Engineering Mathematics

Ali Karimpour

Professor

Ferdowsi University of Mashhad

Content of this course

1. Fourier Series and Fourier Integral.

2. Partial Differential Equation and Its Solutions.

3. Complex Analysis. (The theory of functions of a complex variable)

Complex Analysis (The theory of functions of a complex variable)

- ❑ Fundamentals
- ❑ Analytic Functions and Differentiability
- ❑ Integration in the Complex Plane
- ❑ Complex Series
- ❑ Residue Theory and Calculation of Real Integrals

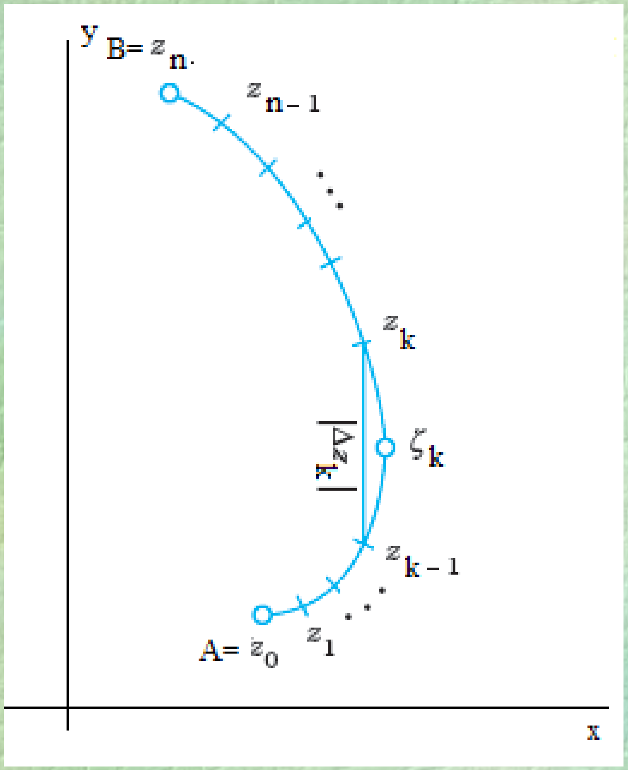
Integration in the Complex Plane

Consider *f*(*z*) as follows (Analytic or non analytic)

$$f(z) = u(x,y) + iv(x,y)$$

Consider following summation

$$I = \sum_{k=1}^{n} f(\xi_k)\Delta z_k \qquad \Delta z_k = z_k - z_{k-1}$$



Integral in the complex plane is as follows:

$$\int_A^B f(z)dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\Delta z_k$$

Integration in the Complex Plane

In a specific case where *A* and *B* coincide, and the path of integration forms a closed curve, the mentioned integral is called a line integral and is denoted by the symbol below.

$$\int_A^B f(z)dz \Longrightarrow \oint f(z)dz$$

Determining the upper limit in complex integration

$$\left|\sum_{k=1}^{n} f(\xi_k)\Delta z_k\right| \le \sum_{k=1}^{n} |f(\xi_k)\Delta z_k| = \sum_{k=1}^{n} |f(\xi_k)||\Delta z_k|$$

$$if \quad n \to \infty \qquad \left|\int_C f(z)dz\right| \le \int_C |f(z)||dz| \le \int_C M|dz| = M\int_C |dz|$$

$$\boxed{\left|\int_C f(z)dz\right| \le ML}$$

Integration in the Complex Plane

If the path of integration is fixed

$$\int_A^B f(z)dz = -\int_B^A f(z)dz$$

$$\int_A^B kf(z)dz = k\int_A^B f(z)dz$$

$$\int_A^B [f(z) \pm g(z)]dz = \int_A^B f(z)dz \pm \int_A^B g(z)dz$$

If D is a point on the arc AB.

$$\int_A^B f(z)dz = \int_A^D f(z)dz + \int_D^B f(z)dz$$

Integration in the Complex Plane

Another form of complex integral:

$$I = \sum_{k=1}^{n} f(\xi_k)\Delta z_k \qquad \Delta z_k = z_k - z_{k-1}$$

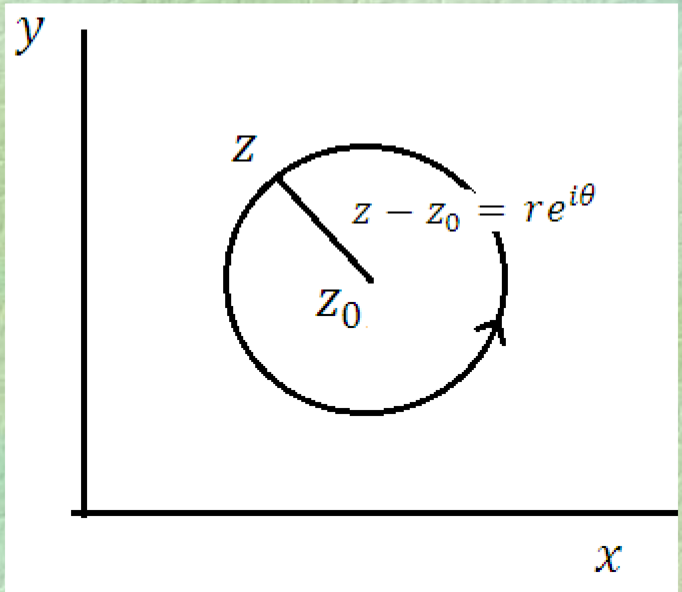
$$\mathrm{I} = \sum_{k=1}^{n} [u(\mu_k, \eta_k) + iv(\mu_k, \eta_k)](\Delta x_k + i\Delta y_k)$$

$$= \sum_{k=1}^{n} [u(\mu_k, \eta_k)\Delta x_k - v(\mu_k, \eta_k)\Delta y_k] + i\sum_{k=1}^{n} [v(\mu_k, \eta_k)\Delta x_k + u(\mu_k, \eta_k)\Delta y_k]$$

$$\int_C f(z)dz = \int_C udx - vdy + i\int_C vdx + udy \;= \int_C (u + iv)(dx + idy)$$

Integration in the Complex Plane

Example 1: If c is a circle with radius r and center Z_0, and if n is an integer, find the value of the following integral (counterclockwise).



$$I = \int_C \frac{dz}{(z - z_0)^{n+1}}$$

$$z - z_0 = re^{i\theta} \qquad dz = rie^{i\theta} d\theta$$

$$I = \int_0^{2\pi} \frac{rie^{i\theta} d\theta}{r^{n+1} e^{i(n+1)\theta}} = \int_0^{2\pi} \frac{ie^{-in\theta} d\theta}{r^n}$$

$$\begin{cases} 2\pi i & n = 0 \\ 0 & n \neq 0 \end{cases}$$

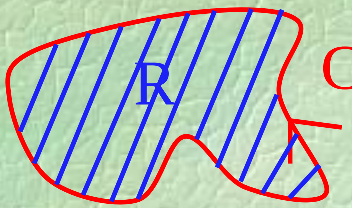
$$I = \int_C \frac{dz}{(z - z_0)^{n+1}} = \begin{cases} 2\pi i & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Very very important

Integration in the Complex Plane

Theorem 1 (Green's Theorem): If $P(x,y)$, $Q(x,y)$ are continuous on a simply connected region R with a piecewise smooth boundary C, and if the partial derivatives $\partial Q/\partial x$ and $\partial P/\partial y$ are continuous on R and C, then:

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$



where the integration is over the closed path C and R is the region to the left of C.

Theorem 2: In any region where the $\int_C Pdx + Qdy$ is path-independent, the partial derivatives of the function $\varphi(x,y) = \int_{a,b}^{x,y} P(x,y)dx + Q(x,y)dy$ are:

$$\frac{\partial \varphi}{\partial y} = Q(x,y) \qquad \frac{\partial \varphi}{\partial x} = P(x,y)$$

Integration in the Complex Plane

Theorem 3: If in all points of a simply connected region we have:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

Then $\int P(x,y)dx + Q(x,y)dy$ in R is independent of the path and vice versa.

Theorem 4 (Cauchy's Theorem): If R is a region (either simply connected or multiply connected) with a piecewise smooth boundary C, and if $f(z)$ is analytic and $f\,'(z)$ is continuous inside and on the boundary of R, then:

$$\int_C f(z)dz = 0$$

Integration in the Complex Plane

Proof: We have

$$\int_C f(z)dz = \int_C udx - vdy + i\int_C vdx + udy$$

Assuming continuity of $f'(z)$ means that the following partial derivatives exist:

$$\frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y} \quad \frac{\partial v}{\partial x} \quad \frac{\partial v}{\partial y}$$

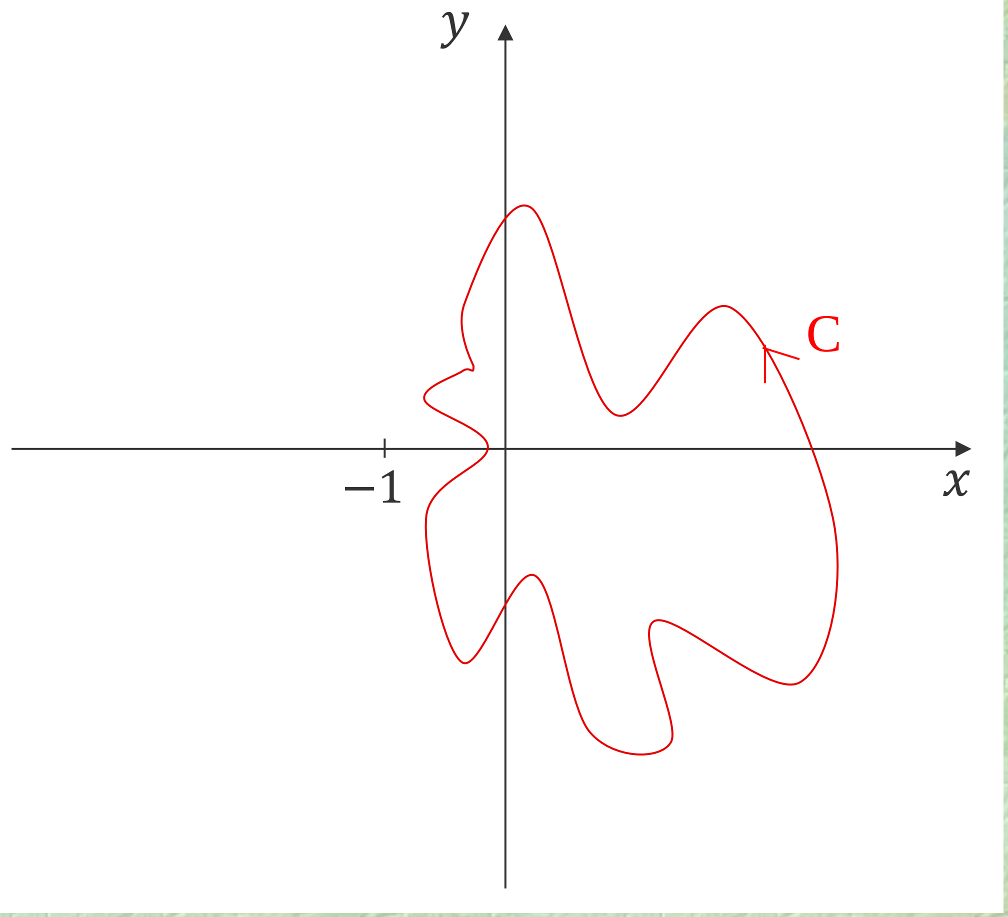
According to Theorem 1, we have:

$$\int_C f(z)dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy \qquad \int_C f(z)dz = 0$$

The more general form of the theorem is known as the **Cauchy-Goursat Theorem**, where the continuity of $f'(z)$ is not required.

