LINEAR CONTROL SYSTEMS

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Lecture 2

Different representations of control systems

Topics to be covered include:

- Differential Equation. (Complete Description)
- Transfer Function Model. (Simplest Description)
- State Space Model. (SS model)
- Function Block Diagram. (FBD)
- Signal Flow Graph Model. (SFG model)

Modeling of Systems

Model: Relationship among observed signals.

Building models

1- **Modeling**
- Split up system into subsystems,
- Joined subsystems mathematically,
- Does not necessarily involve any experimentation on the actual system.

2- **System identification**
- It is directly based on experimentation.
- Input and output signals from the system are recorded.

3- **Combined**

Linear differential equation

$$\frac{d^n y}{dt^n}+a_{n-1}\frac{d^{n-1}y}{dt^{n-1}}+\ldots\ldots+a_1\frac{dy}{dt}+a_0 y=\frac{d^m u}{dt^m}+b_{m-1}\frac{d^{m-1}u}{dt^{m-1}}+\ldots\ldots+b_1\frac{du}{dt}+b_0 u$$

The study of differential equations of the type described above is a rich and interesting subject. Of all the methods available for studying linear differential equations, one particularly useful tool is provided by Laplace Transforms.

External description or transfer function model

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y = \frac{du}{dt} + b_0 u$$

$\textit{Taking laplace transform} \Downarrow$ $\boxed{\textit{zero initial condition}}$

$$s^2 y(s) + a_1 s y(s) + a_0 y(s) = s u(s) + b_0 u(s)$$

$$G(s) = \frac{y(s)}{u(s)} = \frac{s + b_0}{s^2 + a_1 s + a_0}$$

TF model

Or

Input-output model

DE model → TF model

?? ←

Differential Equation Model & Transfer Function Model

Exercise 1: Differential equation and transfer function model of an RC circuit.

. . . .

Exercise 2: Differential equation and transfer function model of a mechanical system.

. . . .

Exercise 3: Differential equation and transfer function model of level control system.

. . . .

Exercise 4: Differential equation and transfer function model of dc motor(speed control with terminal voltage).

. . . .

Exercise 5: Differential equation and transfer function model of dc motor(speed control with field voltage).

. . . .

Internal description or state space model

General form of LTI systems in state space form

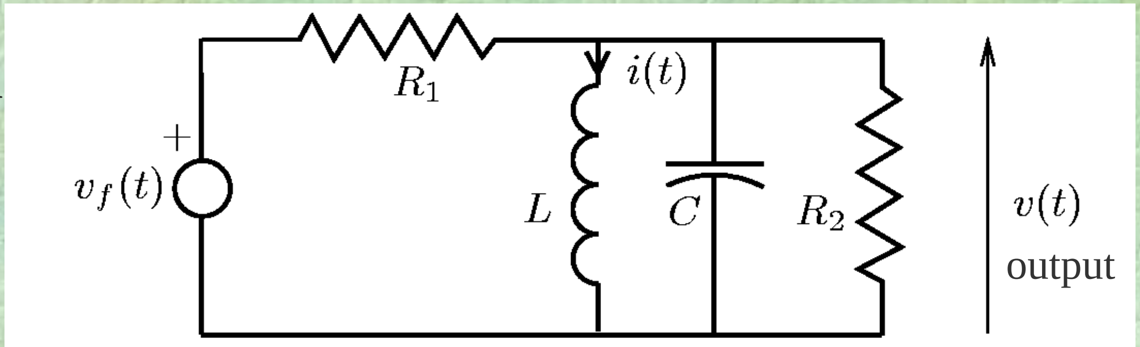
$$\frac{dx(t)}{dt} = \mathbf{A}x(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & . & . & b_{1p} \\ b_{21} & b_{22} & . & . & b_{2p} \\ . & . & . & . & . \\ . & . & . & . & . \\ b_{n1} & b_{n2} & . & . & b_{np} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ u_p \end{bmatrix}$$

SS model

State space model & transfer function model

Exercise 6: Derive SS model for following system.



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SS model ⟶ TF model

?? ⟵

• • • • • • • • •

Exercise 7: Derive transfer function model for above system.

a) Directly.
b) Through SS model.

• • • • • • • • •

Converting differential equation to SS model

DE model	$\longrightarrow$	SS model

. . . .

Exercise 8: a) Derive differential equation model for a pendulum.
b) Derive state space model for pendulum.
c) Is it possible possible to derive transfer function model for pendulum.

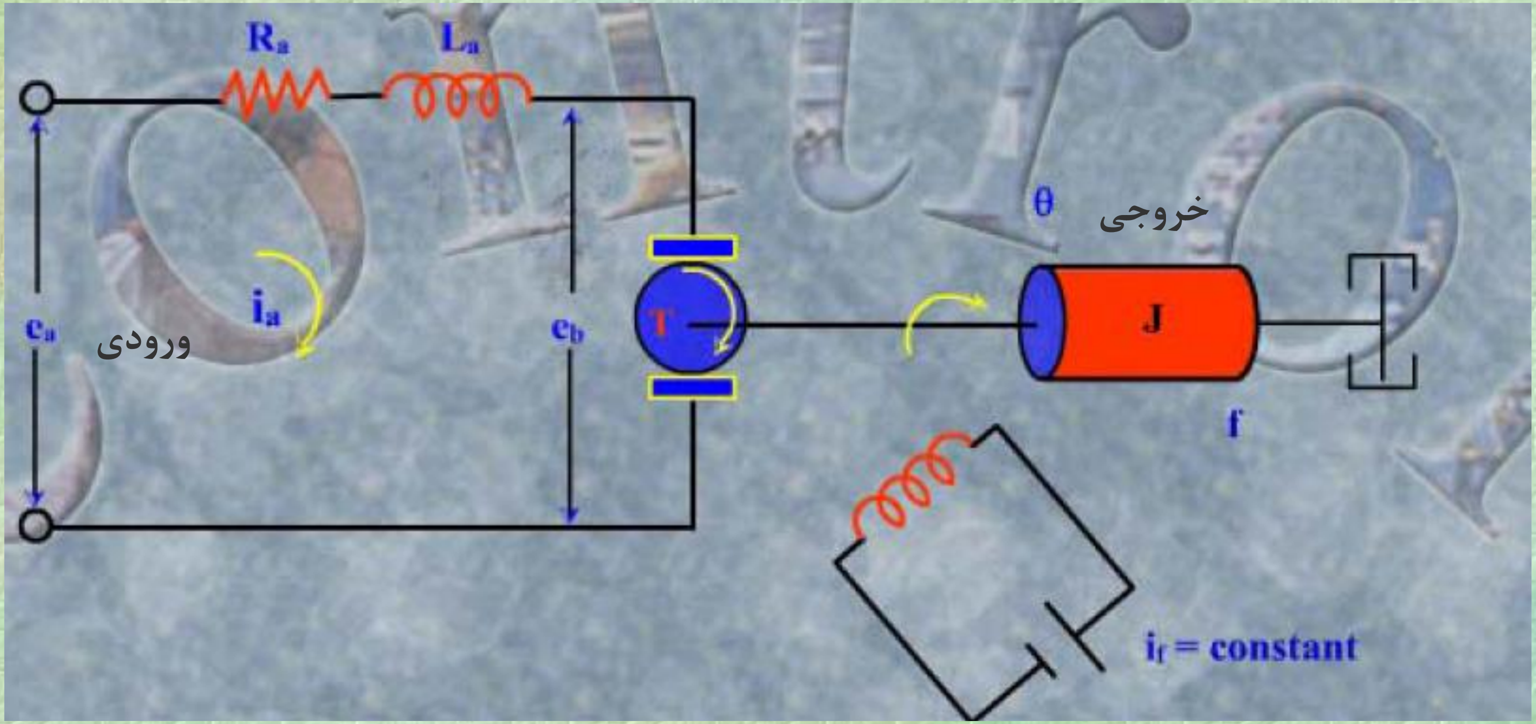
. . . .

Converting high order differential equation to SS

Exercise 9: a) Derive differential equation model for following position control system.

b) Derive state space model for the system.

c) Derive transfer function model for the system.

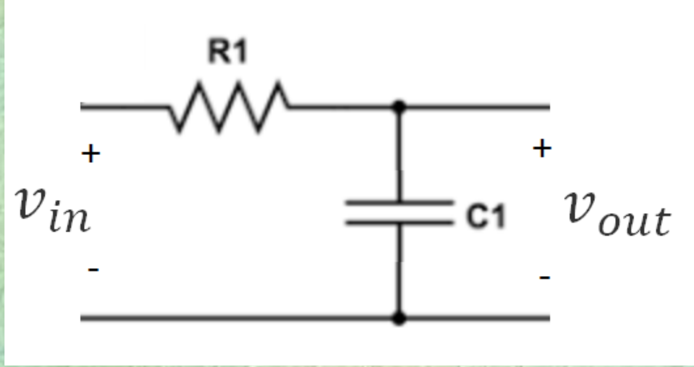


Function Block Diagram

FBD construction

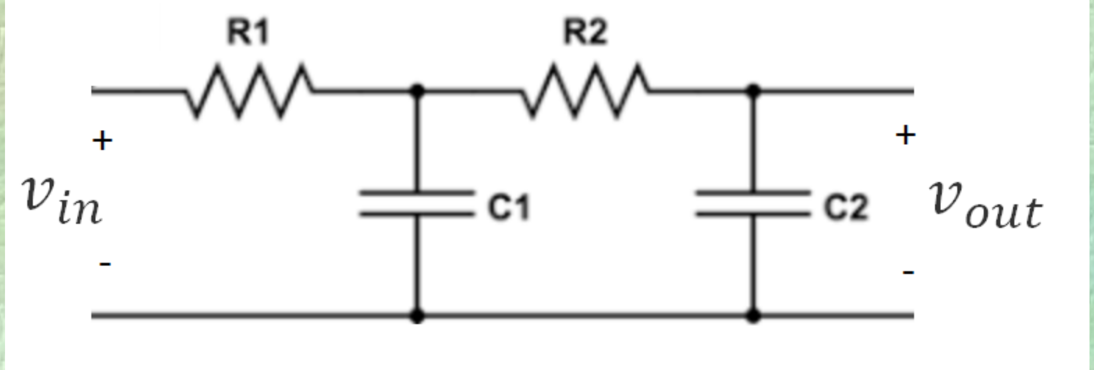
. . . .

Exercise 10: Derive FBD and transfer function for following system.



. . . .

