LINEAR CONTROL SYSTEMS

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Lecture 4

Stability analysis

Topics to be covered include:

- Stability of linear control systems.
    - Bounded input bounded output stability (BIBO).
    - Zero input stability.
- Stability of linear control systems through Routh Hurwitz criterion.

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Stability analysis

System output = zero-state response + zero-input response

$$y(t) = y_{zs}(t) + y_{zi}(t)$$

Input output stability

BIBO stability
(bounded-input bounded-output)

Internal stability
Asymptotic stability
Lyapunov(marginal) stability

Definition ?

Suitable model ?

Stability check?

Definition ?

Suitable model ?

Stability check?

Stability analysis

Input output stability(BIBO) $g(s)$

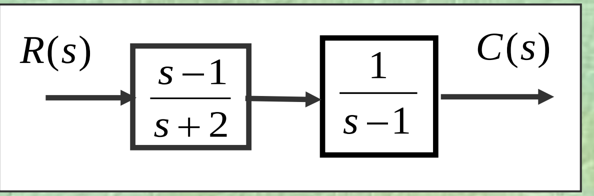
Theorem: A SISO system with proper rational transfer function $g(s)$ is BIBO stable if and only if every pole of $g(s)$ has negative real part.

Internal stability $\dot{x} = Ax$

Theorem:
$\dot{x} = Ax$ is asymptotically stable if and only if all eigenvalues of A have negative real parts.

Relation between BIBO stability and asymptotic stability?

Example 1: Discuss the stability of the system .



BIBO stability:

$$G(s)=\frac{C(s)}{R(s)}=\frac{1}{s+2}$$

There is no RHP root , so system is BIBO stable.

Internal stability:

For internal stability we need state-space model so we have:

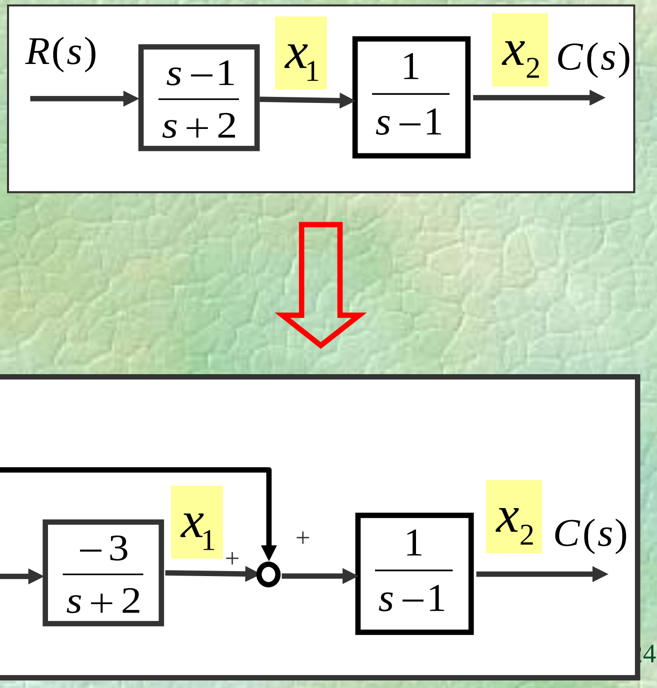
$$x_2(s)=\frac{1}{s-1}x_1(s) \longrightarrow \dot{x}_2=x_1+x_2$$

$$x_1(s)=\frac{s-1}{s+2}R(s) \longrightarrow \dot{x}_1=-2x_1+\dot{r}-r$$

?

$$x_2(s)=\frac{1}{s-1}\big(x_1(s)+R(s)\big) \longrightarrow \dot{x}_2=x_1+x_2+r$$

$$x_1(s)=\frac{-3}{s+2}R(s) \longrightarrow \dot{x}_1=-2x_1-3r$$



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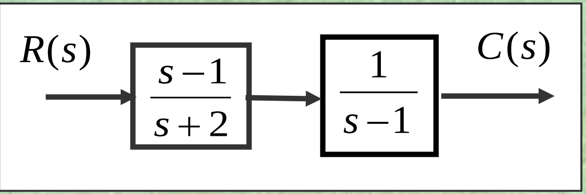