Integration in the Complex Plane

Exercise 1: Evaluate the following integrals along the specified path.



$$\oint \cos z\,dz = ? \qquad \oint e^{\cos z}dz = ?$$

$$\oint (5z^4 + 3z^2 - 7z)dz = ?$$

$$\oint \frac{\sin z}{z^2 + 5z + 6}dz = ?$$

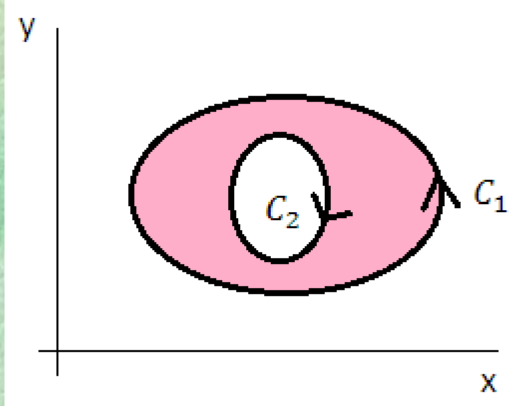
$$\oint \frac{\sinh z}{z^2 + 5z + 6}dz = ?$$

$$\oint \frac{1}{z}dz = ?$$

$$\oint \frac{\cos z}{z(z^2 + 5z + 6)}dz = ?$$

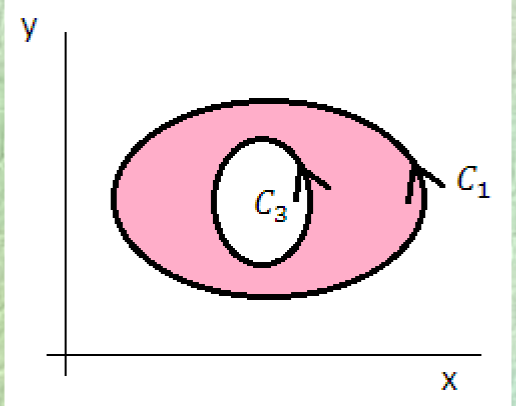
Integration in the Complex Plane

Theorem 5: The line integral of an analytic function over any simple closed curve is equal to the line integral of the same function over any other simple closed curve, provided that the first curve can be continuously deformed into the second curve without passing through any point where *f*(*z*) is non-analytic.



$$\int_C f(z)dz = 0$$

$$\int_{C_1} f(z)dz + \int_{C_2} f(z)dz = 0$$



$$\int_{C_1} f(z)dz = -\int_{C_2} f(z)dz$$

$$\int_{C_1} f(z)dz = \int_{C_3} f(z)dz$$

Integration in the Complex Plane

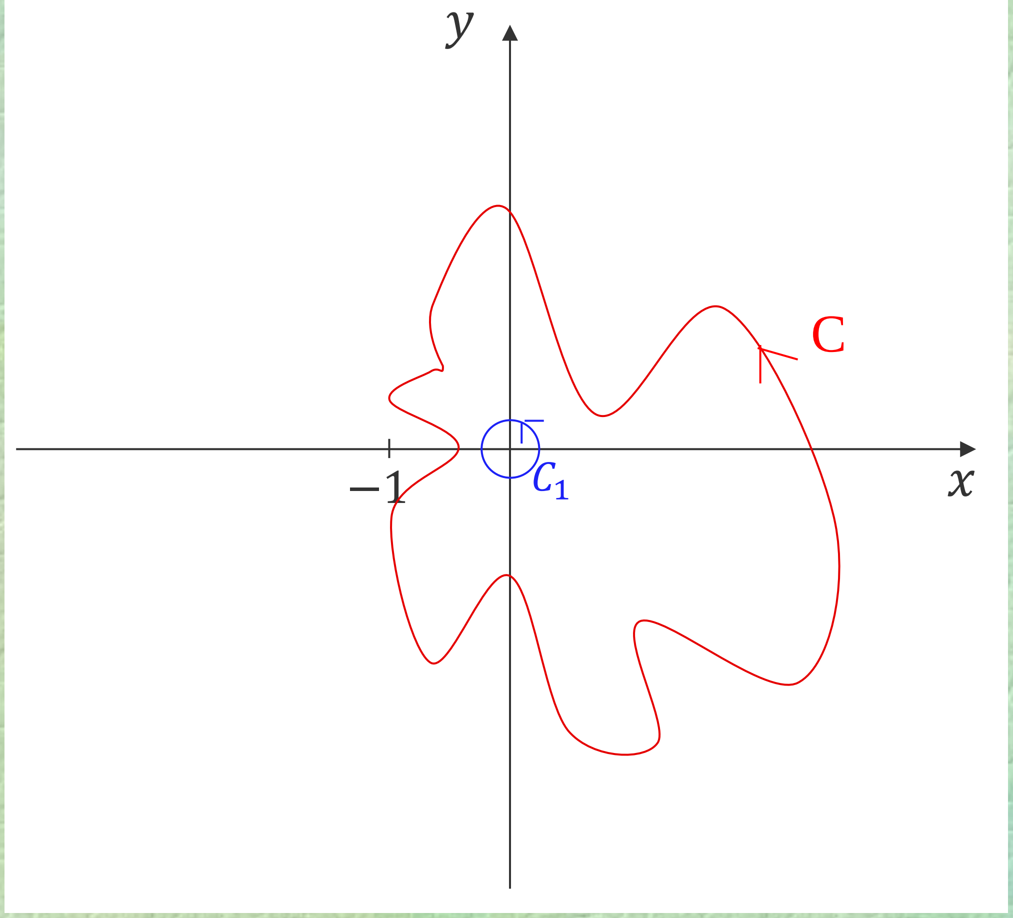
Example 2: Evaluate the following integral along the specified path.

Hint: Use Theorem 5 and Example 1 for assistance.

$$\oint \frac{1}{z}dz = \int_{C_1} \frac{1}{z}dz = 2\pi i$$

$$\oint \frac{cosz}{z(z^2+5z+6)}dz = ?$$

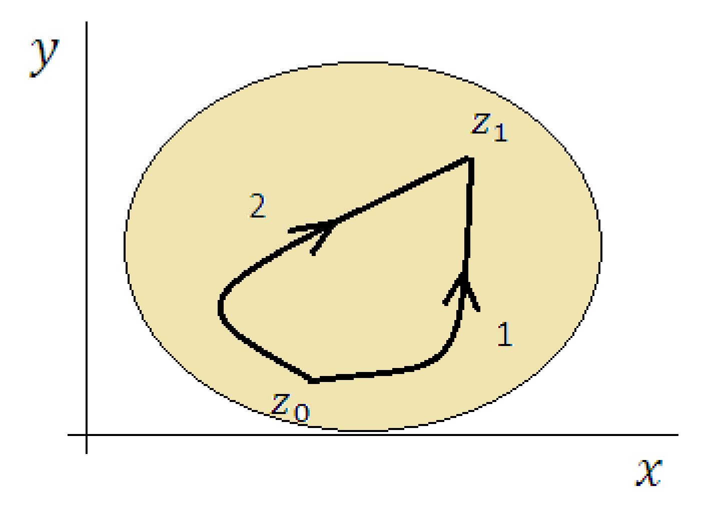
$$= \int_{C_1} \frac{cosz}{z(z^2+5z+6)}dz = \frac{\pi}{3}i$$



Integration in the Complex Plane

Theorem 6: In any simply connected region where $f(z)$ is analytic, the following integral is path-independent:

$$\int_{z_0}^{z_1} f(z)\,dz$$



Proof:

$$\int_C f(z)\,dz = 0$$

....

Integration in the Complex Plane

Theorem 7: If *f*(*z*) is analytic throughout a simply connected domain *R*, then

$$F(z) = \int_{z_0}^{z} f(z)dz$$

Is an analytic function throughout R. Its derivative is *f*(*z*).

Proof: Since *f*(*z*) is analytic in the simply connected domain *R*, the following integral is path-independent:

$$F(z) = \int_{z_0}^{z} f(z)dz$$

Therefore, *F*(*z*) is a function of *z* alone. This integral can be written as follows:

$$F(z) = U + iV = \int_{x_0,y_0}^{x,y} udx - vdy + i\int_{x_0,y_0}^{x,y} vdx + udy$$

Integration in the Complex Plane

$$F(z) = U + iV = \int_{x_0,y_0}^{x,y} udx - vdy + i\int_{x_0,y_0}^{x,y} vdx + udy$$

$$U = \int_{x_0,y_0}^{x,y} udx - vdy \qquad V = \int_{x_0,y_0}^{x,y} vdx + udy$$

According to **Theorem 2**, since the integrals are path-independent

$$\frac{\partial U}{\partial x} = u \qquad \frac{\partial U}{\partial y} = -v \qquad \frac{\partial V}{\partial x} = v \qquad \frac{\partial V}{\partial y} = u$$

$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x} \qquad \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$

Therefore, $F(z)$ is analytic.

$$\boxed{F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = u + iv = f(z)}$$

Integration in the Complex Plane

Theorem 8: If *f*(*z*) is analytic in a simply-connected domain *R*, then provided that the integration path lies entirely within *R*,

$$\int_{z_0}^{z_1} f(z)dz = G(z_1) - G(z_0)$$

In the above relation, *G*(*z*) is an arbitrary antiderivative (primitive) of *f*(*z*).

Proof: Since *f*(*z*) is analytic in *R*, from Theorem 7 we have:

$$F(z) = \int_{z_0}^{z} f(z)dz$$

$$F'(z) - G'(z) = f(z) - f(z) = 0 \longrightarrow F(z) = G(z) + c$$

$$F(z) = \int_{z_0}^{z} f(z)dz = G(z) + c \xrightarrow{z = z_0} c = -G(z_0)$$

$$\int_{z_0}^{z} f(z)dz = G(z) - G(z_0) \qquad \boxed{\int_{z_0}^{z_1} f(z)dz = G(z_1) - G(z_0)}$$

Integration in the Complex Plane

Example 3: It is required to determine the following integral:

$$I = \int_0^{1+\pi i} \left(z^2 + cosh2z\right) dz$$

The integrand is:

$$f(z) = z^2 + cosh2z$$

$f(z)$ is analytic everywhere and an antiderivative (primitive) is given by

$$G(z) = \frac{1}{3} z^3 + \frac{1}{2} sinh2z$$

Therefore, according to **Theorem 8**, the value of this integral is:

$$I = (\frac{1}{3} z^3 + \frac{1}{2} sinh\, 2z)\Big|_0^{1+i\pi} = \frac{1}{3}(1+i\pi)^3 + \frac{1}{2} sinh2(1+i\pi) = \;\; \ldots.$$

Integration in the Complex Plane

Theorem 9: If $u(x,y)$ is a solution to Laplace's equation in a domain such as R, then in R, there exists an analytic function such that u is its real part. In other words, there exists an analytic function $f(z)=u+iv$ such that

$$v(x,y)=\int_{a,b}^{x,y}-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy$$

and the path of integration from (a,b) to (x,y) lies entirely within R.

Proof: We have

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad\qquad \frac{\partial(\frac{\partial u}{\partial x})}{\partial x}=\frac{\partial(-\frac{\partial u}{\partial y})}{\partial y}$$

In this case, the integral of v in the domain R is independent of the path between a fixed point (a,b) and a variable point (x,y) (**Theorem 3**). Now, according to **Theorem 2**, we have:

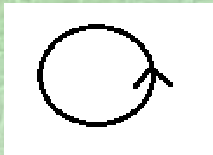
$$\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x} \qquad\qquad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

Integration in the Complex Plane

One of the most important theorems in this section is:

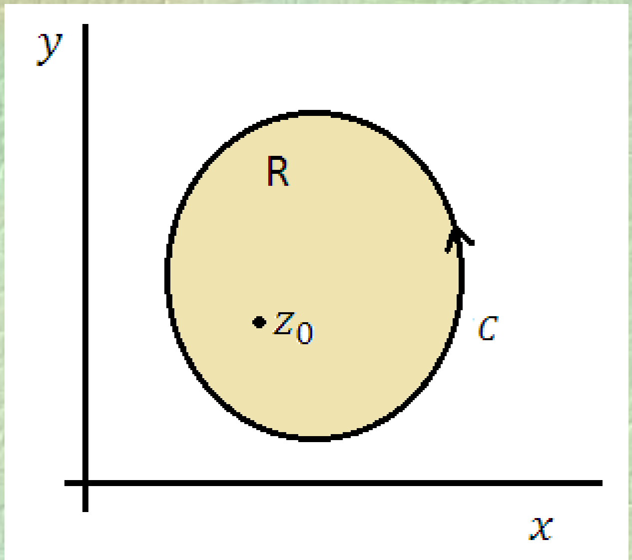
Theorem 10: If *C*, the boundary of a simply-connected domain such as *R*, is piecewise smooth, and if *f*(*z*) is analytic inside and on the boundary *C*, and if z_0 is a point inside *R*, then:

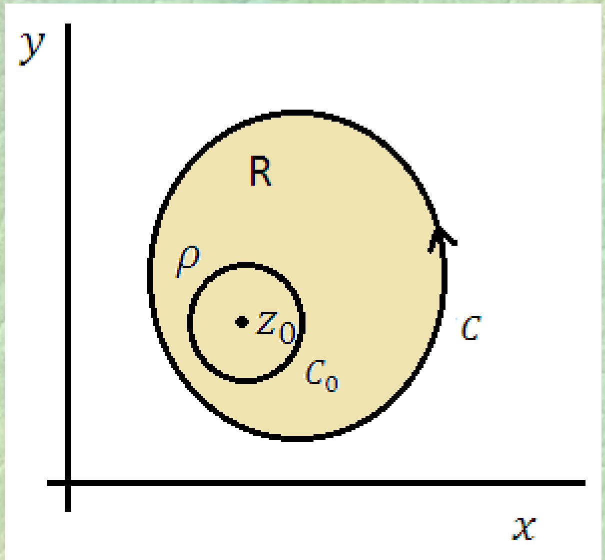
$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz$$



the integration over C is performed in the positive (counter-clockwise) direction.