Exercise 11: Derive FBD and transfer function for following system.



. . . .

Signal Flow Graph Model (SFG)

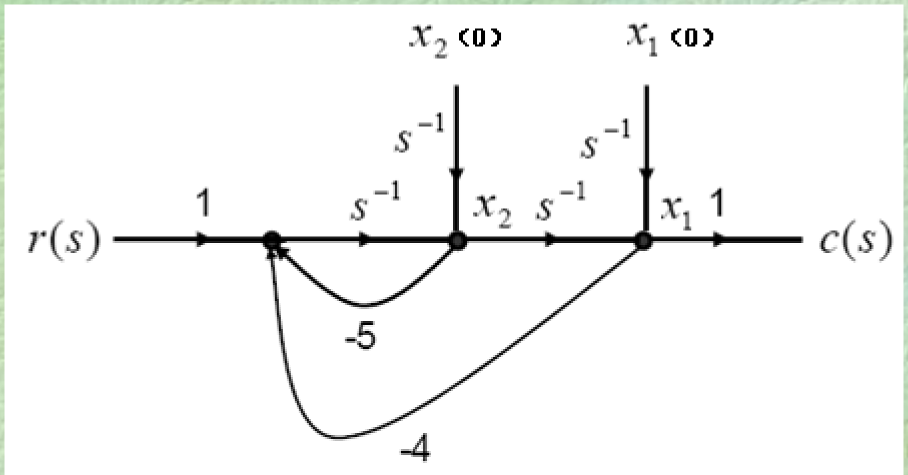
SFG construction ....

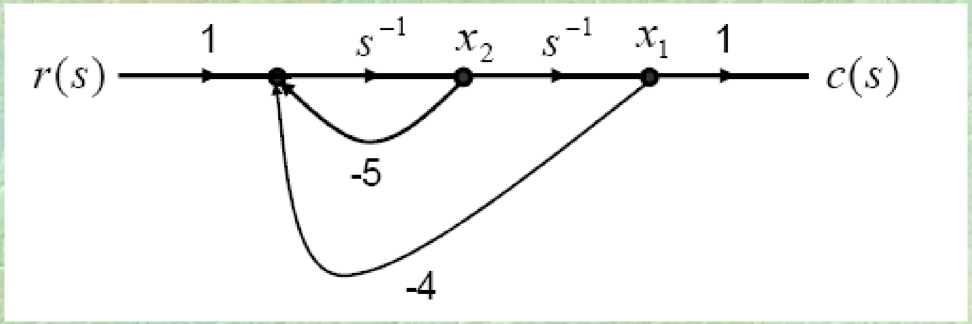
Exercise 12: Derive SFG of following SS model.

....

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + b_1u$$
$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + b_2u$$
$$y = c_1x_1 + c_2x_2 + du$$

Signal Flow Graph to State Space





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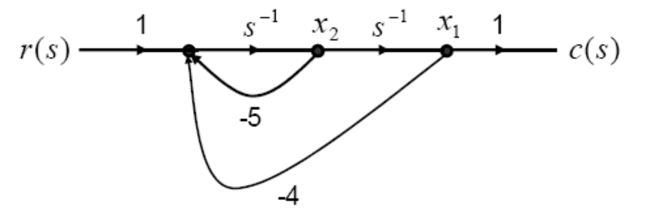
SFG model ⟷ SS model

Mason’s flow graph loop rule

SFG or FBD



Mason’s rule

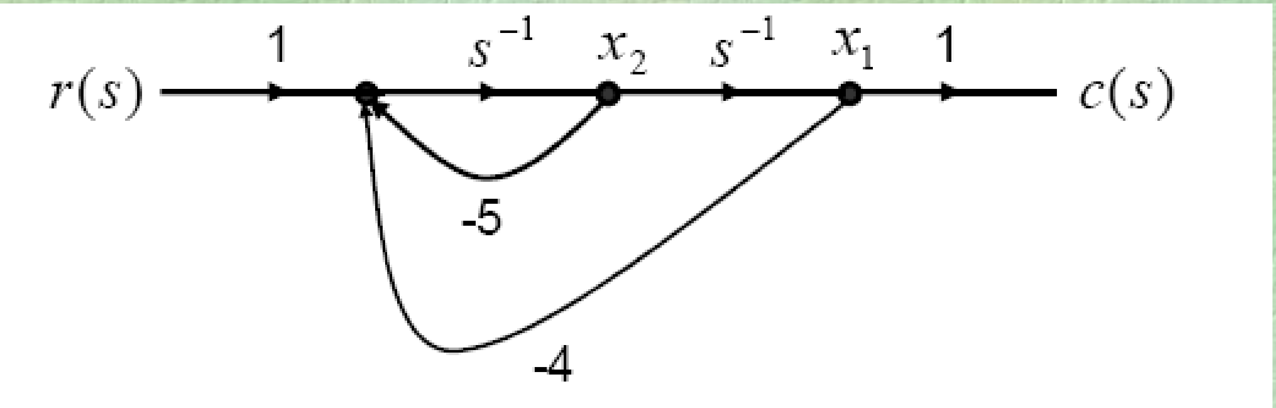
TF

$$G(s)=\frac{c(s)}{r(s)}=\frac{1}{s^2+5s+4}$$

$$\frac{y(s)}{u(s)}=\frac{\sum_{i=1}^{m} M_i\Delta_i}{\Delta}$$

Mason's flow graph lop rule

Exercise 13: Derive c(s)/r(s)

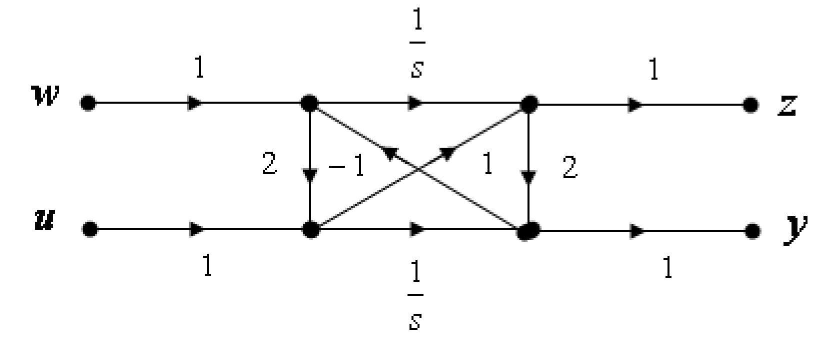


$$\frac{c(s)}{r(s)} = \frac{\sum_{i=1}^{m} M_i \Delta_i}{\Delta}$$

. . . .

Exercise 14: Suppose u=-ky , derive z/w.

It is appeared in university entrance exam 1394.



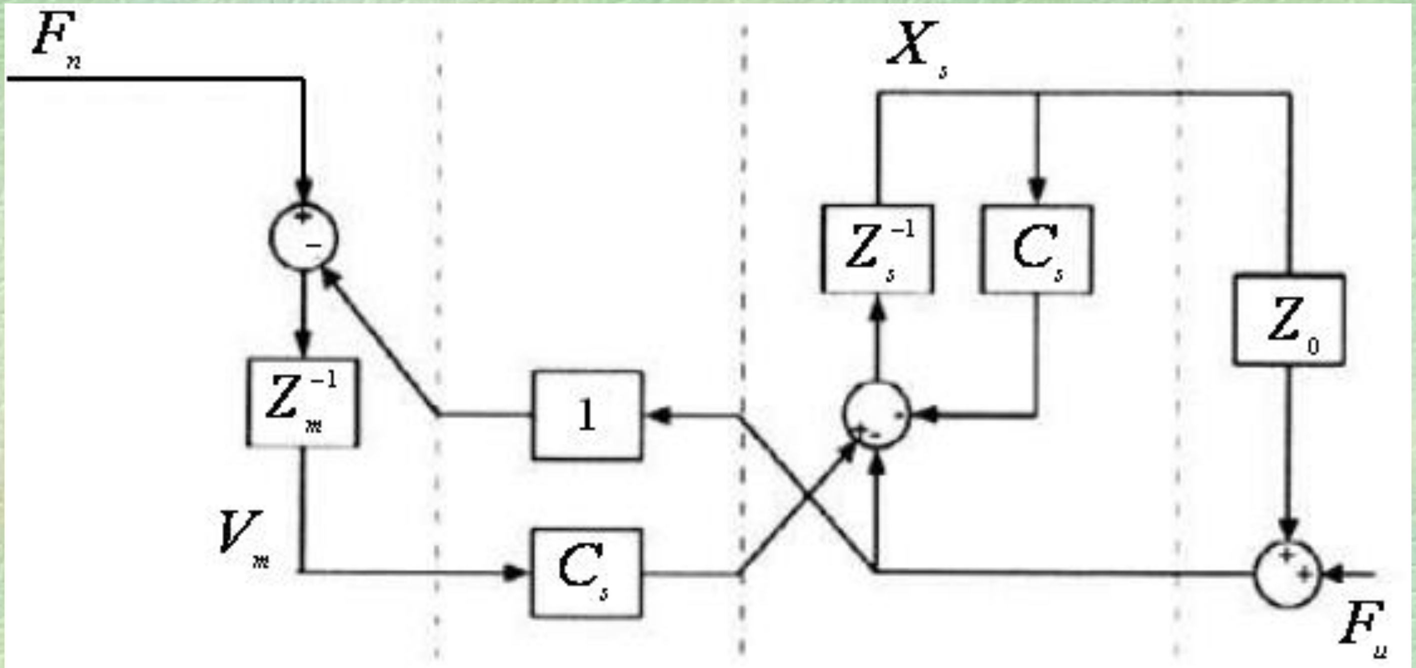
. . . .

$$\frac{z}{w} = \frac{2s^2 + s + k}{(5+2k)s^2 + (k+4)s}$$

Mason's flow graph lop rule

University entrance exam 1393

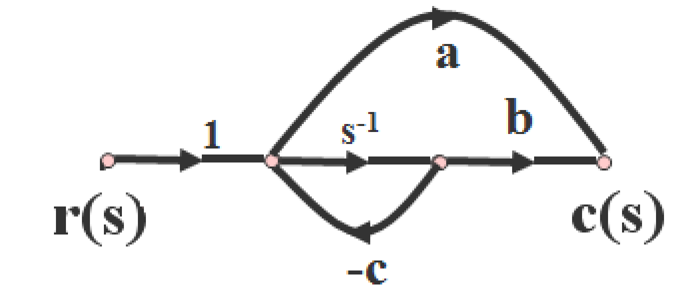
Exercise 15: Consider following tele-operating system and derive V_m/F_n

