ADVANCED CONTROL

Ali Karimpour

Professor

Ferdowsi University of Mashhad

Reference:

Chi-Tsong Chen, “Linear System Theory and Design”, 1999.

I thank my student, Alireza Bemani for his help in correction slides of this lecture.

Lecture 2

Basic Idea of Linear Algebra-Part I

Topics to be covered include:

- Basis, Representation, and Orthonormalization.
- Linear Algebraic Equations.
- Similarity Transformation.
- Diagonal and Jordan Form.

What you will learn after studying this section

- n-dimensional Real Vector Space R^n
- Linearly Dependent and Linearly Independent
- Basis of a Linear Space and Norm
- Orthogonal Vectors and Orthonormal Vectors
- Linear Algebraic Equations
- Range Space, Null Space, Rank and Nullity
- Similarity Transfomation
- Eigenvectors and Generalized Eigenvector
- Canonical Form, Diagonal Form, Modal Form and Jordan Form
- Determinant and Eigenvalues and nilpotent property

n-dimensional real vector space R^n

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad x_i \in R$$

An important property of n-dimensional real vector space R^n

$$\forall \mathbf{x}_1, \mathbf{x}_2 \in R^n \ \ and \ \ \alpha, \beta \in R \qquad \Rightarrow \quad \alpha \mathbf{x}_1 + \beta \mathbf{x}_2 \in R^n$$

For example:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} 0.5 \\ 2.5 \\ 0 \end{bmatrix} \qquad 7\mathbf{x}_1 - 4\mathbf{x}_2 = \begin{bmatrix} 5 \\ -17 \\ 21 \end{bmatrix} \in R^3$$

n-dimensional real vector space R^n

Linear dependence in the real n-dimensional vector space R^n

Definition 1: The set of vectors $x_1, x_2, \ldots, x_m$ the space R^n are linearly dependent if there exists a set of scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$, not all zero, such that:

$$\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \ldots. + \alpha_m \mathbf{x}_m = 0$$

If the above relation holds only when all $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$, then the given vectors are linearly independent.

n-dimensional real vector space R^n

Example 1: Are the following vectors linearly independent? Why?

$$\begin{bmatrix}1\\0\\2\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Example 2: Are the following vectors linearly independent? Why?

$$\begin{bmatrix}1\\0\\2\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Note: The presence of the zero vector in any set of vectors...

n-dimensional real vector space R^n

An interesting property of linearly dependent vectors

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ldots + \alpha_n\mathbf{x}_n = 0$$

If the vectors are linearly dependent, at least one scalar coefficient must be non-zero. so, assume α_i is non-zero, then:

$$\mathbf{x}_i = -\frac{\alpha_1}{\alpha_i}\mathbf{x}_1 - \frac{\alpha_2}{\alpha_i}\mathbf{x}_2 - \ldots - \frac{\alpha_n}{\alpha_i}\mathbf{x}_n$$

Therefore, in any set of linearly dependent vectors, at least one of the vectors can be expressed as a linear combination of the other vectors.

n-dimensional real vector space R^n

Important Note: The maximum number of linearly independent vectors in the space R^n is n.

Example 3: Are the following vectors linearly independent? Why?

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad x_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Example 4: If the following vectors are not linearly independent, express one of them as a linear combination of the remaining ones.

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad x_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

n-dimensional real vector space R^n

Bases of the vector space R^n

Definition 2: A set of linearly independent vectors in R^n is called a basis if every vector in R^n can be uniquely represented as a linear combination of these basis vectors.

Exercise 1: Any n linearly independent vectors in R^n form a basis for R^n. Why?

Example 5: Are the following vectors a basis for the space R^2? Why?

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Example 6: Are the following vectors a basis for the space R^2? Why?

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Note: Bases of a vector space...

Norms

To measure the magnitude of a vector in a vector space R^n, we need the concept of a **norm**.

A **norm** is a function that assigns a non-negative real number to each element of a vector space R^n.

$$\| \, . \, \|: \quad R^n \rightarrow R^+$$

A **norm** must satisfy the following properties:

$$1 - \text{Positivity} \quad \|x\| \geq 0 \ , \forall \ x \in R^n \text{ and } \|x\| = 0 \text{ if and only if } x = 0$$

$$2 - \text{Homogeneity} \quad \|\alpha x\| = |\alpha| \|x\| \ , \forall \ x \in R^n \text{ and } \forall \ \alpha \in R$$

$$3 - \text{Triangle inequality} \quad \|x + y\| \leq \|x\| + \|y\| \ , \forall \ x, y \in R^n$$

Norm of vectors

p-norm is: $$\|x\|_p = \left(\sum_i |a_i|^p \right)^{\frac{1}{p}} \qquad p \geq 1$$