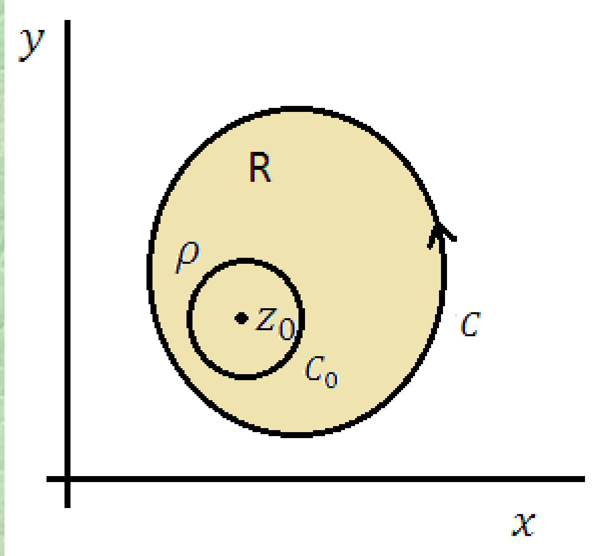
Proof:





Integration in the Complex Plane

Proof:



$$\int_C \frac{f(z)}{z-z_0}dz = \int_{C_0} \frac{f(z)}{z-z_0}dz$$

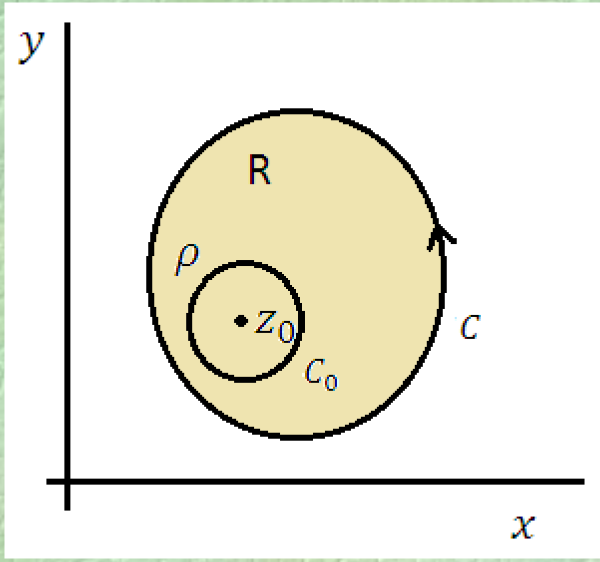
$$\int_C \frac{f(z)}{z-z_0}dz = \int_{C_0} \frac{f(z_0)+[f(z)-f(z_0)]}{z-z_0}dz$$

$$= f(z_0)\int_{C_0} \frac{dz}{z-z_0} + \int_{C_0} \frac{f(z)-f(z_0)}{z-z_0}dz$$

$$\int_C \frac{f(z)dz}{z-z_0} = f(z_0)2\pi i + \int_{C_0} \frac{f(z)-f(z_0)}{z-z_0}dz$$

Integration in the Complex Plane

$$\int_C \frac{f(z)dz}{z-z_0} = f(z_0)2\pi i + \int_{C_0} \frac{f(z)-f(z_0)}{z-z_0}dz$$



$$\left|\int_{C_0} \frac{f(z)-f(z_0)}{z-z_0}dz\right| \le \int_{C_0} \frac{|f(z)-f(z_0)|}{|z-z_0|}|dz|$$

Since $f(z)$ is analytic in R, so it is continuous.

$$|z-z_0| \equiv \rho < \delta \qquad\qquad |f(z)-f(z_0)| < \varepsilon$$

$$\left|\int_{C_0} \frac{f(z)-f(z_0)}{z-z_0}dz\right| < \int_{C_0} \frac{\varepsilon}{\rho}|dz| = \frac{\varepsilon}{\rho}\int_{C_0}|dz| = \frac{\varepsilon}{\rho}2\pi\rho = 2\pi\varepsilon$$

$$\boxed{f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz}$$

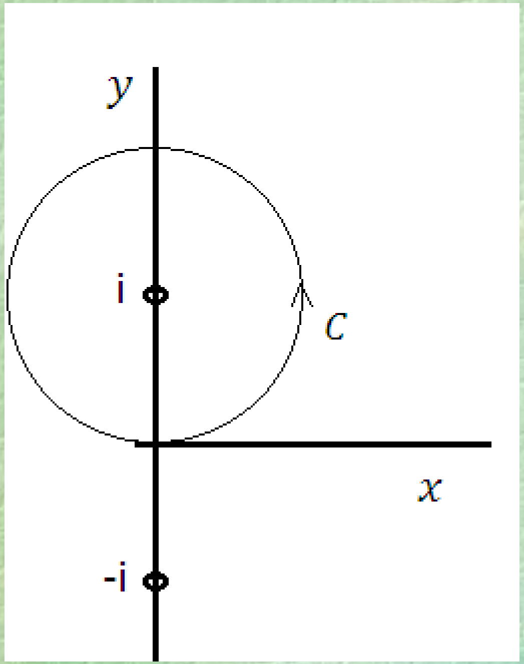
Integration in the Complex Plane

Example 4: It is required to determine the values of

$$I = \int_C \frac{e^z}{z^2+1} dz$$

given that C is a circle with a unit radius centered at (a) $z=i$ and (b) $z=-i$.

Solution (a): The integral is written as follows:

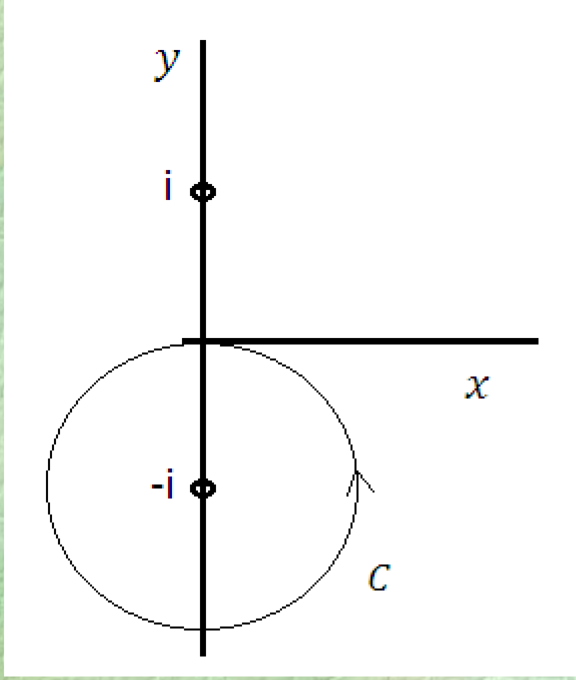


$$I = \int_C \frac{e^z}{z+i} \frac{dz}{z-i}$$

$$I = \int_C \frac{e^z}{z+i} \frac{dz}{z-i} = 2\pi i f(z_0) = 2\pi i f(i) = 2\pi i \frac{e^i}{2i} = \pi(cos1 + isin1) = 1.7 + i2.6$$

Integration in the Complex Plane

Solution (*b*): The integral is written as follows:



$$I = \int_C \frac{e^z}{z^2+1}dz$$

$$I = \int_C \frac{e^z}{z-i}\frac{dz}{z+i}$$

$$I = \int_C \frac{e^z}{z-i}\frac{dz}{z+i} = 2\pi i f(z_0) = 2\pi i f(-i) = 2\pi i \frac{e^{-i}}{-2i} = -\pi(cos1 - isin1) = -1.7 + i2.6$$

Integration in the Complex Plane

Theorem 11: If *f*(*z*) is analytic throughout a closed simply-connected domain *R*, then at every interior point z_0 of *R* derivatives of *f*(*z*) of all orders exist as follows and are analytic, where *C* is the boundary of *R*.

$$f^{n}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}}$$

Theorem 12 (Morera's Theorem): If *f*(*z*) is continuous in a region *R*, and if for every simple closed curve that can be drawn in *R*, we have:

$$\int_C f(z)dz = 0$$

Then *f*(*z*) is analytic in *R*.

Integration in the Complex Plane

Theorem 13 (Inequality Theorem): If $f(z)$ is analytic inside and on a circle C with radius r and center z_0, then:

$$|f^n(z_0)| \leq \frac{n!\,M}{r^n}$$

where M is the maximum value of $|f(z)|$ on C.

In the special case for $n=0$, we have:

$$|f(z_0)| \leq M$$

Proof:

$$|f^n(z_0)| = \left|\frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}\right| \leq \frac{n!}{2\pi}\int_C \frac{|f(z)||dz|}{|z-z_0|^{n+1}} \leq \frac{n!}{2\pi}\frac{M}{r^{n+1}}\int_C |dz|$$

$$\leq \frac{n!}{2\pi}\frac{M}{r^{n+1}}2\pi r \leq \frac{n!\,M}{r^n} \Rightarrow |f^n(z_0)| \leq \frac{n!\,M}{r^n}$$

Integration in the Complex Plane

Theorem 14 (Maximum Modulus Theorem): The modulus of a non-constant function $f(z)$ cannot have a maximum in a region where the function is analytic.

The maximum modulus of a non-constant function $f(z)$ in a region where the function is analytic, is located on the boundary.

The minimum modulus of a non-constant function $f(z)$ in a region where the function is analytic and does not become zero, is located on the boundary.

Integration in the Complex Plane

Example 5: Consider the function $f(z) = z^2 + 2$. The goal is to find the extrema of $|f(z)|$ on the closed region $|z| \le 1$

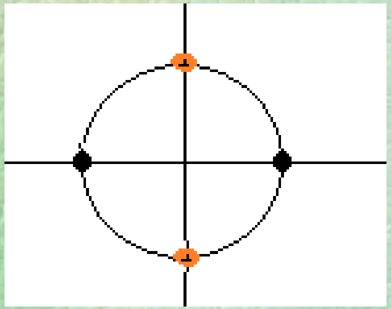
Since $f(z)$ is analytic within the given region and does not become zero, the extrema will be located on the boundary.

Let $y = sin\theta \quad , \quad x = cos\theta$

$$|f(z)| = \left|z^2 + 2\right| = \left|x^2 - y^2 + 2ixy + 2\right| = \sqrt{(x^2 - y^2 + 2)^2 + (2xy)^2}$$

$$= \sqrt{(cos2\theta + 2)^2 + sin^2 2\theta} \; = \sqrt{4cos2\theta + 5}$$

$$\frac{d|f(z)|}{d\theta} = \frac{-8sin2\theta}{2\sqrt{4cos2\theta + 5}} = 0 \qquad \theta = 0, \pi \qquad \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

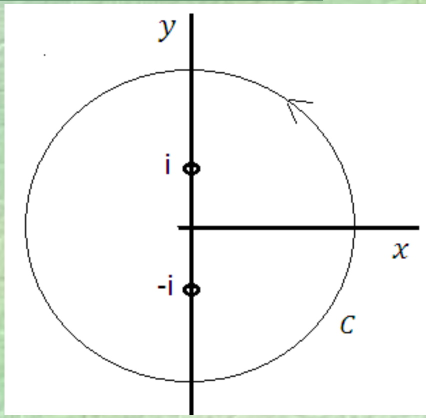


Theorem 15 (Liouville's Theorem): If a function $f(z)$ is entire (analytic everywhere) and bounded in the complex plane, then $f(z)$ is a constant function.

Exercises

Exercise 2: Find the value of the integral along the given path:

$$I = \int_C \frac{e^z}{z^2+1} dz$$



Exercise 3: Determine the value of the integral along the circle centered at the origin with radius 2, oriented counterclockwise.

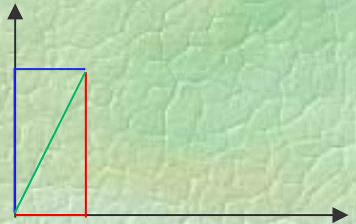
$$\oint \frac{cosz}{z(z^2+4z+3)} dz = ?$$

Exercise 4: Determine the value of the integral along the unit circle centered at the origin, oriented counterclockwise.

$$\oint tanzdz = ?$$

Exercise 5: Determine the value of the integral along the three paths: red, blue, and green.

$$\int_0^{1+\pi i} (e^z + sinh2z)dz$$



Complex Analysis (The theory of functions of a complex variable)

- ❑ Fundamentals
- ❑ Analytic Functions and Differentiability
- ❑ Integration in the Complex Plane
- ❑ Complex Series
- ❑ Residue Theory and Calculation of Real Integrals

Complex Series

Consider the following infinite sequence:

$$z_1, z_2, \dots, z_n, \dots$$

The sequence converges if for every $\varepsilon > 0$, there exists a positive integer n_0 such that:

$$|z_n - z| < \varepsilon \qquad\qquad \forall \quad n > n_0$$

If a sequence is not convergent, then it is divergent.

Exercise 6: Show that every sequence has at most one limit.

Theorem: Suppose for the sequence $z_1, z_2, z_3, \dots z_n, \dots$ we have:

$$z_n = x_n + \mathrm{i}y_n$$

In this case:

$$\lim_{n\to\infty} z_n = z \quad \leftrightarrow \quad \lim_{n\to\infty} x_n = x\,, \qquad \lim_{n\to\infty} y_n = y$$

Complex Series

Suppose the following expression is a series where the terms are functions of a single complex variable *z*:

$$f_1(z) + f_2(z) + \cdots + f_n(z) + \cdots$$

Partial Sums of the Series is:

$$S_1(z) = f_1(z)$$

$$S_2(z) = f_1(z) + f_2(z)$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$S_n(z) = f_1(z) + f_2(z) + \cdots + f_n(z)$$

The series converges to $S(z)$ if:

Complex Series

Consider the following definition:

the remainder after n terms in $S(z)$.

$$S(z) - S_n(z) = R_n(z)$$

Definition of Convergence: A series is called convergent if the limit of $|R_n(z)|$ as n approaches infinity is zero.

Definition of the Region of Convergence: The set of all values of z for which the series is convergent is called the region of convergence of the series.

Definition of a Divergent Series: A series that is not convergent is called a divergent series.

Complex Series

Theorem: A necessary and sufficient condition for the convergence of the series of complex terms below is that the series composed of the real parts and the series composed of the imaginary parts of these terms each converge

$$f_1(z) + f_2(z) + \cdots + f_n(z) + \cdots$$

Moreover, if following series (real parts and the imaginary parts of original series):

$$\sum_{n=1}^{\infty} Re(f_n)\,, \qquad \sum_{n=1}^{\infty} Im(f_n)$$

If the real part and imaginary part of a complex series converge respectively to the functions *Re*(*z*) and *Im*(*z*), then the given series *Re*(*z*) + *iIm*(*z*) also converges.