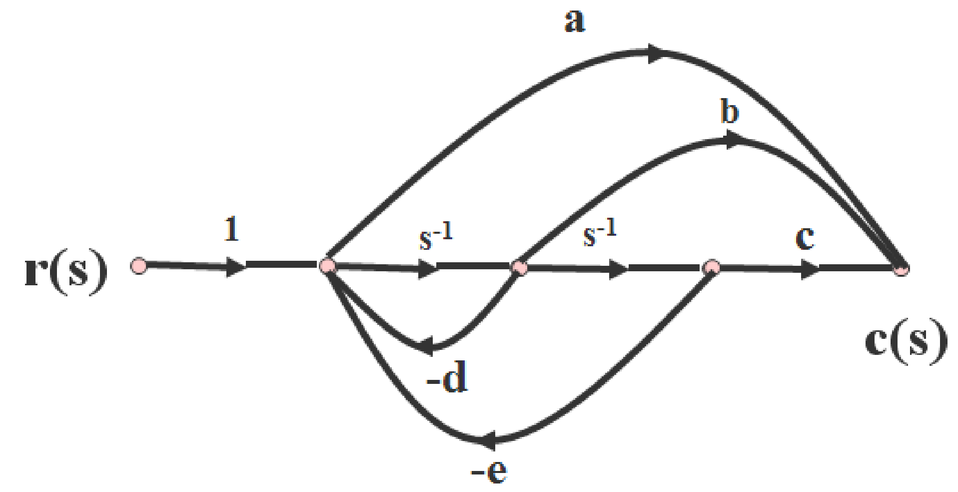


$$\frac{V_m}{F_n}=\frac{Z_m^{-1}\left(1+Z_s^{-1}C_s+Z_s^{-1}Z_0\right)}{1+Z_s^{-1}C_s+Z_s^{-1}Z_0+Z_m^{-1}C_sZ_s^{-1}Z_0}=\frac{Z_s+C_s+Z_0}{Z_m\left(Z_s+C_s+Z_0\right)+C_sZ_0}$$

Mason's flow graph lop rule

Exercise 16: Find the TF model for the following state diagrams.





$$\frac{c(s)}{r(s)} = \frac{as+b}{s+c}$$

$$\frac{c(s)}{r(s)} = \frac{as^2+bs+c}{s^2+ds+e}$$

This is a base form for first order transfer function.

This is a base form for second Order transfer function.

Realization

TF

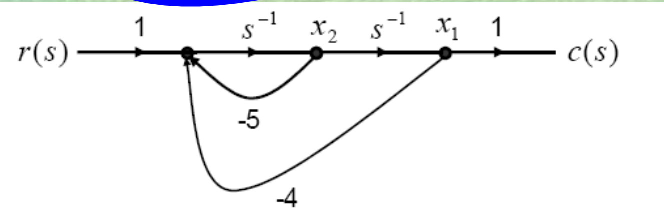
$$G(s) = \frac{1}{s^2 + 5s + 4}$$

Realization

SS

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -4x_1 - 5x_2 + r$$
$$c = x_1$$

SD



Realization

Some Realization methods

1- Direct Realization.

. . . .

2- Series Realization.

. . . .

3- Parallel Realization.

. . . .

Direct Realization

Exercise 17:

Find SS model of following TF by direct method.

$$G(s)=\frac{c(s)}{r(s)}=\frac{s^2+7s+12}{s^3+4s^2+5s+2}$$

. . . .

Exercise 18:

Find SS model of following TF by series method.

$$G(s)=\frac{c(s)}{r(s)}=\frac{2s^2+13s+17}{s^3+6s^2+11s+6}$$

. . . .

Exercise 19:

Find SS model of following TF by parallel method.

$$G(s)=\frac{c(s)}{r(s)}=\frac{s^2+7s+12}{s^3+4s^2+5s+2}$$

. . . .

Converting differential equation to state space

DE

$$\ddot{c}+5\dot{c}+4c=r$$

Explained before … but !?

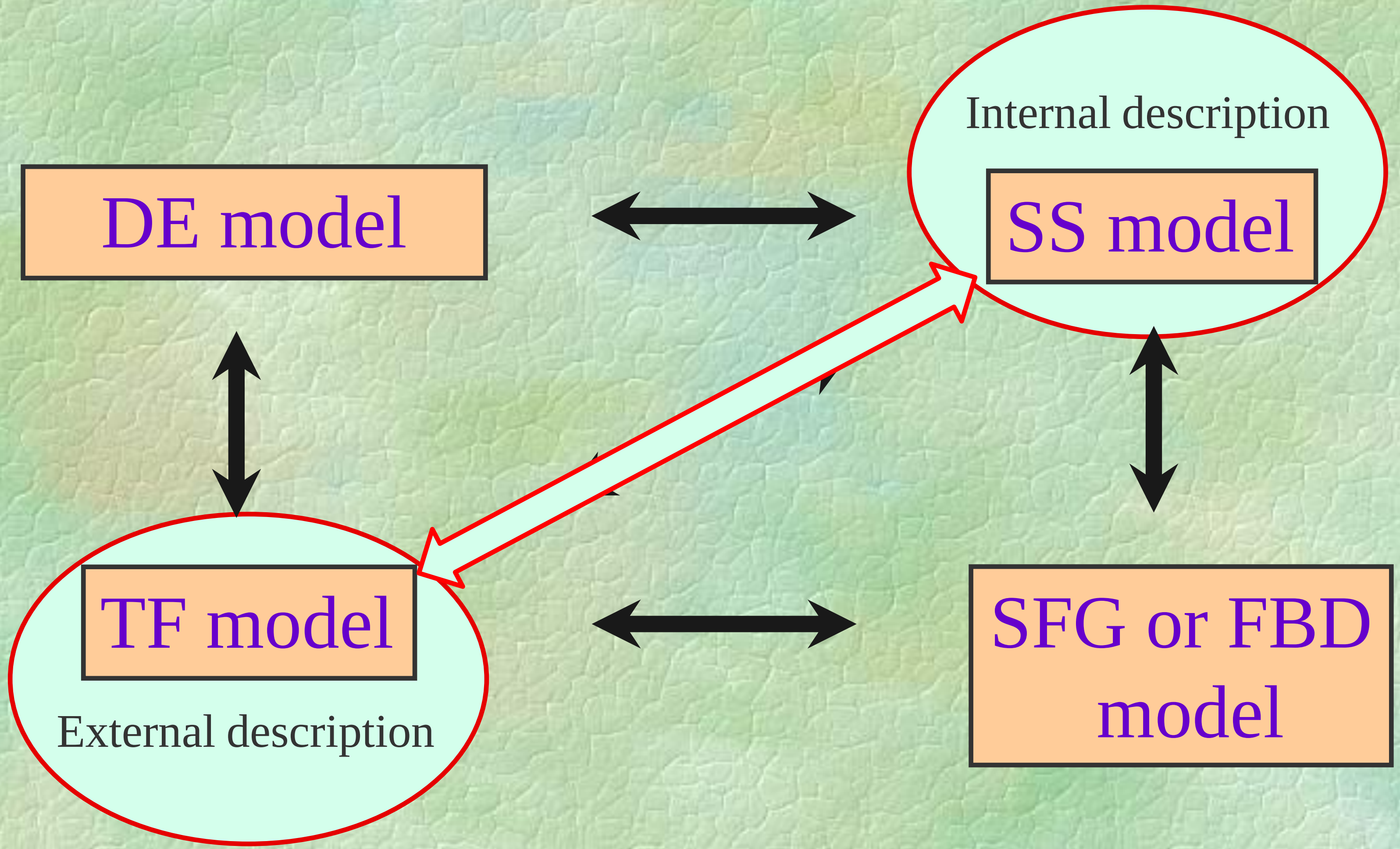
SS

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -4x_1 - 5x_2 + r$$
$$c = x_1$$

TF

$$G(s)=\frac{1}{s^2+5s+4}$$

Different representations (Summary)



Nonlinear systems

Although almost every real system includes nonlinear features, many systems can be reasonably described, at least within certain operating ranges, by linear models.

Nonlinear systems

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = g(x(t), u(t))$$

Say that $\{x_Q(t), u_Q(t), y_Q(t)\}$ is a given set of trajectories that satisfy the above equations, so we have

$$\dot{x}_Q(t) = f(x_Q(t), u_Q(t)); \qquad x_Q(t_o) \text{ given}$$
$$y_Q(t) = g(x_Q(t), u_Q(t))$$

$$\dot{x}(t) \approx f(x_Q, u_Q) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}} (u(t) - u_Q)$$

$$y(t) \approx g(x_Q, u_Q) + \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}} (u(t) - u_Q)$$

Nonlinear systems

$$\dot{x}(t) \approx f(x_Q, u_Q) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}} (u(t) - u_Q)$$

$$y(t) \approx g(x_Q, u_Q) + \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}} (u(t) - u_Q)$$

$$\dot{x}(t) - \dot{x}_Q(t) \approx \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}} (u(t) - u_Q)$$

$$y(t) - y_Q(t) \approx \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}} (u(t) - u_Q)$$

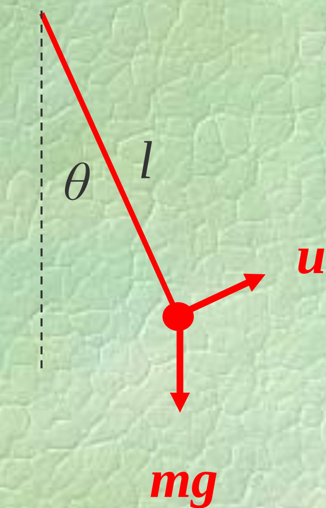
Linearization procedure

$$\dot{\delta x} = A\,\delta x + B\,\delta u \qquad A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} ; \qquad B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}}$$