For p=1 we have 1-norm $$\|x\|_1 = \left(\sum_i |a_i| \right)$$

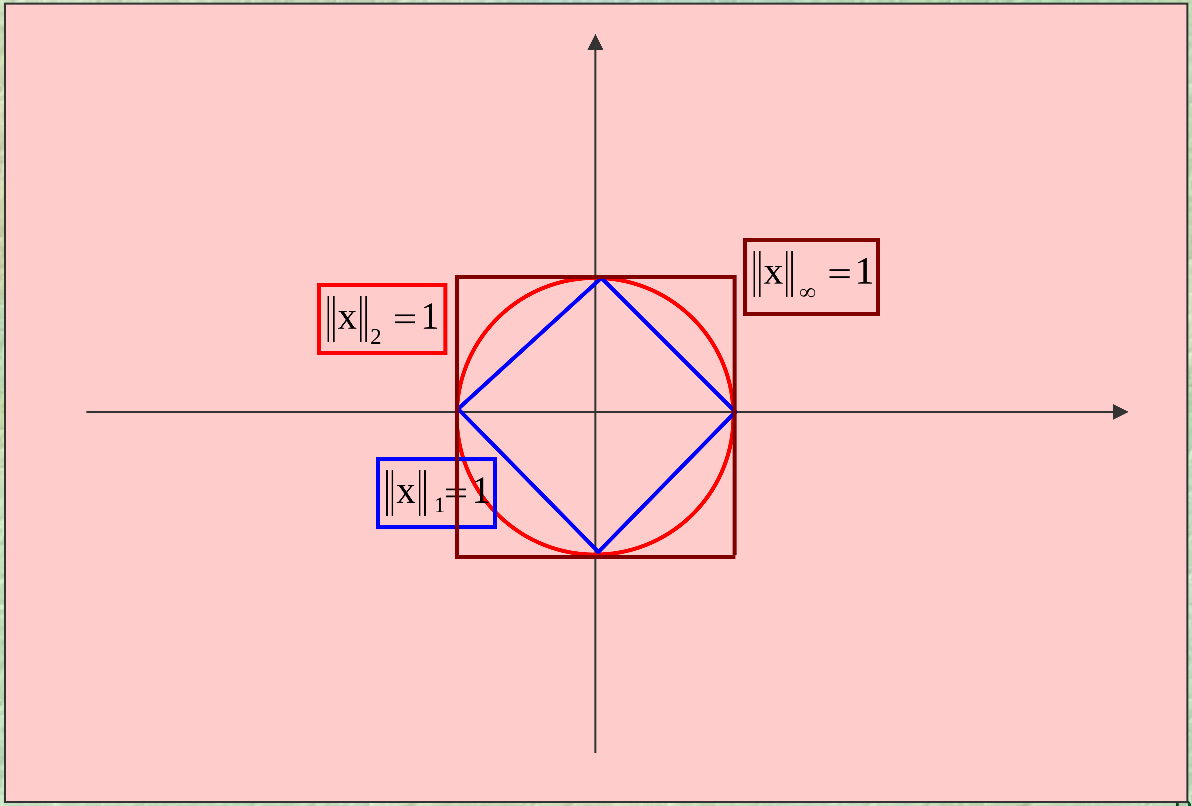
For p=2 we have 2-norm or euclidian norm $$\|x\|_2 = \left(\sum_i |a_i|^2 \right)^{1/2}$$

For p=∞ we have ∞-norm $$\|x\|_\infty = \max_i \left\{ |a_i| \right\}$$

Norm of vectors

Let $x = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

Then
$$\|x\|_1 = (1+1+2) = 4$$
$$\|x\|_2 = \sqrt{1^2+1^2+2^2} = \sqrt{6}$$
$$\|x\|_\infty = \max(1,1,2) = 2$$



Orthogonal and Orthonormal Vectors

Orthogonal Vectors and Orthonormal Vectors

Definition 3: A vector is called a unit vector (or normal vector) if its norm is equal to one.

Example 7: Which of the following vectors are unit vectors?

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \qquad x_3 = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$$

Definition 4: Two vectors x_1 and x_2 are called orthogonal if $x_1^T x_2 = 0$.

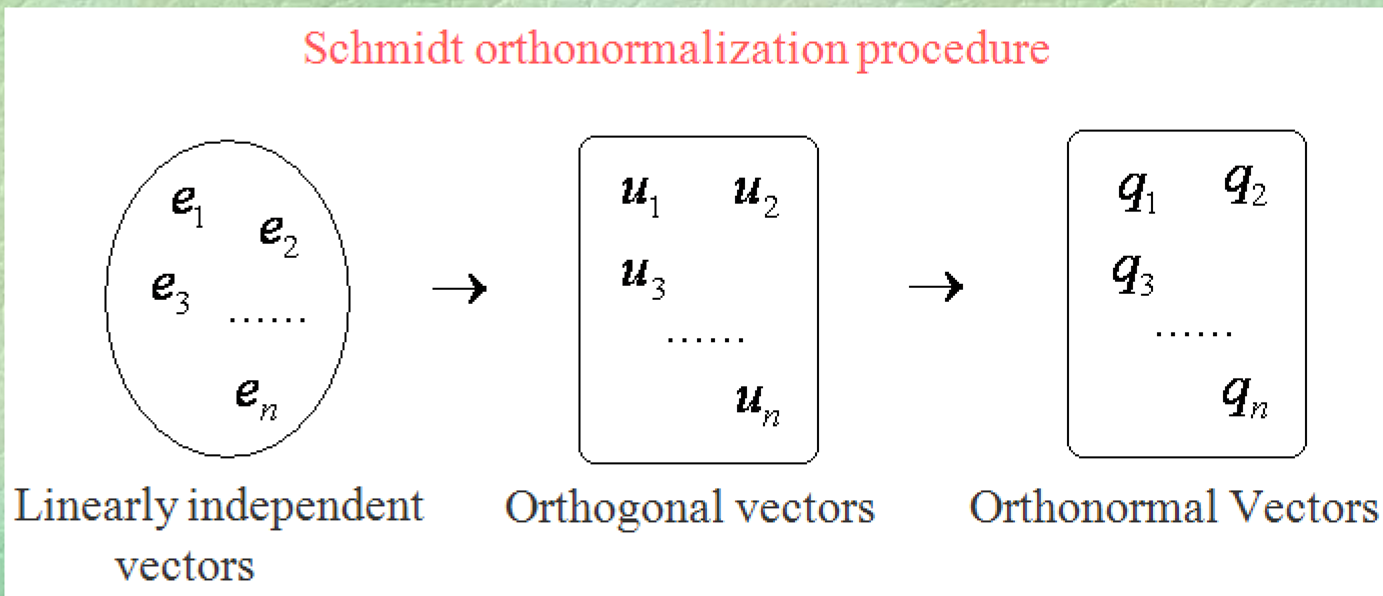
Example 8: Which of the following vectors are orthogonal?

$$x_1 = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix}$$

Definition 5: Two vectors x_1 and x_2 are called orthonormal if $x_1^T x_2 = 0$ and both vectors are unit vectors.

