ADVANCED CONTROL

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Reference:

Chi-Tsong Chen, “Linear System Theory and Design”, 1999.

I thank my student, Alireza Bemani for his help in correction slides of this lecture.

Lecture 4

State Space Solutions and Realization

Topics to be covered include:

- Introduction.
- Solution of State Equations.
- Equivalent State Equations.
- Realizations.
- Solution of Linear Time-Varying (LTV) Equations.
- Equivalence Time-Varying Equations.
- Time-Varying Realizations.

What you will learn after studying this section

- **Solution of LTI state equations**
- **Equivalent(algebraic) state equations**
- **Zero state equivalent**
- **Realizable state equations**
- **Some different realization**
- **Solution of LTV state equation**
- **Fundamental matrix and stste transition matrix and their properties**
- **State Space Representation for LTV Systems**
- **Realization of LTV Systems**

Introduction

General forms of **state-space** equations:

$$\frac{dx}{dt} = f(x(t), u(t), t)$$
$$y(t) = g(x(t), u(t), t)$$

State-space equation for a **linear time-varying** (LTV) system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

State-space equation for a linear time invariant (LTI) system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

If the initial condition and input are defined, then x(t), y(t) ?

Solution of LTI state equation

First method: $\dot{x}(t) = Ax(t) + Bu(t)$ $\quad x(t)\big|_{t=t_0} = x_{t_0}$

We saw in the previous section: $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$

By multiplying the state equation by e^{-At}, we have:

$$e^{-At}\dot{x}(t) = e^{-At}Ax(t) + e^{-At}Bu(t)$$

$$e^{-At}\dot{x}(t) - e^{-At}Ax(t) = e^{-At}Bu(t)$$

$$\frac{d}{dt}\left(e^{-At}x(t)\right) = e^{-At}Bu(t)$$

Integrating both sides leads to: $e^{-A\tau}x(\tau)\Big|_0^t = \int_0^t e^{-A\tau}Bu(\tau)d\tau$

And finally

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

State transition equation $x(t) = e^{A(t-t_0)}x_{t_0} + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau$

Solution of LTI state equation

Second method: $\dot{x}(t) = Ax(t) + Bu(t)$ $x(t)|_{t=t_0} = x_{t_0}$

By Laplace transform we have:

$$sx(s) - x_0 = Ax(s) + Bu(s)$$

$$x(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}Bu(s)$$

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Convolution integral

State transition equation $x(t) = e^{A(t-t_0)}x_{t_0} + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau$

Solution of LTI state equation

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(t)\big|_{t=t_0} = x_{t_0}$$

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Methods for calculation e^{At}

1- Exponential series: $$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \ldots + \frac{t^n}{n!}A^n + \ldots$$

2- Finding a polynomial of order n−1 that is equivalent to e^{At} with respect to the spectrum of *A*.

$$e^{At} = h(A)$$

3- Using Jordan form of A and…

$$e^{At} = Qe^{\hat{A}t}Q^{-1}$$

4. Using the inverse Laplace transform $$e^{At} = L^{-1}\left((sI - A)^{-1}\right)$$

Solution of LTI state equation

Example 1: Consider the following system.

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Determine $x(t)$

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

First, we need to find e^{At}.

$$e^{At} = L^{-1}\left((sI - A)^{-1}\right) = L^{-1}\left(\begin{bmatrix} s & 1 \\ -1 & s+2 \end{bmatrix}^{-1}\right) \qquad e^{At} = L^{-1}\left(\begin{bmatrix} \frac{s+2}{s^2+2s+1} & \frac{-1}{s^2+2s+1} \\ \frac{1}{s^2+2s+1} & \frac{s}{s^2+2s+1} \end{bmatrix}\right)$$

$$e^{At} = \begin{bmatrix} (1+t)e^{-t} & -te^{-t} \\ te^{-t} & (1-t)e^{-t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} (1+t)e^{-t} & -te^{-t} \\ te^{-t} & (1-t)e^{-t} \end{bmatrix} x(0) + \begin{bmatrix} -\int_0^t (t-\tau)e^{-(t-\tau)} u(\tau) d\tau \\ \int_0^t (1-(t-\tau))e^{-(t-\tau)} u(\tau) d\tau \end{bmatrix}$$

Solution of LTI state equation

Example 2: Drive $x(t)$ for a unit step applied as the input.

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$e^{At} = L^{-1}\left((sI - A)^{-1}\right) = L^{-1}\left(\begin{bmatrix} s+2 & -1 \\ 0 & s+3 \end{bmatrix}^{-1}\right) = L^{-1}\left(\begin{bmatrix} \frac{1}{s+2} & \frac{1}{(s+2)(s+3)} \\ 0 & \frac{1}{s+3} \end{bmatrix}\right)$$

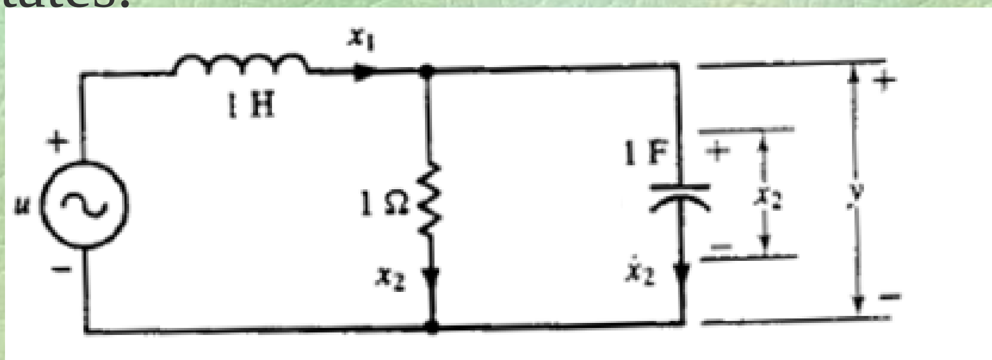
$$e^{At} = \begin{bmatrix} e^{-2t} & e^{-2t} - e^{-3t} \\ 0 & e^{-3t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} e^{-2t} & e^{-2t} - e^{-3t} \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-2(t-\tau)} & e^{-2(t-\tau)} - e^{-3(t-\tau)} \\ 0 & e^{-3(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(\tau) d\tau$$

$$x(t) = \begin{bmatrix} e^{-2t} & e^{-2t} - e^{-3t} \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-2(t-\tau)} - e^{-3(t-\tau)} \\ e^{-3(t-\tau)} \end{bmatrix} d\tau = \begin{bmatrix} \cdots\cdots \\ \cdots\cdots \end{bmatrix}$$