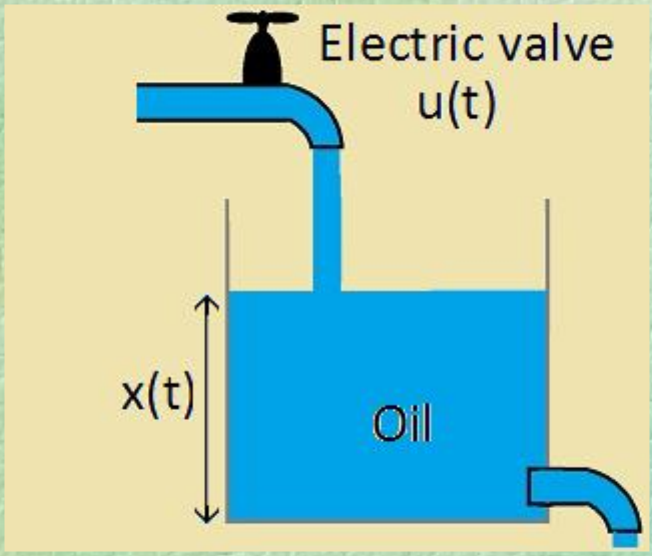
$$\delta y = C\,\delta x + D\,\delta u \qquad C = \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} ; \qquad D = \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}}$$

Linear model around equilibrium point

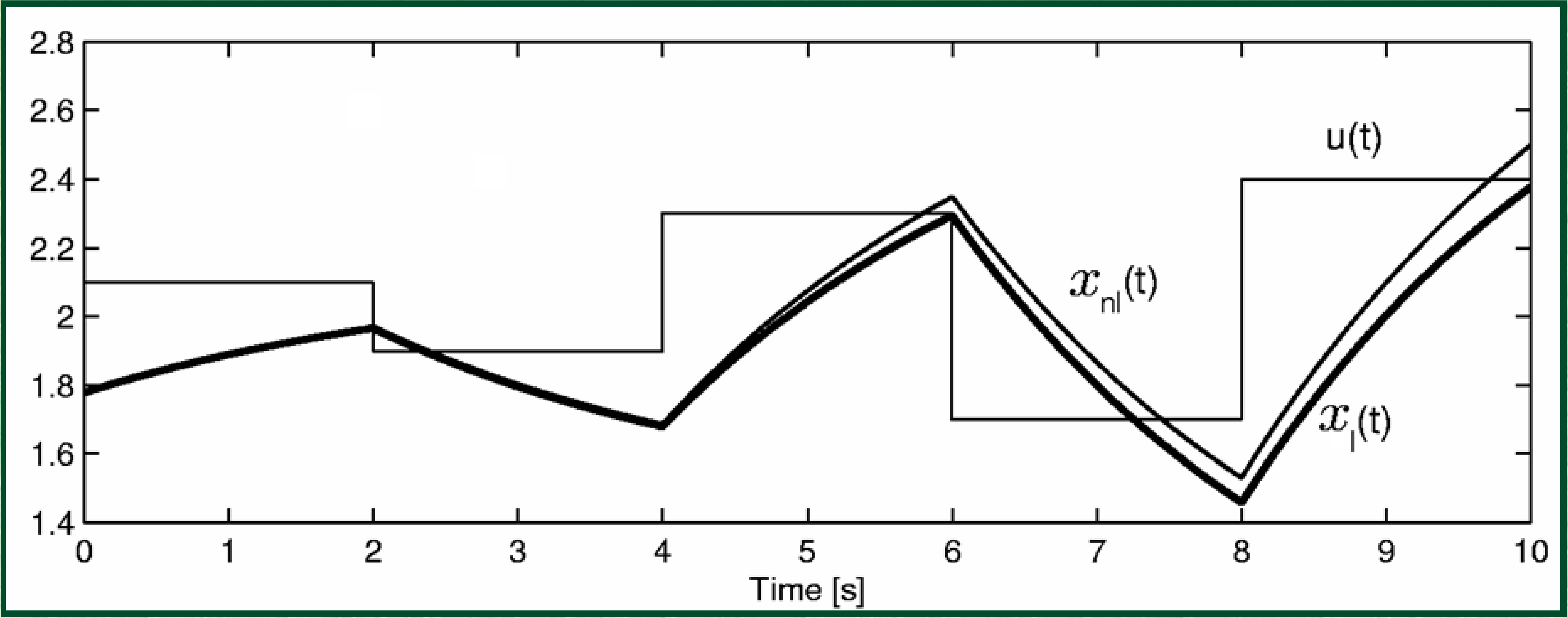
Exercise 20: Derive linear model for inverted pendulum around its equilibrium points.



Exercise 21: Derive linear model for following level control system. Suppose out put flow of valve is $Au(t)^2/3$ and output flow is $Ax(t)^{1/2}$.

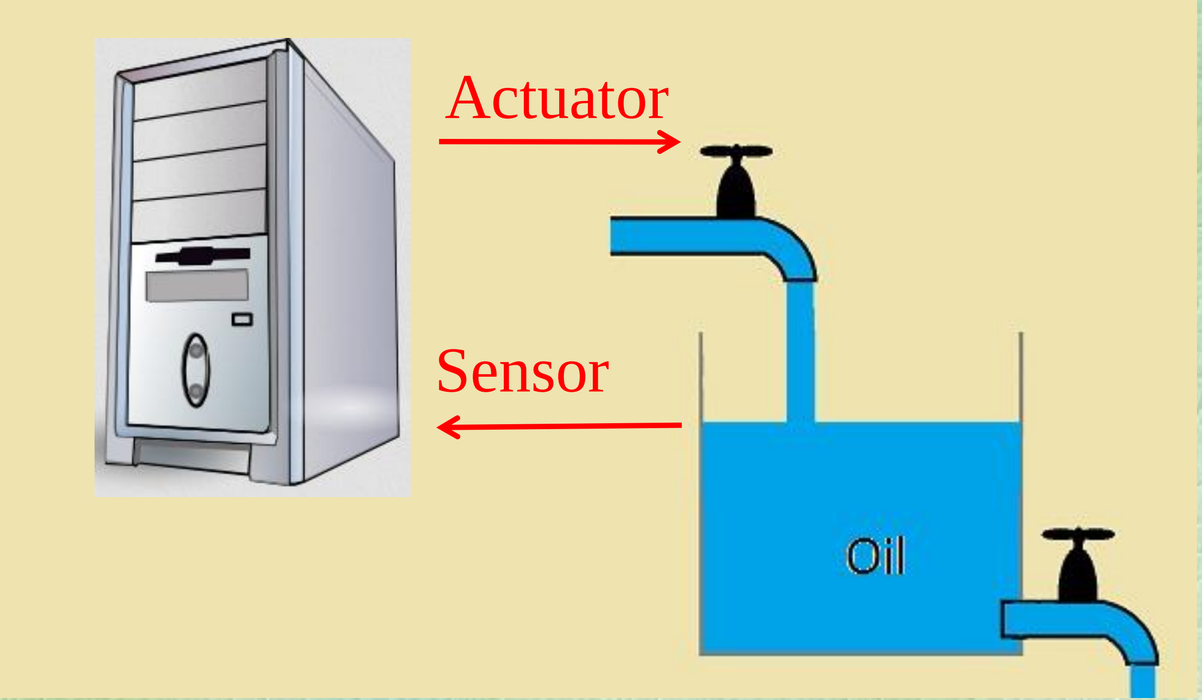


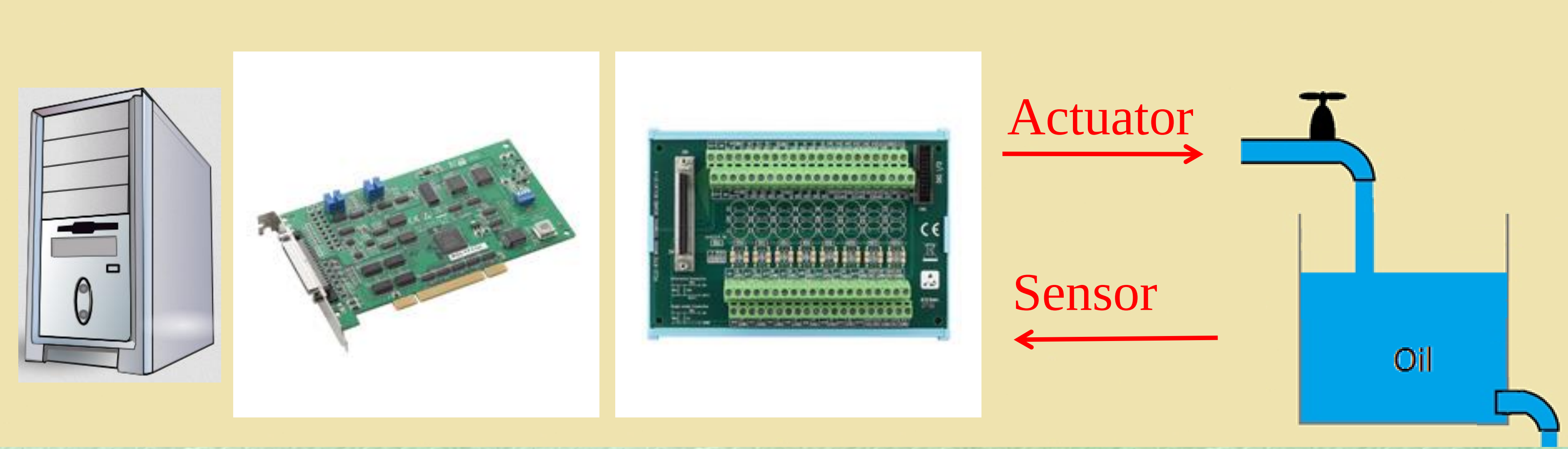
Simulation of level control system



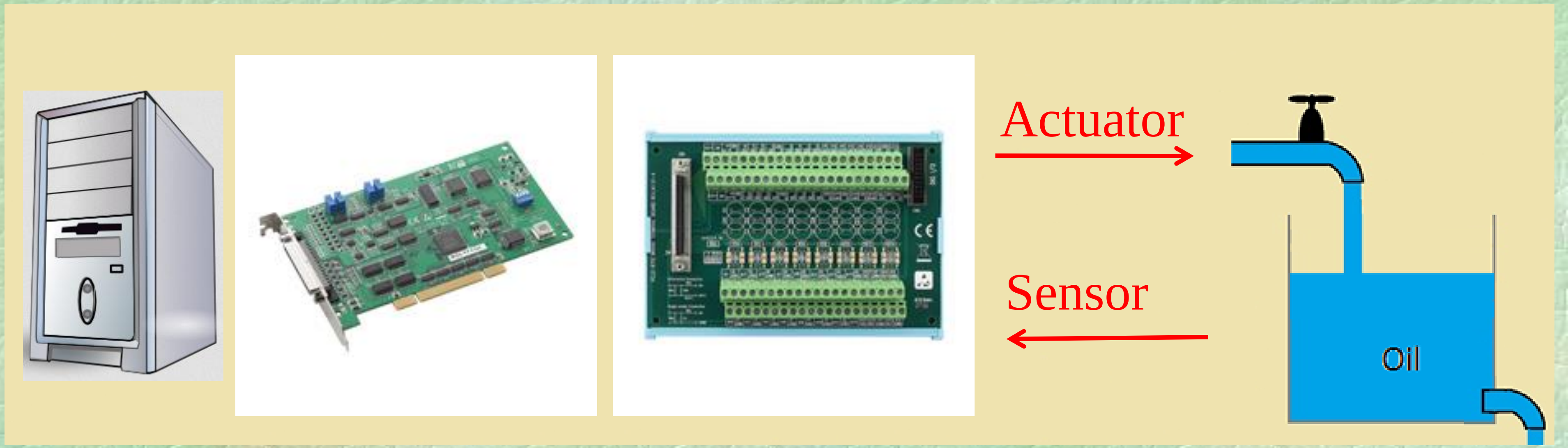
Disturbance

What must one do?

