determine the Euler coefficients without integration.

$$f(t) = sint \qquad 0<t<\pi$$



$$a_n = -\frac{T}{2n\pi} b'_n - \frac{1}{n\pi}\sum_{k=1}^{m} F_k \sin\frac{2n\pi}{T} t_k$$

$$b_n = \frac{T}{2n\pi} a'_n + \frac{1}{n\pi}\sum_{k=1}^{m} F_k \cos\frac{2n\pi}{T} t_k$$

$$b_n = 0, \qquad a_n = \frac{4}{\pi(1-4n^2)}$$

Calculate the Fourier series coefficients without integration

Exercise 6: A periodic function f(t) is defined over one period as follows. Determine the Euler coefficients without integration.

$$f(t)=\begin{cases}0 & -\pi<t\leq 0\\ t & 0<t<\pi\end{cases}$$

Convergence rate in Fourier series

Theorem: For sufficiently large n, the Fourier series coefficients of a periodic function that satisfies the Dirichlet conditions always tend to zero with a rate of at least c/n, where c is a constant.

If the function is continuous, then the Fourier series coefficients tend to zero with a rate of at least c/n^2.

If the function has one or more discontinuities, then at least one of the Fourier series coefficients tends to zero with a rate of c/n.

If both the function and its derivative are continuous, then the Fourier series coefficients tend to zero with a rate of at least c/n^3.

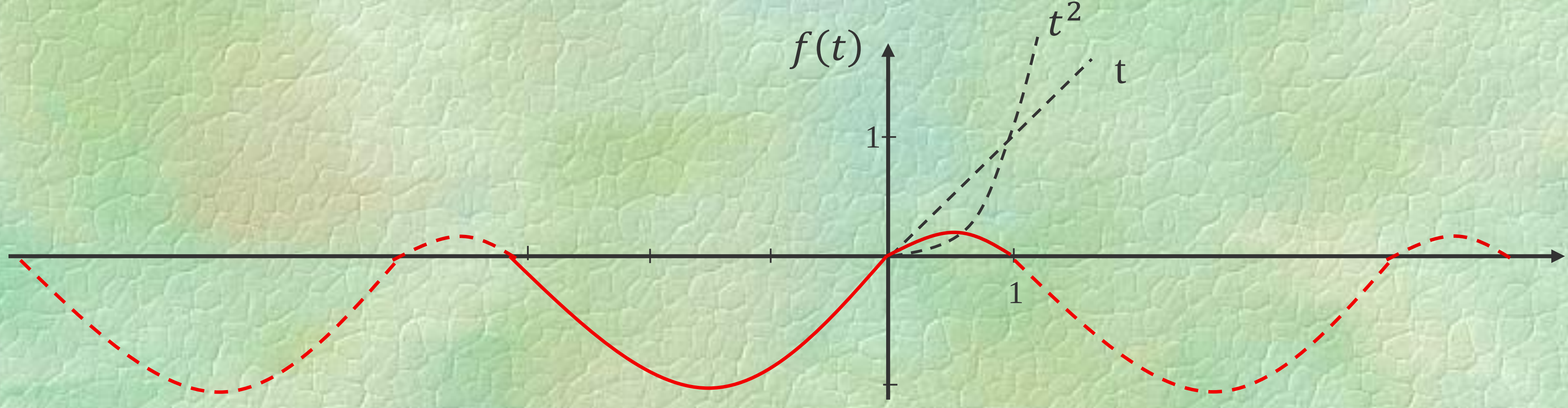
If the derivative of the function has one or more discontinuities, then at least one of the Fourier series coefficients tends to zero with a rate of c/n^2.

Convergence rate in Fourier series

Example 8: Determine the rate of decrease of the coefficients a_n and b_n for the following function.

$$f(t) = \begin{cases} \sin(t) & -\pi \le t < 0 \\ t - t^2 & 0 \le t < 1 \end{cases}$$

Solution:



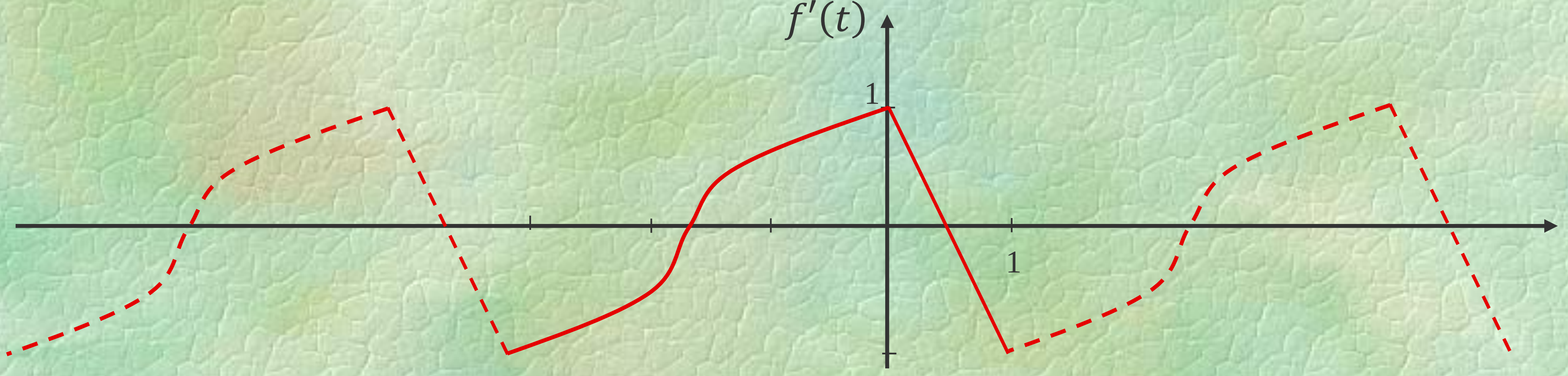
$f(-\pi) = f(1) = 0$ $\quad$ $f(0^-) = f(0^+) = 0$

$\dfrac{c}{n^2}$ *Minimum rate of decrease for large n*

Convergence rate in Fourier series

$$f(t) = \begin{cases} \sin(t) & -\pi \le t < 0 \\ t - t^2 & 0 \le t < 1 \end{cases}$$

$$f'(t) = \begin{cases} \cos(t) & -\pi \le t < 0 \\ 1 - 2t & 0 \le t < 1 \end{cases}$$



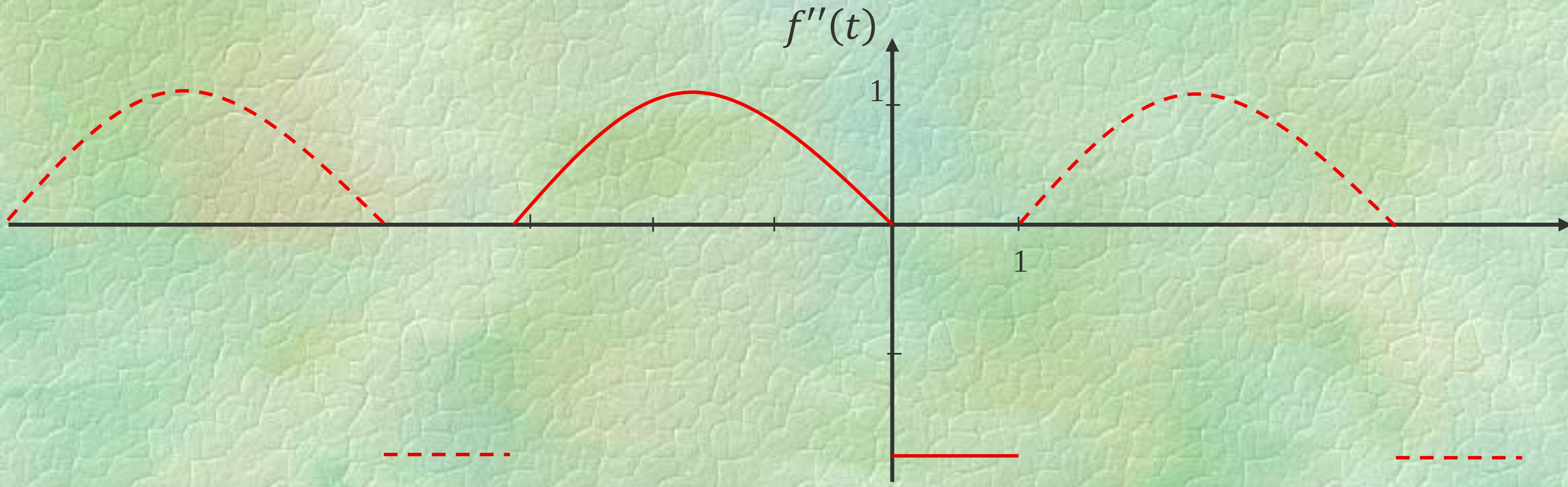
$f'(-\pi) = f'(1) = -1 \quad f'(0^-) = f'(0^+) = 1$

$\dfrac{c}{n^3}$ *Minimum rate of decrease for large n*

Convergence rate in Fourier series

$$f(t)=\begin{cases}\sin(t) & -\pi\le t<0\\ t-t^2 & 0\le t<1\end{cases}$$

$$f'(t)=\begin{cases}\cos(t) & -\pi\le t<0\\ 1-2t & 0\le t<1\end{cases}\qquad f''(t)=\begin{cases}-\sin(t) & -\pi\le t<0\\ -2 & 0\le t<1\end{cases}$$



$f''(-\pi)\neq f''(1)$ $\quad$ $f''(0^-)\neq f''(0^+)$ $\quad$ $\dfrac{c}{n^3}$

At least one of the coefficients has a rate of convergence for large *n*.