Equivalent state equation

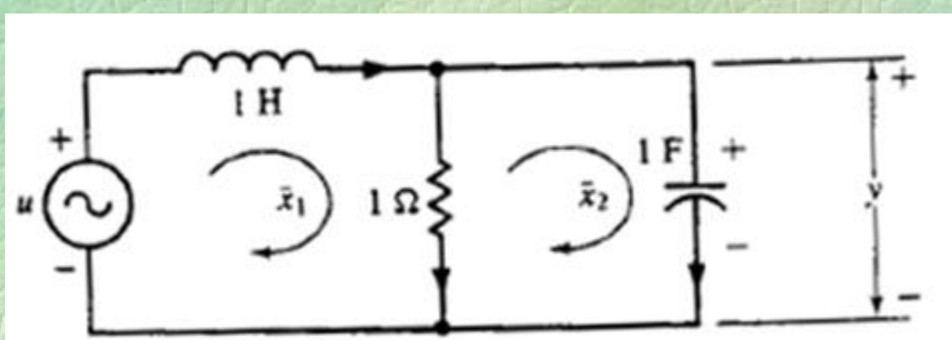
Example 3: a) Derive the state-space equation according to the chosen states.



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}u$$

$$y=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

b) Derive the state-space equation according to the chosen states.



$$\begin{bmatrix}\dot{\bar{x}}_1\\ \dot{\bar{x}}_2\end{bmatrix}=\begin{bmatrix}-1 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix}$$

Equivalent state equation

$$\dot{x} = Ax + bu$$
$$y = cx + du$$

Similarity transformation

$$\underset{w=Px}{\Rightarrow}$$

$$\hat{A} = PAP^{-1} \qquad \hat{b} = Pb$$
$$\hat{c} = cP^{-1} \qquad \hat{d} = d$$

$$\dot{w} = \hat{A}w + \hat{b}u$$
$$y = \hat{c}w + \hat{d}u$$

1- It can lead to a simpler system.

2- It doesn’t change the eigenvalues.

3- Similar transfer function.

4- It doesn’t change observability.

5- It doesn’t change controllability.

Equivalent state equation

Invariance of eigenvalues

$$\dot{x} = Ax + bu$$
$$y = cx + du$$

$$w = Px$$

$$\hat{A} = PAP^{-1} \qquad \hat{b} = Pb$$

$$\hat{c} = cP^{-1} \qquad \hat{d} = d$$

$$\dot{w} = \hat{A}w + \hat{b}u$$
$$y = \hat{c}w + \hat{d}u$$

$$\left|sI - A\right| = 0$$

$$\left|sI - \hat{A}\right| = 0$$

$$\left|sI - \hat{A}\right| = \left|sPP^{-1} - PAP^{-1}\right| = \left|P(sI - A)P^{-1}\right| = \left|P\right|\left|sI - A\right|\left|P^{-1}\right|$$

$$= \left|sI - A\right|$$

Similarity transform doesn't change the eigenvalues

Equivalent state equation

Similar transfer function

$$\dot{x} = Ax + bu$$
$$y = cx + du$$

$$w = Px$$
$$\hat{A} = PAP^{-1} \qquad \hat{b} = Pb$$
$$\hat{c} = cP^{-1} \qquad \hat{d} = d$$

$$\dot{w} = \hat{A}w + \hat{b}u$$
$$y = \hat{c}w + \hat{d}u$$

$$\hat{g}(s) = \hat{c}(sI - \hat{A})^{-1}\hat{b} + \hat{d} = cP^{-1}(sI - PAP^{-1})^{-1}Pb + d$$

$$= cP^{-1}P(sI - A)^{-1}P^{-1}Pb + d = c(sI - A)^{-1}b + d$$

$$= g(s)$$

Similarity transform doesn't change the transfer function

Equivalent state equation

The application of similarity transformations.

- Finding simpler similar systems.

  Canonical form, Jordan form, modal form, ….

- Scaling for better implementation.

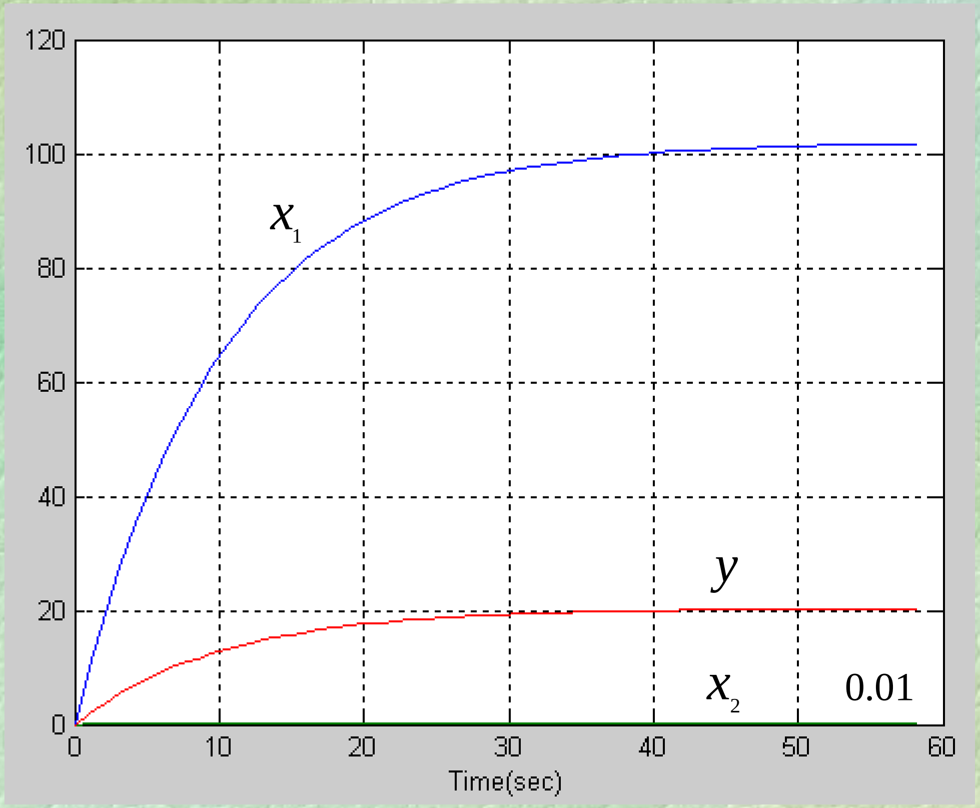
Equivalent state equation

The application of similarity transformations

Example 4: Consider following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 0.1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0.2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$




```
[y,x,t]=step(A,b,c,d);
plot(t,x,t,y)
grid on
xlabel('Time(sec)')
```