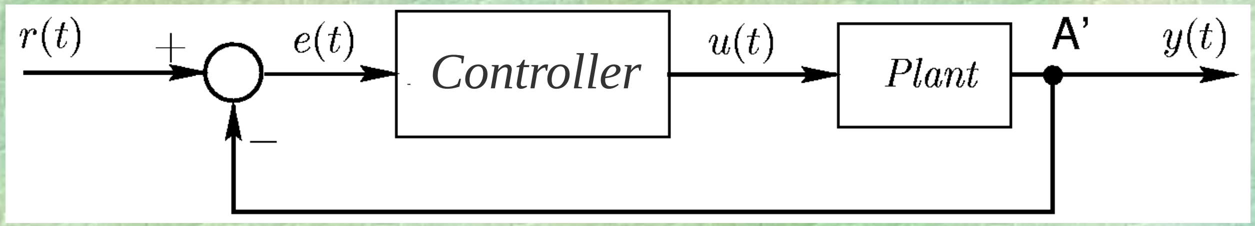




Open loop and closed loop systems



Closed-loop
Control system



Linear model for time delay

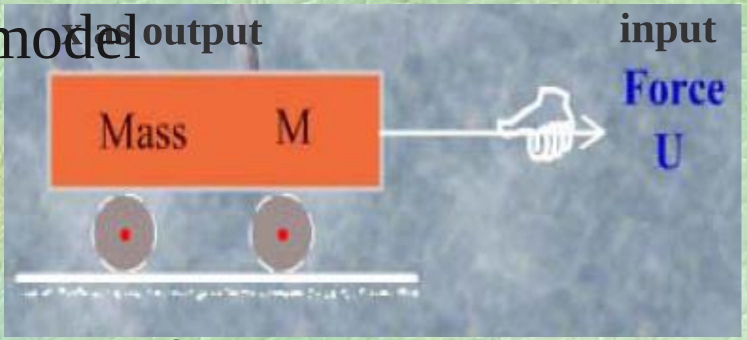
$$e^{-\tau s}=\frac{1}{e^{\tau s}}=\frac{1}{1+\tau s+\frac{\tau^2}{2}s^2+\frac{\tau^3}{6}s^3+........}\cong\frac{1}{1+\tau s+\frac{\tau^2}{2}s^2}$$

$$e^{-\tau s}=\frac{e^{-\frac{\tau}{2}s}}{e^{\frac{\tau}{2}s}}\cong\frac{1-\frac{\tau}{2}s}{1+\frac{\tau}{2}s}$$

1^{st} Order Pade approximation

Exercises

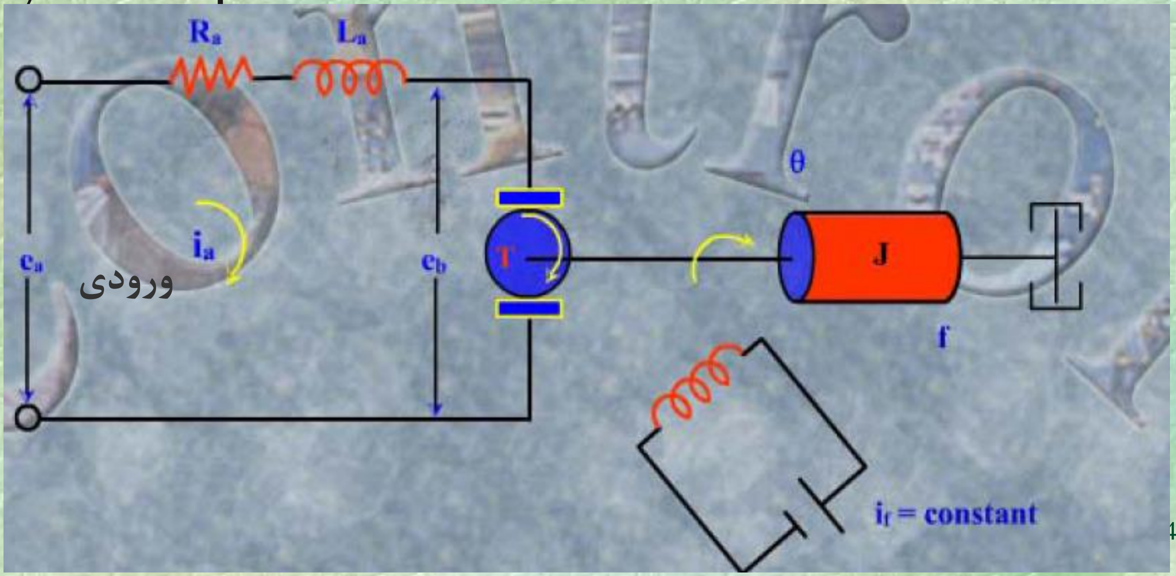
Exercise 22: Find the TF function and SS model and differential equation model for following system.



Exercise 23: Find the TF function and SS model for position control system.

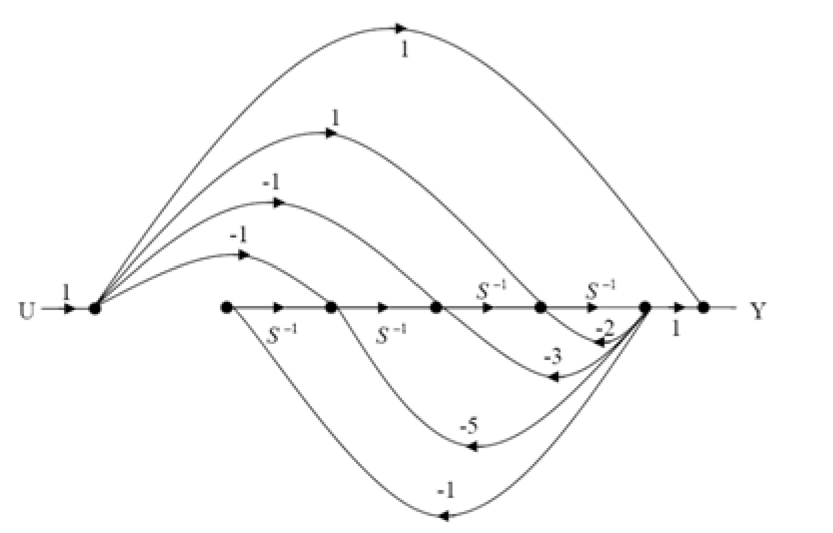
a) Suppose angular position as output.

b) Suppose angular velocity as output.



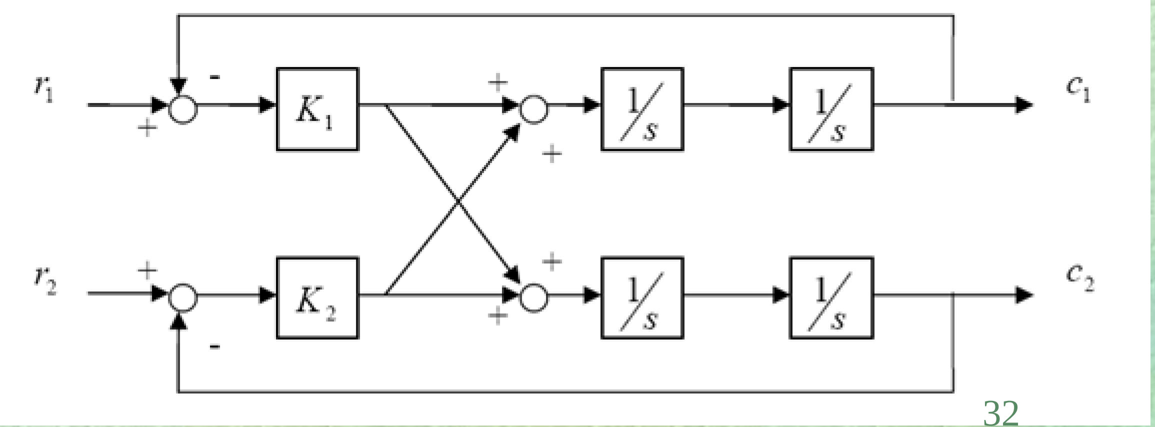
Exercises

Exercise 24: Find Y(s)/U(s) for following system.



$$\text{Answer}: \quad \frac{Y(s)}{U(s)} = \frac{s^4+3s^3+2s^2+4s+1}{s^4+2s^3+3s^2+5s+1}$$

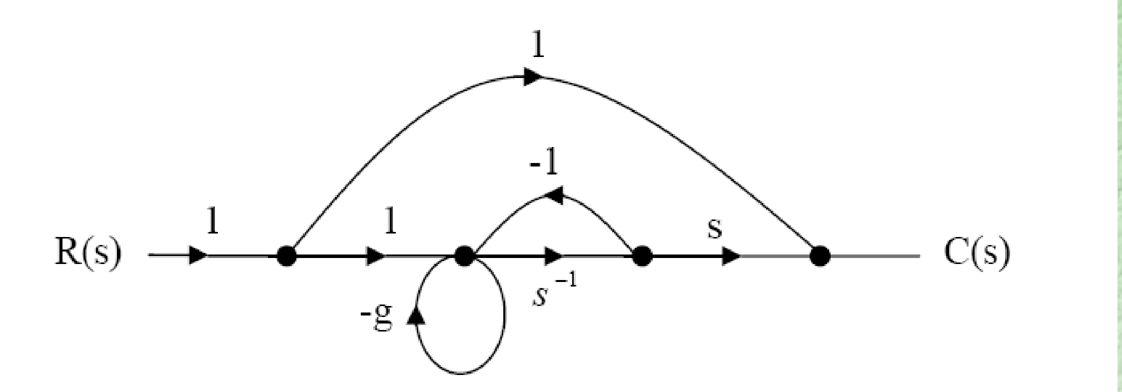
Exercise 25: Find $c_1(s)/r_2(s)$ for following system.