Convergence rate in Fourier series

Example 9: For the periodic function *f*(*t*), the Fourier series coefficients are determined by the following relations. Discuss the continuity of the function and its derivatives.

$$a_n = \frac{n\pi}{n^4 + \pi^4} \qquad b_n = \frac{\pi}{n^2 + n\pi + \pi^2}$$

$$\frac{\pi}{n^3} \qquad \frac{\pi}{n^2}$$

The rate of convergence of one of the coefficients is c/n^2, therefore:

$f(t)$ *is Continues*

$f'(t)$ *has at least one discontinuity*

Exercises

Exercise 7: A function is defined as follows. Write a Fourier series that contains only sine terms and converges to the function in the defined interval. Is the requested series unique?

$$f(t) = 1 \qquad 0 < t < 1$$

Exercise 8: The functions *f* and *g* are given in the interval 0 to *l*. Find the half-range sine series expansion of the function f and the half-range cosine series expansion of the function f. (To which function does each series converge? Draw the corresponding function.)

$$f(x) = sin\left(\frac{\pi}{l}x\right) \qquad g(x) = x^2$$

Exercise 9: A periodic function *f*(t) is defined over one period as follows. Determine the Euler coefficients without integration.

$$f(x) = \begin{cases} \frac{\pi}{2} + x & -\pi < t \leq 0 \\ \frac{\pi}{2} - x & 0 < t \leq \pi \end{cases}$$

Exercise 10: A periodic function with known Fourier coefficients is given. What can you analyze about the continuity of the function and its derivatives?

$$a_n = \frac{2n}{\sqrt{(n^2-1)^2 + 4n^2}} \qquad b_n = \frac{2}{\sqrt{(n^2-1)^2 + 4n^2}}$$

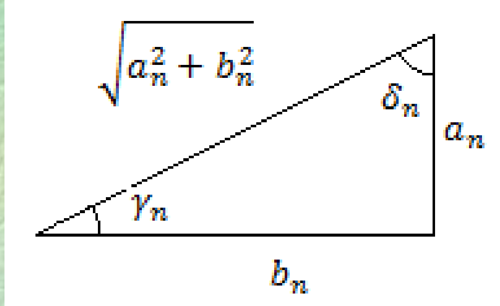
Different Representations of Fourier Series

Fourier Series Representation Using Only Sine and Cosine Functions

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos\left(\frac{2\pi n}{T} t\right) + b_n \sin\left(\frac{2\pi n}{T} t\right)\}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}[\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos\left(\frac{2\pi n}{T} t\right) + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin\left(\frac{2\pi n}{T} t\right)]$$

$$\sin(\gamma_n) = \frac{a_n}{\sqrt{a_n^2 + b_n^2}} = \cos(\delta_n) \qquad \cos(\gamma_n) = \frac{b_n}{\sqrt{a_n^2 + b_n^2}} = \sin(\delta_n)$$



$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \sin(\frac{2n\pi}{T} t + \gamma_n)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \cos(\frac{2n\pi}{T} t - \delta_n)$$

Different Representations of Fourier Series

Fourier Series Representation Using Only Sine and Cosine Functions

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \sin(\frac{2n\pi}{T} t + \gamma_n) \qquad f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \cos(\frac{2n\pi}{T} t - \delta_n)$$

$$\frac{a_0}{2} = A_0 \qquad \sqrt{a_n^2 + b_n^2} = A_n$$

$$\boxed{f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{2n\pi}{T} t - \delta_n)} \qquad \boxed{f(t) = A_0 + \sum_{n=1}^{\infty} A_n \sin(\frac{2n\pi}{T} t + \gamma_n)}$$

Different Representations of Fourier Series

Exponential Fourier series:

$$\cos(\theta) = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \qquad \sin(\theta) = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos\left(\frac{2\pi n}{T}t\right) + b_n \sin\left(\frac{2\pi n}{T}t\right)\}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{\frac{a_n}{2}\left[e^{j\frac{2n\pi}{T}t} + e^{-j\frac{2n\pi t}{T}}\right] + \frac{b_n}{2j}\left[e^{j\frac{2n\pi}{T}t} - e^{-j\frac{2n\pi}{T}t}\right]\}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{2}(a_n - jb_n)e^{j\frac{2n\pi}{T}t} + \frac{1}{2}(a_n + jb_n)e^{-j\frac{2n\pi}{T}t}\right]$$

Different Representations of Fourier Series

Exponential Fourier series:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos\left(\frac{2\pi n}{T}t\right) + b_n \sin\left(\frac{2\pi n}{T}t\right)\}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{2}(a_n - jb_n)e^{j\frac{2n\pi}{T}t} + \frac{1}{2}(a_n + jb_n)e^{-j\frac{2n\pi}{T}t}\right]$$

$$\frac{a_0}{2} = C_0 \qquad \frac{1}{2}(a_n - jb_n) = C_n \qquad \frac{1}{2}(a_n + jb_n) = C_{-n}$$

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} C_n e^{j\frac{2n\pi}{T}t}}$$

Different Representations of Fourier Series

Exponential Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{j\frac{2n\pi}{T}t}$$

$$\frac{a_0}{2} = C_0 \qquad \frac{1}{2}(a_n - jb_n) = C_n \qquad \frac{1}{2}(a_n + jb_n) = C_{-n}$$

$$\boxed{a_n = \frac{2}{T}\int_d^{d+T} f(t)\cos\left(\frac{2n\pi}{T}t\right)dt} \qquad \boxed{b_n = \frac{2}{T}\int_d^{d+T} f(t)\sin\left(\frac{2n\pi}{T}t\right)dt}$$

$$C_n = \frac{1}{2}(a_n - jb_n) \qquad \boxed{C_n = \frac{1}{T}\int_d^{d+T} f(t)e^{-j\frac{2n\pi}{T}t}dt}$$

Different Representations of Fourier Series

Exponential and trigonometric Fourier series

$$f(t)=\sum_{n=-\infty}^{\infty} C_n e^{j\frac{2n\pi}{T}t}$$

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\left(\frac{2\pi n}{T}t\right)+b_n\sin\left(\frac{2\pi n}{T}t\right)\right\}$$

$$a_n=\frac{2}{T}\int_d^{d+T} f(t)\cos\left(\frac{2n\pi}{T}t\right)dt$$

$$C_n=\frac{1}{T}\int_d^{d+T} f(t)e^{-j\frac{2n\pi}{T}t}dt$$

$$b_n=\frac{2}{T}\int_d^{d+T} f(t)\sin\left(\frac{2n\pi}{T}t\right)dt$$

$$\boxed{C_n=\frac{1}{2}(a_n-jb_n)}$$

$$\boxed{a_n=C_n+C_{-n}}$$

$$\boxed{b_n=j(C_n-C_{-n})}$$

Different Representations of Fourier Series

Example 10: Compute the complex exponential Fourier coefficients for the following periodic function. Then, use these coefficients to find the trigonometric Fourier coefficients.

$$f(t) = e^{-t} \qquad -1 < t < 1 \qquad T = 2$$

Solution:

$$C_n = \frac{1}{T}\int_{d}^{d+T} f(t)e^{-j\frac{2n\pi}{T}t}dt$$

$$C_n = \frac{1}{2}\int_{-1}^{1} e^{-t}e^{-jn\pi t}dt = \qquad = \frac{(-1)^n \sinh(1)}{1+jn\pi}$$

$$\boxed{a_n = C_n + C_{-n} = \qquad = \frac{(-1)^n 2\sinh(1)}{1+n^2\pi^2}}$$

$$\boxed{b_n = j(C_n - C_{-n}) = \qquad = \frac{(-1)^n 2n\pi\sinh(1)}{1+n^2\pi^2}}$$

Different Representations of Fourier Series

Exercise 11: Compute the complex exponential Fourier coefficients for the following periodic function. Then, use these coefficients to find the trigonometric Fourier coefficients.

$$f(t) = 1 \qquad -1 < t < 1 \qquad T = 2$$

Exercse 12: Compute the complex exponential Fourier coefficients for the following periodic function. Then, use these coefficients to find the trigonometric Fourier coefficients.

$$f(t) = t \qquad 0 < t < 1 \qquad T = 1$$

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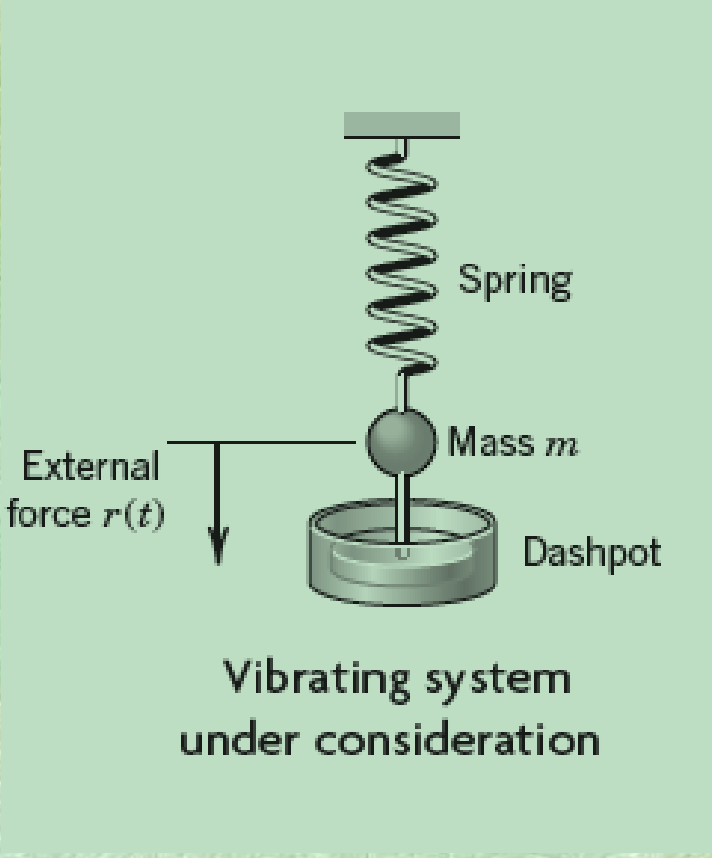
Applications of Fourier Series in Engineering

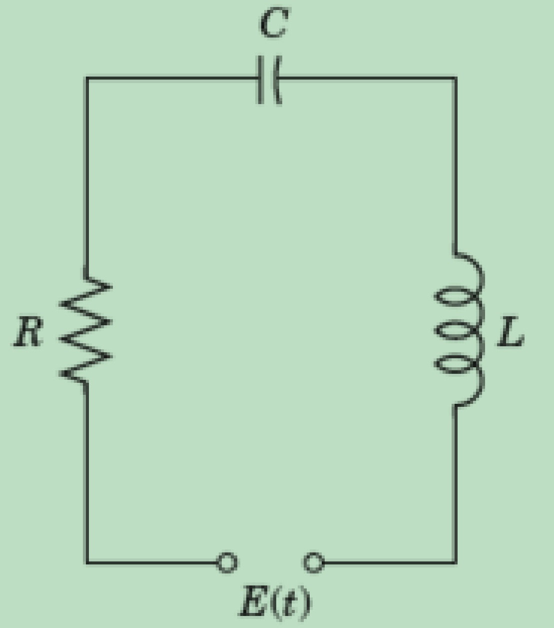
Consider the following system. Given the input r(t), determine y(t) in the steady state.



$$my'' + cy' + ky = r(t)$$

Applications of Fourier Series in Engineering

Consider the following system. Given the input *E*(t), determine the current *i*(t) in the steady state.