Suppose we need states to be within the range of ± 10

$$\hat{x}_1 = ?\, x_1, \quad \hat{x}_2 = ?\, x_2$$

$$\hat{x} = Px = \begin{bmatrix} ? & 0 \\ 0 & ? \end{bmatrix} x$$

Equivalent state equation

The application of similarity transformations in scaling

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-0.1 & 2\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}10\\ 0.1\end{bmatrix}u$$

$$y=\begin{bmatrix}0.2 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

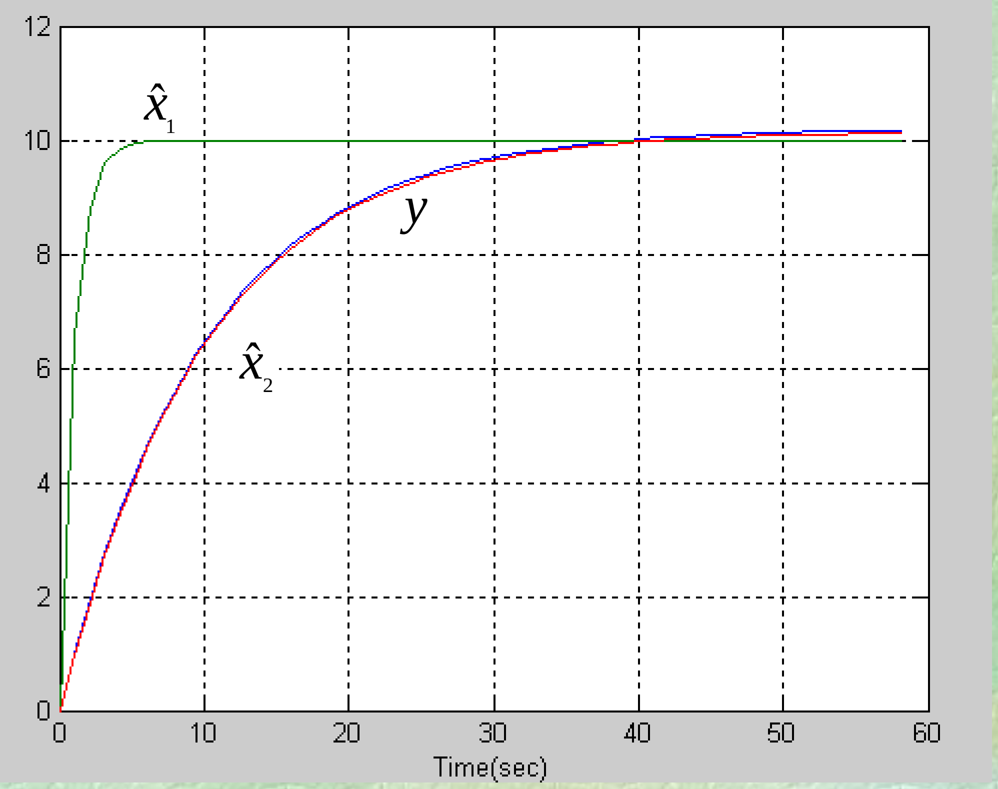
$$\hat{x}=Px$$

$$\hat{A}=PAP^{-1} \qquad \hat{b}=Pb$$

$$\hat{c}=cP^{-1} \qquad \hat{d}=d$$

$$\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix}=\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix}+\begin{bmatrix} \\ \end{bmatrix}u$$

$$y=\begin{bmatrix} & \end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix}$$



All states are in the range within ± 10

Zero state equivalent

Definition 1: Two sets of state-space equations are zero-state equivalent if there transfer functions are similar.

Example 5: a) Are the following state-space equations similar? b) Are they zero-state equivalent?

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x} = -2x + u$$

$$y = x$$

It is clear that they aren’t similar

$$\Downarrow$$

$$g(s) = \frac{1}{s+2}$$

$$\Downarrow$$

$$g(s) = \frac{1}{s+2}$$

So the mentioned state-space equations are zero-state equivalent.

Zero state equivalent

$$\dot{x} = Ax + bu$$
$$y = cx + du$$

$$\dot{w} = \bar{A}w + \bar{b}u$$
$$y = \bar{c}w + \bar{d}u$$

Theorem 1: Two sets of state-space equations are zero-state equivalent if and only if the following relations hold.

$$d = \bar{d}$$

$$cA^m b = \bar{c}\bar{A}^m\bar{b} \qquad m = 0, 1, 2, \ldots.$$

Proof: Since two sets of state-space equations are zero-state equivalent, their transfer functions must be the same:

$$d + c(sI - A)^{-1}b = \bar{d} + \bar{c}(sI - \bar{A})^{-1}\bar{b}$$

With the use of series expansion, we have:

$$d + cbs^{-1} + cAbs^{-2} + cA^2bs^{-3} + \ldots = \bar{d} + \bar{c}\bar{b}s^{-1} + \bar{c}\bar{A}\bar{b}s^{-2} + \bar{c}\bar{A}^2\bar{b}s^{-3} + \ldots$$

Realization

State-space equations

$$\dot{x} = Ax + Bu$$
$$y = Cx + Eu$$

This transformation → **is unique**

Input-output description
(Transfer function)

$$G(s) = C(sI - A)^{-1}B + E$$

Input-output description
(Transfer function)

$$G(s) = C(sI - A)^{-1}B + E$$

Realization → **This transformation is not unique**

State-space equations

$$\dot{x} = Ax + Bu$$
$$y = Cx + Eu$$

Important note: For which types of systems does a state-space description exist?

Realization

Theorem 2: The transfer matrix $G_{q\times p}$(s) is realizable if and only if G(s) is a proper matrix.

Proof: It is evident that to prove the theorem, both sides must be shown.

G(s) is realizable ⟹ G(s) is a proper matrix

G(s) is a proper matrix ⟹ G(s) is realizable

First, we prove the first part.

Since G(s) is realizable, there exists a state-space representation.

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

So, the corresponding transfer function is:

$$G(s) = C(sI - A)^{-1}B + D = C\frac{adj(sI - A)}{|sI - A|}B + D \quad \dots\dots\dots\dots$$

Realization

Theorem 2: The transfer matrix $G_{q\times p}(s)$ is realizable if and only if $G(s)$ is a proper matrix.

G(s) is a proper matrix ⇒ G(s) is realizable

Now, we prove the second part.