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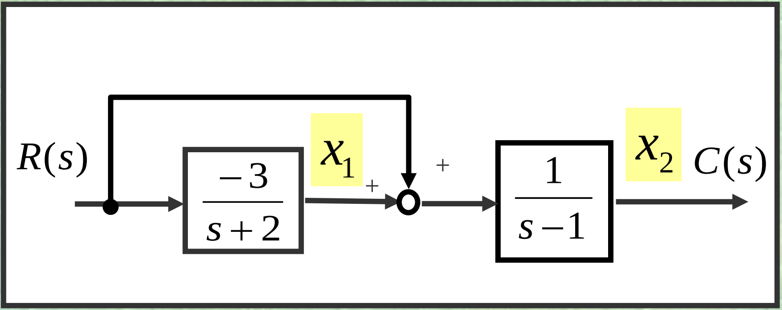

$$\dot{x}_1 = -2x_1 - 3r \qquad \dot{x}_2 = x_1 + x_2 + r$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix} r$$

$$c = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$|sI - A| = \begin{vmatrix} s+2 & 0 \\ -1 & s-1 \end{vmatrix} = (s-1)(s+2)$$

$$\rightarrow \lambda_1 = +1, \lambda_2 = -2$$

The system is not internally stable (neither asymptotic nor Lyapunov stable).

Very important note: If RHP poles and zeros between different part of system omitted then the system is internally unstable although it may be BIBO stable.

Example 2: Discuss the BIBO stability of a closed loop system with following transfer function. Proof your answer. (Olympiad 1394)

$$T(s) = \frac{1}{s^2 + 1}$$

Example 3: a) Find the eigenvalues of following system.

b) Check the BIBO stability.

c) Check the Asymptotic stability.

d) Check the Lyapunov stability.

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Example 4: a) Find the eigenvalues of following system.

b) Check the BIBO stability.

c) Check the Asymptotic stability.

d) Check the Lyapunov stability.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Stability analysis

Input output stability(BIBO) and asymptotic stability are the same if ….

$G(s)=\dfrac{n(s)}{d(s)}$ $\longrightarrow$ Let $d(s)=0$ $\longrightarrow$ Find poles $p_1, p_2, ..., p_n$

Stability analysis

Input output stability(BIBO) $g(s)$

$$G(s)=\frac{n(s)}{d(s)} \longrightarrow \text{Let } d(s)=0$$

$\longrightarrow$ Find poles $p_1, p_2, \ldots, p_n$

Internal stability $\dot{x}=Ax$

Let $|sI-A|=0$

$\longrightarrow$ Find eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

Input output stability(BIBO) and asymptotic stability are the same if ….

Very important note!

For both kind of stability we need to compute the zero of some polynomial

Stability of linear control systems through Routh Hurwitz criterion.

Stability analysis

Topics to be covered include:

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Stability and Polynomial Analysis

Consider a polynomial of the following form:

$$p(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$$

The problem to be studied deals with the question of whether that polynomial has any root in RHP or on the jw axis.

Some Polynomial Properties of Special Interest

Property 1: The coefficient $a_{n\text{-}1}$ satisfies

$$a_{n-1} = -\sum_{i=1}^{n} \lambda_i$$

Property 2: The coefficient a_0 satisfies

$$a_0 = (-1)^n \prod_{i=1}^{n} \lambda_i$$

Property 3: If all roots of p(s) have negative real parts, it is necessary that $a_i > 0$, $i \in \{0, 1, \ldots, n\text{-}1\}$.

Property 4: If any of the polynomial coefficients is nonpositive (negative or zero), then, one or more of the roots have nonnegative real plant.

Routh Hurwitz Algorithm

$$p(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$$

The Routh Hurwitz algorithm is based on the following numerical table.

s^n	1	a_{n-2}	a_{n-4}	....
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	....
s^{n-2}	$\gamma_{2,1}$	$\gamma_{2,2}$	$\gamma_{2,3}$	....
s^{n-3}	$\gamma_{3,1}$	$\gamma_{3,2}$	$\gamma_{3,3}$	....
.				
.				
.				
s^0	$\gamma_{n,1}$			

Routh's table

Number of sign changes =number of roots in RHP

Routh Hurwitz Algorithm

Example 5: Check the position of roots of following system.

$$p(s) = 2s^4 + s^3 + 3s^2 + 5s + 10 = 0$$

……………….   Two roots in RHP

Routh Hurwitz properties:

- Multiply each row by ………….
- s → 1/s no change in ………….