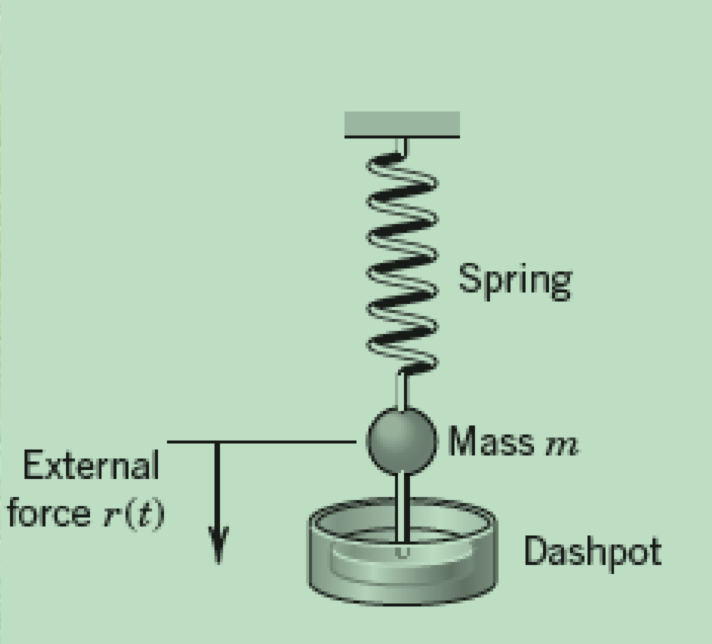


Electrical analog of the system (*RLC-circuit*)

$$LI'' + RI' + \frac{1}{C}I = E'(t)$$

Applications of Fourier Series in Engineering

Example 11: Consider the following system. Given the input *r*(t), determine *y*(t) in the steady state.



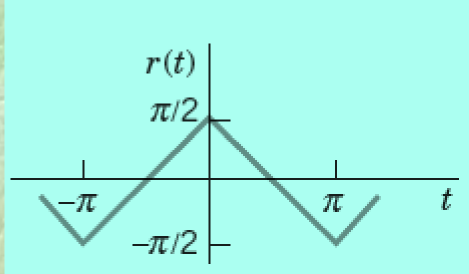
Vibrating system under consideration

$$my'' + cy' + ky = r(t)$$

$y(t) = y_h(t) + y_p(t)$

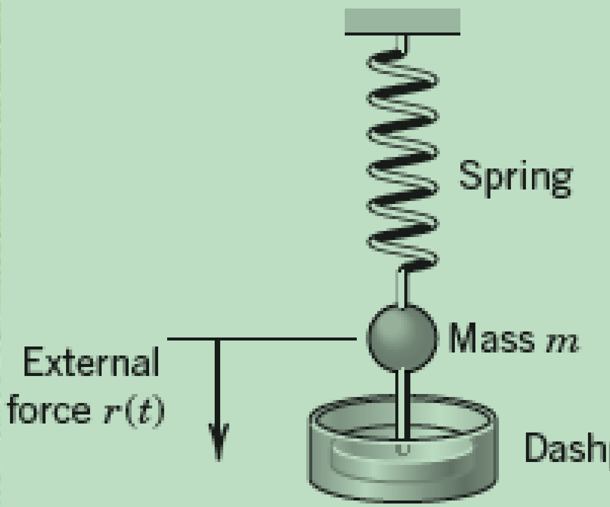
$y_h(t)$ *is easy to find.*

$y_p(t)$ *depends on* $r(t)$



Applications of Fourier Series in Engineering

Example 11: Consider the following system. Given the input *r*(t), determine *y*(t) in the steady state.

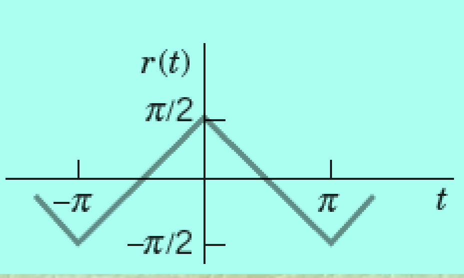


Vibrating system under consideration

$$my'' + cy' + ky = r(t)$$

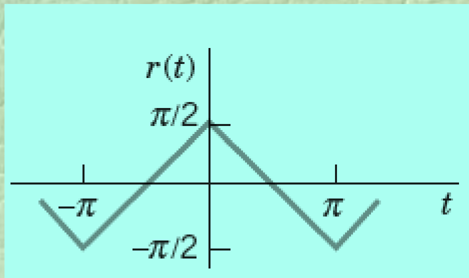
$$m = 1\ g \quad c = 0.05\ g/\sec \quad k = 25\ g/\sec^2$$

$$y'' + 0.05y' + 25y = r(t)$$



Applications of Fourier Series in Engineering

Example 12: Consider the following system. Given the input *r*(t), determine *y*(t) in the steady state.



$$y'' + 0.05y' + 25y = r(t)$$

$$r(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \ldots.\right)$$

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \quad n = 1, 3, 5 \ldots$$

$$y_n = A_n \cos nt + B_n \sin nt$$

Applications of Fourier Series in Engineering

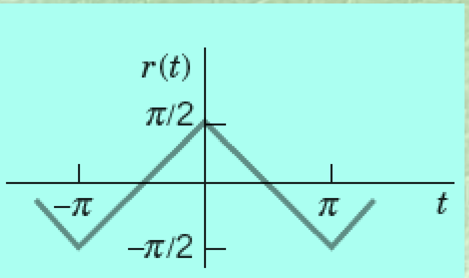
$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \quad n = 1, 3, 5\ldots$$

$$y_n = A_n \cos nt + B_n \sin nt$$

$$A_n = \frac{4\left(25 - n^2\right)}{n^2\pi\left((25 - n^2)^2 + (0.05)^2\right)} \qquad B_n = \frac{0.2}{n\pi\left((25 - n^2)^2 + (0.05)^2\right)}$$

Applications of Fourier Series in Engineering

Example 12: Consider the following system. Given the input *r*(t), determine *y*(t) in the steady state.



$$y'' + 0.05y' + 25y = r(t)$$

$$r(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \dots\right)$$

$$y_n = A_n \cos nt + B_n \sin nt$$

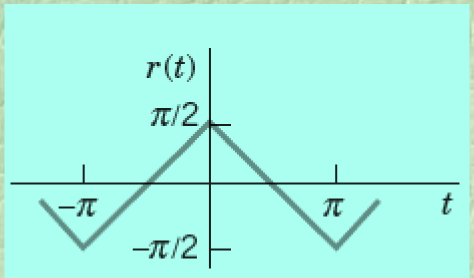
$$y(t) = \sum_{n=1}^{\infty}(A_n \cos(nt) + B_n \sin(nt))$$

$$y(t) = \sum_{n=1}^{\infty} \sqrt{A_n^2 + B_n^2}\sin(nt + \gamma_n) = \sum_{n=1}^{\infty} C_n \sin(nt + \gamma_n)$$

$C_1 = 0.0531 \quad C_3 = 0.0088 \quad C_5 = 0.2037 \quad C_7 = 0.0011 \quad C_9 = 0.00003$

Applications of Fourier Series in Engineering

Example 12: Consider the following system. Given the input r(t), determine y(t) in the steady state.



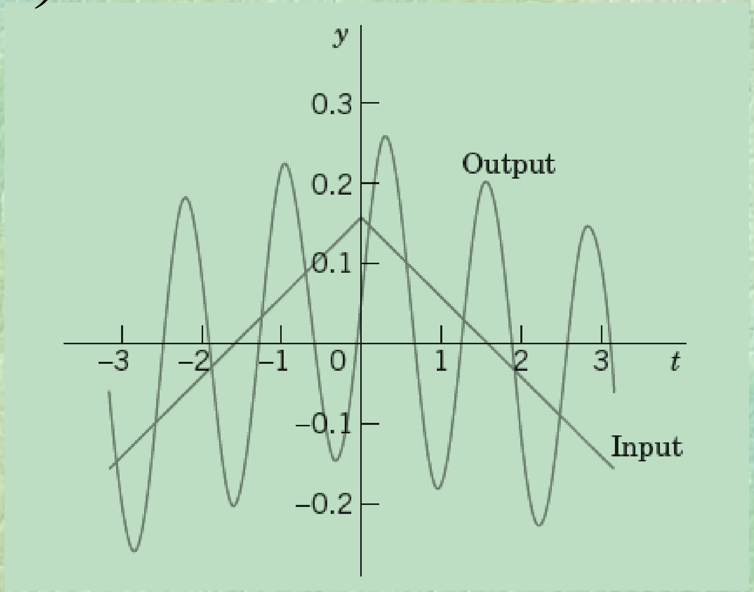
$$y'' + 0.05y' + 25y = r(t)$$

$$r(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + ...\right)$$

$$y(t) = \sum_{n=1}^{\infty} C_n \ \sin(nt + \gamma_n)$$

$$C_1 = 0.0531 \quad C_3 = 0.0088 \quad C_5 = 0.2037$$

$$C_7 = 0.0011 \qquad C_9 = 0.00003$$

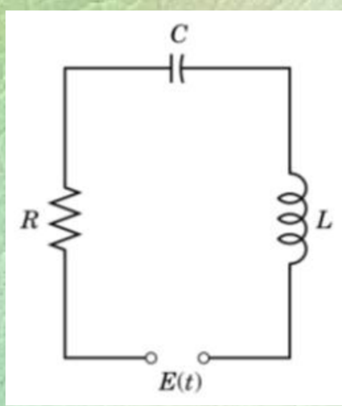


Exercises

Exercise 13: Compute the complex exponential Fourier coefficients for the following periodic function. Then, use these coefficients to find the trigonometric Fourier coefficients.

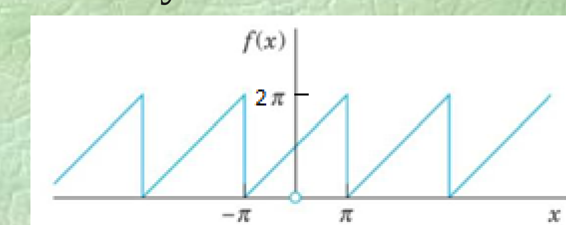
$$f(x) = 2e^{-x}$$
$$-1 < x < 1$$

Exercise 14: Compute the steady-state current for the following RLC circuit.



$$E(t) = \begin{cases} 100(\pi t + t^2) & -\pi < t < 0 \\ 100(\pi t - t^2) & 0 < t < \pi \end{cases}$$

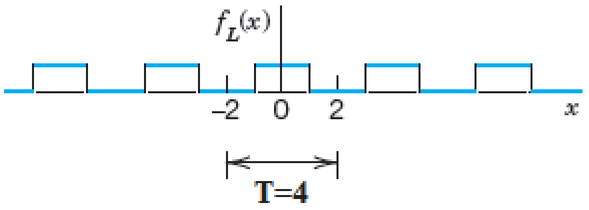
Exercise 15: Using MATLAB, plot and analyze the Fourier series of the following function for the first 5 and 20 terms.