Complex Series

Theorem (Ratio Test): For the series

$$f_1(z) + f_2(z) + \cdots + f_n(z) + \cdots$$

Suppose:

$$\lim_{n\to\infty} \left| \frac{f_{n+1}(z)}{f_n(z)} \right| = |r(z)|$$

In this case, the given series:

$$\begin{cases} 0 \le |r(z)| < 1 & \text{the series converges.} \\ |r(z)| > 1 & \text{the series diverges.} \\ |r(z)| = 1 & \text{is the boundary of convergence, and the ratio test is inconclusive there.} \end{cases}$$

Complex Series

Example 6: Determine the region of convergence of the following series.

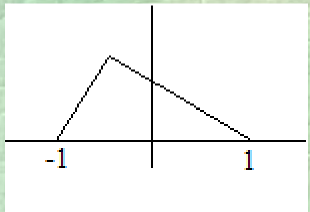
$$1+\frac{1}{2^2}\frac{z+1}{z-1}+\frac{1}{3^2}\left(\frac{z+1}{z-1}\right)^2+\frac{1}{4^2}\left(\frac{z+1}{z-1}\right)^3+\cdots$$

Using the ratio test, we have:

$$\left|\frac{f_{n+1}(z)}{f_n(z)}\right|=\left|\frac{\frac{1}{(n+1)^2}\left(\frac{z+1}{z-1}\right)^n}{\frac{1}{n^2}\left(\frac{z+1}{z-1}\right)^{n-1}}\right|=\left|\frac{n^2}{(n+1)^2}\frac{z+1}{z-1}\right| \xrightarrow{n\to\infty} \left|\frac{z+1}{z-1}\right|$$

$$\left|\frac{z+1}{z-1}\right|<1 \longleftarrow$$

The region of convergence of the series



$$|z+1|<|z-1|$$

Complex Series

Example 6: Determine the region of convergence of the following series.

$$1+\frac{1}{2^2}\frac{z+1}{z-1}+\frac{1}{3^2}\left(\frac{z+1}{z-1}\right)^2+\frac{1}{4^2}\left(\frac{z+1}{z-1}\right)^3+\cdots$$

Using the test on the boundary of the imaginary axis is problematic:

$$|z+1|=|z-1|$$

In this case, the series on the imaginary axis has an absolute value of:

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$$

This series is absolutely convergent, so…

Complex Series

Uniform Convergence Definition: A series of functions $f_n(z)$ in a region R, whether closed or open, converges uniformly to a function $S(z)$ if for every positive integer ε, there exists a corresponding integer N such that, independent of z, for every z in R, we have:

$$\forall n > N \quad \rightarrow \quad |S(z) - S_n(z)| < \varepsilon$$

Complex Series

Theorem (M-Test or Weierstrass Test): If there exists a sequence of positive constants M_n such that for all positive integers n and for all values of z in a given region D, we have:

$$|f_n(z)| \leq M_n$$

and if the series:

$$M_1 + M_2 + M_3 + \cdots + M_n + \cdots$$

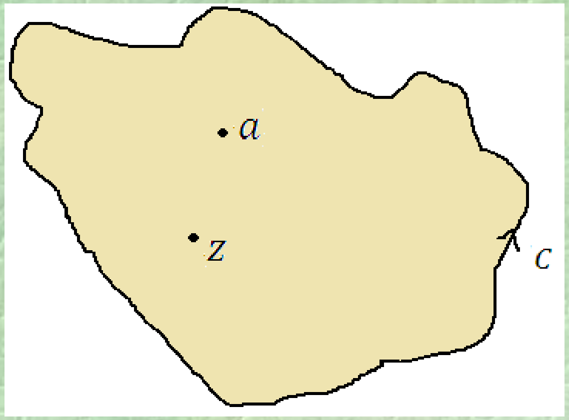
converges, then the original series

$$f_1(z) + f_2(z) + \cdots + f_n(z) + \cdots$$

converges uniformly in D.

Complex Series

Theorem (Taylor Series): If $f(z)$ is analytic throughout a bounded region enclosed by a simple closed curve C, and if z and a are both inside C, then:



$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + \cdots + f^{n-1}(a)\frac{(z-a)^{n-1}}{(n-1)!} + R_n$$

In which
$$R_n = \frac{(z-a)^n}{2\pi i}\int_C \frac{f(t)dt}{(t-a)^n(t-z)}$$

Proof: We know that (according to Theorem 10 in the Integral Section):

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z}$$

Complex Series

Proof: (continue)

$$f(z)=\frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z}$$

$$f(z)=\frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z}=\frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-a}\frac{1}{1-\frac{z-a}{t-a}}dt$$

$$1\ \big|\underline{\,1-u\,}$$

$$\frac{1}{1-u}=1+u+u^2+\cdots+\frac{u^n}{1-u}$$

$$f(z)=\frac{1}{2\pi i}\int_C \frac{f(t)}{t-a}\left[1+\frac{z-a}{t-a}+\left(\frac{z-a}{t-a}\right)^2+\cdots+\left(\frac{z-a}{t-a}\right)^{n-1}+\frac{\left(\frac{z-a}{t-a}\right)^n}{1-\frac{z-a}{t-a}}\right]dt$$

Complex Series

Proof: (continue)

$$f(z)=\frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z}$$

$$1 \;\big|\underline{\,1-u\,}$$

$$f(z)=\frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z}=\frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-a}\frac{1}{1-\dfrac{z-a}{t-a}}dt$$

$$\frac{1}{1-u}=1+u+u^2+\cdots+\frac{u^n}{1-u}$$

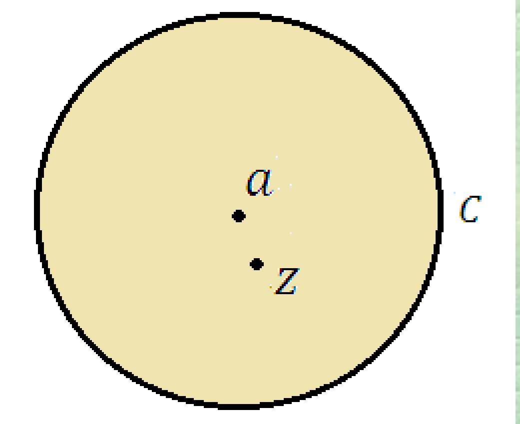
$$f(z)=\frac{1}{2\pi i}\int_C \frac{f(t)}{t-a}\left[1+\frac{z-a}{t-a}+\left(\frac{z-a}{t-a}\right)^2+\cdots+\left(\frac{z-a}{t-a}\right)^{n-1}+\frac{\left(\dfrac{z-a}{t-a}\right)^n}{1-\dfrac{z-a}{t-a}}\right]dt$$

$$f(z)=f(a)+f'(a)(z-a)+\cdots+f^{(n-1)}(a)\frac{(z-a)^{n-1}}{(n-1)!}+\frac{(z-a)^n}{2\pi i}\int_C \frac{f(t)dt}{(t-a)^n(t-z)}$$

Complex Series

Theorem: Taylor Series

$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + f'''(a)\frac{(z-a)^3}{3!} \ldots$$



In every point inside any circle centered at a, where $f(z)$ is analytic within the circle, the Taylor series converges to $f(z)$.

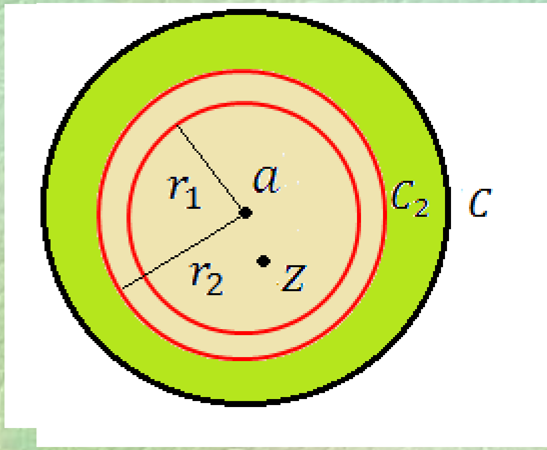
Proof: In the previous theorem, we saw that:

$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + \cdots + f^{n-1}(a)\frac{(z-a)^{n-1}}{(n-1)!} + R_n$$

We need to show that as n becomes sufficiently large, the term R_n tends to zero.

Complex Series

Proof: (continue)



$$R_n = \frac{(z-a)^n}{2\pi i}\int_C \frac{f(t)dt}{(t-a)^n(t-z)}$$

$$R_n = \frac{(z-a)^n}{2\pi i}\int_{C_2} \frac{f(t)dt}{(t-a)^n(t-z)}$$

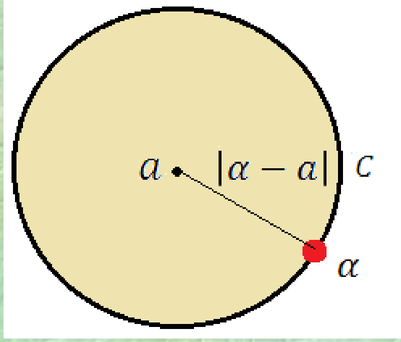
$|t-a| = r_2$ $\qquad |t-z| > r_2 - r_1$ $\qquad |z-a| < r_1$

$$|R_n(z)| = \left|\frac{(z-a)^n}{2\pi i}\int_{C_2} \frac{f(t)dt}{(t-a)^n(t-z)}\right| \le \frac{|(z-a)|^n}{|2\pi i|}\int_{C_2} \frac{|f(t)||dt|}{|t-a|^n|t-z|}$$

$$< \frac{r_1^n}{2\pi}\int_{C_2} \frac{M|dt|}{{r_2}^n(r_2-r_1)} \quad = M\left(\frac{r_1}{r_2}\right)^n \frac{r_2}{r_2-r_1}$$

Complex Series

- If a is zero, the series is called the **Maclaurin series**, and the statements mentioned also apply to it.
- **Definition of Radius and Circle of Convergence:** The largest circle that can be drawn around the point $z=a$ such that the Taylor series $f(z)$ converges everywhere inside it is called the circle of convergence, and the radius of this circle is called the radius of convergence.
- In the special case where α is the nearest singularity such that $f(z)$ approaches infinity as z approaches α, then the radius of convergence is equal to $|\alpha - a|$.



- If the function is not analytic but bounded at the point α, then the radius of convergence may be greater than $|\alpha - a|$. Refer to the footnotes in your reference books for more details.

Complex Series

Theorem: If $f(z)$ is represented by a series in the neighborhood of $z=a$ in the following form, then this representation is unique.

$$\sum_{n=1}^{\infty} a_n\,(z-a)^n$$

Theorem (Binomial Theorem): The following series converges under the given conditions.

$$(s+t)^n = s^n + ns^{n-1}t + \frac{n(n-1)}{2!}s^{n-2}t^2 + \frac{n(n-1)(n-2)}{3!}s^{n-3}t^3 + \cdots$$

Absolutely crucial: $|s| > |t| \qquad \forall\, n \in Z$

$|s| \le |t| \qquad \forall\, n \in Z \;\; and \;\; n \ge 0$

Complex Series

Example 7: Find the Taylor series expansion of the following function around z=1.

$$f(z) = \frac{3}{(3z - z^2)}$$

$$f(z) = \frac{1}{z} + \frac{1}{3 - z} \quad = [1 + (z - 1)]^{-1} + [2 - (z - 1)]^{-1}$$

Using the **binomial theorem**:

$$(s + t)^n = s^n + ns^{n-1}t + \frac{n(n-1)}{2!}s^{n-2}t^2 + \frac{n(n-1)(n-2)}{3!}s^{n-3}t^3 + \cdots$$

$$[1 + (z - 1)]^{-1} = 1 - (z - 1) + (z - 1)^2 - \cdots + (-1)^n(z - 1)^n + \cdots$$

$$[2 - (z - 1)]^{-1} = 2^{-1} + 2^{-2}(z - 1) + 2^{-3}(z - 1)^2 + \cdots + 2^{-(n+1)}(z - 1)^n + \cdots$$

Complex Series

Example 7: Find the Taylor series expansion of the following function around z=1.

$$f(z) = \frac{3}{(3z - z^2)}$$

$$[1 + (z-1)]^{-1} = 1 - (z-1) + (z-1)^2 - \cdots + (-1)^n (z-1)^n + \cdots$$

$$[2 - (z-1)]^{-1} = 2^{-1} + 2^{-2}(z-1) + 2^{-3}(z-1)^2 + \cdots + 2^{-(n+1)}(z-1)^n + \cdots$$

$$|z-1| < 1$$

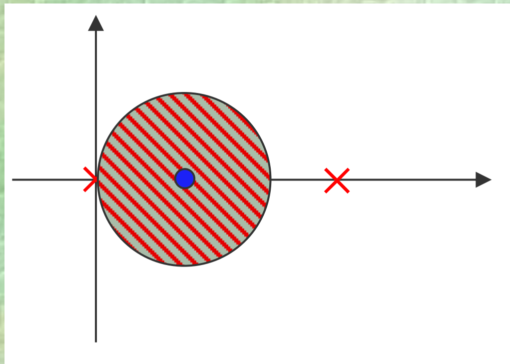
?? $|z-1| < 1$

$$|z-1| < 2$$

$$f(z) = \frac{3}{2} - \frac{3}{4}(z-1) + \frac{9}{8}(z-1)^2 - \cdots + \left[\frac{1}{2^{n+1}} + (-1)^n\right](z-1)^n + \cdots$$

The radius of convergence is given by:

$$|z-1| < 1$$



Complex Series

Theorem (Liouville's Theorem): If $f(z)$ is bounded and analytic for all values of z, then $f(z)$ is constant.