Orthonormal lization

Constructing an Orthonormal Set from a Set of Linearly Independent Vectors (**Gram-Schmidt Process**)



$$u_1 = e_1, \qquad q_1 = u_1 / \|u_1\|$$

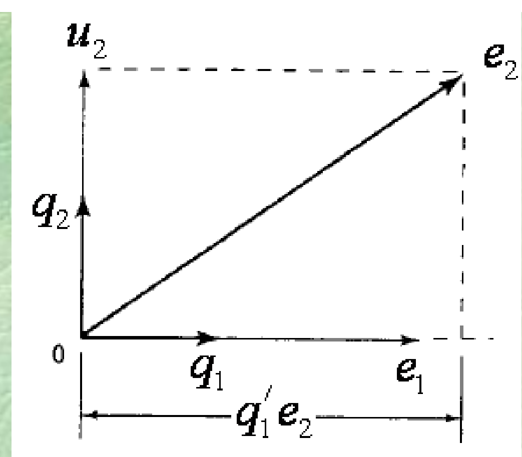
$$u_2 = e_2 - (q_1' e_2) q_1$$

$$q_2 = u_2 / \|u_2\|$$

$$u_3 = e_3 - (q_1' e_3) q_1 - (q_2' e_3) q_2$$

$$q_3 = u_3 / \|u_3\|$$

$$\vdots$$



Orthonormal lization

Exercise 2: Find the orthonormal vectors corresponding to the following vectors.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Linear Algebraic Equation

The concept of multiplying a matrix by a vector:

$$A\mathbf{x}=\begin{bmatrix}1 & 0 & 3\\ 2 & 1 & -1\\ 4 & 0 & 5\end{bmatrix}\begin{bmatrix}2\\ 3\\ -5\end{bmatrix}=2\begin{bmatrix}1\\ 2\\ 4\end{bmatrix}+3\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}-5\begin{bmatrix}3\\ -1\\ 5\end{bmatrix}$$

Multiplying matrix *A* by vector *x* means determining how the columns of *A* are combined.

The elements of vector *x* determine how the columns of *A* are combined.

The concept of multiplying a vector by a matrix:

$$\mathbf{z}^T A=\begin{bmatrix}2 & 3\end{bmatrix}\begin{bmatrix}1 & 3 & 6\\ 2 & 4 & 2\end{bmatrix}=2\begin{bmatrix}1 & 3 & 6\end{bmatrix}+3\begin{bmatrix}2 & 4 & 2\end{bmatrix}$$

Multiplying the transpose of vector *z* by matrix *A* means determining how the rows of *A* are combined.

The elements of vector *x* determine how the rows of *A* are combined.

Linear Algebraic Equation

In this section, we will study the following relationship:

$$A\mathbf{x} = \mathbf{y}$$

$$A_{m\times n} : \mathbf{x}_{n\times 1} \to \mathbf{y}_{m\times 1} \quad or \quad A: \ R^n \mapsto R^m$$

The **range space** of matrix A:

The maximum number of linearly independent vectors in the range space of matrix A is called the rank of A, and it is denoted by $\rho(A)$

The **null space** of matrix A:

The maximum number of linearly independent vectors where Ax=0 is called the nullity of A, and it is denoted by $N(A)$

Linear Algebraic Equation

Example 9: Consider the given matrix.

$$A=\begin{bmatrix}0 & 1 & 1 & 2\\ 1 & 2 & 3 & 4\\ 2 & 0 & 2 & 0\end{bmatrix}$$

a) Determine five elements of the range space of matrix *A*.

$$\begin{bmatrix}0\\0\\0\end{bmatrix}\quad\begin{bmatrix}0\\1\\2\end{bmatrix}\quad\begin{bmatrix}1\\2\\0\end{bmatrix}\quad\begin{bmatrix}1\\3\\2\end{bmatrix}\quad\begin{bmatrix}4\\10\\4\end{bmatrix}$$

b) Determine the rank of *A*.

Rank of *A* is 2.

c) Determine five elements of the range null space of matrix *A*.

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\quad\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}\quad\begin{bmatrix}0\\2\\0\\-1\end{bmatrix}\quad\begin{bmatrix}10\\10\\-10\\0\end{bmatrix}\quad\begin{bmatrix}1\\3\\-1\\-1\end{bmatrix}$$

d) Determine the nullity of *A*.

Nullity of *A* is 2.

Linear Algebraic Equation

The rank of matrix *A* is the dimension of the range space, or the maximum number of linearly independent columns of matrix *A*.

Important Note: The rank of matrix *A* is the dimension of the range space or the maximum number of linearly independent rows of matrix *A*.

So:

The maximum number of linearly independent rows of matrix *A* = The maximum number of linearly independent columns of matrix *A*.

Important Note: Assume *A* is an m×n matrix.

$$\rho(A) + N(A) = n$$

Important Note: Suppose *A* is an m×n matrix.

$$\rho(A) \le \min\{m, n\}$$

Linear Algebraic Equation

Theorem 1:

1- For a matrix A of dimensions m×n and a vector y of dimensions m×1, a solution x of dimensions n×1 exists for the equation

$$Ax=y$$

if and only if y is in the range space of A. This means:

$$\rho(A)=\rho([A\ y])$$

2- For a matrix A of dimensions m×n, a solution x of dimensions n×1 exists for every vector y of dimensions m×1 in the equation

$$Ax=y$$

if and only if A has rank mmm (full row rank).

Linear Algebraic Equation

Theorem 2: All solutions can be expressed in parametric form.

For a matrix A of dimensions m×n and a vector y of dimensions m×1, suppose x is a solution to the equation

$$Ax=y$$

and let the nullity of A be k (where $k=n-\rho(A)$).

If A has rank nnn (i.e., $k=0$), then the given solution x_p is unique.

If A has rank less than n (i.e., $k>0$), then all solutions to the equation

$$Ax=y$$

can be obtained from the following relation:

$$x = x_p + \alpha_1 n_1 + \alpha_2 n_2 + \ldots + \alpha_k n_k$$

where α_i are arbitrary constants and n_i are linearly independent vectors in the null space of A.

Linear Algebraic Equation

Example 10: For the given system, it is desirable to obtain all solutions.

$$\begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 0 & 2 & 0 \end{bmatrix} x = \begin{bmatrix} -4 \\ -8 \\ 0 \end{bmatrix}$$

It is clear that one solution to the above equation is:

$$x_p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}$$

and all solutions to the above equation are given by:

$$x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

Linear Algebraic Equation

Theorem 3: Consider a square matrix A and the following equation:

$$Ax=y$$

1- If A is non-singular (invertible), then the equation has a unique solution, and the solution is given by:

$$x=A^{-1}y.$$

2- The homogeneous equation $Ax=0$ has a non-zero solution if and only if A is singular. The number of linearly independent solutions is equal to the nullity of A.

Similarity Transformation

Consider a matrix A of dimensions $n\times n$ linearly independent vectors:

$$A \ \ and \ \ \boxed{\begin{matrix} q_1 & q_2 \\ q_3 & \\ \ldots\ldots & \\ & q_n \end{matrix}} \ \ \rightarrow \ \ \hat{A}$$

Linearly independent vectors

$\widehat{\boldsymbol{A}}$ is a similarity transformation of A and is calculated as follows.