Routh Hurwitz special cases:

1- The first element of a row is zero.

2- All elements of a row are zero.

Routh Hurwitz Algorithm

Example 6: Check the position of roots of following system.

$$p(s) = s^4 + s^3 + 2s^2 + 2s + 3 = 0$$

……………… Two roots in RHP

Example 7: Check the position of roots of following system.

$$p(s) = s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$$

……………… Two roots on imaginary axis

Remarks when a complete row is zero …….

Routh Hurwitz Algorithm

Example 8: Check the position of roots of following system.

$$p(s) = s^6 + 5s^5 + s^4 - 25s^3 - 26s^2 + 20s + 24 = 0$$

……………. Two roots in RHP

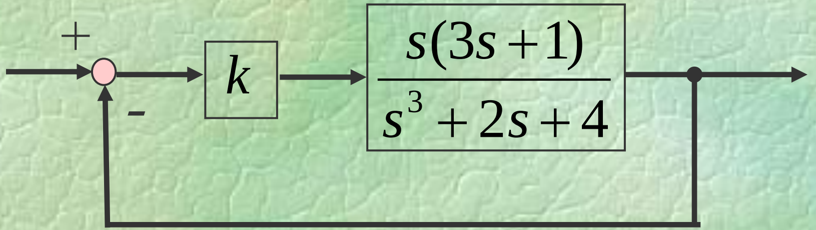
Example 9: Check the position of roots of following system.

$$p(s) = s^6 - s^5 - 3s^4 - 3s^3 - 22s^2 + 4s + 24 = 0$$

……. Two RHO roots + Two roots on imaginary axis

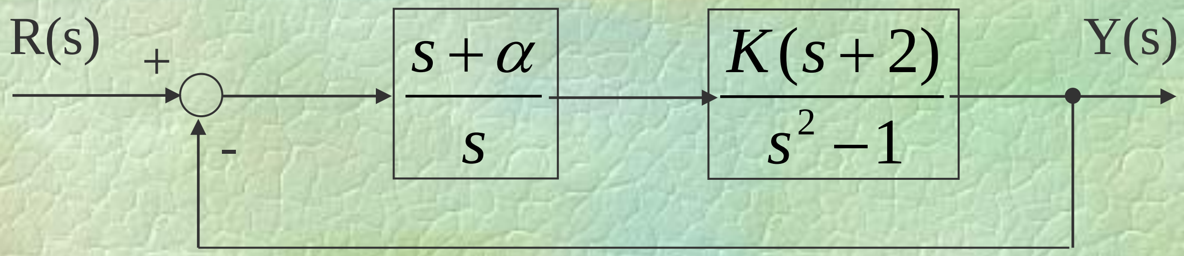
Example 10: Check the stability of system for different values of k