Since $f(z)$ is analytic everywhere, the series below converges at all points in the complex plane:

$$f(z) = f(0) + f'(0)z + \cdots + \frac{f^{n}(0)}{n!} z^{n} + \cdots$$

Now, if C is an arbitrary circle centered at the origin with an infinite radius, from Cauchy's inequality we have:

$$\left|f^{(n)}(0)\right| \leq \frac{n!\, M_c}{r^n} \qquad \rightarrow \qquad f^{(n)}(0) = 0$$

$$f(z) = f(0)$$

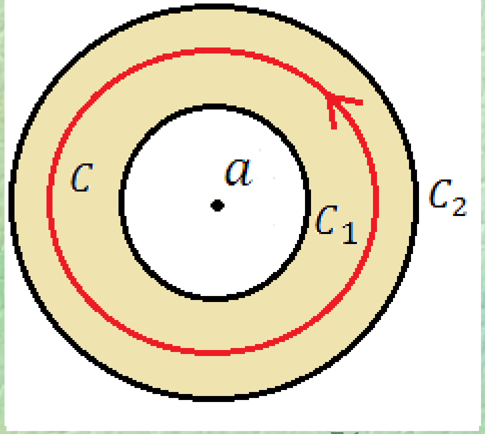
Complex Series

Laurent Series Expansion: In many applications, it is necessary to expand a function around points where the function is not analytic at those points or in their neighborhood. Clearly, in such cases, the Taylor series method cannot be used, and a new type of series, known as the Laurent series, is required.

Theorem: If $f(z)$ is analytic and bounded within a closed region R between two concentric circles, then $f(z)$ can be represented by a series in every point of the annular region bounded by the two concentric circles.

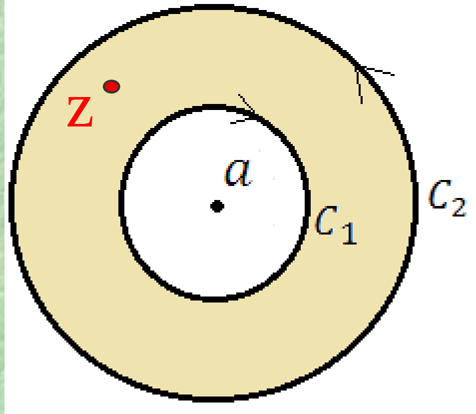
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(t)dt}{(t-a)^{n+1}}$$

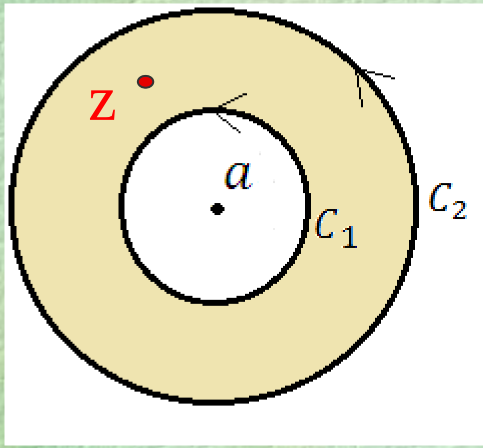


Complex Series

Proof: According to Cauchy's integral formula, for any z belonging to the annulus, given that $f(z)$ is analytic, we have:



$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z}$$



$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(t)dt}{t-z} + \frac{1}{2\pi i}\int_{C_1} \frac{f(t)dt}{t-z}$$

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(t)dt}{t-z} + \frac{1}{2\pi i}\int_{C_1} \frac{f(t)dt}{z-t}$$

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(t)dt}{t-a}\frac{1}{1-\frac{z-a}{t-a}}dt + \frac{1}{2\pi i}\int_{C_1} \frac{f(t)dt}{z-a}\frac{1}{1-\frac{t-a}{z-a}}dt$$

Complex Series

$$f(z)=\frac{1}{2\pi i}\int_{C_2}\frac{f(t)dt}{t-a}\frac{1}{1-\frac{z-a}{t-a}}dt+\frac{1}{2\pi i}\int_{C_1}\frac{f(t)dt}{z-a}\frac{1}{1-\frac{t-a}{z-a}}dt$$

$$\frac{1}{1-u}=1+u+u^2+\cdots+\frac{u^n}{1-u}$$

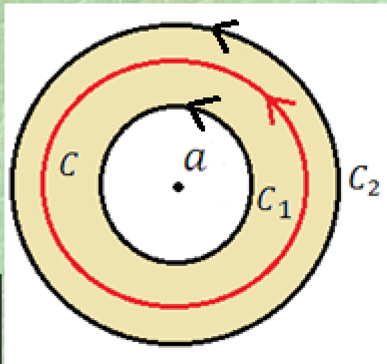
$$f(z)=\frac{1}{2\pi i}\int_{C_2}\frac{f(t)}{t-a}\left[1+\frac{z-a}{t-a}+\cdots+\left(\frac{z-a}{t-a}\right)^{n-1}+\frac{\left(\frac{z-a}{t-a}\right)^n}{1-\frac{z-a}{t-a}}\right]dt$$

$$+\frac{1}{2\pi i}\int_{C_1}\frac{f(t)}{z-a}\left[1+\frac{t-a}{z-a}+\cdots+\left(\frac{t-a}{z-a}\right)^{n-1}+\frac{\left(\frac{t-a}{z-a}\right)^n}{1-\frac{t-a}{z-a}}\right]dt$$

Complex Series

Proof: (continue)

$$f(z)=\frac{1}{2\pi i}\int_{C_2}\frac{f(t)}{t-a}\left[1+\frac{z-a}{t-a}+\cdots+\left(\frac{z-a}{t-a}\right)^{n-1}+\frac{\left(\frac{z-a}{t-a}\right)^{n}}{1-\frac{z-a}{t-a}}\right]dt$$



$$+\frac{1}{2\pi i}\int_{C_1}\frac{f(t)}{z-a}\left[1+\frac{t-a}{z-a}+\cdots+\left(\frac{t-a}{z-a}\right)^{n-1}+\frac{\left(\frac{t-a}{z-a}\right)^{n}}{1-\frac{t-a}{z-a}}\right]dt$$

a_0 $\qquad a_1 \qquad a_{n-1}$

$$f(z)=\frac{1}{2\pi i}\int_{C_2}\frac{f(t)dt}{t-a}+\frac{(z-a)}{2\pi i}\int_{C_2}\frac{f(t)dt}{(t-a)^2}+\cdots+\frac{(z-a)^{n-1}}{2\pi i}\int_{C_2}\frac{f(t)dt}{(t-a)^n}+R_{n2}$$

a_{-1}

$$+\frac{1}{(z-a)2\pi i}\int_{C_1}f(t)dt+\cdots+\frac{1}{(z-a)^n 2\pi i}\int_{C_1}(t-a)^{n-1}f(t)dt+R_{n1}$$

a_{-n}

$$a_n=\frac{1}{2\pi i}\int_{C}\frac{f(t)dt}{(t-a)^{n+1}}$$

Complex Series

Proof: (continue)

$$f(z) = R_{n1} + \frac{a_{-n}}{(z-a)^n} + \cdots + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + \cdots + a_{n-1}(z-a)^{n-1} + R_{n2}$$

$$R_{n2} = \frac{(z-a)^n}{2\pi i}\int_{C_2} \frac{f(t)dt}{(t-a)^n(t-z)} \qquad R_{n1} = \frac{1}{2\pi i(z-a)^n}\int_{C_1} \frac{(t-a)^n f(t)dt}{z-t}$$

$$\lim_{n\to\infty} R_{n1} = 0 \quad , \qquad \lim_{n\to\infty} R_{n2} = 0$$

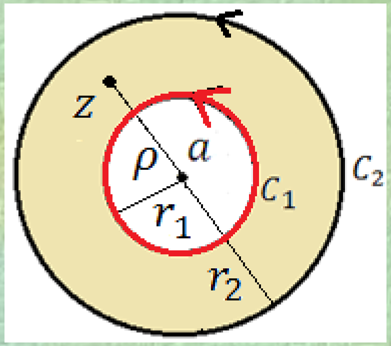
$$a_n = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{(t-a)^{n+1}}$$

$$f(z) = \cdots + \frac{a_{-n}}{(z-a)^n} + \cdots + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + \cdots + a_{n-1}(z-a)^{n-1} + \cdots$$

Complex Series

Proof: (continue)

$$R_{n1} = \frac{1}{2\pi i(z-a)^n}\int_{C_1} \frac{(t-a)^n f(t)dt}{z-t}$$



$$|R_{n1}| = \left|\frac{1}{2\pi i(z-a)^n}\int_{C_1} \frac{(t-a)^n f(t)dt}{z-t}\right| \leq \frac{1}{|2\pi i||(z-a)|^n}\int_{C_1} \frac{|t-a|^n |f(t)||dt|}{|z-t|}$$

$$|t-a| = r_1 \quad |z-a| = \rho \qquad \rho > r_1 \quad |z-t| \geq \rho - r_1$$

$$|R_{n1}| \leq \frac{1}{2\pi\rho^n}\int_{C_1} \frac{{r_1}^n M|dt|}{\rho - r_1} = M\left(\frac{r_1}{\rho}\right)^n \frac{r_1}{\rho - r_1} \qquad \lim_{n\to\infty} R_{n1} = 0$$

Complex Series

Proof: (continue)

$R_{n2} = ?$

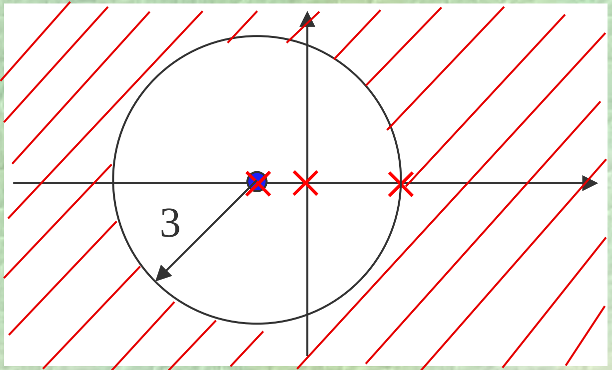
Complex Series

Example 8: Find all acceptable Laurent series expansions for the following function around z=−1 and determine the radius of convergence for each.

$$f(z)=\frac{7z-2}{(z+1)z(z-2)}$$

Solution:

$$f(z)=\frac{-3}{z+1}+\frac{1}{z}+\frac{2}{z-2}$$



$$f(z)=\frac{-3}{z+1}+[(z+1)-1]^{-1}+2[(z+1)-3]^{-1}$$

$$(s+t)^n=s^n+ns^{n-1}t+\frac{n(n-1)}{2!}s^{n-2}t^2+\frac{n(n-1)(n-2)}{3!}s^{n-3}t^3+\cdots$$

$$[(z+1)-1]^{-1}=\quad(z+1)^{-1}+(z+1)^{-2}+(z+1)^{-3}+\cdots$$

$$[(z+1)-3]^{-1}=\quad(z+1)^{-1}+3(z+1)^{-2}+9(z+1)^{-3}+\cdots$$

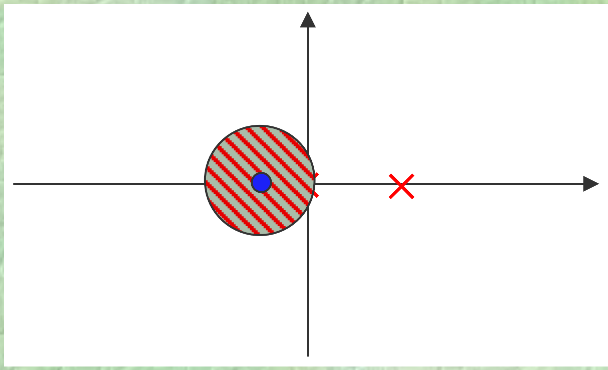
$$f(z)=\frac{7}{(z+1)^2}+\frac{19}{(z+1)^3}+\cdots\qquad\qquad |z+1|>3$$

Complex Series

$$f(z)=\frac{-3}{z+1}+[(z+1)-1]^{-1}+2[(z+1)-3]^{-1}$$

$$f(z)=\frac{-3}{z+1}+[-1+(z+1)]^{-1}+2[-3+(z+1)]^{-1}$$

$$0<|z+1|<1 \qquad 0<|z+1|<3$$



In this case, the use of the binomial series expansion is valid for the convergence radius 0<|z+1|<1.