The i-th column of $\widehat{\boldsymbol{A}}$ is the representation of Aq_i in the given basis.

Similarity Transformation

$$A \text{ and } \begin{pmatrix} q_1 & q_2 \\ q_3 & \\ \ldots\ldots & \\ & q_n \end{pmatrix} \rightarrow \hat{A}$$

Linearly independent vectors

The i-th column of $\hat{A}$ is the representation of Aq_i in the given basis.

Example 11: Find the similarity transformation of matrix A with respect to the given basis.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 3 \\ 2 & 0 & 1 \end{bmatrix} \quad q_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ q_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} and \ q_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

The first column of $\hat{A}$ is the representation of Aq_1 in the given basis.

$$Aq_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \quad \hat{A} = \begin{bmatrix} -1 & & \\ 0 & & \\ 0 & & \end{bmatrix}$$

The second column of $\hat{A}$ is the representation of Aq_1 in the given basis.

$$Aq_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \hat{A} = \begin{bmatrix} -1 & -3 & \\ 0 & 1 & \\ 0 & 2 & \end{bmatrix}$$

The third column of $\hat{A}$ is the representation of Aq_1 in the given basis.

$$Aq_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad \hat{A} = \begin{bmatrix} -1 & -3 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

Similarity Transformation

$$A \ and \ \begin{pmatrix} q_1 & q_2 \\ q_3 & \\ \cdots\cdots & \\ & q_n \end{pmatrix} \rightarrow \hat{A}$$

Linearly independent vectors

The i-th column of $\hat{A}$ is the representation of Aq_i in the given basis.

Example 12: Find the similarity transformation of matrix A with respect to the given basis.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 3 \\ 2 & 0 & 1 \end{bmatrix} \quad q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ q_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} and \ q_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

The first column of $\hat{A}$ is the representation of Aq_1 in the given basis.

$$Aq_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \hat{A} = \begin{bmatrix} 1 & & \\ 1 & & \\ 2 & & \end{bmatrix}$$

The second column of $\hat{A}$ is the representation of Aq_1 in the given basis.

$$Aq_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \quad \hat{A} = \begin{bmatrix} 1 & 0 & \\ 1 & -1 & \\ 2 & 0 & \end{bmatrix}$$

The third column of $\hat{A}$ is the representation of Aq_1 in the given basis.

$$Aq_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \quad \hat{A} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 3 \\ 2 & 0 & 1 \end{bmatrix}$$

Similarity Transformation

The i-th column of $\hat{A}$ is the representation of Aq_i in the given basis.

$$A \text{ and } \boxed{\begin{matrix} q_1 & q_2 \\ q_3 & \\ & \dots\dots \\ & q_n \end{matrix}} \rightarrow \hat{A}$$

Linearly independent vectors

The relationship between matrix A and $\hat{A}$

$$A[q_1 \quad q_2 \quad \dots \quad q_n] = [q_1 \quad q_2 \quad \dots \quad q_n]\hat{A}$$

$$AQ = Q\hat{A}$$

$$A = Q\hat{A}Q^{-1} \qquad \hat{A} = Q^{-1}AQ$$

Similarity Transformation

$$A \text{ and } \begin{pmatrix} q_1 & q_2 \\ q_3 & \\ \cdots\cdots & \\ & q_n \end{pmatrix} \rightarrow \hat{A}$$

Linearly independent vectors

The i-th column of $\hat{A}$ is the representation of Aq_i in the given basis.

Example 13: Find the similarity transformation of matrix A with respect to b, Ab, A2b.

$$A = \begin{bmatrix} 3 & 2 & -1 \\ -2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad q_1 = b,\ q_2 = Ab,\ q_3 = A^2b$$

The first column of $\hat{A}$ is the representation of Aq_1 in the given basis

$$Aq_1 = Ab \qquad \hat{A} = \begin{bmatrix} 0 & & \\ 1 & & \\ 0 & & \end{bmatrix}$$

The second column of $\hat{A}$ is the representation of Aq_2 in the given basis

$$Aq_2 = A^2b \quad \hat{A} = \begin{bmatrix} 0 & 0 & \\ 1 & 0 & \\ 0 & 1 & \end{bmatrix}$$

The third column of $\hat{A}$ is the representation of Aq_3 in the given basis

$$Aq_3 = \begin{bmatrix} -5 \\ 10 \\ -13 \end{bmatrix} \quad \hat{A} = \begin{bmatrix} 0 & 0 & 17 \\ 1 & 0 & -15 \\ 0 & 1 & 5 \end{bmatrix}$$

Similarity Transformation

A *and* $\boxed{\begin{aligned} q_1 &= b, \\ q_2 &= Ab, \\ &\cdots \\ q_n &= A^{n-1}b \end{aligned}}$ $\rightarrow$ $\hat{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \beta_n \\ 1 & 0 & \dots & 0 & \beta_{n-1} \\ 0 & 1 & \dots & 0 & \beta_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 1 & \beta_1 \end{bmatrix}$

Linearly independent vectors

$$\Delta(\lambda) = \det(\lambda I - \hat{A}) \qquad \Delta(\lambda) = \lambda^n - \beta_1\lambda^{n-1} - \beta_2\lambda^{n-2} - \dots - \beta_{n-1}\lambda - \beta_n$$

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \beta_n & \beta_{n-1} & \dots & \beta_2 & \beta_1 \end{bmatrix} \quad \hat{A} = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 1 & 0 \end{bmatrix} \quad \hat{A} = \begin{bmatrix} \beta_1 & 1 & 0 & \dots & 0 \\ \beta_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{n-1} & 0 & 0 & \dots & 1 \\ \beta_n & 0 & 0 & \dots & 0 \end{bmatrix}$$

Diagonal Form

$$A \;\; and \;\; \boxed{\begin{array}{c} q_1 = b, \\ q_2 = Ab, \\ \cdots \\ q_n = A^{n-1}b \end{array}} \;\; \rightarrow \;\; \hat{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \beta_n \\ 1 & 0 & \dots & 0 & \beta_{n-1} \\ 0 & 1 & \dots & 0 & \beta_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 1 & \beta_1 \end{bmatrix}$$

Linearly independent vectors

$$A \;\; and \;\; \boxed{\begin{array}{c} q_1 = ?, \\ q_2 = ?, \\ \cdots \\ q_n = ? \end{array}} \;\; \rightarrow \;\; \hat{A} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Linearly independent vectors

Simply choose the eigenvectors of matrix A as the basis, provided the eigenvalues are distinct.