$$\text{Answer}: \frac{c_1}{r_2} = \frac{k_2}{s^2+k_1+k_2}$$

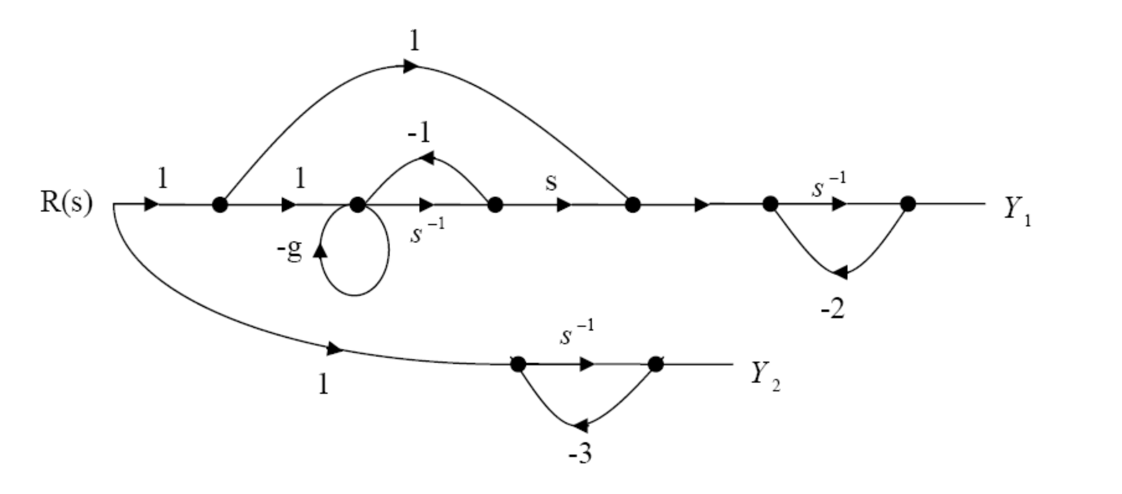
Exercises

Exercise 26: Find g such that C(s)/R(s) for following system be (2s+2)/s+2



Answer $g = \frac{1}{s}$

Exercise 27: Find $Y_1(s)/R(s)$ for following system

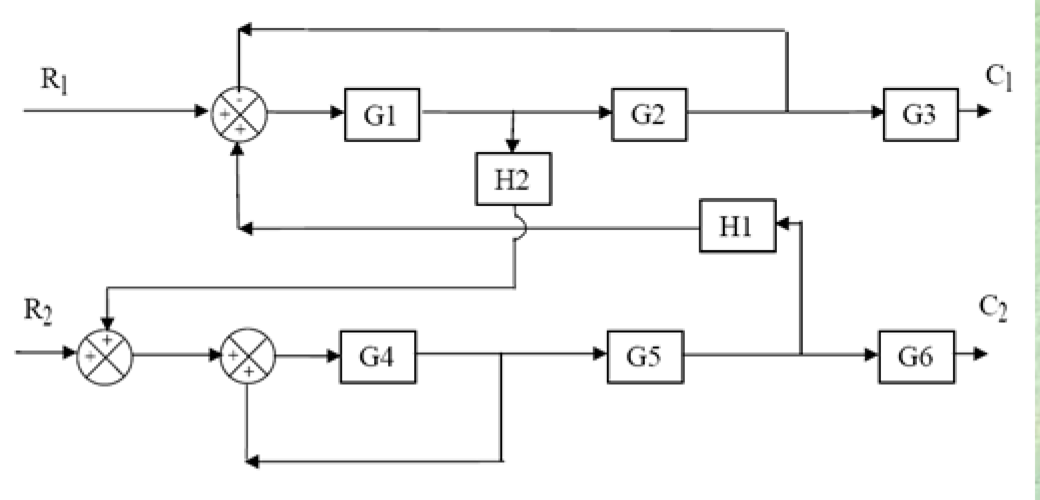


Answer

$$\frac{(g+2)s^2+(3g+7)s+3}{(g+1)s^3+(5g+6)s^2+(6g+11)s+6}$$

Exercises

Exercise 28: Find $C_1(s)/R_1(s)$ and $C_2(s)/R_2(s)$ for following system



Answer

$$\frac{C_2(S)}{R_2(S)} = \frac{P_2\Delta_2}{\Delta} = \frac{G_4G_5G_6(1+G_1G_2)}{1-G_4+G_1G_2-G_1H_2G_4G_5H_1-G_1G_2G_4}$$

$$\frac{C_1(S)}{R_1(S)} = \frac{P_1\Delta_1}{\Delta} = \frac{G_1G_2G_3(1-G_4)}{1-G_4+G_1G_2-G_1H_2G_4G_5H_1-G_1G_2G_4}$$

Exercise 29: Find the SS model for following system (Final Exams).

$$y'' + 5y' + 6y = u' + u$$

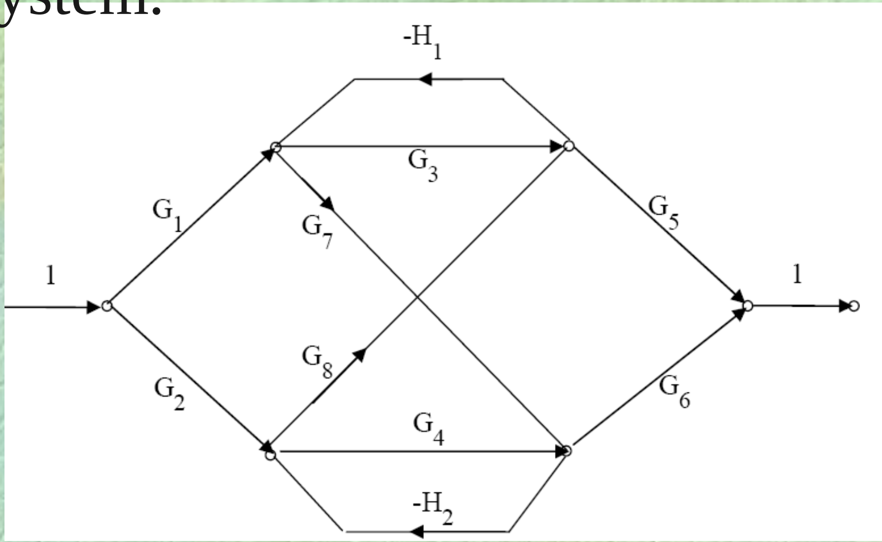
Exercise 30: Find the TF model for following system without any inverse manipulation.

$$\dot{x} = \begin{bmatrix} -1 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$c = [1 \quad 0 \quad 0]x$$

Exercises

Exercise 31: Find C/R for following system.



Answer

$$\frac{C}{R}=\frac{G_1G_3G_5(1+G_4H_2)+G_2G_4G_6(1+G_3H_1)+G_1G_7G_6+G_2G_8G_5-G_1G_7H_2G_8G_5-G_2G_8H_1G_7G_6}{1+G_4H_2+G_3H_1-G_7H_2G_8H_1+G_4H_2G_3H_1}$$

Exercise 32: Find the SS model for following system.

$$g(s)=\frac{s^4+3s^3+2s^2+4s+1}{s^4+2s^3+3s^2+5s+1}$$

Exercises

Exercise 33: Find direct realization, series realization and parallel realization for following system:

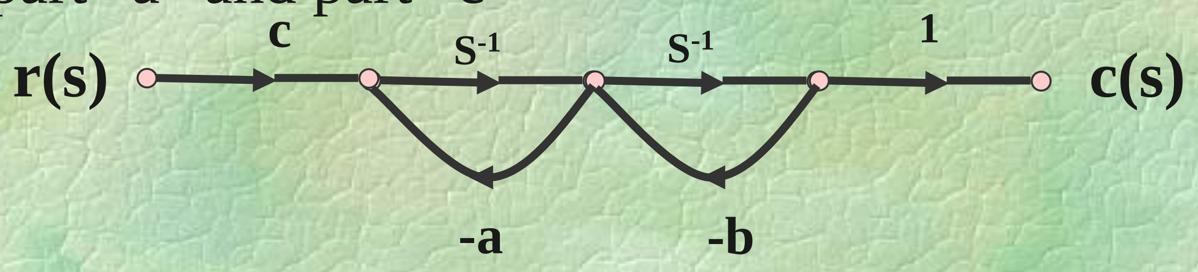
$$G(s)=\frac{s+1}{(s^2+2s+2)(s+3)}$$

Exercise 34: a) Find the transfer function of following system by Masson formula (Final Exam / Olampiad).

b) Find the state space model of system.

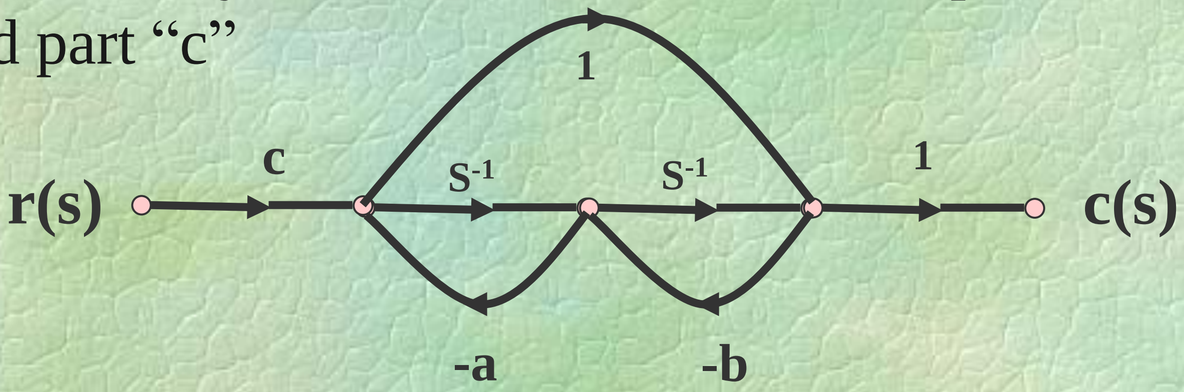
c) Find the transfer function of system from the SS model in part b

d) Compare part "a" and part "c"

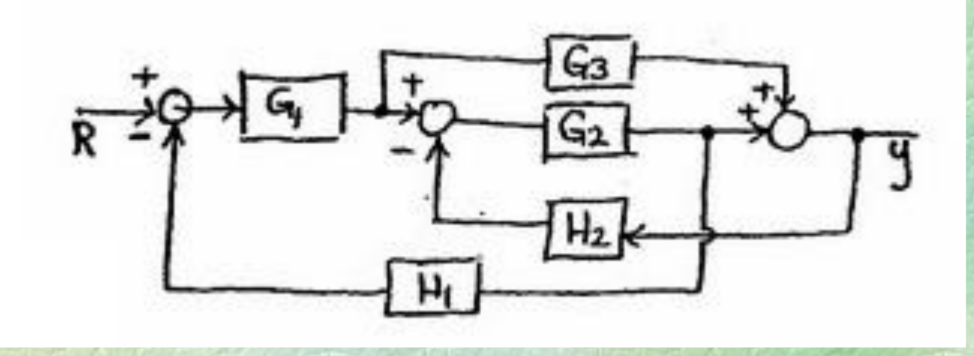


Exercises

Exercise 35: a) Find the transfer function of following system by Masson formula (Olymiad 2008).
b) Find the state space model of system.
c) Find the transfer function of system from the SS model in part b
d) Compare part "a" and part "c"



Exercise 36: Find the characteristic equation of following system. (University entrance exam 1390).