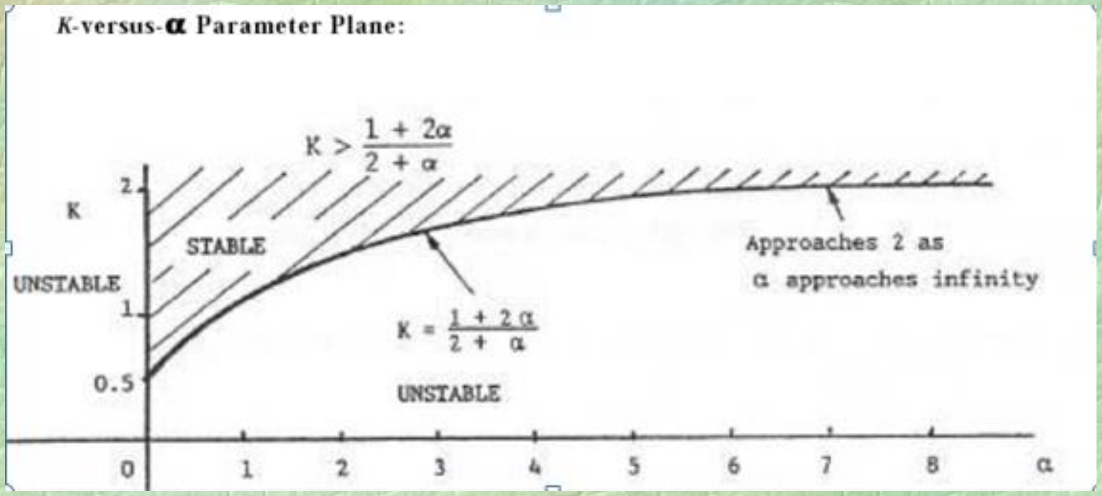
……. Stability for k>0.528



Routh Hurwitz Algorithm

Example 11: The block Diagram of a control system is depicted in the following figure. Find the region in α-K plane concluding the system stable.





Exercises

1- Check the internal stability of following system.

$$\dot{x} = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -3 & 2 \\ 1 & 4 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = [1 \quad 2 \quad 0] x$$

2- a) Check the internal stability of following system.
b) Check the BIBO stability of following system.

$$\dot{x} = \begin{bmatrix} -3 & -6 & -4 \\ 1 & 2 & 2 \\ -1 & -6 & -6 \end{bmatrix} x + \begin{bmatrix} 6 \\ -3 \\ 4 \end{bmatrix} u$$

$$y = [2 \quad 2 \quad -1] x$$

3- Check the number of zeros in the RHP in following systems

$$a) \quad p(s) = s^6 + 5s^5 + 4s^4 - 10s^3 - 11s^2 + 5s + 6$$

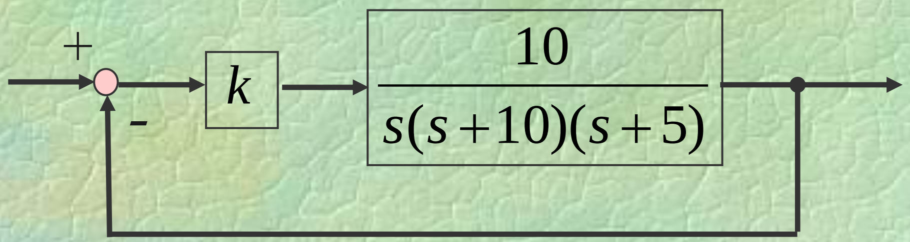
$$b) \quad p(s) = s^6 + 5s^5 + 9s^4 + 15s^3 + 14s^2 - 20s - 24$$

4- Are the real parts of all roots of following system less than -1.

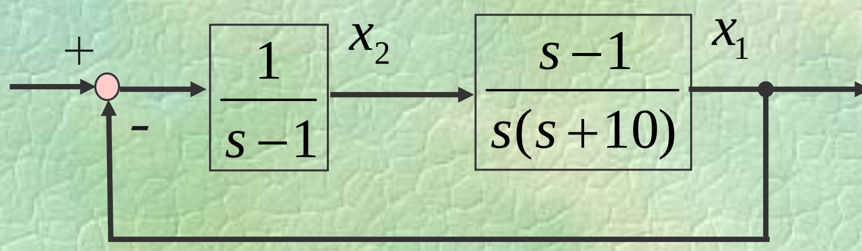
$$p(s) = s^5 + 4s^4 + 5s^3 + 6s^2 + 2s + 4$$

Exercises

5- Check the internal stability of following system versus k.



6- a)Check the BIBO stability of following system.
b) Check the internal stability of following system.



7- The eigenvalues of a system are -3,4,-5 and the poles of its transfer function are -3 and -5.(Midterm spring 2008)
a) Check the BIBO stability of following system.
b) Check the internal stability of following system.