Diagonal Form

If the eigenvalues of A are distinct and the eigenvectors of A are chosen as the basis, then:

$$A \;\; and \;\; \boxed{\begin{matrix} q_1 = ?, \\ q_2 = ?, \\ \cdots \\ q_n = ? \end{matrix}} \;\; \rightarrow \;\; \hat{A} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Linearly independent vectors

How to compute the eigenvalues of A:

$$\Delta(\lambda) = |\lambda I - A| = 0$$

If the eigenvalues of A are distinct:

$$\Delta(\lambda) = \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n = 0 \;\Rightarrow\; \lambda_1, \lambda_2, \ldots, \lambda_n$$

How to compute the eigenvector corresponding to each eigenvalue of A

$$(A - \lambda_i I)v_i = 0$$

$$v_i \neq 0$$

The property of an eigenvector corresponding to an eigenvalue: $Av_i = \lambda_i v_i$

Diagonal Form

$$A \text{ and } \begin{pmatrix} q_1 & q_2 \\ q_3 & \\ & \cdots\cdots \\ & q_n \end{pmatrix} \rightarrow \hat{A}$$

Linearly independent vectors

The i-th column of $\hat{\boldsymbol{A}}$ is the representation of Aq_i in the given basis.

The similarity transformation of matrix A in terms of the bases $v_1, v_2, \ldots, v_n$

The property of an eigenvector corresponding to an eigenvalue: $Av_i = \lambda_i v_i$

The first column of $\hat{A}$ is the representation of Av_1 in the given basis

$$Av_1 = \lambda_1 v_1 \qquad \hat{A} = \begin{bmatrix} \lambda_1 & \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix}$$

The second column of $\hat{A}$ is the representation of Av_2 in the given basis

$$Av_2 = \lambda_2 v_2 \qquad \hat{A} = \begin{bmatrix} \lambda_1 & 0 & \\ 0 & \lambda_2 & \\ \vdots & \vdots & \\ 0 & 0 & \end{bmatrix} \qquad \Rightarrow \quad \hat{A} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

Diagonal Form

Example 14: If possible, determine the diagonal form of matrix A and the transformation that diagonalizes matrix A.

$$A=\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 2\\ 0 & 1 & 1\end{bmatrix}$$

First, calculate the eigenvalues of A:

$$\Delta(\lambda)=\left|\lambda I-A\right|=\lambda(\lambda-2)(\lambda+1) \qquad \lambda_1=0,\ \lambda_2=2\ and\ \ \lambda_3=-1$$

Now, calculate the eigenvector corresponding to each eigenvalue of A:

$$\lambda_1=0,\quad (A-\lambda_1 I)v_1=\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 2\\ 0 & 1 & 1\end{bmatrix}v_1=0 \qquad v_1=\begin{bmatrix}2\\ 1\\ -1\end{bmatrix}$$

$$Q=[v_1\ \ v_2\ \ v_3]=\begin{bmatrix}2 & 0 & 0\\ 1 & 1 & -2\\ -1 & 1 & 1\end{bmatrix}$$

$$\lambda_2=2,\quad (A-\lambda_2 I)v_2=\begin{bmatrix}-2 & 0 & 0\\ 1 & -2 & 2\\ 0 & 1 & -1\end{bmatrix}v_2=0 \qquad v_2=\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$$

$$\hat{A}=Q^{-1}AQ=\begin{bmatrix}0 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & -1\end{bmatrix}$$

$$\lambda_3=-1,\quad (A-\lambda_3 I)v_3=\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 2\\ 0 & 1 & 2\end{bmatrix}v_3=0 \qquad v_3=\begin{bmatrix}0\\ -2\\ 1\end{bmatrix}$$

Diagonal Form

Example 15: If possible, determine the diagonal form of matrix *A* and the transformation that diagonalizes matrix *A*.

$$A=\begin{bmatrix}-1 & 1 & 1\\ 0 & 4 & -13\\ 0 & 1 & 0\end{bmatrix}$$

First, calculate the eigenvalues of *A*:

$$\Delta(\lambda)=\left|\lambda I-A\right|=(\lambda+1)(\lambda^2-4\lambda+13) \qquad \lambda_1=-1,\ \lambda_2,\ \lambda_3=2\pm 3j$$

Now, calculate the eigenvector corresponding to each eigenvalue of *A*:

$$\lambda_1=-1,\quad (A-\lambda_1 I)v_1=\begin{bmatrix}0 & 1 & 1\\ 0 & 5 & -13\\ 0 & 1 & 1\end{bmatrix}v_1=0 \qquad v_1=\begin{bmatrix}2\\ 0\\ 0\end{bmatrix}$$

$$\lambda_2=2+3j,\quad (A-\lambda_2 I)v_2=\begin{bmatrix}-3-3j & 1 & 1\\ 0 & 2-3j & -13\\ 0 & 1 & -2-3j\end{bmatrix}v_2=0\ \ v_2=\begin{bmatrix}1\\ 2+3j\\ 1\end{bmatrix}\ \ v_3=v_2^{*}=\begin{bmatrix}1\\ 2-3j\\ 1\end{bmatrix}$$

$$Q=[v_1\ \ v_2\ \ v_3]=\begin{bmatrix}2 & 1 & 1\\ 0 & 2+3j & 2-3j\\ 0 & 1 & 1\end{bmatrix} \qquad \hat{A}=Q^{-1}AQ=\begin{bmatrix}-1 & 0 & 0\\ 0 & 2+3j & 0\\ 0 & 0 & 2-3j\end{bmatrix}$$

Modal Form

Example 16: Determine the modal form of matrix *A* and the transformation that converts matrix *A* into its modal form.

$$A=\begin{bmatrix}-1 & 1 & 1\\ 0 & 4 & -13\\ 0 & 1 & 0\end{bmatrix}$$

First, calculate the eigenvalues of *A*:

$$\Delta(\lambda)=\left|\lambda I-A\right|=(\lambda+1)(\lambda^2-4\lambda+13) \qquad \lambda_1=-1,\ \lambda_2,\ \lambda_3=2\pm 3j$$