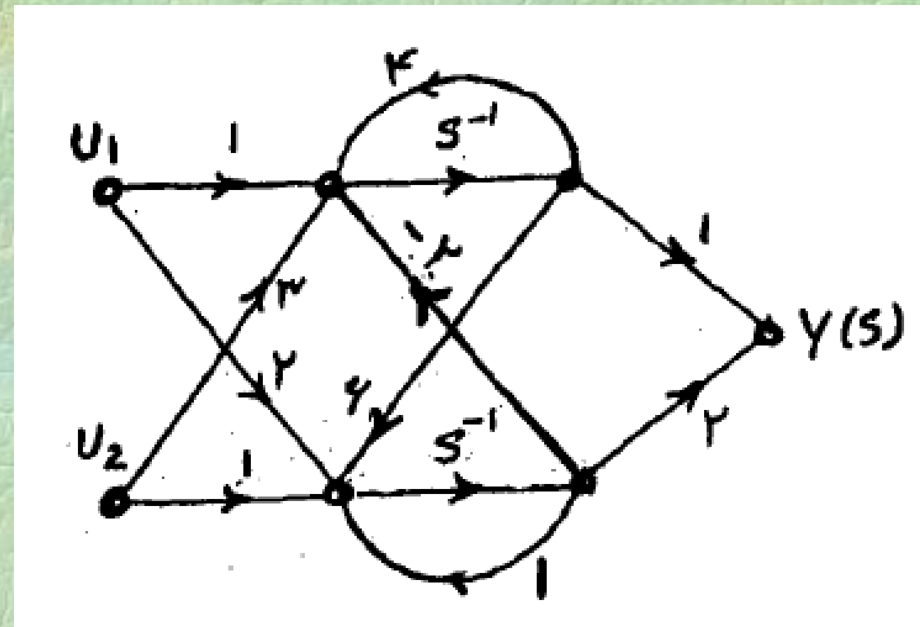


$Answer: 1 + G_2H_2 + G_1G_2H_1 - G_1G_2G_3H_1H_2$

Exercises

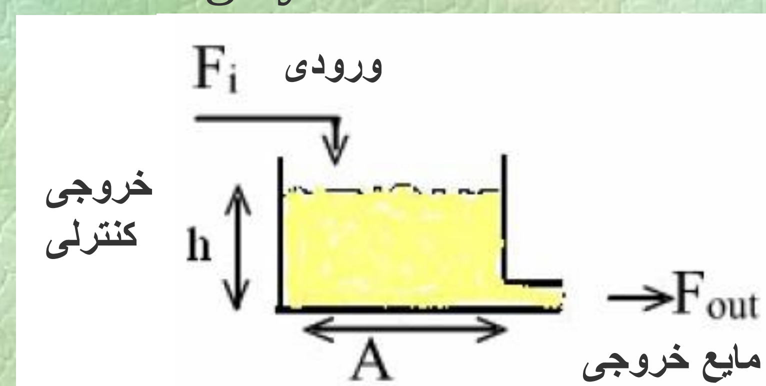
Exercise 37: Find $\left.\dfrac{Y(s)}{U_1(s)}\right|_{U_2(s)=0}$

(University entrance exam 1391).



Exercise 38: Find the linear model of the following system around h=2.

$$3\frac{dh}{dt}+0.5\sqrt{h}=F_i$$



Appendix

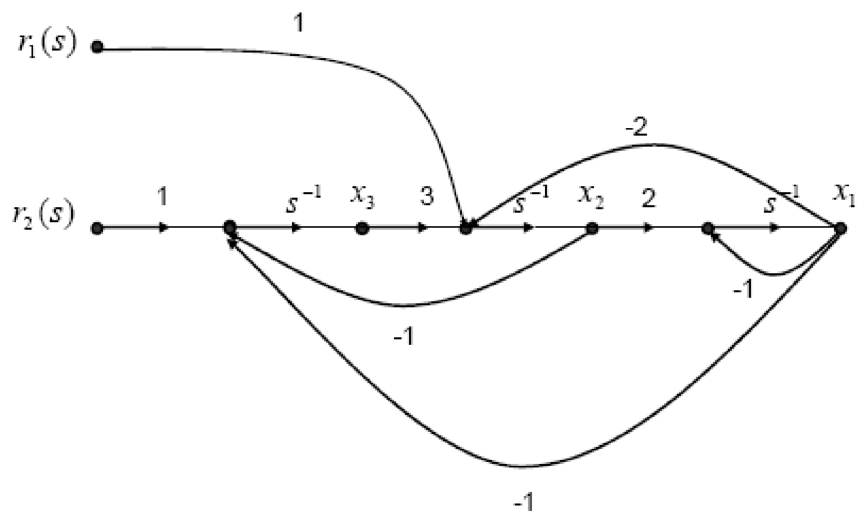
Example 1: Express the following set of differential equations in the form of $\dot{X}(t) = AX(t) + Br(t)$ and draw corresponding state diagram.

$$\dot{x}_1 = -x_1(t) + 2x_2(t)$$
$$\dot{x}_2 = -2x_1(t) + 3x_3(t) + r_1(t)$$
$$\dot{x}_3 = -x_1(t) - x_2(t) + r_2(t)$$

Solution: Clearly

$$\dot{X} = \begin{bmatrix} -1 & 2 & 0 \\ -2 & 0 & 3 \\ -1 & -1 & 0 \end{bmatrix} X + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}$$

State diagram is:

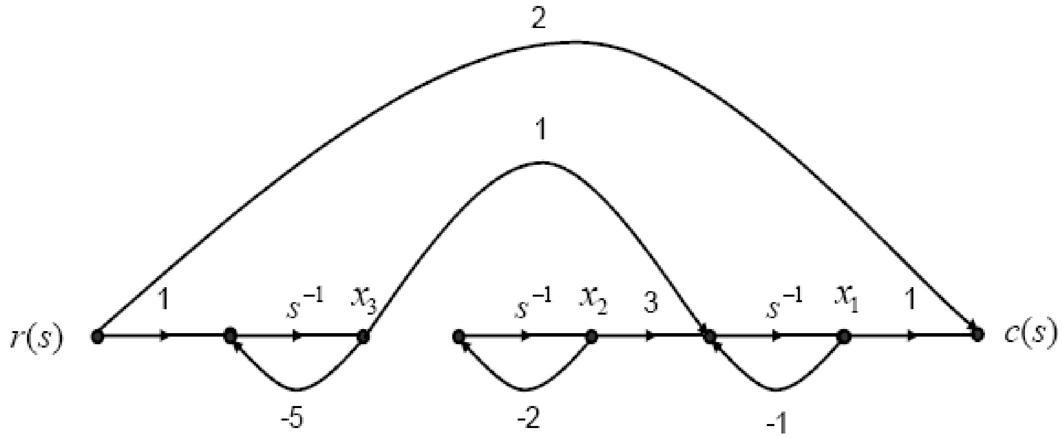


Appendix

Example 2: Determine the transfer function of following system without using any inverse manipulation.

$$\dot{X}(t)=\begin{bmatrix}-1 & 3 & 1\\ 0 & -2 & 0\\ 0 & 0 & -5\end{bmatrix}X(t)+\begin{bmatrix}0\\0\\1\end{bmatrix}r(t) \qquad c(t)=[1 \quad 0 \quad 0]X(t)+2r(t)$$

Solution: Transfer function without inverse manipulation is possible by using state diagram, state diagram of system is:



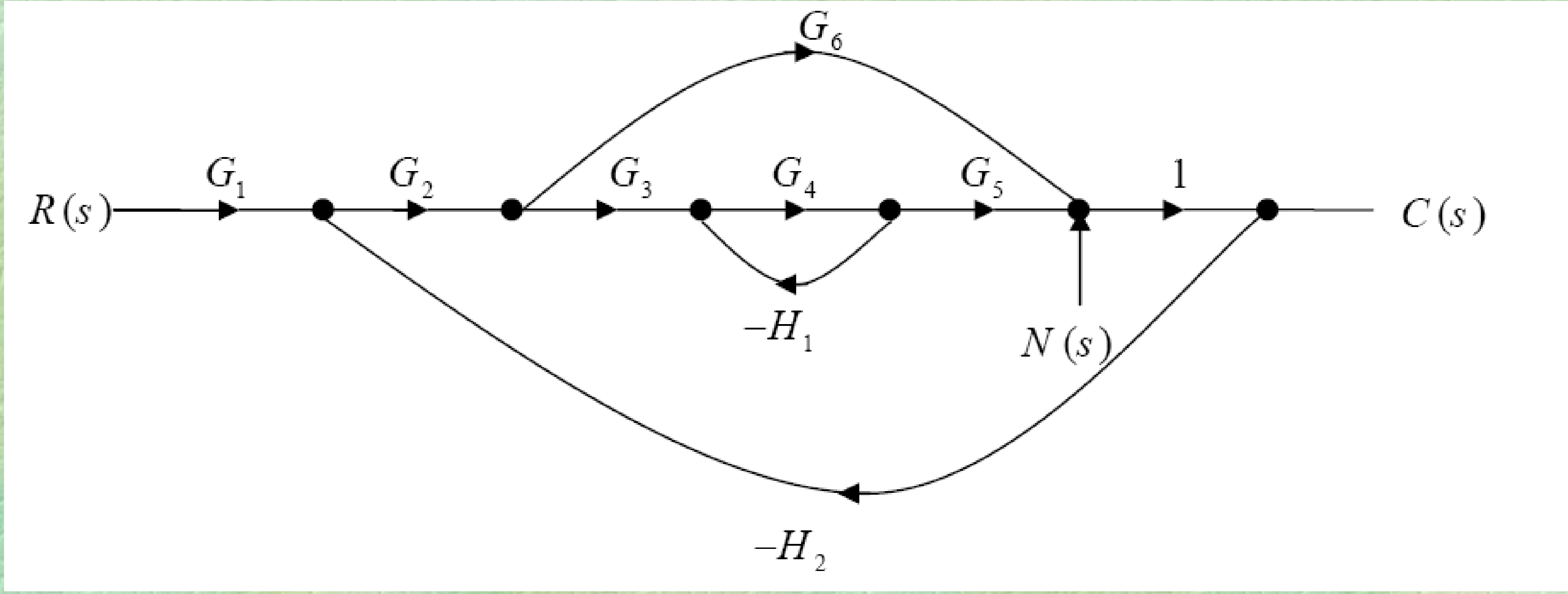
By using general gain formula, transfer function is:

$$\frac{c(s)}{r(s)}=\frac{M_1\Delta_1+M_2\Delta_2}{\Delta}=\frac{2\times\left(1-(-5s^{-1}-2s^{-1}-s^{-1})+(10s^{-2}+5s^{-2}+2s^{-2})-(-10s^{-3})\right)+s^{-2}\times\left(1-(-2s^{-1})\right)}{1-(-5s^{-1}-2s^{-1}-s^{-1})+(10s^{-2}+5s^{-2}+2s^{-2})-(-10s^{-3})}$$

$$=\frac{2s^3+16s^2+35s+22}{s^3+8s^2+17s+10}$$

Appendix

Example 3: Derive C(s)/N(s)



$$m=1$$

$$M_1=1 \quad \Delta_1=1+G_4H_1$$

$$\Delta=1-(-H_1G_4-G_2G_6H_2-G_2G_3G_4G_5H_2)+\left((-G_4H_1)(-G_2G_6H_2)\right)$$

$$=1+H_1G_4+G_2G_6H_2+G_2G_3G_4G_5H_2+G_4H_1G_2G_6H_2$$

$$\frac{C(s)}{N(s)}=\frac{1.(1+H_1G_4)}{1+H_1G_4+G_2G_6H_2+G_2G_3G_4G_5H_2+G_4H_1G_2G_6H_2}$$

Appendix

Example 4: Determine the modal form of this transfer function. One particular useful canonical form is called the Modal Form.

It is a diagonal representation of the state-space model.
Assume for now that the transfer function has distinct real poles p_i (but this easily extends to the case with complex poles.)

$$\begin{aligned} G(s) &= \frac{N(s)}{D(s)} = \frac{N(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \\ &= \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots + \frac{r_n}{s-p_n} \end{aligned}$$

Now define a collection of first order systems, each with state x_i

Appendix

$$\frac{X_1}{U(s)} = \frac{r_1}{s-p_1} \Rightarrow \dot{x}_1 = p_1 x_1 + r_1 u$$
$$\frac{X_2}{U(s)} = \frac{r_2}{s-p_2} \Rightarrow \dot{x}_2 = p_2 x_2 + r_2 u$$
$$\vdots$$
$$\frac{X_n}{U(s)} = \frac{r_n}{s-p_n} \Rightarrow \dot{x}_n = p_n x_n + r_n u$$

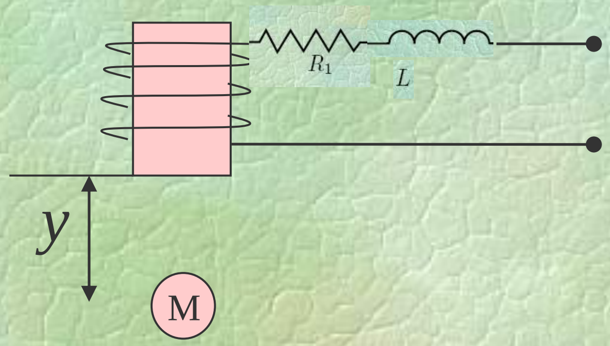
Which can be written as:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + Bu(t) \\ y(t) &= C\mathbf{x}(t) + Du(t) \end{aligned}$$

With :

$$A = \begin{bmatrix} p_1 & & \\ & \ddots & \\ & & p_n \end{bmatrix} \quad B = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \quad C = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}^T$$

Example 5 Suppose electromagnetic force is i^2/y and find linearzed model around $y=y_0$



$$M\frac{d^2y}{dt^2}=Mg-\frac{i^2(t)}{y}$$

$$e(t)=R_1 i+L\frac{di}{dt}$$

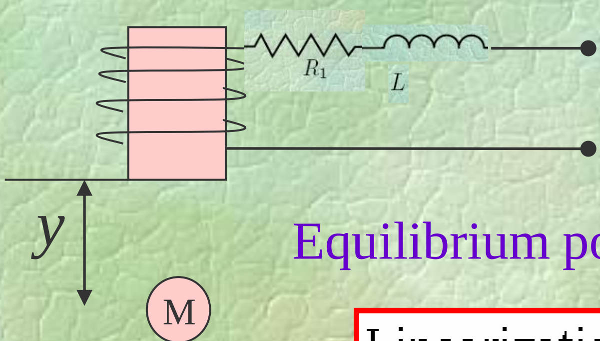
$$x_1=y\,,x_2=\dot{y}\,,x_3=i$$

$$\dot{x}_1=x_2 \qquad \dot{x}_2=g-\frac{1}{M}\frac{{x_3}^2}{x_1} \qquad \dot{x}_3=-\frac{R_1}{L}x_3+\frac{e(t)}{L}$$

Equilibrium point:

$$x_{1Q}=y_0 \qquad x_{2Q}=0 \qquad x_{3Q}=\mathrm{i}_Q=\sqrt{Mgy_0} \qquad e_Q=R_1\sqrt{Mgy_0}$$

Example 5 Suppose electromagnetic force is i^2/y and find linearzed model around $y=y_0$



$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = g - \frac{1}{M}\frac{x_3^{\,2}}{x_1} \qquad \dot{x}_3 = -\frac{R_1}{L}x_3 + \frac{e(t)}{L}$$

Equilibrium point: $\left(x_{1Q}, x_{2Q}, x_{3Q}, e_Q\right) = \left(y_0, 0, \sqrt{Mgy_0}, R_1\sqrt{Mgy_0}\right)$

Linearization procedure

$$\delta\dot{x} = A\,\delta x + B\,\delta u \qquad A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} ; \qquad B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}}$$

$$\delta y = C\,\delta x + D\,\delta u \qquad C = \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} ; \qquad D = \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}}$$

$$\delta\ddot{x}_1 = \delta x_2$$

$$\delta\ddot{x}_2 = \frac{g}{y_0}\delta x_1 - 2\sqrt{\frac{g}{My_0}}\delta x_3$$

$$\delta\ddot{x}_3 = -\frac{R_1}{L}\delta x_3 + \frac{\delta e(t)}{L}$$

$$\begin{bmatrix} \delta\ddot{x}_1 \\ \delta\ddot{x}_2 \\ \delta\ddot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{g}{y_0} & 0 & -2\sqrt{\frac{g}{My_0}} \\ 0 & 0 & -\frac{R_1}{L} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} \delta e(t)$$

Example 6 Consider the following nonlinear system. Suppose $u(t)=0$ and initial condition is $x_{10}=x_{20}=1$. Find the linearized system around response of system.

$$\dot{x}_1(t)=\frac{-1}{x_2(t)^2}$$

$$\dot{x}_2(t)=u(t)x_1(t)$$

$$\dot{x}_2(t)=0.x_1(t)=0 \quad \Rightarrow \quad x_2(t)=a=1$$

$$\dot{x}_1(t)=-1 \quad \Rightarrow \quad x_1(t)=-t+b=-t+1$$

Linearization procedure

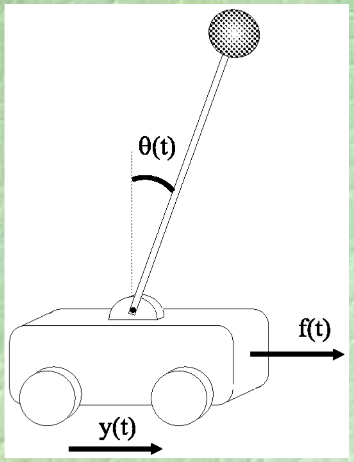
$$\dot{\delta} x = A\,\delta x + B\,\delta u \qquad A=\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_g\\u=u_g}}\ ; \qquad B=\left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_g\\u=u_g}}$$

$$\delta y = C\,\delta x + D\,\delta u \qquad C=\left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_g\\u=u_g}}\ ; \qquad D=\left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_g\\u=u_g}}$$

$$\begin{bmatrix}\delta\dot{x}_1\\ \delta\dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix}+\begin{bmatrix}0\\ 1-t\end{bmatrix}\delta u(t)$$

Appendix

Example 7(Inverted pendulum)



In Figure , we have used the following notation:

$y(t)$	-	distance from some reference point
$\theta(t)$	-	angle of pendulum
M	-	mass of cart
m	-	mass of pendulum (assumed concentrated at tip)
l	-	length of pendulum
$f(t)$	-	forces applied to pendulum

Appendix

Example 7(Inverted pendulum)

Appendix

Application of Newtonian physics to this system leads to the following model:

$$\ddot{y} = \frac{1}{\lambda_m + \sin^2\theta(t)}\left[\frac{f(t)}{m} + \dot{\theta}^2(t)\ell\sin\theta(t) - g\cos\theta(t)\sin\theta(t)\right]$$

$$\ddot{\theta} = \frac{1}{\ell\lambda_m + \sin^2\theta(t)}\left[-\frac{f(t)}{m}\cos\theta(t) + \dot{\theta}^2(t)\ell\sin\theta(t)\cos\theta(t) + (1-\lambda_m)g\sin\theta(t)\right]$$

where $\lambda_m = (M/m)$

This is a linear state space model in which A, B and C are:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{M\ell} & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{M\ell} \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$