G(s) is a proper matrix transfer function, so we have:

$$G(s)=G(\infty)+G_{sp}(s)\ =G(\infty)+\frac{1}{d(s)}\left[N_1 s^{r-1}+N_2 s^{r-2}+\ldots+N_{r-1}s+N_r\right]$$

In above relation: $d(s)=s^r+\alpha_1 s^{r-1}+\ldots+\alpha_{r-1}s+\alpha_r$

Now we claim that the state-space equations of the system are given by:

$$\dot{x}=\begin{bmatrix} -\alpha_1 I_p & -\alpha_2 I_p & \ldots & -\alpha_{r-1} I_p & -\alpha_r I_p \\ I_p & 0_p & \ldots & & 0_p \\ 0_p & I_p & \ldots & 0_p & 0_p \\ \vdots & \vdots & & \vdots & \vdots \\ 0_p & 0_p & \ldots & I_p & 0_p \end{bmatrix} x+\begin{bmatrix} I_p \\ 0_p \\ 0_p \\ \vdots \\ 0_p \end{bmatrix} u$$

$$y=\begin{bmatrix} N_1 & N_2 & \ldots & N_{r-1} & N_r \end{bmatrix} x+G(\infty)u$$

$$C(sI-A)^{-1}B+D=\ldots.=G(s)$$

Realization

Theorem 2: The transfer matrix $G_{q\times p}(s)$ is realizable if and only if $G(s)$ is a proper matrix.

Now we claim that the state-space equations of the system are given by:

$$\dot{x}=\begin{bmatrix}-\alpha_1 I_p & -\alpha_2 I_p & \dots & -\alpha_{r-1} I_p & -\alpha_r I_p\\ I_p & 0_p & \dots & & 0_p\\ 0_p & I_p & \dots & 0_p & 0_p\\ \vdots & \vdots & & \vdots & \vdots\\ 0_p & 0_p & \dots & I_p & 0_p\end{bmatrix}x+\begin{bmatrix}I_p\\0_p\\0_p\\\vdots\\0_p\end{bmatrix}u$$

$$y=\begin{bmatrix}N_1 & N_2 & \dots & N_{r-1} & N_r\end{bmatrix}x+G(\infty)u$$

$$C(sI-A)^{-1}B+D=\dots.=G(s)$$

$$(sI-A)^{-1}B=\begin{bmatrix}Z_1\\Z_2\\\vdots\\Z_r\end{bmatrix}$$

$$B=(sI-A)\begin{bmatrix}Z_1\\Z_2\\\vdots\\Z_r\end{bmatrix}$$

$$sZ_2=Z_1$$
$$sZ_3=Z_2$$
$$sZ_4=Z_3$$
$$sZ_r=Z_{r-1}$$

$$sZ_1=-\alpha_1 Z_1-\alpha_2 Z_2-\dots-\alpha_r Z_r+I_P$$

$$sZ_1=\left(-\alpha_1-\frac{\alpha_2}{s}-\dots-\frac{\alpha_r}{s^{r-1}}\right)Z_1+I_P$$

Realization

Theorem 2: The transfer matrix $G_{q\times p}(s)$ is realizable if and only if $G(s)$ is a proper matrix.

$$\dot{x}=\begin{bmatrix} -\alpha_1 I_p & -\alpha_2 I_p & \dots & -\alpha_{r-1} I_p & -\alpha_r I_p \\ I_p & 0_p & \dots & & 0_p \\ 0_p & I_p & \dots & 0_p & 0_p \\ \vdots & \vdots & & \vdots & \vdots \\ 0_p & 0_p & \dots & I_p & 0_p \end{bmatrix} x + \begin{bmatrix} I_p \\ 0_p \\ 0_p \\ \vdots \\ 0_p \end{bmatrix} u$$

$$y=\begin{bmatrix} N_1 & N_2 & \dots & N_{r-1} & N_r \end{bmatrix} x + G(\infty)u$$

$$C(sI-A)^{-1}B+D=\dots=G(s)$$

$$sZ_2=Z_1 \qquad sZ_3=Z_2 \qquad \dots \qquad sZ_r=Z_{r-1}$$

$$sZ_1=\left(-\alpha_1-\frac{\alpha_2}{s}-\dots-\frac{\alpha_r}{s^{r-1}}\right)Z_1+I_P$$

$$Z_1=\frac{s^{r-1}}{d(s)}I_P \qquad Z_2=\frac{s^{r-2}}{d(s)}I_P \qquad \dots \qquad Z_r=\frac{1}{d(s)}I_P$$

$$C(sI-A)^{-1}B+G(\infty)=C\begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_r \end{bmatrix}+D=\frac{1}{d(s)}\left[N_1 s^{r-1}+N_2 s^{r-2}+\dots+N_r\right]+G(\infty)$$

Realization

Example 6: Find the state-space of following system:

$$G(s)=\begin{bmatrix} \dfrac{4s-10}{2s+1} & \dfrac{3}{s+2} \\ \dfrac{1}{(2s+1)(s+2)} & \dfrac{s+1}{(s+2)^2} \end{bmatrix}$$

$$G(s)=\begin{bmatrix} \dfrac{4s-10}{2s+1} & \dfrac{3}{s+2} \\ \dfrac{1}{(2s+1)(s+2)} & \dfrac{s+1}{(s+2)^2} \end{bmatrix}=\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} \dfrac{-12}{2s+1} & \dfrac{3}{s+2} \\ \dfrac{1}{(2s+1)(s+2)} & \dfrac{s+1}{(s+2)^2} \end{bmatrix}$$

$$G(s)=\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}+\frac{1}{s^3+4.5s^2+6s+2}\begin{bmatrix} -6(s+2)^2 & 3(s+2)(s+0.5) \\ 0.5(s+2) & (s+1)(s+0.5) \end{bmatrix}$$

$$G(s)=\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}+\frac{1}{s^3+4.5s^2+6s+2}\left\{\begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix}s^2+\begin{bmatrix} -24 & 7.5 \\ 0.5 & 1.5 \end{bmatrix}s+\begin{bmatrix} -24 & 3 \\ 1 & 0.5 \end{bmatrix}\right\}$$

Realization

Example 6: Find the state-space of following system:

$$G(s)=\begin{bmatrix} \dfrac{4s-10}{2s+1} & \dfrac{3}{s+2} \\ \dfrac{1}{(2s+1)(s+2)} & \dfrac{s+1}{(s+2)^2} \end{bmatrix}$$

$$G(s)=\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}+\frac{1}{s^3+4.5s^2+6s+2}\left\{\begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix}s^2+\begin{bmatrix} -24 & 7.5 \\ 0.5 & 1.5 \end{bmatrix}s+\begin{bmatrix} -24 & 3 \\ 1 & 0.5 \end{bmatrix}\right\}$$

$$\dot{x}=\left[\begin{array}{cc|cc|cc} -4.5 & 0 & -6 & 0 & -2 & 0 \\ 0 & -4.5 & 0 & -6 & 0 & -2 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]x+\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right]u$$

$$y=\left[\begin{array}{cc|cc|cc} -6 & 3 & -24 & 7.5 & -24 & 3 \\ 0 & 1 & 0.5 & 1.5 & 1 & 0.5 \end{array}\right]x+\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}u$$