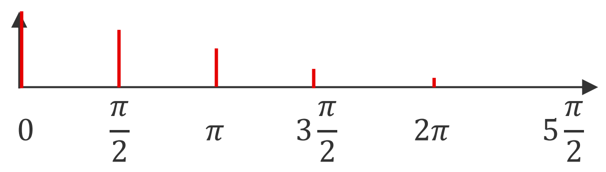
Part One: Fourier Series and Fourier Integral

- Introduction to Fourier Series
- Determining Fourier Series Coefficients and Related Theorems
- Half-Range Expansions
- Different Representations of Fourier Series
- Applications of Fourier Series in Engineering
- Fourier Integral
- Applications of Fourier Integral in Engineering

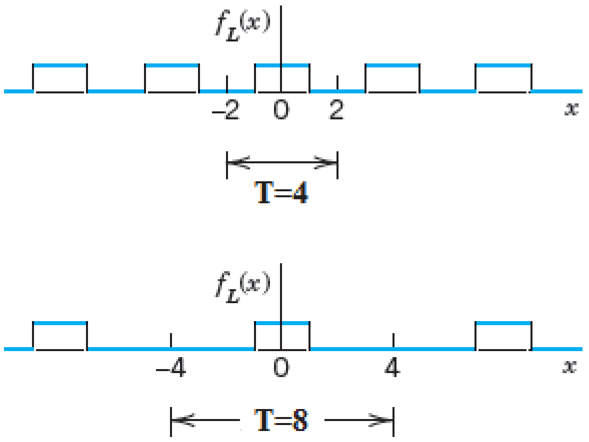
Fourier Integral



$$\frac{2n\pi}{T}: \frac{\pi}{2}, 2\frac{\pi}{2}, 3\frac{\pi}{2}, \ldots$$

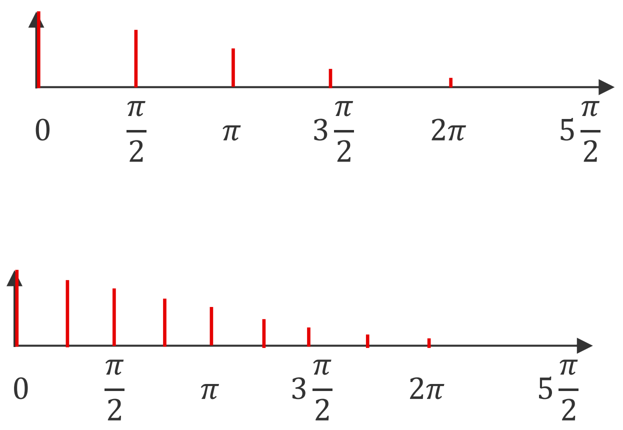


Fourier Integral

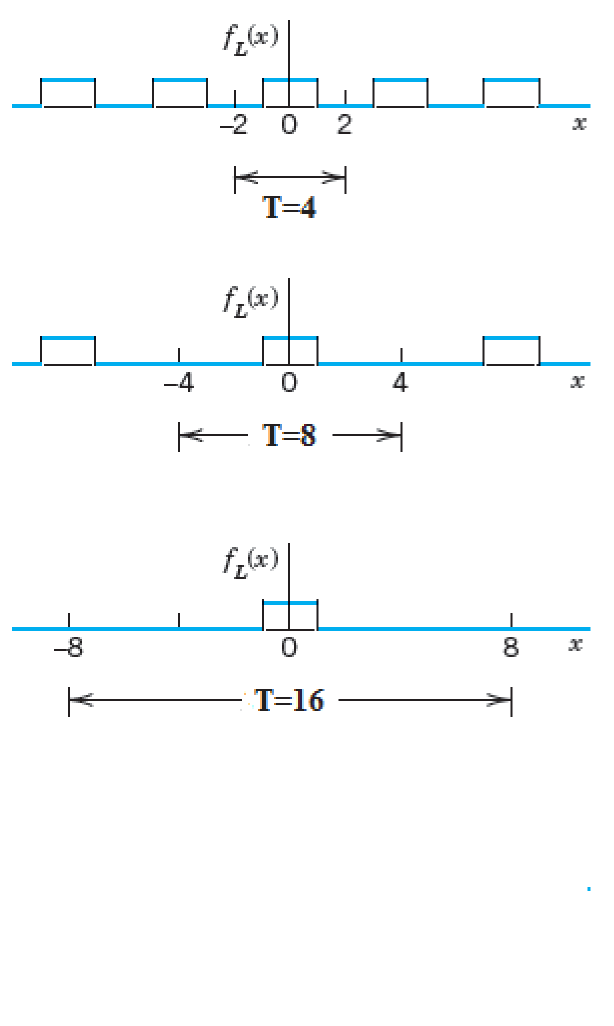


$$\frac{2n\pi}{T}: \frac{\pi}{2}, 2\frac{\pi}{2}, 3\frac{\pi}{2}, \ldots$$

$$\frac{2n\pi}{T}: \frac{\pi}{4}, 2\frac{\pi}{4}, 3\frac{\pi}{4}, \ldots$$



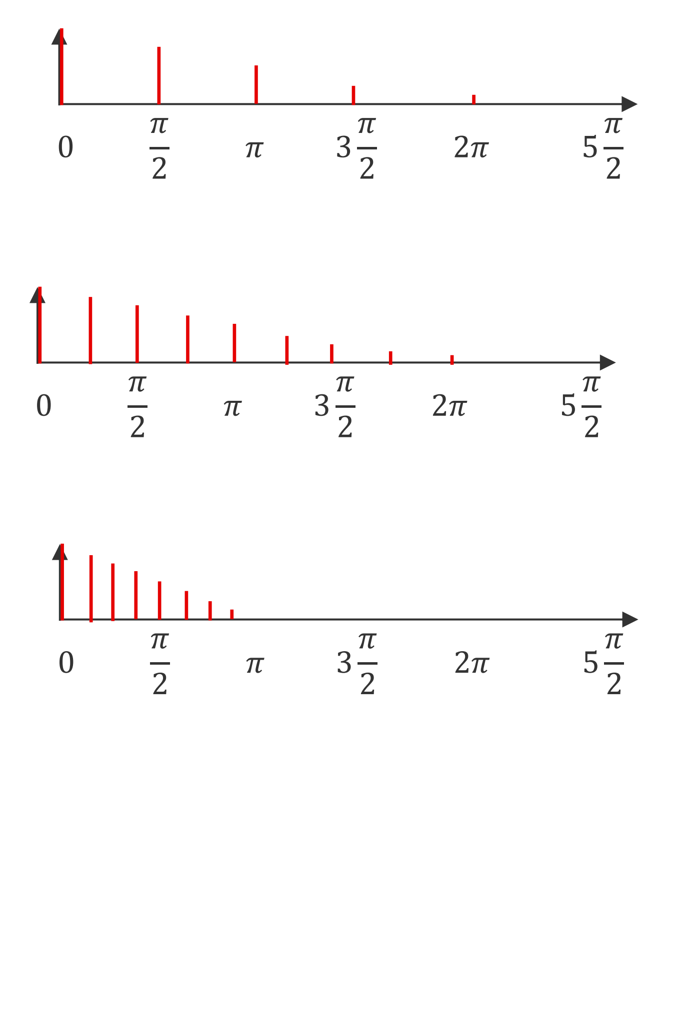
Fourier Integral



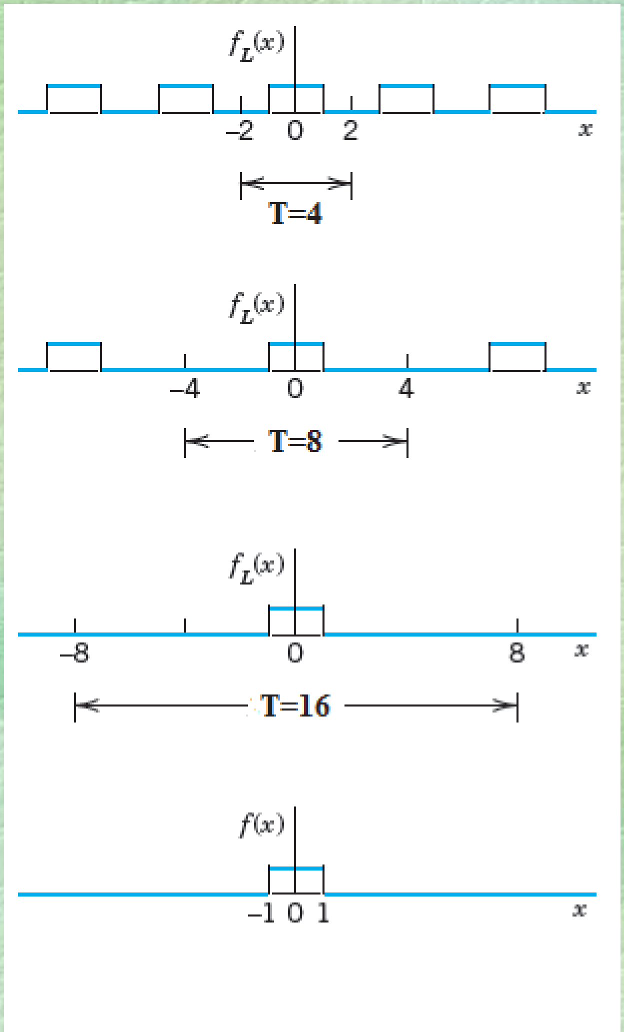
$$\frac{2n\pi}{T}:\ \frac{\pi}{2},2\frac{\pi}{2},3\frac{\pi}{2},\ldots$$