$$(s+t)^n=s^n+ns^{n-1}t+\frac{n(n-1)}{2!}s^{n-2}t^2+\frac{n(n-1)(n-2)}{3!}s^{n-3}t^3+\cdots$$

$$[-1+(z+1)]^{-1}=(-1)^{-1}-(z+1)-(z+1)^2-\cdots$$

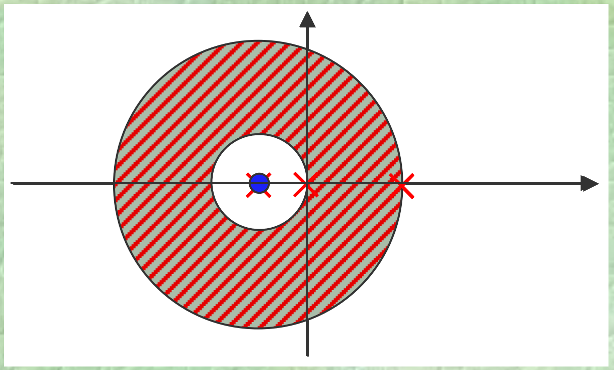
$$[-3+(z+1)]^{-1}=(-3)^{-1}-\frac{1}{9}(z+1)\quad-\frac{1}{27}(z+1)^2+\cdots$$

$$f(z)=\frac{-3}{z+1}-\frac{5}{3}-\frac{11}{9}(z+1)-\frac{29}{27}(z+1)^2\ldots \qquad 0<|z+1|<1$$

Complex Series

$$f(z)=\frac{-3}{z+1}+[(z+1)-1]^{-1}+2[(z+1)-3]^{-1}$$

$$f(z)=\frac{-3}{z+1}+[(z+1)-1]^{-1}+2[-3+(z+1)]^{-1}$$



$|z+1|>1$ $\qquad$ $0<|z+1|<3$

In this case, the use of the binomial series expansion is valid for the convergence radius 1<|z+1|<3.

$$(s+t)^n=s^n+ns^{n-1}t+\frac{n(n-1)}{2!}s^{n-2}t^2+\frac{n(n-1)(n-2)}{3!}s^{n-3}t^3+\cdots$$

$$[(z+1)-1]^{-1}=\quad(z+1)^{-1}+(z+1)^{-2}+(z+1)^{-3}+\cdots$$

$$[-3+(z+1)]^{-1}=\ (-3)^{-1}-\frac{1}{9}(z+1)\quad-\frac{1}{27}(z+1)^2+\cdots$$

$$f(z)=\cdots+\frac{1}{(z+1)^2}-\frac{2}{z+1}-\frac{2}{3}-\frac{2}{9}(z+1)-\cdots\qquad 1<|z+1|<3$$

Complex Series

$$f(z) = \frac{-3}{z+1} + [-1 + (z+1)]^{-1} + 2[(z+1) - 3]^{-1}$$

$1 > |z+1|$

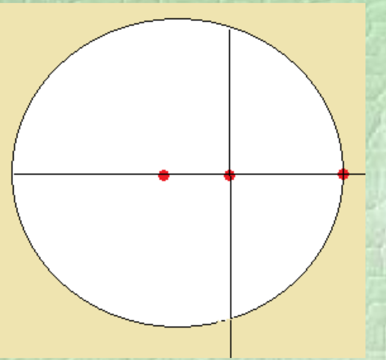
$|z+1| > 3$

Unacceptable

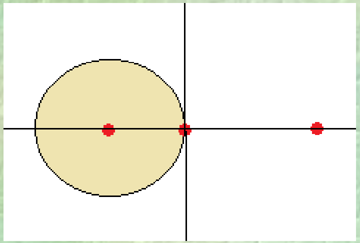
Complex Series

$$f(z) = \frac{7z-2}{(z+1)z(z-2)}$$

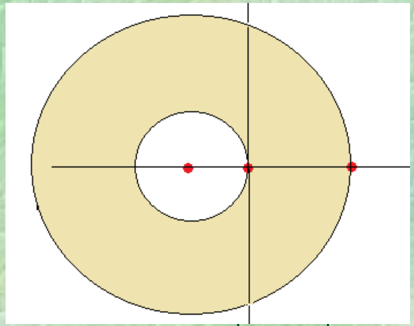
$$f(z) = \frac{7}{(z+1)^2} + \frac{19}{(z+1)^3} + \cdots \qquad |z+1| > 3$$



$$f(z) = \frac{-3}{z+1} - \frac{5}{3} - \frac{11}{9}(z+1) - \frac{29}{27}(z+1)^2 \ldots \qquad 0 < |z+1| < 1$$



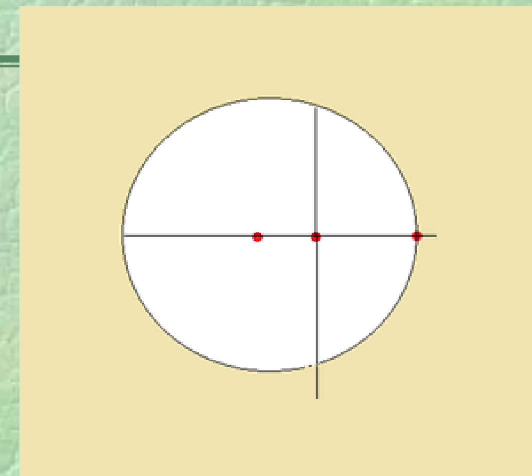
$$f(z) = \cdots + \frac{1}{(z+1)^2} - \frac{2}{z+1} - \frac{2}{3} - \frac{2}{9}(z+1) - \cdots \qquad 1 < |z+1| < 3$$



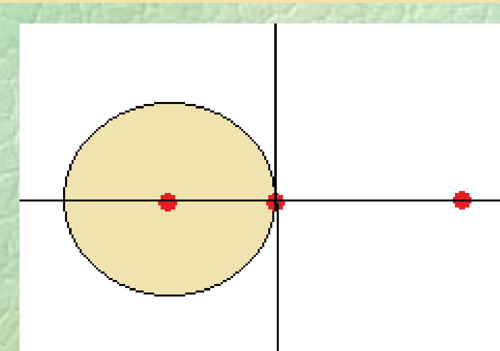
Integration by Complex Series

$$f(z)=\frac{7z-2}{(z+1)z(z-2)}$$

$$f(z)=\frac{7}{(z+1)^2}+\frac{19}{(z+1)^3}+\cdots$$



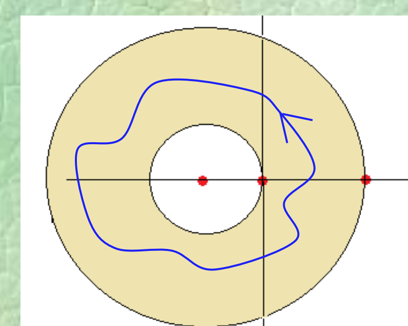
$$f(z)=\frac{-3}{z+1}-\frac{5}{3}-\frac{11}{9}(z+1)-\frac{29}{27}(z+1)^2$$



residue f(z)

$$f(z)=\cdots+\underbrace{\frac{1}{(z+1)^2}+\frac{-2}{z+1}}_{principal\ part\ f(z)}-\frac{2}{3}-\frac{2}{9}(z+1)-\cdots$$

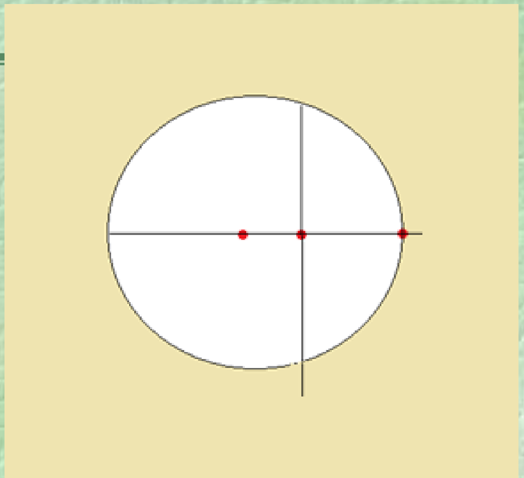
$$\int_{C_1} f(z)dz=?\qquad \int_{C_1} f(z)dz=2\pi i(-2)$$



Integration by Complex Series

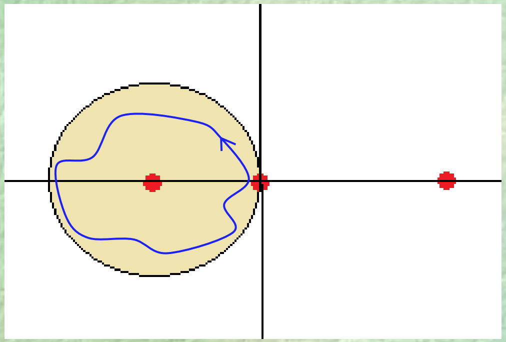
$$f(z)=\frac{7z-2}{(z+1)z(z-2)}$$

$$f(z)=\frac{7}{(z+1)^2}+\frac{19}{(z+1)^3}+\cdots$$

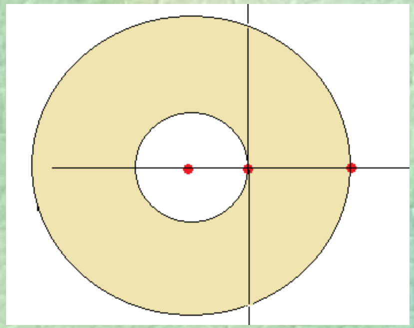


residue f(z)

$$f(z)=\underbrace{\frac{-3}{z+1}}_{\textit{principal part } f(z)}-\frac{5}{3}-\frac{11}{9}(z+1)-\frac{29}{27}(z+1)^2$$



$$f(z)=\cdots+\frac{1}{(z+1)^2}-\frac{2}{z+1}-\frac{2}{3}-\frac{2}{9}(z+1)-\cdots$$



$$\int_{C_1} f(z)dz=?$$

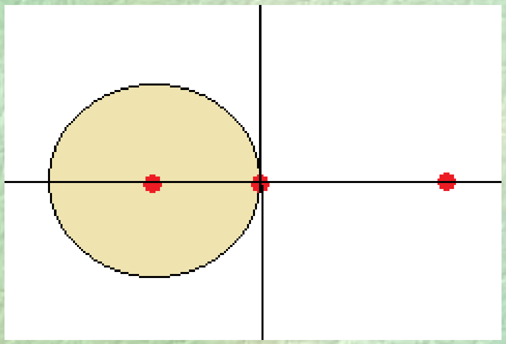
$$\int_{C_1} f(z)dz=2\pi i(-3)$$

Integration by Complex Series

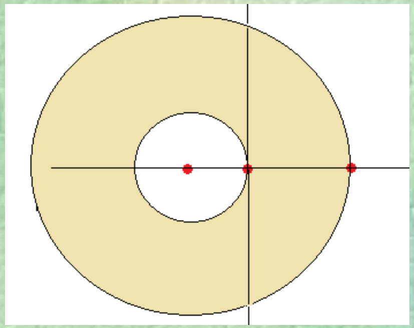
$$f(z)=\frac{7z-2}{(z+1)z(z-2)}$$

$$f(z)=\frac{7}{(z+1)^2}+\frac{19}{(z+1)^3}+\cdots$$

$$f(z)=\frac{-3}{z+1}-\frac{5}{3}-\frac{11}{9}(z+1)-\frac{29}{27}(z+1)^2$$



$$f(z)=\cdots+\frac{1}{(z+1)^2}-\frac{2}{z+1}-\frac{2}{3}-\frac{2}{9}(z+1)-\cdots$$



$$\int_{C_1} f(z)dz = ? \qquad \int_{C_1} f(z)dz = 0$$

Integration by Complex Series

$$f(z) = \ldots + \underbrace{\frac{a_{-m}}{(z-z_1)^m} + \cdots + \frac{a_{-1}}{z-z_1}}_{\text{principal part } f(z)} + a_0 + a_1(z-z_1) + \cdots + a_m(z-z_1)^m + \cdots$$

The point z_1 is a pole of the function $f(z)$

a_{-1} is the residue of $f(z)$ around the point z_1.

$$\int_{C_1} f(z)dz = 2\pi i(a_{-1})$$

Exercises

Exercise 7: Show that every sequence has at most one limit.

Exercise 8: Find all acceptable Laurent and Taylor series expansions for the given function around the point -2 and determine the radius of convergence for each.

$$f(z) = \frac{7z-2}{(z+1)z(z-2)}$$

Exercise 9: Find the Taylor series expansions of the following functions around the given point and determine the radius of convergence for each.

$a)\ f(z) = cosz \qquad z_0 = 0$

$b)\ f(z) = sinhz \qquad z_0 = \pi$

$c)\ f(z) = e^z \qquad z_0 = 0$

Exercise 10: Find the Laurent series expansions of the following functions around the given point and determine the radius of convergence for each.