Realization

$$y(s) = G(s)u(s) = G(s)\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}$$

$$y(s) = G(s)u(s) = G_{c1}(s)u_1 + G_{c2}(s)u_2 + \cdots + G_{cp}(s)u_p$$

$$y(s) = G(s)u(s) = y_{c1}(s) + y_{c2}(s) + \cdots y_{cp}(s)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = y_{c1} + y_{c2} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} d_1 & d_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Realization

Example 7: Find the state-space of following system:

$$G(s)=\begin{bmatrix}\frac{4s-10}{2s+1} & \frac{3}{s+2}\\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2}\end{bmatrix}$$

Column 1

$$G_{:,1}(s)=\begin{bmatrix}\frac{4s-10}{2s+1}\\ \frac{1}{(2s+1)(s+2)}\end{bmatrix}$$

$$G_{:,1}(s)=\begin{bmatrix}2\\0\end{bmatrix}+\frac{1}{s^2+2.5s+1}\begin{bmatrix}-6(s+2)\\0.5\end{bmatrix}$$

$$G_{:,1}(s)=\begin{bmatrix}2\\0\end{bmatrix}+\frac{1}{s^2+2.5s+1}\left\{\begin{bmatrix}-6\\0\end{bmatrix}s+\begin{bmatrix}-12\\0.5\end{bmatrix}\right\}$$

$$\dot{x}_1=\begin{bmatrix}-2.5 & -1\\ 1 & 0\end{bmatrix}x_1+\begin{bmatrix}1\\0\end{bmatrix}u$$

$$y_{c1}=\begin{bmatrix}-6 & -12\\ 0 & 0.5\end{bmatrix}x_1+\begin{bmatrix}2\\0\end{bmatrix}u_1$$

Column 2

$$G_{:,2}(s)=\begin{bmatrix}\frac{3}{s+2}\\ \frac{s+1}{(s+2)^2}\end{bmatrix}$$

$$G_{:,2}(s)=\begin{bmatrix}0\\0\end{bmatrix}+\frac{1}{s^2+4s+4}\begin{bmatrix}3(s+2)\\ s+1\end{bmatrix}$$

$$G_{:,2}(s)=\begin{bmatrix}0\\0\end{bmatrix}+\frac{1}{s^2+4s+4}\left\{\begin{bmatrix}3\\1\end{bmatrix}s+\begin{bmatrix}6\\1\end{bmatrix}\right\}$$

$$\dot{x}_2=\begin{bmatrix}-4 & -4\\ 1 & 0\end{bmatrix}x_2+\begin{bmatrix}1\\0\end{bmatrix}u_2$$

$$y_{c2}=\begin{bmatrix}3 & 6\\ 1 & 1\end{bmatrix}x_2+\begin{bmatrix}0\\0\end{bmatrix}u_2$$

Realization

Example 7: Find the state-space of following system:

$$G(s)=\begin{bmatrix} \dfrac{4s-10}{2s+1} & \dfrac{3}{s+2} \\ \dfrac{1}{(2s+1)(s+2)} & \dfrac{s+1}{(s+2)^2} \end{bmatrix}$$

Column 1

$$\dot{x}_1=\begin{bmatrix} -2.5 & -1 \\ 1 & 0 \end{bmatrix}x_1+\begin{bmatrix} 1 \\ 0 \end{bmatrix}u$$

$$y_{c1}=\begin{bmatrix} -6 & -12 \\ 0 & 0.5 \end{bmatrix}x_1+\begin{bmatrix} 2 \\ 0 \end{bmatrix}u_1$$

Column 2

$$\dot{x}_2=\begin{bmatrix} -4 & -4 \\ 1 & 0 \end{bmatrix}x_2+\begin{bmatrix} 1 \\ 0 \end{bmatrix}u_2$$

$$y_{c2}=\begin{bmatrix} 3 & 6 \\ 1 & 1 \end{bmatrix}x_2+\begin{bmatrix} 0 \\ 0 \end{bmatrix}u_2$$

$$\dot{x}=\begin{bmatrix} -2.5 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix}x+\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y=\begin{bmatrix} -6 & -12 & 3 & 6 \\ 0 & 0.5 & 1 & 1 \end{bmatrix}x+\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Solution of LTV state equation

In this section, we aim to solve an LTV system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

First, we solve the homogeneous part.

$$\dot{x}(t) = A(t)x(t)$$

Definition 2(Fundamental matrix): Let A be an $n \times n$ matrix. Consider n linearly independent initial conditions $x_1(t_0)$, $x_2(t_0)$, … , $x_n(t_0)$ and their corresponding responses $x_1(t)$, $x_2(t)$, … , $x_n(t)$. The fundamental matrix is then defined as:

$$X(t) = \begin{bmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \end{bmatrix}$$

Solution of LTV state equation

Example 8: Find the fundamental matrix of following system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} x(t)$$

Consider following initial condition:

$$\mathbf{x}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{x}_1(t) = \begin{bmatrix} 1 \\ 0.5t^2 \end{bmatrix}$$

$$\mathbf{x}_2(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \mathbf{x}_2(t) = \begin{bmatrix} 1 \\ 0.5t^2 + 2 \end{bmatrix}$$

$$\Rightarrow X(t) = \begin{bmatrix} 1 & 1 \\ 0.5t^2 & 0.5t^2 + 2 \end{bmatrix}$$

Now let following initial condition:

$$\mathbf{x}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{x}_1(t) = \begin{bmatrix} 1 \\ 0.5t^2 \end{bmatrix}$$

$$\mathbf{x}_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{x}_2(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow X(t) = \begin{bmatrix} 1 & 0 \\ 0.5t^2 & 1 \end{bmatrix}$$

Note: Fundamental matrix is …..

Solution of LTV state equation

Note: The fundamental matrix $X(t)$ satisfies the following homogeneous equation.

$$\dot{x}(t) = A(t)x(t)$$

So we have:

$$\dot{X}(t) = A(t)X(t)$$

Lemma1: The fundamental matrix $X(t)$ is non-singular for all times.