$$\frac{2n\pi}{T}:\ \frac{\pi}{4},2\frac{\pi}{4},3\frac{\pi}{4},\ldots$$

$$\frac{2n\pi}{T}:\ \frac{\pi}{8},2\frac{\pi}{8},3\frac{\pi}{8},\ldots$$



Fourier Integral

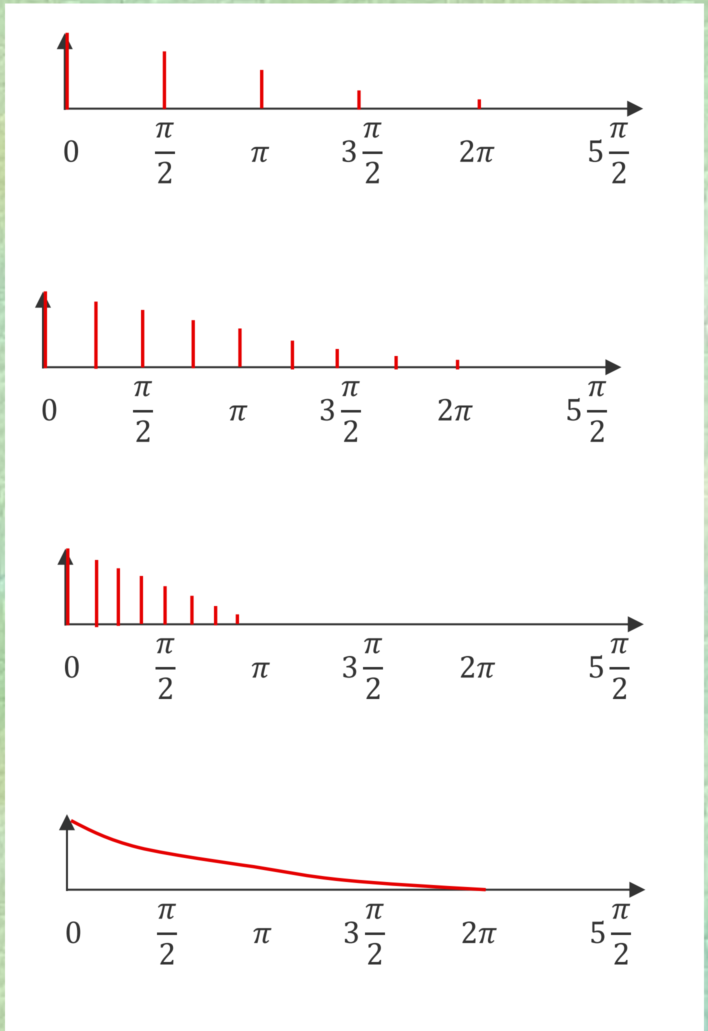


$$\frac{2n\pi}{T}: \frac{\pi}{2}, 2\frac{\pi}{2}, 3\frac{\pi}{2}, \ldots$$

$$\frac{2n\pi}{T}: \frac{\pi}{4}, 2\frac{\pi}{4}, 3\frac{\pi}{4}, \ldots$$

$$\frac{2n\pi}{T}: \frac{\pi}{8}, 2\frac{\pi}{8}, 3\frac{\pi}{8}, \ldots$$

$$T \to \infty \qquad \to$$



All frequencies $\to$ *Fourier Integral*

Fourier Integral

$T \to \infty$ *Fourier Series* $\to$ *Fourier Integral*

$$a_n = \frac{2}{T}\int_d^{d+T} f(t)\cos\left(\frac{2n\pi}{T}t\right)dt \longrightarrow A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)dt$$

$$b_n = \frac{2}{T}\int_d^{d+T} f(t)\sin\left(\frac{2n\pi}{T}t\right)dt \longrightarrow B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega t)dt$$

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2k\pi}{T}t\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2k\pi}{T}t\right)$$

$$f(t) = \int_0^{\infty}\left[A(\omega)\cos(\omega t) + B(\omega)\sin(\omega t)\right]d\omega$$

Fourier Integral

Trigonometric Fourier Series

$$a_n = \frac{2}{T}\int_d^{d+T} f(t)\cos\left(\frac{2n\pi}{T}t\right)dt \qquad b_n = \frac{2}{T}\int_d^{d+T} f(t)\sin\left(\frac{2n\pi}{T}t\right)dt$$

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2k\pi}{T}t\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2k\pi}{T}t\right)$$

Trigonometric Fourier Integral

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)dt \qquad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega t)dt$$

$$f(t) = \int_0^{\infty}\left[A(\omega)\cos(\omega t) + B(\omega)\sin(\omega t)\right]d\omega$$

Fourier Integral

Conditions for Convergence of the Fourier Integral:

a) The function *f*(t) must satisfy Dirichlet conditions in any finite interval.

b) The following integral must exist:

$$\int_{-\infty}^{\infty} |f(t)|dt$$

Fourier Integral

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)dt$$

$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega t)dt$$

$$f(t)=\int_0^{\infty}\left[A(\omega)\cos(\omega t)+B(\omega)\sin(\omega t)\right]d\omega$$

If *f*(t) is an even function:

$$A(\omega)=\frac{2}{\pi}\int_0^{\infty} f(t)\cos(\omega t)dt$$

$$B(\omega)=0$$

$$\longrightarrow \quad f(t)=\int_0^{\infty} A(\omega)\cos(\omega t)d\omega$$

If *f*(t) is an odd function:

$$A(\omega)=0$$

$$B(\omega)=\frac{2}{\pi}\int_0^{\infty} f(t)\sin(\omega t)dt$$

$$\longrightarrow \quad f(t)=\int_0^{\infty} B(\omega)\sin(\omega t)d\omega$$

Exponential Fourier Integral

Exponential Fourier Series

$$C_n = \frac{1}{T}\int_{d}^{d+T} f(t)e^{-j\frac{2n\pi}{T}t}dt$$

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{j\frac{2n\pi}{T}t}$$

Exponential Fourier Integral

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$f(t) = \int_{-\infty}^{\infty} g(\omega)e^{j\omega t}d\,\omega$$

Exponential Fourier Integral

Exponential Fourier Integral

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$f(t) = \int_{-\infty}^{\infty} g(\omega)e^{j\omega t}d\omega$$

Fourier Transform Pair

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\left[\int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}d\tau\right]}_{F(j\omega)}e^{j\omega t}d\omega$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

Fourier Transform Pair

Fourier Transform Pair

$$\mathcal{F}[f(t)] = F(j\omega)$$

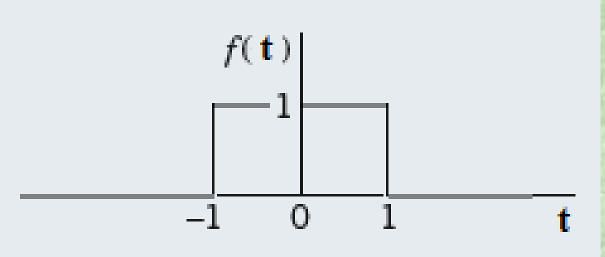
$$\mathcal{F}^{-1}[F(j\omega)] = f(t)$$

$$f(t) \xleftrightarrow{\mathcal{F}} F(j\omega)$$

Fourier Integral

Example 13: Determine the Fourier integral representation of the following function.

$$f(t)=\begin{cases}1 & \text{for } -1\le t\le 1\\ 0 & \text{for } |t|>1\end{cases}$$



$$B(\omega)=0$$

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)dt = \qquad = \frac{2\sin(\omega)}{\omega\pi}$$

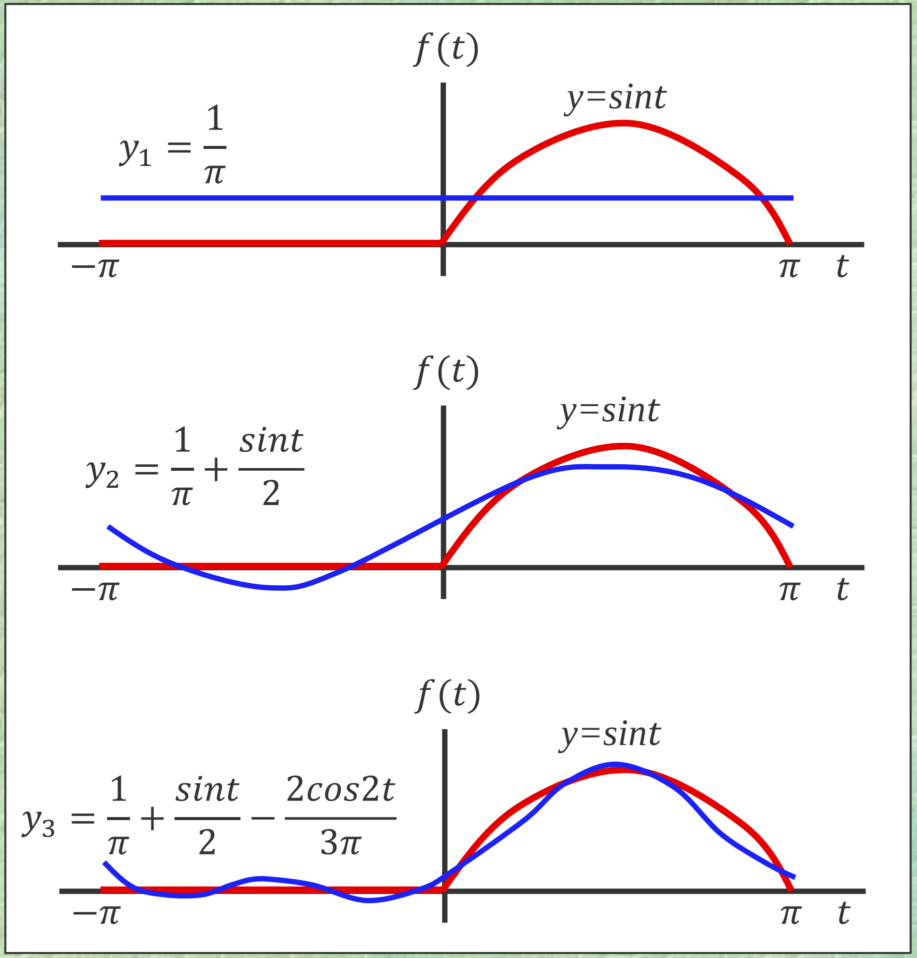
Therefore, the Fourier integral representation of this function is obtained as follows:

$$f(t)=\int_0^{\infty} A(\omega)\cos(\omega t)d\omega=\frac{2}{\pi}\int_0^{\infty}\frac{\sin(\omega)}{\omega}\cos(\omega t)d\omega$$

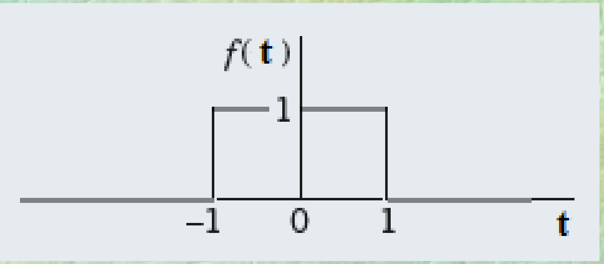
Approximation in Fourier Integral

Approximating the function using the first few terms of its Fourier series

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\sin t - \frac{2}{3\pi}\cos 2t - \ldots$$