$a)\ f(z) = tanz \qquad z_0 = 0$

$b)\ f(z) = tanz \qquad z_0 = \pi$

$c)\ f(z) = e^{\frac{1}{z}} \qquad z_0 = 0$

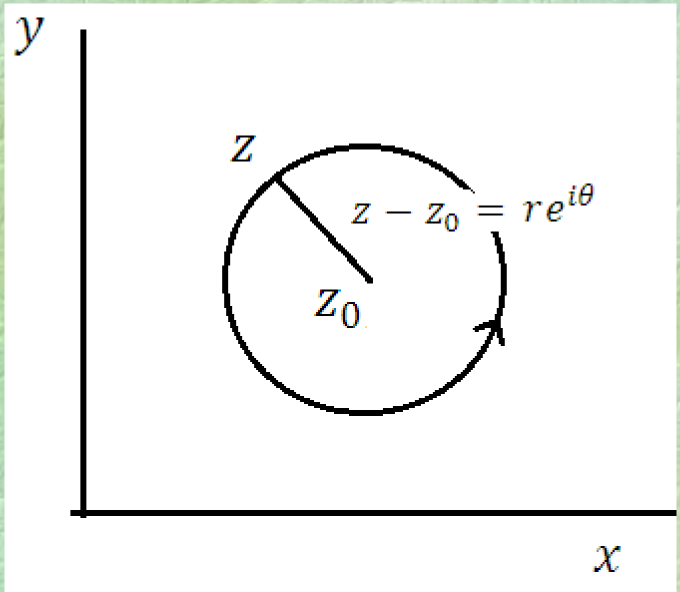
$d)\ f(z) = \dfrac{1-e^z}{z} \qquad z_0 = 0$

Complex Analysis (The theory of functions of a complex variable)

- ❑ Fundamentals
- ❑ Analytic Functions and Differentiability
- ❑ Integration in the Complex Plane
- ❑ Complex Series
- ❑ Residue Theory and Calculation of Real Integrals

Residue Theory and Calculation of Real Integrals

Example 9: If C is a circle with radius r centered at z_0 and n is an integer, find the value of the following integral (counterclockwise):



$$I = \int_C \frac{dz}{(z-z_0)^{n+1}}$$

$$I = \int_C \frac{dz}{(z-z_0)^{n+1}} = \begin{cases} 2\pi i & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Very very important

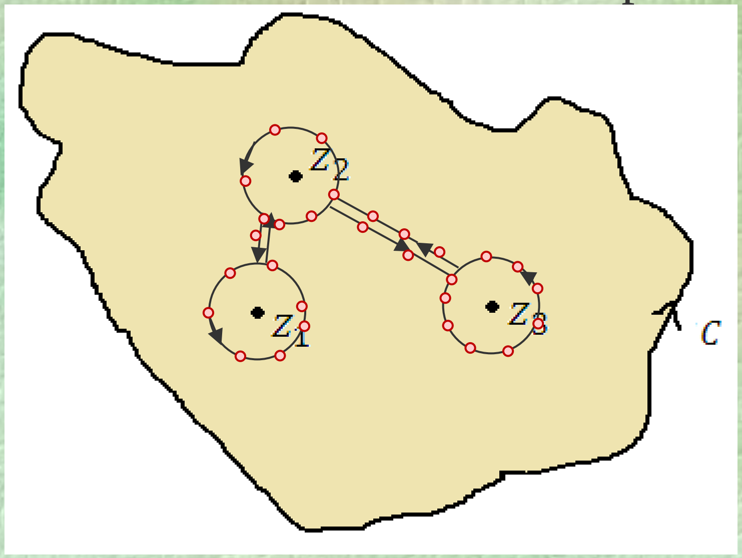
Residue Theory and Calculation of Real Integrals

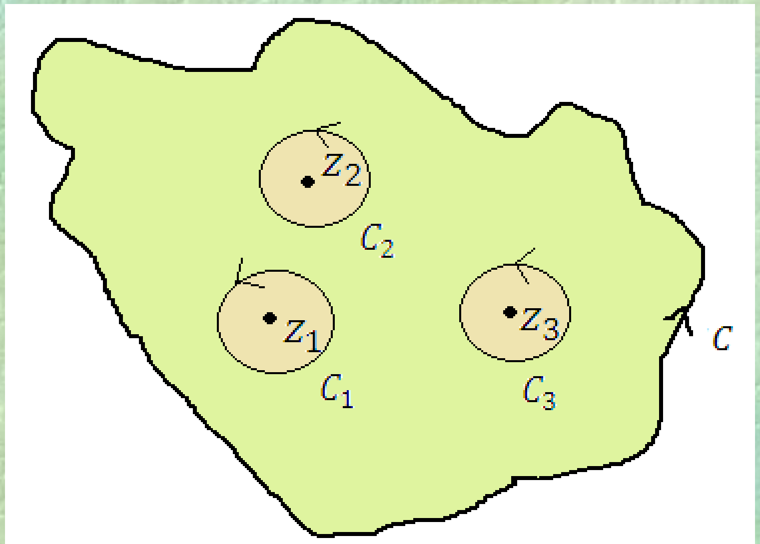
The goal of this section is to: $$\int_C f(z)dz = ?$$

Case (a): The function $f(z)$ is analytic inside and on the closed contour C.

According to Cauchy-Goursat Theorem $$\int_C f(z)dz = 0$$

Case (b): The function $f(z)$ is analytic at all points inside and on the closed contour C except at a finite number of points.

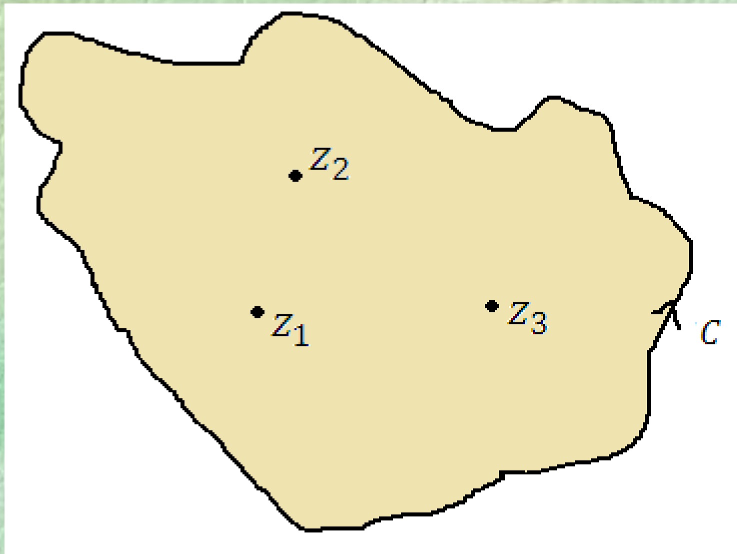


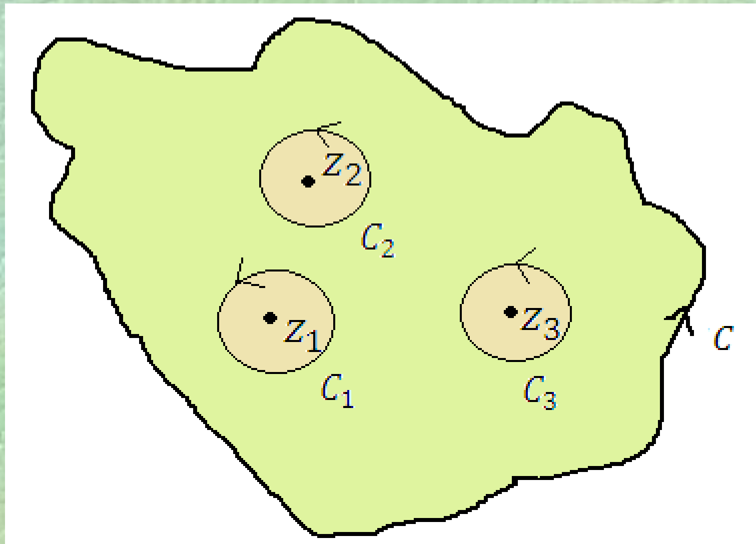


Residue Theory and Calculation of Real Integrals

The goal of this section is to: $\int_C f(z)dz = ?$

Case (b): The function $f(z)$ is analytic at all points inside and on the closed contour C except at a finite number of points.



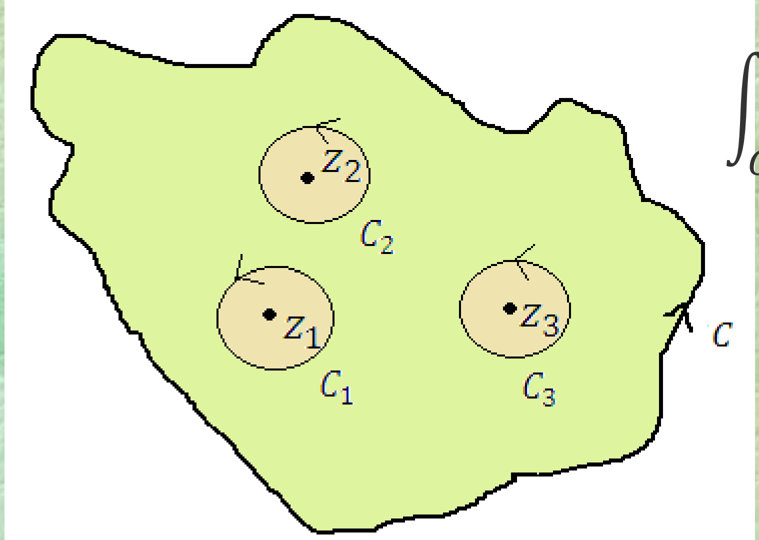


$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz$$

Residue Theory and Calculation of Real Integrals

The goal of this section is to: $\int_C f(z)dz = ?$

Case (b): The function $f(z)$ is analytic at all points inside and on the closed contour C except at a finite number of points.



$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz$$

$$\int_{C_1} f(z)dz = ?$$

Residue Theory and Calculation of Real Integrals

The goal of this section is to: $\int_C f(z)dz = ?$

Case (b): The function $f(z)$ is analytic at all points inside and on the closed contour C except at a finite number of points.

$$\int_{C_1} f(z)dz = ?$$

$$f(z) = \ldots + \frac{a_{-m}}{(z-z_1)^m} + \cdots + \frac{a_{-1}}{z-z_1} + a_0 + a_1(z-z_1) + \cdots + a_m(z-z_1)^m + \cdots$$

$$\int_{C_1} f(z)dz = 2\pi i(a_{-1})$$

residues

Residue Theory and Calculation of Real Integrals

$$\int_{C_1} f(z)dz = ?$$

$$f(z) = \ ... + \frac{a_{-m}}{(z-z_1)^m} + \cdots + \frac{a_{-1}}{z-z_1} + a_0 + a_1(z-z_1) + \cdots + a_m(z-z_1)^m + \cdots$$

Region of Validity of the Series

Principal Part of $f(z)$

Pole or Singularity of Order m

Essential Singularity

Residue of $f(z)$ around the point z_1

$$\int_{C_1} f(z)dz = 2\pi i(a_{-1})$$

residues

Residue Theory and Calculation of Real Integrals

Example 10: Determine the type of the pole of the following function at z=1 and find the residue.

$$f(z) = \frac{1}{z(z-1)^2}$$

Solution: First, obtain the Laurent series expansion of $f(z)$ around z=1.

$$f(z) = \frac{1}{z(z-1)^2} = \frac{1}{(z-1)^2} z^{-1} = \frac{1}{(z-1)^2}(1+z-1)^{-1}$$

$$f(z) = \frac{1}{(z-1)^2}\left\{(1)^{-1}+(-1)(1)^{-2}(z-1)^1 + \frac{(-1)(-2)}{2!}(1)^{-3}(z-1)^2+\cdots\right\}$$

$$f(z) = \frac{1}{(z-1)^{\boxed{2}}} + \frac{\boxed{-1}}{z-1} + 1 - (z-1) + \cdots \qquad |z-1| < 1$$

Residue Theory and Calculation of Real Integrals

Example 11: Determine the type of the pole of the following function at z=0 and find the residue.

$$f(z) = e^{\frac{1}{z}}$$

Solution: First, obtain the Laurent series expansion of $f(z)$ around z=0.

$$e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$e^{\frac{1}{z}} = 1 + \frac{\boxed{1}}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \boxed{\cdots}$$

Residue Theory and Calculation of Real Integrals

Example 12: Determine the poles of the following function, find their types, and compute the residues.

$$f(z) = \frac{1 - e^z}{z}$$

Solution: It is clear that the function should be examined at z=0

$$f(z) = \frac{1}{z}\left\{1 - \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)\right\}$$

$$f(z) = -1 - \frac{z}{2!} - \frac{z^2}{3!} - \cdots$$

Residue Theory and Calculation of Real Integrals

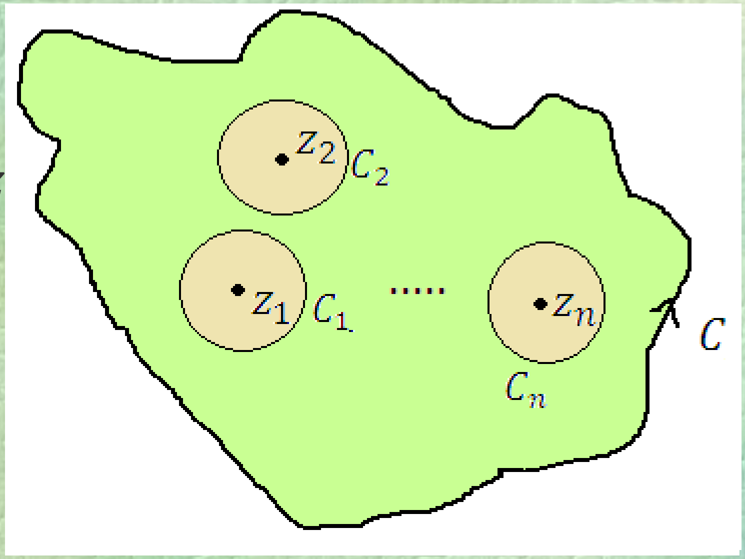
Theorem: If C is a closed contour and $f(z)$ is analytic inside and on C except at a finite number of points $z_1, z_2, \dots$ inside C, then

$$\int_C f(z)dz = 2\pi i(r_1 + r_2 + \cdots r_n)$$

$r_1, r_2, \dots$ are the residues of the function f(z)at the singular points .$z_1, z_2, \dots\dots$