Solution of LTV state equation

Definition 3: (State Transition Matrix):

Let $X(t)$ be any fundamental matrices of the following homogenous system:

$$\dot{x}(t) = A(t)x(t)$$

So, the **state transition matrix** defined as:

$$\Phi(t, t_0) = X(t)X^{-1}(t_0)$$

The state transition matrix is the unique solution of the following equation:

$$\frac{\partial}{\partial t}\Phi(t, t_0) = A(t)\Phi(t, t_0)$$
$$\Phi(t_0, t_0) = I$$

Solution of LTV state equation

Example 9: Derive the state transition matrix for the following system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} x(t)$$

According to previous example the fundamental matrix ot the system is:

$$X(t) = \begin{bmatrix} 1 & 1 \\ 0.5t^2 & 0.5t^2 + 2 \end{bmatrix}$$

$$\Phi(t,t_0) = X(t)X^{-1}(t_0) = \begin{bmatrix} 1 & 1 \\ 0.5t^2 & 0.5t^2 + 2 \end{bmatrix} \begin{bmatrix} 0.25t_0^2 + 1 & -0.5 \\ -0.25t_0^2 & 0.5 \end{bmatrix}$$

$$\Phi(t,t_0) = \begin{bmatrix} 1 & 0 \\ 0.5(t^2 - t_0^2) & 1 \end{bmatrix}$$

And another fundamental matrix is:

$$X(t) = \begin{bmatrix} 1 & 0 \\ 0.5t^2 & 1 \end{bmatrix}$$

$$\Phi(t,t_0) = X(t)X^{-1}(t_0) = \begin{bmatrix} 1 & 0 \\ 0.5t^2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.5t_0^2 & 1 \end{bmatrix}$$

$$\Phi(t,t_0) = \begin{bmatrix} 1 & 0 \\ 0.5(t^2 - t_0^2) & 1 \end{bmatrix}$$

Note: The state transition matrix …………….

Property of state transition matrix

1- $\Phi(t,t) = I$

2- $\Phi^{-1}(t,t_0)= \Phi(t_0,t)$

3- $\Phi(t_2,t_1)\Phi(t_1,t_0) = \Phi(t_2,t_0)$

Solution of LTV state equation

In this section, we aimed to solve an LTV system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

We propose that the solution to the above system is:

$$x(t) = \Phi(t,t_0)x_{t_0} + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau$$
$$= \Phi(t,t_0)\left(x_{t_0} + \int_{t_0}^{t} \Phi(t_0,\tau)B(\tau)u(\tau)d\tau\right)$$

To prove our proposal, we must show it meets the initial condition:

$$x(t_0) = \Phi(t_0,t_0)x_{t_0} + \int_{t_0}^{t_0} \Phi(t,\tau)B(\tau)u(\tau)d\tau = x_{t_0}$$

It must also satisfy the mentioned equation:

$$\dot{x}(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = A(t)x(t) + B(t)u(t)$$

So, the output is:

$$y(t) = C(t)\Phi(t,t_0)x_{t_0} + C(t)\int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t)$$

Solution of LTV state equation

In this section, we aimed to solve an LTV system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

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$$= \Phi(t,t_0)\left(x_{t_0} + \int_{t_0}^{t} \Phi(t_0,\tau)B(\tau)u(\tau)d\tau\right)$$

So, the output is:

$$y(t) = C(t)\Phi(t,t_0)x_{t_0} + C(t)\int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t)$$

Zero-input response is:

$$y(t) = C(t)\Phi(t,t_0)x_{t_0} \qquad x(t) = \Phi(t,t_0)x_{t_0}$$

Zero-state response is:

$$y(t) = C(t)\int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t)$$

$$y(t) = \int_{t_0}^{t} \left(C(t)\Phi(t,\tau)B(\tau) + D(t)\delta(t-\tau)\right)u(\tau)d\tau$$

Solution of LTV state equation

In this section, we aimed to solve an LTV system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

We propose that the solution to the above system is:

$$x(t) = \Phi(t,t_0)x_{t_0} + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau$$
$$= \Phi(t,t_0)\left(x_{t_0} + \int_{t_0}^{t} \Phi(t_0,\tau)B(\tau)u(\tau)d\tau\right)$$

So, the output is:

$$y(t) = C(t)\Phi(t,t_0)x_{t_0} + C(t)\int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t)$$

Zero-state response is:

$$y(t) = \int_{t_0}^{t} \left(C(t)\Phi(t,\tau)B(\tau) + D(t)\delta(t-\tau)\right)u(\tau)d\tau$$

We previously saw:

$$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$$

$$G(t,\tau) = C(t)\Phi(t,\tau)B(\tau) + D(t)\delta(t-\tau) = C(t)X(t)X^{-1}(\tau)B(\tau) + D(t)\delta(t-\tau)$$

Equivalent state equation for LTV systems

$$\dot{x} = Ax + bu$$
$$y = cx + du$$

Similarity transformation

$$\underset{w=Px}{\Longrightarrow}$$

$$\hat{A} = PAP^{-1} \qquad \hat{b} = Pb$$
$$\hat{c} = cP^{-1} \qquad \hat{d} = d$$

$$\dot{w} = \hat{A}w + \hat{b}u$$
$$y = \hat{c}w + \hat{d}u$$

Key properties: Similar eigenvalues and transfer functions.

But in LTV system:

$$\dot{x} = A(t)x + b(t)u$$
$$y = c(t)x + d(t)u$$

Similarity LTV transformation

$$\underset{w=P(t)x}{\Longrightarrow}$$

$$\dot{w} = \hat{A}(t)w + \hat{b}(t)u$$
$$y = \hat{c}(t)w + \hat{d}(t)u$$

$$\hat{A}(t) = ...$$

$$\hat{b}(t) = P(t)b,\ \hat{c}(t) = cP^{-1}(t),\ \hat{d} = d$$

Key properties: Similar impulse response.

Equivalent state equation for LTV systems

$$\dot{x} = A(t)x + b(t)u$$
$$y = c(t)x + d(t)u$$

Similarity LTV transformation $\Longrightarrow$ w=P(t)x

$$\dot{w} = \hat{A}(t)w + \hat{b}(t)u$$
$$y = \hat{c}(t)w + \hat{d}(t)u$$

Theorem 3: Let A_0 be an arbitrary constant matrix. There exists a transformation matrix $P(t)$ such that:

$$\hat{A}(t) = A_0$$

Proof:

$$\dot{x} = A(t)x + b(t)u$$
$$y = c(t)x + d(t)u$$

Similarity LTV transformation $\Longrightarrow$ w=P(t)x

$$\dot{w} = A_0 w + \hat{b}(t)u$$
$$y = \hat{c}(t)w + \hat{d}(t)u$$

$\downarrow$ Fundumental matrix

$$X(t)$$

$\downarrow$ Fundumental matrix

$$W(t) = e^{A_0 t}$$

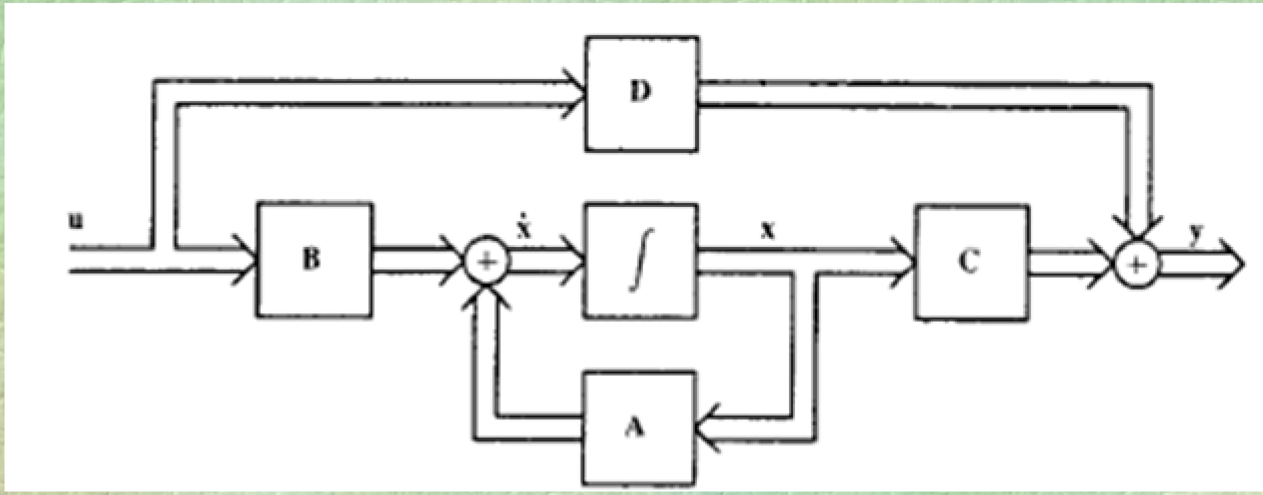
$$W(t) = e^{A_0 t} = P(t)X(t)$$

$$P(t) = e^{A_0 t} X^{-1}(t)$$

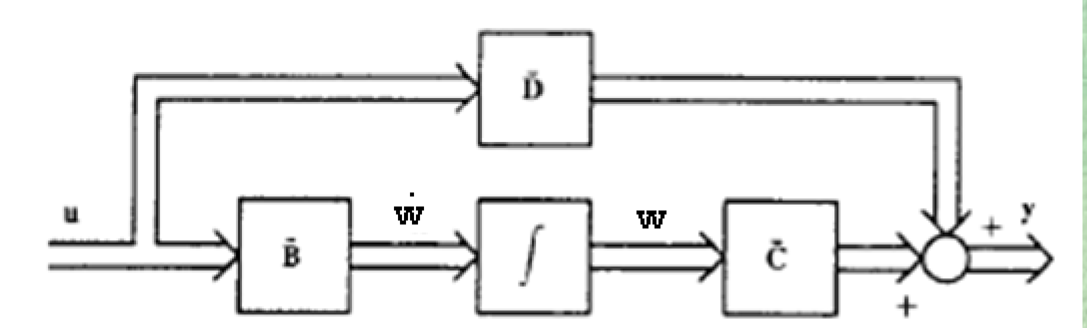
$$\hat{A}(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = A_0$$

Equivalent state equation for LTV systems

In the special case of the previous theorem where $A_0=0$, we have:



$$W = P(t)x \qquad \Downarrow$$



Equivalent state equation for LTV systems

$$\dot{x} = A(t)x + b(t)u$$
$$y = c(t)x + d(t)u$$

Similarity LTV transformation

$$\Longrightarrow$$

w=P(t)x

$$\dot{w} = \hat{A}(t)w + \hat{b}(t)u$$
$$y = \hat{c}(t)w + \hat{d}(t)u$$

Definition 4: A matrix $P(t)$ is called a Lyapunov transformation if:

1- $P(t)$ is nonsingular.

2- $P(t)$ and $P'(t)$ are continuos.

3- $P(t)$ and $P^{-1}(t)$ are bounded for all t.

Equivalent state equation for LTV systems

Example 10: For following system find a similarity transformation such that:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$\hat{A}(t)=A_0=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Show the new system. Is it a Lyapunov transformation?

The desired similarity transformation is: $P(t)=e^{A_0 t}X^{-1}(t)$

$$P(t)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}e^{-2t} & 0\\ 0 & e^{-t}\end{bmatrix}^{-1}=\begin{bmatrix}e^{2t} & 0\\ 0 & e^{t}\end{bmatrix}$$

Is it not a Lyapunov transformation.

$$\hat{A}(t)=\left(P(t)A+\dot{P}(t)\right)P^{-1}(t)=\left(\begin{bmatrix}e^{2t} & 0\\ 0 & e^{t}\end{bmatrix}\begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}+\begin{bmatrix}2e^{2t} & 0\\ 0 & e^{t}\end{bmatrix}\right)P^{-1}(t)=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\hat{b}(t)=P(t)b(t)=\begin{bmatrix}e^{2t} & 0\\ 0 & e^{t}\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}e^{2t}\\ 0\end{bmatrix}$$

$$\hat{c}(t)=c(t)P^{-1}(t)=\begin{bmatrix}e^{-2t} & e^{-t}\end{bmatrix}\qquad \hat{d}(t)=0$$

Realization for LTV systems

State-space equation

$$\dot{x} = A(t)x + B(t)u$$
$$y = C(t)x + D(t)u$$

This transformation → **is unique**

Impulse response

$$G(t,\tau) = C(t)X(t)X^{-1}(\tau)B(\tau) + D(t)\delta(t-\tau)$$

Impulse response

$$G(t,\tau) = C(t)X(t)X^{-1}(\tau)B(\tau) + D(t)\delta(t-\tau)$$

Realization → **This transformation is not unique**

State-space equation

$$\dot{x} = A(t)x + B(t)u$$
$$y = C(t)x + D(t)u$$

Important note: For which types of systems does a state-space description exist?

Realization

Theorem 4: The impulse response matrix $G_{q\times p}(t,\tau)$ is realizable if and only if $G(t,\tau)$ can be decomposed as follows:.

Proof: It is evident that to prove the theorem, both sides must be shown.

impulse response matrix G(t,τ) is realizable ⟹ $G(t,\tau) = M(t)N(\tau) + D(t)\delta(t-\tau)$

$G(t,\tau) = M(t)N(\tau) + D(t)\delta(t-\tau)$ ⟹ impulse response matrix G(t,τ) is realizable

First, we prove the first part.

Since G(t,τ) is realizable, there exists a state-space representation.

$$\dot{x} = A(t)x + B(t)u$$
$$y = C(t)x + D(t)u$$

So, the corresponding impulse response matrix is:

$$G(t,\tau) = C(t)\Phi(t,\tau)B(\tau) + D(t)\delta(t-\tau) = \ldots\ldots\ldots = M(t)N(\tau) + D(t)\delta(t-\tau)$$

Realization

Theorem 4: The impulse response matrix $G_{q\times p}(t,\tau)$ is realizable if and only if $G(t,\tau)$ can be decomposed as follows:.

Proof: It is evident that to prove the theorem, both sides must be shown.

$$G(t,\tau)=M(t)N(\tau)+D(t)\delta(t-\tau)$$ ⇒ impulse response matrix G(t,τ) is realizable

Now, we prove the second part.