Approximation in Fourier Integral



$$f(t) = \int_0^\infty A(\omega)\cos(\omega t)\,d\omega = \frac{2}{\pi}\int_0^\infty \frac{\sin(\omega)}{\omega}\cos(\omega t)\,d\omega = \begin{cases} 1 & -1 < t < 1 \\ 0 & |t| > 1 \end{cases}$$

$$f_{\omega_0}(t) = \frac{2}{\pi}\int_0^{\omega_0} \frac{\sin(\omega)}{\omega}\cos(\omega t)\,d\omega = ?$$

Define Si(x) as:

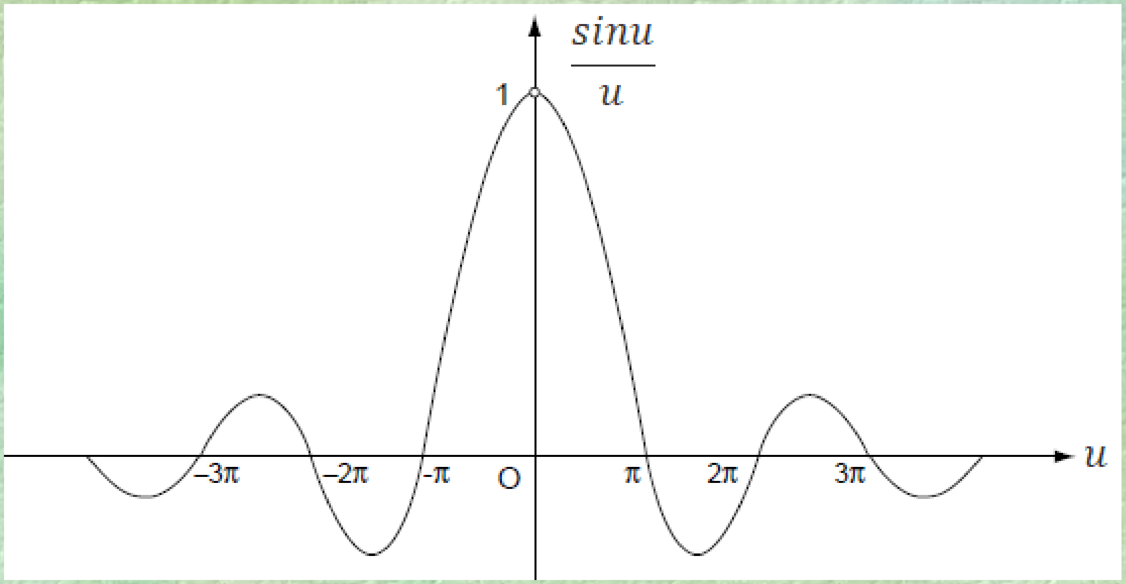
$$Si(x) = \int_0^x \frac{sinu}{u}\,du$$

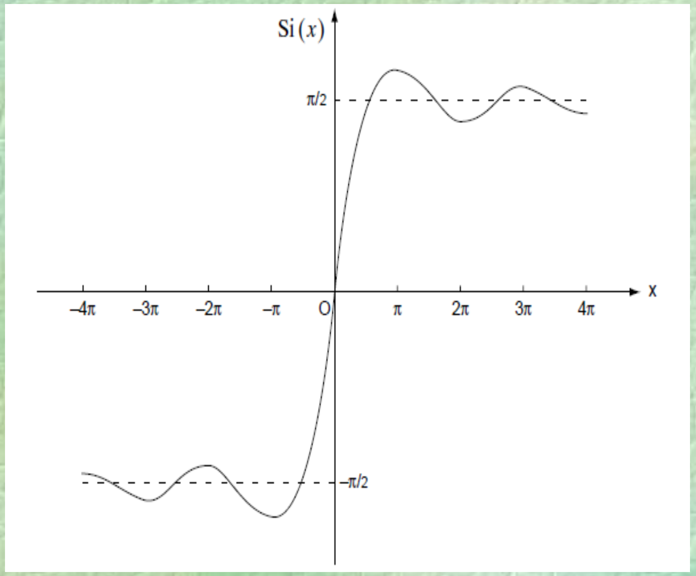
How do we plot this function?

Approximation in Fourier Integral

Define *Si*(*x*) as:

$$Si(x) = \int_0^x \frac{\sin u}{u} du$$





Approximation in Fourier Integral

$$f_{\omega_0}(t)=\frac{2}{\pi}\int_0^{\omega_0}\frac{\sin(\omega)}{\omega}\cos(\omega t)d\omega=?$$

$$Si(x)=\int_0^x\frac{sinu}{u}du$$

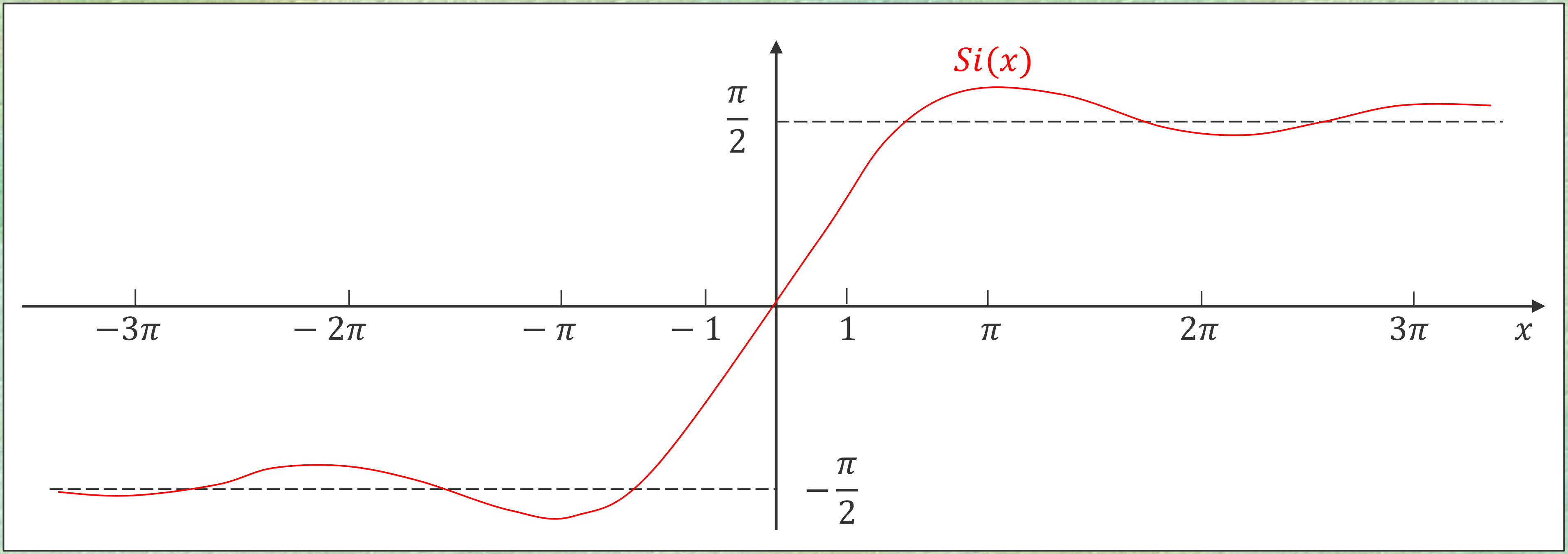
$$f_{\omega_0}(t)=\frac{1}{\pi}\int_0^{\omega_0}\left(\frac{\sin\omega(t+1)}{\omega}\right)d\omega-\frac{1}{\pi}\int_0^{\omega_0}\left(\frac{\sin\omega(t-1)}{\omega}\right)d\omega$$

$$f_{\omega_0}(t)=\frac{1}{\pi}\int_0^{\omega_0(t+1)}\left(\frac{\sin u}{u}\right)du-\frac{1}{\pi}\int_0^{\omega_0(t-1)}\left(\frac{\sin u}{u}\right)du$$

$$f_{\omega_0}(t)=\frac{1}{\pi}Si(\omega_0(t+1))-\frac{1}{\pi}Si(\omega_0(t-1))$$

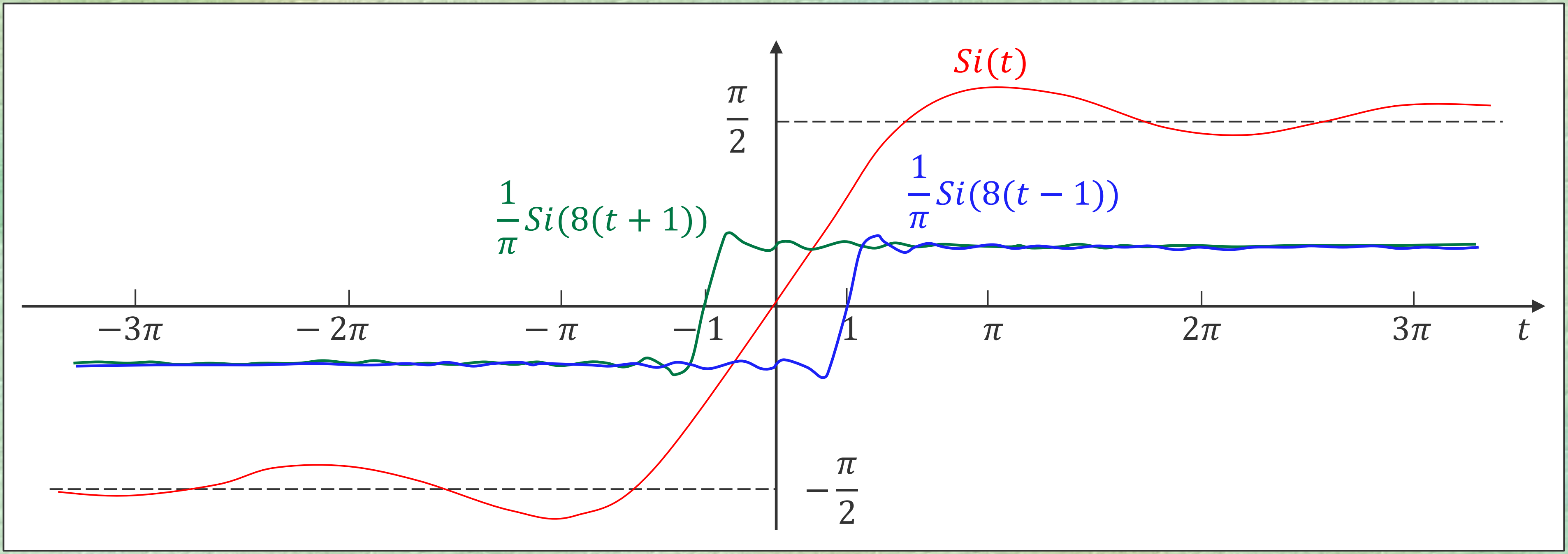
Approximation in Fourier Integral

$$f_8(t) = \frac{2}{\pi} \int_0^8 \frac{\sin(\omega)}{\omega} \cos(\omega t) d\omega = \frac{1}{\pi}\left(\underline{Si(8(t+1))} - Si(8(t-1))\right)$$