Now, calculate the eigenvector corresponding to each eigenvalue of *A*:

$$\lambda_1=-1,\quad (A-\lambda_1 I)v_1=\begin{bmatrix}0 & 1 & 1\\ 0 & 3 & -13\\ 0 & 1 & -1\end{bmatrix}v_1=0 \qquad v_1=\begin{bmatrix}2\\ 0\\ 0\end{bmatrix}$$

$$\lambda_2=2+3j,\quad (A-\lambda_2 I)v_2=\begin{bmatrix}-3-3j & 1 & 1\\ 0 & 2-3j & -13\\ 0 & 1 & -2-3j\end{bmatrix}v_2=0 \quad v_2=\begin{bmatrix}1\\ 2+3j\\ 1\end{bmatrix} \quad v_2=\begin{bmatrix}1\\ 2\\ 1\end{bmatrix},\ v_3=\begin{bmatrix}0\\ 3\\ 0\end{bmatrix}$$

$$Q=[v_1\ \ v_2\ \ v_3]=\begin{bmatrix}2 & 1 & 0\\ 0 & 2 & 3\\ 0 & 1 & 0\end{bmatrix} \qquad \hat{A}=Q^{-1}AQ=\begin{bmatrix}-1 & 0 & 0\\ 0 & 2 & 3\\ 0 & -3 & 2\end{bmatrix}$$

Jordan Form

If the eigenvalues of A are non-repeated and the eigenvectors of matrix A are chosen as the bases, then:

$$A \ \ and \ \ \boxed{\begin{array}{l} q_1 = v_1 \\ q_2 = v_2 \\ \quad \cdots \\ q_n = v_n \end{array}} \ \rightarrow \ \hat{A} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Linearly independent vectors

If the eigenvalues of A are repeated, then:

$$A \ \ and \ \ \boxed{\begin{array}{l} q_1 = \\ q_2 = \\ \quad \cdots \\ q_n = \end{array}} \ \rightarrow \ \hat{A} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \quad \hat{A} = \begin{bmatrix} \lambda_1 & 1 & \dots & 0 \\ 0 & \lambda_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Linearly independent vectors

Jordan Form

Example 17: If possible, determine the diagonal form of matrix A and the transformation that diagonalizes matrix A.

$$A=\begin{bmatrix}1 & 0 & 12\\ 0 & 1 & 1\\ 0 & 0 & 4\end{bmatrix}$$

First, calculate the eigenvalues of A:

$$\Delta(\lambda)=\left|\lambda I-A\right|=(\lambda-1)^2(\lambda-4) \qquad \lambda_1=4,\ \lambda_2=1\ and\ \ \lambda_3=1$$

Now, calculate the eigenvector corresponding to each eigenvalue of A:

$$\lambda_1=4, \quad (A-\lambda_1 I)v_1=\begin{bmatrix}-3 & 0 & 12\\ 0 & -3 & 1\\ 0 & 0 & 0\end{bmatrix}v_1=0 \qquad v_1=\begin{bmatrix}12\\ 1\\ 3\end{bmatrix}$$

$$\lambda_2=1, \quad \boxed{(A-\lambda_2 I)}\,v_2=\boxed{\begin{bmatrix}0 & 0 & 12\\ 0 & 0 & 1\\ 0 & 0 & 3\end{bmatrix}}v_2=0 \qquad v_2=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix} \quad v_3=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$$

$$Q=[v_1\ \ v_2\ \ v_3]=\begin{bmatrix}12 & 1 & 0\\ 1 & 0 & 1\\ 3 & 0 & 0\end{bmatrix} \qquad \hat{A}=Q^{-1}AQ=\begin{bmatrix}4 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Jordan Form

Example 18: If possible, determine the diagonal form of matrix A and the transformation that diagonalizes matrix A.

$$A=\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 4\\ 0 & 0 & 4\end{bmatrix}$$

First, calculate the eigenvalues of A:

$$\Delta(\lambda)=\left|\lambda I-A\right|=(\lambda-1)^2(\lambda-4) \qquad \lambda_1=4,\ \lambda_2=1 \ and \ \lambda_3=1$$

Now, calculate the eigenvector corresponding to each eigenvalue of A:

$$\lambda_1=4, \quad (A-\lambda_1 I)v_1=\begin{bmatrix}-3 & 2 & 3\\ 0 & -3 & 4\\ 0 & 0 & 0\end{bmatrix}v_1=0 \qquad v_1=\begin{bmatrix}17/3\\ 4\\ 3\end{bmatrix}$$

$$\lambda_2=1, \quad \boxed{(A-\lambda_2 I)}\, v_2=\boxed{\begin{bmatrix}0 & 2 & 3\\ 0 & 0 & 4\\ 0 & 0 & 3\end{bmatrix}}v_2=0 \qquad v_2=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix} \qquad v_3=?$$

Diagonalization is not possible, so we arrive at the Jordan form. Therefore, for this form, we need the generalized eigenvectors.

Diagonal Form

How to compute the eigenvalues of A: $\Delta(\lambda) = |\lambda I - A| = 0$

How to calculate an eigenvector of the matrix A.

$$(A - \lambda_i I)v_i = 0$$

$$v_i \neq 0$$

How to calculate a generalized eigenvector of order 2 of the matrix A.

$$(A - \lambda_i I)^2 v_i = 0$$
$$(A - \lambda_i I)v_i \neq 0$$
$$v_{i2} = v_i \qquad v_{i1} = (A - \lambda_i I)v_{i2}$$

How to calculate a generalized eigenvector of order 3 of the matrix A.

$$(A - \lambda_i I)^3 v_i = 0$$
$$(A - \lambda_i I)^2 v_i \neq 0$$
$$v_{i3} = v_i \qquad v_{i2} = (A - \lambda_i I)v_{i3} \qquad v_{i1} = (A - \lambda_i I)v_{i2}$$

Jordan Form

Example 19: Determine the diagonal or Jordan form of matrix A and the transformation that converts matrix A into its diagonal or Jordan form.

$$A=\begin{bmatrix}1&2&3\\0&1&4\\0&0&4\end{bmatrix}$$

First, calculate the eigenvalues of A:

$$\Delta(\lambda)=\left|\lambda I-A\right|=(\lambda-1)^2(\lambda-4)\qquad \lambda_1=4,\ \lambda_2=1\ and\ \ \lambda_3=1$$