Proof:

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots \int_{C_n} f(z)dz$$



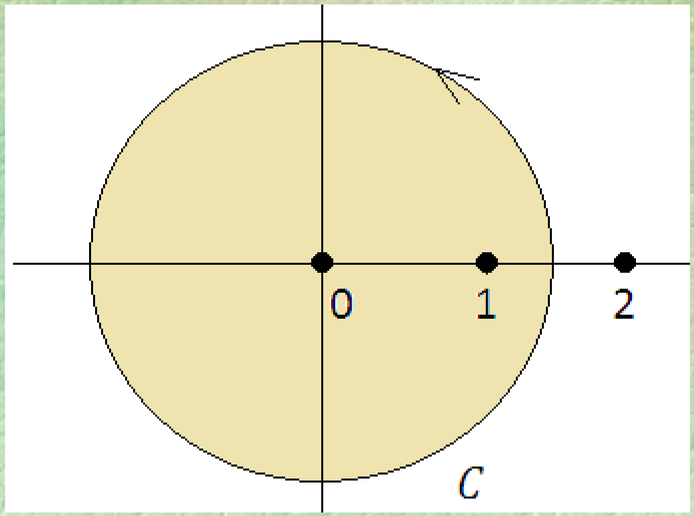
$$\int_C f(z)dz = 2\pi i r_1 + 2\pi i r_2 + \cdots + 2\pi i r_n$$

Residue Theory and Calculation of Real Integrals

Example 13: Compute the following integral around a circle of radius 1.5 centered at the origin, in the counterclockwise direction:

$$\int_C \frac{-3z+4}{z(z-1)(z-2)}dz$$

Solution: The given function is analytic on the contour C except at z=0 and z=1 Thus:



$$\int_C f(z)dz = 2\pi i(r_1 + r_2)$$

$$\frac{-3z+4}{z(z-1)(z-2)} = \frac{2}{z} + \frac{-1}{z-1} + \frac{-1}{z-2}$$

$$\int_C f(z)dz = 2\pi i(2-1) = 2\pi i$$

Residue Theory and Calculation of Real Integrals

Method for Computing the Remainder(a_{-1}):

1- We have pole of order 1

$$f(z) = \frac{a_{-1}}{z - z_1} + a_0 + a_1(z - z_1) + \cdots + a_m(z - z_1)^m + \cdots$$

$$a_{-1} = \lim_{z \to z_1}\big((z - z_1)f(z)\big)$$

2- We have pole of order 2

$$f(z) = \frac{a_{-2}}{(z - z_1)^2} + \frac{a_{-1}}{z - z_1} + a_0 + a_1(z - z_1) + \cdots + a_m(z - z_1)^m + \cdots$$

$$a_{-1} = \lim_{z \to z_1}\left(\frac{d}{dz}(z - z_1)^2 f(z)\right)$$

Residue Theory and Calculation of Real Integrals

Method for Computing the Remainder(a_{-1}):

2- We have pole of order m

$$f(z) = \frac{a_{-m}}{(z-z_1)^m} + \cdots + \frac{a_{-1}}{z-z_1} + a_0 + a_1(z-z_1) + \cdots + a_m(z-z_1)^m + \cdots$$

$$a_{-1} = \frac{1}{(m-1)!}\lim_{z\to z_1}\left(\frac{d}{dz^{m-1}}(z-z_1)^m f(z)\right)$$

Residue Theory and Calculation of Real Integrals

Example 14: Determine the residue of the following function at the origin (z=0):

$$f(z) = \frac{1+z}{1-cosz}$$

Solution: First, assume that z=0 is a simple pole (pole of order 1).

$$a_{-1} = \lim_{z \to z_1}\big((z-z_1)f(z)\big)$$

$$zf(z) = \frac{z(1+z)}{1-cosz} = \frac{z(1+z)}{1-(1-\frac{z^2}{2}+\frac{z^4}{24}-\cdots)} = \frac{z(1+z)}{\frac{z^2}{2}-\frac{z^4}{24}+\cdots} = \frac{z+1}{\frac{z}{2}-\frac{z^3}{24}+\cdots}$$

$$a_{-1} = \lim_{z \to 0}(zf(z))$$ Not acceptable

Residue Theory and Calculation of Real Integrals

Now, assume that z=0 is a pole of order 2.

$$a_{-1} = \lim_{z\to 0}\left(\frac{d}{dz}(z-0)^2 f(z)\right)$$

$$zf(z) = \frac{z+1}{\frac{z}{2} - \frac{z^3}{24} + \cdots}$$

$$z^2 f(z) = \frac{z+1}{\frac{1}{2} - \frac{z^2}{24} + \cdots}$$

$$a_{-1} = \lim_{z\to 0}\left(\frac{d}{dz} z^2 f(z)\right) = 2$$

Residue Theory and Calculation of Real Integrals

Method for Computing the Remainder(a_{-1}):

Suppose f(z) is given as the quotient of two analytic functions:

$$f(z) = \frac{p(z)}{q(z)}$$

1- We have pole of order 1

$$a_{-1} = \frac{p\,(z_1)}{q'(z_1)}$$

2- We have pole of order 2

$$a_{-1} = \frac{2p'(z_1)}{q''(z_1)} - \frac{2p(z_1)q'''(z_1)}{3(q''(z_1))^2}$$

Residue Theory and Calculation of Real Integrals

Example 15: Determine the residue of the following function at the origin (z=0):

$$f(z) = \frac{1+z}{1-cosz}$$

Solution: First, assume that z=0 is a simple pole (pole of order 1).

$$a_{-1} = \frac{p\,(z_1)}{q'(z_1)} \quad = \frac{1+0}{sin0} \qquad \textit{Not acceptable}$$

Now, assume that z=0 is a pole of order 2.

$$a_{-1} = \frac{2p'(z_1)}{q''(z_1)} - \frac{2p(z_1)q'''(z_1)}{3(q''(z_1))^2}$$

$$= \frac{2}{cos0} - \frac{2(1+0)sin0}{3(cos0)^2} \quad = 2$$

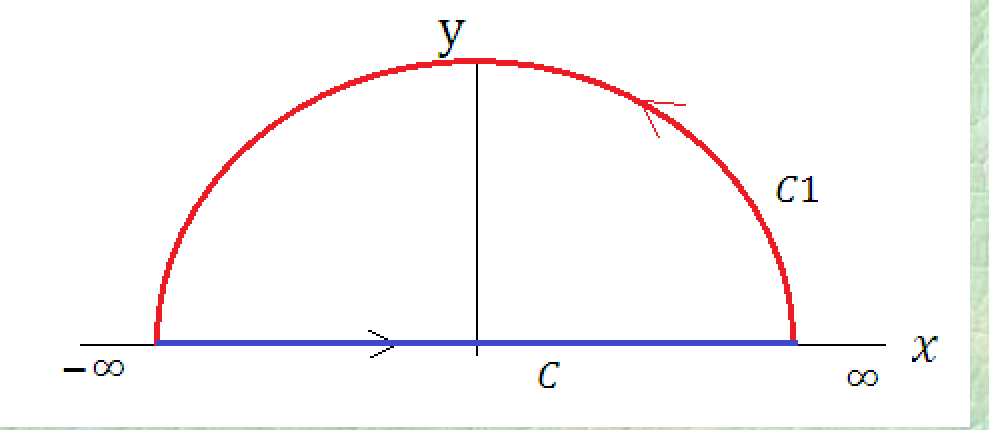
Residue Theory and Calculation of Real Integrals

The goal of this section is to:

$$\int_{-\infty}^{\infty} f(x)dx = ?$$

Suppose $f(x)$ has no poles on the x-axis.

.



$$\int_C f(z)dz \;=\; \int_{-\infty}^{\infty} f(x)dx + \int_{C1} f(z)dz \;=\; 2\pi i\left(\begin{matrix}\text{The residues of f(z) in}\\ \text{the upper half-plane}\end{matrix}\right)$$

If the denominator of $f(z)$ has a degree that is greater than the numerator by at least two, then:

$$\int_{-\infty}^{\infty} f(x)dx \;=\; 2\pi i\left(\begin{matrix}\text{The residues of f(z) in}\\ \text{the upper half-plane}\end{matrix}\right)$$

Residue Theory and Calculation of Real Integrals

Example 16: Compute the following integral:

$$I = \int_0^{\infty} \frac{2x^2 - 1}{x^4 + 5x^2 + 4} dx = ?$$

$$I = 0.5 \int_{-\infty}^{\infty} \frac{2x^2 - 1}{x^4 + 5x^2 + 4} dx \quad = \pi i \left(\begin{array}{c} \text{The residues of f(z) in} \\ \text{the upper half-plane} \end{array} \right)$$

The function has poles at i and $2i$ in the upper half-plane. Therefore:

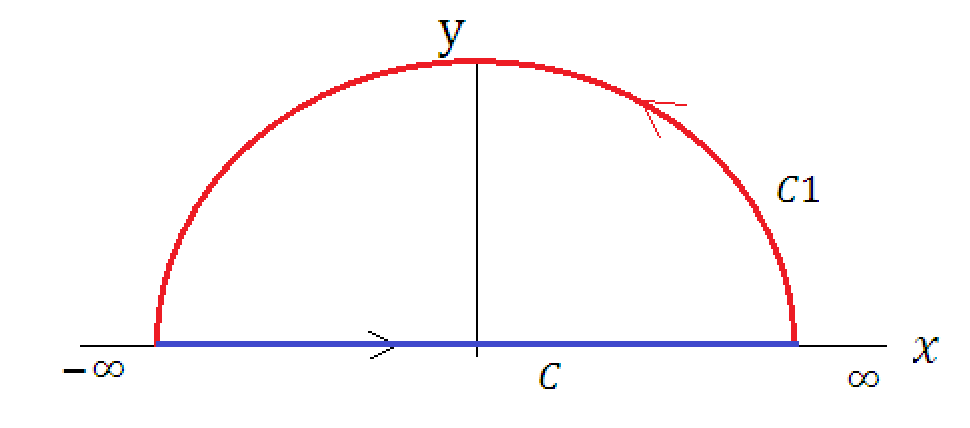
$$I = \int_0^{\infty} \frac{2x^2 - 1}{x^4 + 5x^2 + 4} dx = \pi i (0.5i \; - 0.75i) = \frac{\pi}{4}$$

Residue Theory and Calculation of Real Integrals

The goal of this section is to: $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} cosxdx = ?$ $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} sinxdx = ?$

Suppose $q(x)$ has no poles on the x-axis.

.



$$\int_C \frac{p(z)}{q(z)} e^{iz} dz = \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} e^{ix} dx + \int_{C1} \frac{p(z)}{q(z)} e^{iz} dz = 2\pi i\left(\begin{array}{l}\text{The residues of } \frac{p(z)}{q(z)} e^{iz} \\ \text{in the upper half-plane}\end{array} \right)$$

If $q(x)$ has two more roots than $p(x)$, then:

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} e^{ix} dx = 2\pi i\left(\begin{array}{l}\text{The residues of } \frac{p(z)}{q(z)} e^{iz} \\ \text{in the upper half-plane}\end{array} \right)$$

Residue Theory and Calculation of Real Integrals

The goal of this section is to: $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} cosxdx = ?$ $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} sinxdx = ?$

Suppose $q(x)$ has no poles on the x-axis.

.

If $q(x)$ has two more roots than $p(x)$, then:

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} e^{ix} dx = 2\pi i(\text{The residues of } \tfrac{p(z)}{q(z)} e^{iz} \text{ in the upper half-plane})$$

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} cosxdx = Re\{2\pi i(\text{The residues of } \tfrac{p(z)}{q(z)} e^{iz} \text{ in the upper half-plane})\}$$

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} sinxdx = Im\{2\pi i(\text{The residues of } \tfrac{p(z)}{q(z)} e^{iz} \text{ in the upper half-plane})\}$$

Residue Theory and Calculation of Real Integrals

Example 17: Find the value of:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} cosmxdx$$

Since $q(x)$ has no roots on the real axis and has two more roots than $p(x)$, then:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} cosmxdx \quad = Re\{2\pi i(\text{ The residues of } \frac{1}{1+x^2} cosmx \text{ in the upper half-plane })\}$$

$$= \cdots$$

$$= \pi e^{-m}$$

Residue Theory and Calculation of Real Integrals

The goal of this section is to: $\int_0^{2\pi} R(sin\theta, cos\theta)d\theta = ?$

Suppose R does not have any poles in the interval of integration.

Consider the following variable substitution:

$$z = e^{i\theta}$$

Then we have:

$$cos\theta = \frac{z+\bar{z}}{2} \qquad sin\theta = \frac{z-\bar{z}}{2i}$$

Finally

$$\int_0^{2\pi} R(sin\theta, cos\theta)d\theta = \int_C R\left(\frac{z-\bar{z}}{2i}, \frac{z+\bar{z}}{2}\right)\frac{dz}{iz}$$

where C is the unit circle.

Exercises

Exercise 11: Compute the following:

$$\int_0^{2\pi} \frac{cos2\theta d\theta}{1-2acos\theta+a^2} \qquad -1<a<1$$

Exercise 12: Compute the following:

$$\int_0^{2\pi} \frac{d\theta}{1+asin\theta} \qquad -1<a<1$$

References

- Advanced Engineering Mathematics , E. Kreyszig
- Advanced Engineering Mathematics, C. R. Wylie
- Complex Variables and Applications, J. Brown and R. Churchill