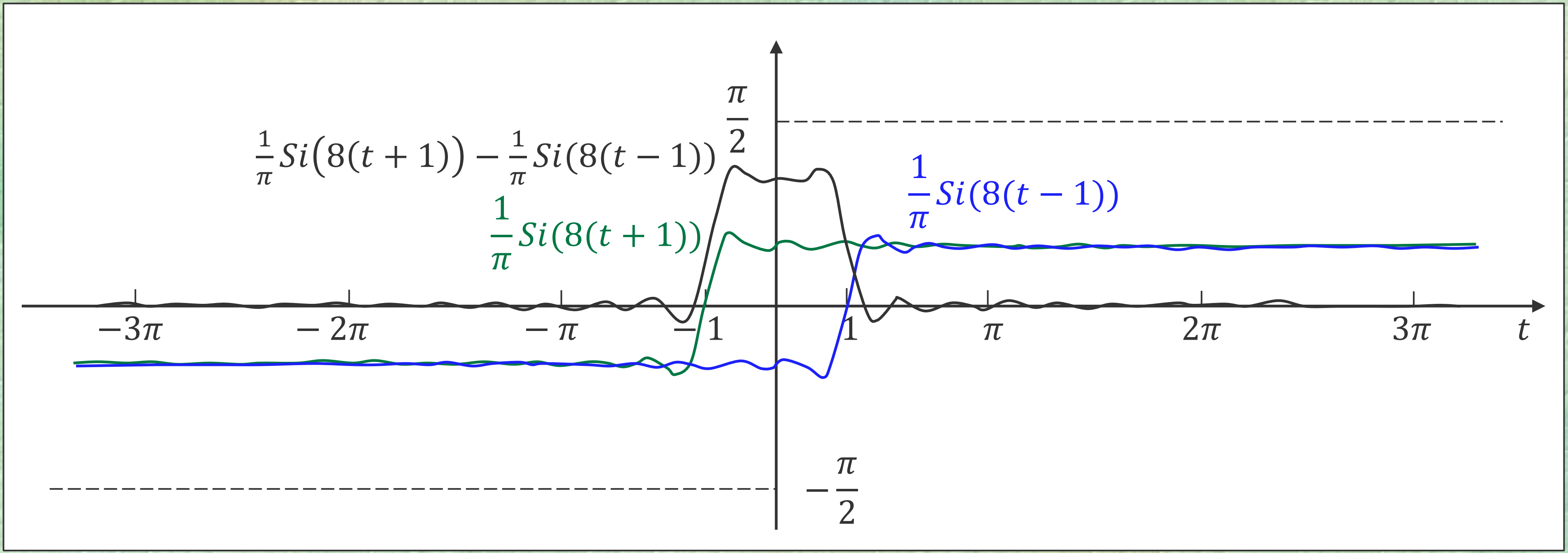
Approximation in Fourier Integral

$$f_8(t)=\frac{2}{\pi}\int_0^8 \frac{\sin(\omega)}{\omega}\cos(\omega t)\,d\omega=\frac{1}{\pi}\left(\underline{Si(8(t+1))}-\underline{Si(8(t-1))}\right)$$



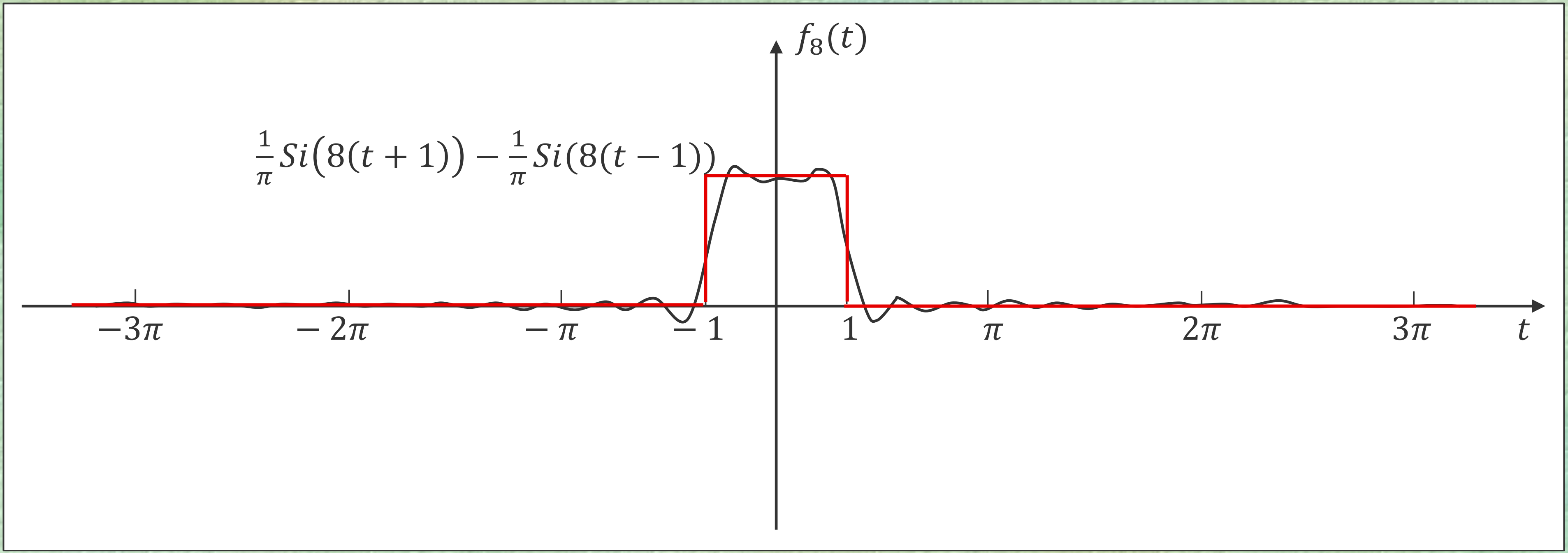
Approximation in Fourier Integral

$$f_8(t) = \frac{2}{\pi}\int_0^8 \frac{\sin(\omega)}{\omega}\cos(\omega t)\,d\omega = \frac{1}{\pi}\left(\underline{Si(8(t+1))} - \underline{Si(8(t-1))}\right)$$



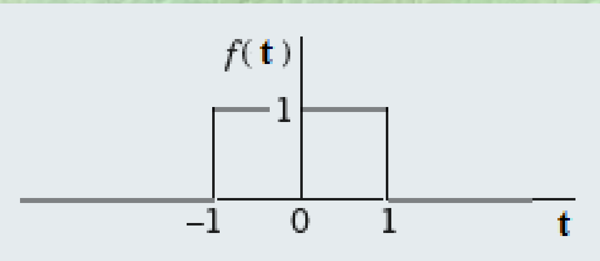
Approximation in Fourier Integral

$$f_8(t)=\frac{2}{\pi}\int_0^8\frac{\sin(\omega)}{\omega}\cos(\omega t)d\omega=\frac{1}{\pi}\left(\underline{Si(8(t+1))}-\underline{Si(8(t-1))}\right)$$



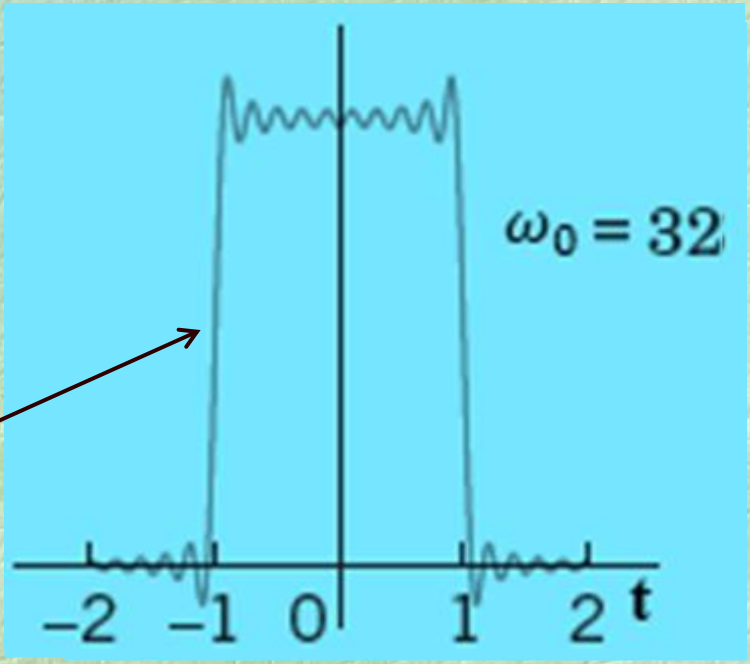
Approximation in Fourier Integral

$$f(t)=\frac{2}{\pi}\int_0^{\infty}\frac{\sin(\omega)}{\omega}\cos(\omega t)\,d\omega=\begin{cases}1 & -1<t<1\\ 0 & |t|>1\end{cases}$$



$$f_{\omega_0}(t)=\frac{2}{\pi}\int_0^{\omega_0}\frac{\sin(\omega)}{\omega}\cos(\omega t)\,d\omega$$

$$=\frac{1}{\pi}\left(Si(\omega_0(t+1))-Si(\omega_0(t-1))\right)$$



$$f(-1)=f(1)=\frac{2}{\pi}\int_0^{\infty}\frac{\sin(\omega)}{\omega}\cos(\omega)\,d\omega=0.5$$

Part One: Fourier Series and Fourier Integral

- ❑ Introduction to Fourier Series
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- ❑ Different Representations of Fourier Series
- ❑ Applications of Fourier Series in Engineering
- ❑ Fourier Integral
- ❑ Applications of Fourier Integral in Engineering

Applications of Fourier Integral in Engineering

Find the particular solution of the following equation.

$$y'' + 3y' + 2y = f(t) \qquad f(t) = \begin{cases} 1 & \text{for } -1 \le t \le 1 \\ 0 & \text{for } |t| > 1 \end{cases}$$