Now, calculate the eigenvector corresponding to each eigenvalue of A:

$$\lambda_1=4,\quad (A-\lambda_1 I)v_1=\begin{bmatrix}-3&2&3\\0&-3&4\\0&0&0\end{bmatrix}v_1=0\qquad v_1=\begin{bmatrix}17/3\\4\\3\end{bmatrix}$$

$$\lambda_2=1,\quad (A-\lambda_2 I)v_2=\begin{bmatrix}0&2&3\\0&0&4\\0&0&3\end{bmatrix}v_2=0\qquad v_2=\begin{bmatrix}1\\0\\0\end{bmatrix}\qquad v_3=?$$

$$\lambda_2=1,\quad (A-\lambda_2 I)^2v_2=\begin{bmatrix}0&0&17\\0&0&12\\0&0&9\end{bmatrix}v_2=0\qquad (A-\lambda_2 I)v_2=\begin{bmatrix}0&2&3\\0&0&4\\0&0&3\end{bmatrix}v_2\neq 0\quad v_{22}=\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Jordan Form

Example 19: Determine the diagonal or Jordan form of matrix A and the transformation that converts matrix A into its diagonal or Jordan form.

$$A=\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 4\\ 0 & 0 & 4\end{bmatrix}$$

First, calculate the eigenvalues of A:

$$\Delta(\lambda)=\left|\lambda I-A\right|=(\lambda-1)^2(\lambda-4) \qquad \lambda_1=4,\ \lambda_2=1 \ and \ \lambda_3=1$$

Now, calculate the eigenvector corresponding to each eigenvalue of A:

$$\lambda_1=4, \quad (A-\lambda_1 I)v_1=\begin{bmatrix}-3 & 2 & 3\\ 0 & -3 & 4\\ 0 & 0 & 0\end{bmatrix}v_1=0 \qquad v_1=\begin{bmatrix}17/3\\ 4\\ 3\end{bmatrix}$$

$$\lambda_2=1, \quad (A-\lambda_2 I)^2 v_2=\begin{bmatrix}0 & 0 & 17\\ 0 & 0 & 12\\ 0 & 0 & 9\end{bmatrix}v_2=0 \qquad (A-\lambda_2 I)v_2=\begin{bmatrix}0 & 2 & 3\\ 0 & 0 & 4\\ 0 & 0 & 3\end{bmatrix}v_2\neq 0 \quad v_{22}=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$$

$$v_3=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix} \qquad v_2=(A-\lambda_2 I)v_3=\begin{bmatrix}2\\ 0\\ 0\end{bmatrix} \qquad Q=[v_1 \ \ v_2 \ \ v_3]=\begin{bmatrix}17/3 & 2 & 0\\ 4 & 0 & 1\\ 3 & 0 & 0\end{bmatrix} \qquad \hat{A}=Q^{-1}AQ=\begin{bmatrix}4 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$$

Jordan Form

Example 20: Determine the diagonal or Jordan form of Matrix A and the transformation that converts matrix A into its diagonal or Jordan form.

$$A=\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ -0.2 & 0.5 & -2 & 0 \\ -0.5 & 0.5 & 0 & -2 \end{bmatrix}$$

First, calculate the eigenvalues of A:

$$\Delta(\lambda)=|\lambda I-A|=(\lambda+2)^4 \qquad \lambda_1=\lambda_2=\lambda_3=\lambda_4=-2$$

Now, calculate the eigenvector corresponding to each eigenvalue of A:

$$\lambda_1=-2, \quad (A-\lambda_1 I)v_1=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.2 & 0.5 & 0 & 0 \\ -0.5 & 0.5 & 0 & 0 \end{bmatrix}v_1=0 \qquad Nullity=k=4-2=2$$

Now, given the nullity 2, two possible cases may occur:

- Finding two generalized eigenvectors of order 2. $Q=[q_1 \quad \boxed{\bar{q}_2} \quad q_3 \quad \boxed{\bar{q}_4}]$
- Finding two generalized eigenvectors of order 3. $Q=[q_1 \quad q_2 \quad \boxed{\bar{q}_3} \quad q_4]$

Jordan Form

Example 20: Determine the diagonal or Jordan form of Matrix A and the transformation that converts matrix A into its diagonal or Jordan form.

$$A=\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ -0.2 & 0.5 & -2 & 0 \\ -0.5 & 0.5 & 0 & -2 \end{bmatrix}$$

$$\lambda_1=-2, \quad (A-\lambda_1 I)v_1=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.2 & 0.5 & 0 & 0 \\ -0.5 & 0.5 & 0 & 0 \end{bmatrix}v_1=0$$

$$Nullity=k=4-2=2$$

Finding a generalized eigenvector.

$$(A-\lambda_1 I)=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.2 & 0.5 & 0 & 0 \\ -0.5 & 0.5 & 0 & 0 \end{bmatrix} \quad (A-\lambda_1 I)^2=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (A-\lambda_1 I)^3=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We do not have a generalized eigenvector of order 3, but we have two of order 2.

$$\overline{v}_2=\begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} \quad \overline{v}_4=\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Jordan Form

Example 20: Determine the diagonal or Jordan form of Matrix A and the transformation that converts matrix A into its diagonal or Jordan form.

$$A=\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ -0.2 & 0.5 & -2 & 0 \\ -0.5 & 0.5 & 0 & -2 \end{bmatrix}$$

Finding a generalized eigenvector of order 2.

$$\bar{v}_2=\begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} \quad \bar{v}_4=\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad v_1=(A-\lambda_1 I)\bar{v}_2=\begin{bmatrix} 0 \\ 0 \\ 0.4 \\ -0.5 \end{bmatrix} \quad v_3=(A-\lambda_1 I)\bar{v}_4=\begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 0.5 \end{bmatrix}$$

$$Q=[q_1 \quad \bar{q}_2 \quad q_3 \quad \bar{q}_4]=\begin{bmatrix} 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0.4 & 0 & 0.5 & 1 \\ -0.5 & 0 & 0.5 & 0 \end{bmatrix}$$

$$\hat{A}=Q^{-1}AQ=\begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

The number of blocks corresponding to λ=-2 is?