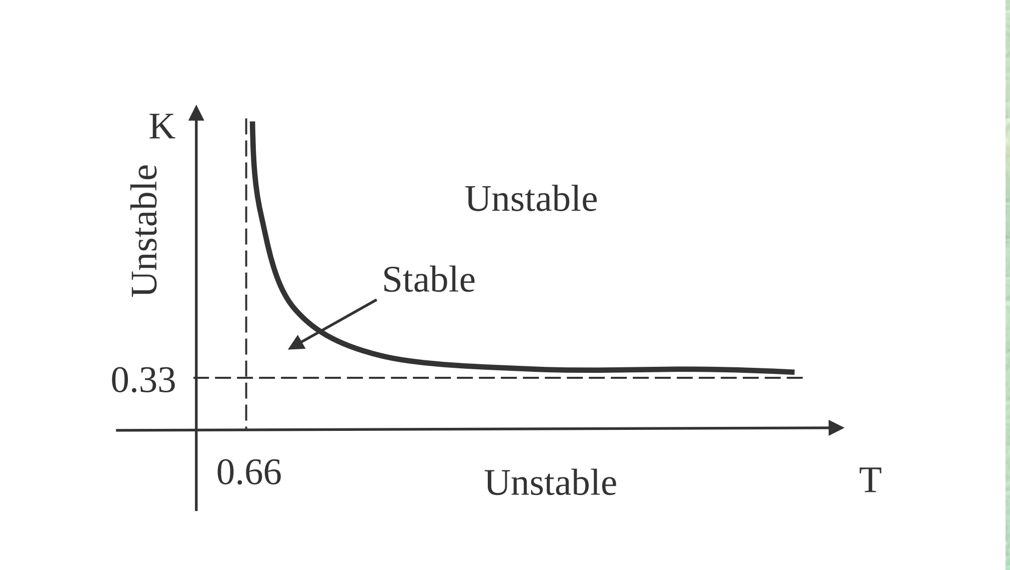
Exercises

8- The open-loop transfer function of a control system with negative unit feedback is:

$$G(s)H(s)=\frac{K(s+2)}{s(1+Ts)(1+2s)}$$

Find the region in K-T plane concluding the system stable.

Answer:

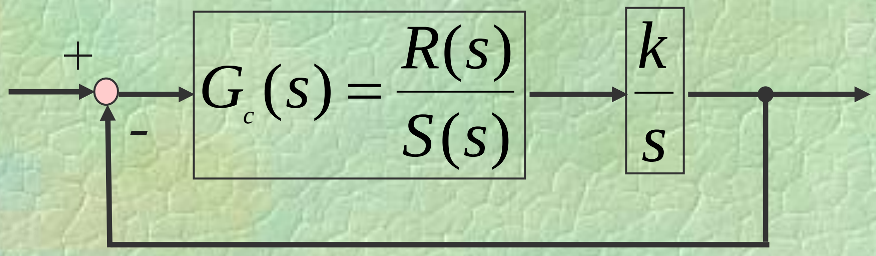


9- Find the number of roots that the real parts are in the region [-2,2] by use of Routh Hurwitz criteria.

$$s^3+5s^2+11s+15=0$$

Supplementary Exercises

10- In the following system k can be positive or negative.



$G_c(s)$ is a proper controller. Show that the system can not be stabilized by any R(s) and S(s). (Nice Problem)

11- The closed loop transfer function of a system is :

$$M(s)=\frac{k(s+2)}{s^4+7s^3+15s^2+(k+25)s+2k}$$

For the stability of the system which one is true? (k>0)

1) $0 \le k \le 28.12$

2) $0 < k < 28.12$

3) $0 \le k < 28.12$

4) $0 < k \le 28.12$

Answer: (4)

Supplementary Exercises

12- Which of the systems with the following characteristic equations can be stable for large values of k? (Master's Entrance Exam 1394)

1) $s^5 + 18s^4 + 108s^3 + (278 + 13k)s^2 + 467s + 280 + 60k = 0$

2) $s^5 + 18s^4 + 108s^3 + (278 + 13k)s^2 + (467 + 25k)s + 280 + 60k = 0$

3) $s^5 + 18s^4 + (108 + k)s^3 + (278 + 13k)s^2 + (467 + 25k)s + 280 + 60k = 0$

4) Each of the three options has five poles. Therefore, it will definitely be unstable for large positive gains.

Answer: (3)

Master's Entrance Exam 1394