According to the previous section, the Fourier integral representation of the function f(t) is:

$$f(t) = \int_0^\infty A(\omega)\cos(\omega t)\,d\omega = \frac{2}{\pi}\int_0^\infty \frac{\sin(\omega)}{\omega}\cos(\omega t)\,d\omega$$

Therefore, the above differential equation can be written as follows:

$$y'' + 3y' + 2y = \frac{2}{\pi}\int_0^\infty \frac{\sin(\omega)}{\omega}\cos(\omega t)\,d\omega$$

Applications of Fourier Integral in Engineering

$$y'' + 3y' + 2y = \frac{2}{\pi}\int_0^{\infty} \frac{\sin(\omega)}{\omega} \cos(\omega t)\, d\omega$$

$$dy'' + 3dy' + 2dy = \frac{2}{\pi}\frac{\sin(\omega)}{\omega} \cos(\omega t)\, d\omega$$

$$dy = a\cos(\omega t) + b sin(\omega t)$$

$$\left(2 - \omega^2\right)a + 3\omega b = \frac{2}{\pi}\frac{sin\omega}{\omega} d\omega$$

$$-3\omega a + \left(2 - \omega^2\right)b = 0$$

According to these two equations, the coefficients are obtained as follows:

$$a = \frac{(2-\omega^2)}{(2-\omega^2)^2 + 9\omega^2}\frac{2}{\pi}\frac{sin\omega}{\omega} d\omega \qquad b = \frac{3\omega}{(2-\omega^2)^2 + 9\omega^2}\frac{2}{\pi}\frac{sin\omega}{\omega} d\omega$$

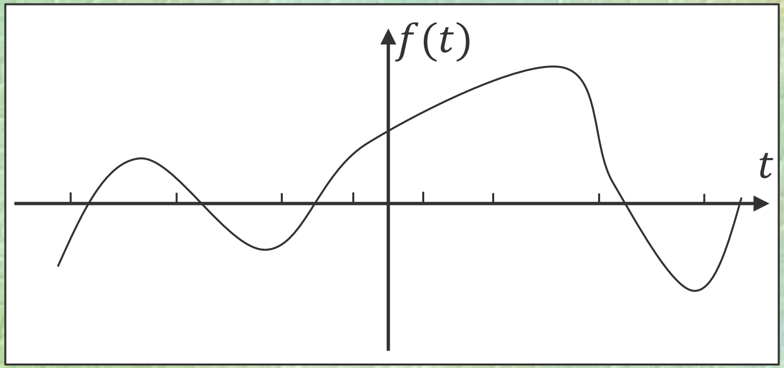
Applications of Fourier Integral in Engineering

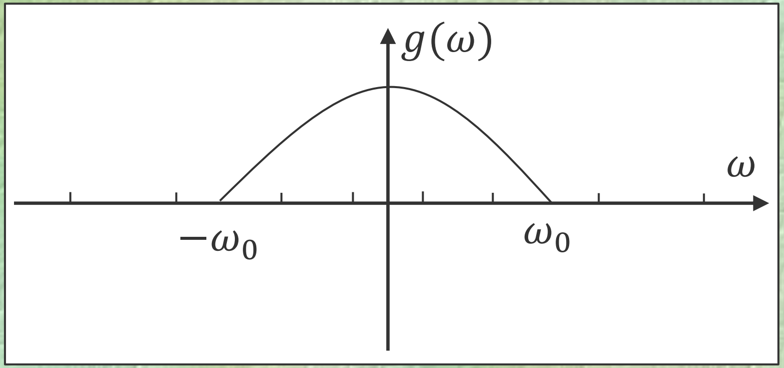
And the final particular solution is:

$$dy = \frac{\left(2-\omega^2\right)cos\omega t + 3\omega sin\omega t}{(2-\omega^2)^2 + 9\omega^2}\frac{2}{\pi}\frac{sin\omega}{\omega}d\omega$$

$$y(t) = \frac{2}{\pi}\int_0^\infty \frac{\left(2-\omega^2\right)cos\omega t + 3\omega sin\omega t}{(2-\omega^2)^2 + 9\omega^2}\frac{sin\omega}{\omega}d\omega$$

Applications of Fourier Integral in Engineering

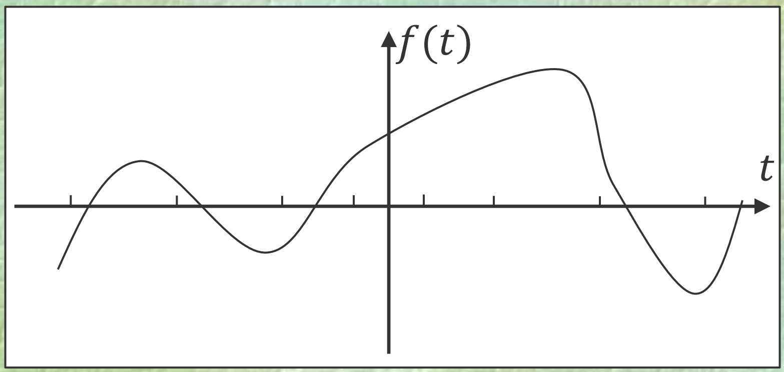


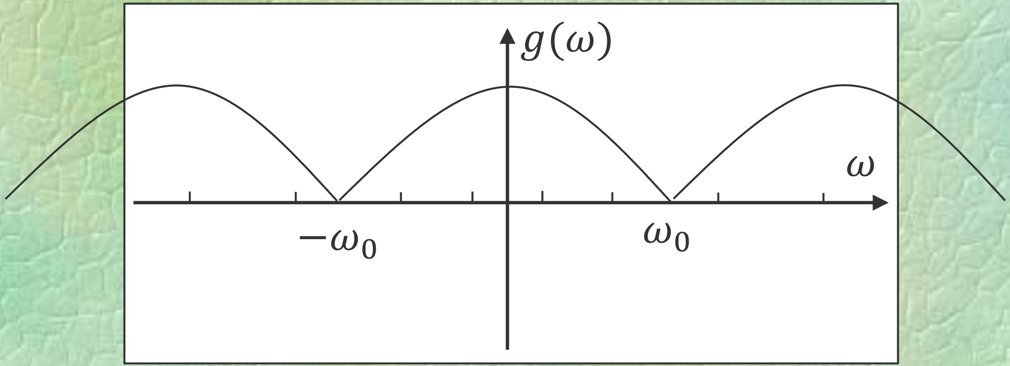


$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$f(t) = \int_{-\infty}^{\infty} g(\omega)e^{j\omega t}d\omega$$

Applications of Fourier Integral in Engineering





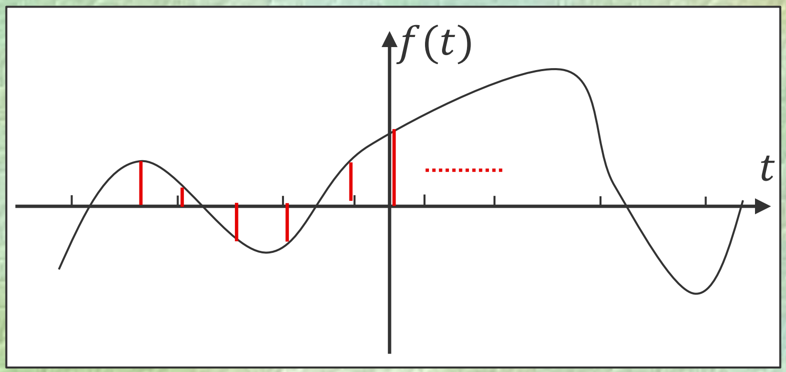
$$g(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

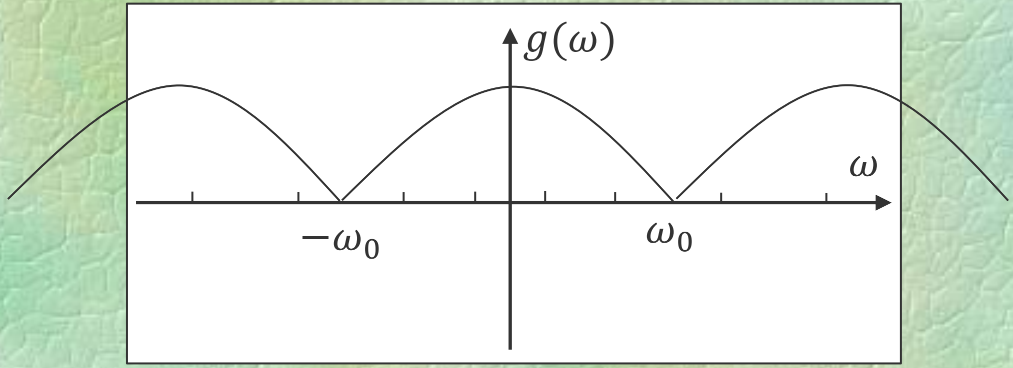
$$f(t)=\int_{-\infty}^{\infty} g(\omega)e^{j\omega t}d\omega$$

$$\downarrow$$

$$g(\omega)=\sum_{n=-\infty}^{\infty} C_n e^{j\frac{2n\pi}{2\omega_0}\omega}$$

Applications of Fourier Integral in Engineering





$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$f(t) = \int_{-\infty}^{\infty} g(\omega)e^{j\omega t}d\omega$$

$$\downarrow$$

$$g(\omega) = \sum_{n=-\infty}^{\infty} C_n e^{j\frac{2n\pi}{2\omega_0}\omega}$$

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT)\ldots$$

Exercises

Exercise 16: Show that the integral of the specified function represents:

$$\int_0^\infty \frac{\cos(wx)+w\sin(wx)}{1+w^2}\,dw = \begin{cases} 0 & x<0 \\ \pi/2 & x=0 \\ \pi e^{-x} & x>0 \end{cases}$$

Exercise 17: Find the Fourier sine integral of the function f.

$$f(x) = \begin{cases} \pi/2 & 0<x<\pi \\ 0 & x>\pi \end{cases}$$

Exercise 18: Show that if f(x)f(x)f(x) has a Fourier transform, then the function f(x−a)f(x - a)f(x−a) also has a Fourier transform, and we have:

$$F\{f(x-a)\} = e^{-iwa} F\{f(x)\}$$

Exercise 19: Does the Fourier cosine and sine transforms exist for the function $f(x)=e^x$? Why?

Exercise 20: Find the particular solution of the equation.

$$y'' + ay' + by = f(t)$$

$$f(t) = \begin{cases} t & t^2 < 1 \\ 0 & 1 < t^2 \end{cases}$$

References

- Advanced Engineering Mathematics , E. Kreyszig

- Advanced Engineering Mathematics, C. R. Wylie