Jordan Form

Example 21: Determine the diagonal or Jordan form of Matrix A and the transformation that converts matrix A into its diagonal or Jordan form.

$$A=\begin{bmatrix}-2 & 1 & 11 & -4\\ 0 & -2 & 1 & 0\\ 0 & 0 & -2 & 0\\ 0 & 0 & 0 & -2\end{bmatrix}$$

First, calculate the eigenvalues of A:

$$\Delta(\lambda)=|\lambda I-A|=(\lambda+2)^4 \qquad \lambda_1=\lambda_2=\lambda_3=\lambda_4=-2$$

Now, calculate the eigenvector corresponding to each eigenvalue of A:

$$\lambda_1=-2, \quad (A-\lambda_1 I)v_1=\begin{bmatrix}0 & 1 & 11 & -4\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}v_1=0 \qquad Nullity=k=4-2=2$$

Now, given the nullity 2, two possible cases may occur:

- Finding two generalized eigenvectors of order 2. $\quad Q=[q_1 \quad \boxed{\bar{q}_2} \quad q_3 \quad \boxed{\bar{q}_4}]$
- Finding two generalized eigenvectors of order 3. $\quad Q=[q_1 \quad q_2 \quad \boxed{\bar{q}_3} \quad q_4]$

Jordan Form

Example 21: Determine the diagonal or Jordan form of Matrix A and the transformation that converts matrix A into its diagonal or Jordan form.

$$A=\begin{bmatrix} -2 & 1 & 11 & -4 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$\lambda_1=-2, \quad (A-\lambda_1 I)v_1=\begin{bmatrix} 0 & 1 & 11 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}v_1=0$$

$$Nullity = k = 4-2=2$$

Finding a generalized eigenvector.

$$(A-\lambda_1 I)=\begin{bmatrix} 0 & 1 & 11 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (A-\lambda_1 I)^2=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (A-\lambda_1 I)^3=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have a generalized eigenvector of order 3.

$$\bar{q}_3=\begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \end{bmatrix}$$

Jordan Form

Example 21: Determine the diagonal or Jordan form of Matrix A and the transformation that converts matrix A into its diagonal or Jordan form.

$$A=\begin{bmatrix}-2 & 1 & 11 & -4\\ 0 & -2 & 1 & 0\\ 0 & 0 & -2 & 0\\ 0 & 0 & 0 & -2\end{bmatrix}$$

We have a generalized eigenvector of order 3.

$$\bar{q}_3=\begin{bmatrix}0\\0\\5\\0\end{bmatrix} \qquad q_2=(A-\lambda_1 I)\bar{q}_3=\begin{bmatrix}55\\5\\0\\0\end{bmatrix} \qquad q_1=(A-\lambda_1 I)q_2=\begin{bmatrix}5\\0\\0\\0\end{bmatrix}$$

Now, to find the last eigenvector, this vector must be independent of the other vectors and satisfy $(A-\lambda_1 I)q_4=0$. Thus:

$$(A-\lambda_1 I)q_4=\begin{bmatrix}0 & 1 & 11 & -4\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}q_4=0 \quad q_4=\begin{bmatrix}0\\4\\0\\1\end{bmatrix}$$

$$\hat{A}=Q^{-1}AQ=\begin{bmatrix}-2 & 1 & 0 & 0\\ 0 & -2 & 1 & 0\\ 0 & 0 & -2 & 0\\ 0 & 0 & 0 & -2\end{bmatrix}$$

The number of blocks corresponding to λ=-2 is?

Determinant and Eigenvalues

The relationship between matrix A and its Jordan form:

$$\hat{A} = Q^{-1} A Q \qquad A = Q \hat{A} Q^{-1}$$

The determinant of matrix A and the determinant of its Jordan form:

$$|A| = |Q\hat{A}Q^{-1}| = |Q||\hat{A}||Q^{-1}| = |\hat{A}|$$

The determinant of matrix A and its eigenvalues:

$$|A| = \prod_{i=1}^{n} \lambda_i$$

Nilpotent Property

Consider the following Jordan block:

$$J = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

The nilpotent property concerning Jordan blocks:

$$(J-\lambda I) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (J-\lambda I)^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (J-\lambda I)^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(J-\lambda I)^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercises

Exercise 3: Find the orthogonal vectors corresponding to the following vectors.

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

Exercise 4: Determine the norm 1, norm 2 and ∞-norm of the following vectors:

$$a = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Exercise 5: Determine the nullity and rank of the following matrices:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 4 & 1 & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercises

Exercise 6: Determine the bases of the range space and the bases of the null space of the following matrices.

$$A_1=\begin{bmatrix}0&1&0\\0&0&0\\0&0&1\end{bmatrix}\qquad A_2=\begin{bmatrix}4&1&-1\\3&2&0\\1&1&0\end{bmatrix}\qquad A_3=\begin{bmatrix}1&2&3&4\\0&-1&-2&2\\0&0&0&1\end{bmatrix}$$

Exercise 7: Consider the following algebraic equation. Is there a solution x for the above equations? Is the solution unique? Is there a solution for $y=[1\ 1\ 1]^T$?

$$\begin{bmatrix}2&-1\\-3&3\\-1&2\end{bmatrix}x=\begin{bmatrix}1\\0\\1\end{bmatrix}=y$$

Exercise 8: Find all solutions to the following algebraic equation.

$$\begin{bmatrix}1&2&3&4\\0&-1&-2&2\\0&0&0&1\end{bmatrix}x=\begin{bmatrix}3\\2\\1\end{bmatrix}$$

Exercises

Exercise 9: Determine the similarity transformation of A in terms of the bases b, Ab, A^2b, and A^3b.

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Exercise 10: Determine the diagonal form of matrix A and the transformation that diagonalizes matrix A.

$$A = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Exercise 11: Determine the diagonal form of matrix A and the transformation that diagonalizes matrix A.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}$$

Exercises

Exercise 12: Determine the determinant of the following matrices without performing calculations.

$$A_1 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Exercise 13: Show that in a square matrix with distinct eigenvalues, the eigenvectors are independent of each other. (Hint: Proof by contradiction and formation of $(A-\lambda_2 I)....(A-\lambda_n I)\sum_{k=1}^{n}\alpha_i v_i$)

Answers to selected problems

Answer 5: The ranks are 2, 3, and 3, and the nullities are 1, 0, and 1, respectively.

Answer 7: There is one unique solution x=[1 1]^T and no solution for the given *y*.

Answer 9:

$$\bar{A} = \begin{bmatrix} 0 & 0 & 0 & -8 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$