We propose that the state-space equation is:

$$\dot{x}=\begin{bmatrix}0 & \cdots & 0\\ \vdots & & \vdots\\ 0 & \cdots & 0\end{bmatrix}x+N(t)u$$

$$y=M(t)x+D(t)u$$

$$\Rightarrow G(t,\tau)=C(t)\Phi(t,\tau)B(\tau)+D(t)\delta(t-\tau)$$

$$=M(t)IN(\tau)+D(t)\delta(t-\tau)$$

$$=M(t)N(\tau)+D(t)\delta(t-\tau)$$

Realization for LTV systems

Example 11: Consider the impulse response of an LTI system given by $g(t)=te^{\lambda t}$. If possible, find one LTI realization and one LTV realization.

LTI realization: $g(t)=te^{\lambda t}$

$$g(s)=\frac{1}{(s-\lambda)^2}=\frac{1}{s^2-2\lambda s+\lambda^2}$$

$$\dot{x}=\begin{bmatrix}2\lambda & -\lambda^2\\ 1 & 0\end{bmatrix}x+\begin{bmatrix}1\\0\end{bmatrix}u$$

$$y=\begin{bmatrix}0 & 1\end{bmatrix}x$$

LTV realization:

$$g(t-\tau)=(t-\tau)e^{\lambda(t-\tau)}$$

$$g(t-\tau)=\begin{bmatrix}e^{\lambda t} & te^{\lambda t}\end{bmatrix}\begin{bmatrix}-\tau e^{-\lambda\tau}\\ e^{-\lambda\tau}\end{bmatrix}$$

$$\dot{x}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}x+\begin{bmatrix}-te^{-\lambda t}\\ e^{-\lambda t}\end{bmatrix}u$$

$$y=\begin{bmatrix}e^{\lambda t} & te^{\lambda t}\end{bmatrix}x$$

Exercises

Exercise 1: Derive the response of system to initial condition $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$

Exercise 2: Derive the response to unit step. (zero initial condition)

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 3 \end{bmatrix} x$$

Exercise 3: Derive the Jordan and modal canonical forms for following system.

$$\dot{x} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} x$$

Exercises

Exercise 4: Find a similarity transformation such that the range of state variables is the same as the output for the given system. If a step input with amplitude *a* is applied, adjust *a* such that all states and the output remain within the range of ±10.

$$\dot{x} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} x$$

Exercise 5: Consider the following systems. Are they similar? Are they zero-state equivalent?

$$\dot{x} = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \qquad \dot{x} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} x \qquad y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} x$$

Exercise 6: Find a realization for the Given system.

$$G(s) = \begin{bmatrix} \dfrac{2}{s+1} & \dfrac{2s-3}{(s+1)(s+2)} \\ \dfrac{s-2}{s+1} & \dfrac{s}{s+2} \end{bmatrix}$$

Exercises

Exercise 7: Find the realization by determining a realization for each column and then augmenting them.

$$G(s)=\begin{bmatrix} \dfrac{2}{s+1} & \dfrac{2s-3}{(s+1)(s+2)} \\ \dfrac{s-2}{s+1} & \dfrac{s}{s+2} \end{bmatrix}$$

Exercise 8: Derive the fundamental matrix and the state transition matrix for the given systems.

$$\dot{x}=\begin{bmatrix} 0 & 1 \\ 0 & t \end{bmatrix}x \qquad \dot{x}=\begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix}x \qquad \dot{x}=\begin{bmatrix} -\sin t & 0 \\ 0 & -\cos t \end{bmatrix}x$$

Exercise 9: Derive an LTI realization for the given systems.

$$\dot{x}=\begin{bmatrix} -\sin t & 0 \\ 0 & -\cos t \end{bmatrix}x$$

Exercises

Exercise 10: Derive an LTI realization and an LTV realization if possible.

$$g(t) = t^2 e^{\lambda t}$$

Exercise 11: Derive an LTI realization and an LTV realization if possible.

$$g(t,\tau) = \sin t (e^{-(t-\tau)}) \cos\tau$$

Exercise 12: For the matrix $A = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}$ show that:

$$\det \Phi(t,t_0) = \exp\left[\int_{t_0}^{t} \left(a_{11}(\tau) + a_{22}(\tau)\right) d\tau\right]$$

Exercise 13: Show that $X(t)=e^{At}Ce^{Bt}$ is a solution for following system.

$$\dot{X} = AX + XB \quad X(0) = C$$

Exercises

Exercise 14: Let $\Phi(t,t_0)=\begin{bmatrix}\Phi_{11}(t,t_0) & \Phi_{12}(t,t_0)\\ \Phi_{21}(t,t_0) & \Phi_{22}(t,t_0)\end{bmatrix}$ is the state transition

matrix of following system

$$\dot{x}=\begin{bmatrix}A_{11}(t) & A_{12}(t)\\ 0 & A_{22}(t)\end{bmatrix}x$$

Shoe that:

$$\Phi_{21}(t,t_0)=0 \quad for\ all\ t\ and\ t_0$$

$$\left(\partial/\partial t\right)\Phi_{ii}(t,t_0)=A_{ii}\Phi_{ii}(t,t_0) \quad for\ \ i=1,2$$

Exercise 15: show that the solution of

$$\dot{X}(t)=A_1X(t)-X(t)A_1$$

is:

$$X(t)=e^{A_1t}X(0)e^{-A_1t}$$

and eigenvalues of *X*(*t*) is independent of t.

Exercises

Exercise 16: Consider the following system(Final 2014):

a) Using a similarity transformation, convert the system to one where the eigenvalues are on the main diagonal.

b) Using an appropriate variable change, convert the system to one where the matrix A=0. What is the relationship between the impulse response of the new system and the original system?

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 2 \quad 3] x$$

Exercise 17: Consider the following system(Final 2014):

a) Using a similarity transformation, convert the system to the controllable canonical form if possible.

a) Using an appropriate variable change, convert the system to one where the matrix A=0.

$$x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 2 \quad 3] x$$

Exercises

Exercise 18: Derive the fundamental matrix and the state transition matrix for the given systems(Fianl 2014).

$$\dot{x} = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} x$$

Answers to selected problems

Answer 1:

$$x(t)=\begin{bmatrix}\cos t+\sin t\\ \cos t-\sin t\end{bmatrix}$$

Answer 2:

$$y(t)=5e^{-t}\sin t \quad for\ t\geq 0$$

Answer 5: Not similar but zero state equivalent.

Answer 6:

$$\dot{x}=\begin{bmatrix}-3 & 0 & -2 & 0\\ 0 & -3 & 0 & -2\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}x+\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 0\end{bmatrix}u$$

$$y=\begin{bmatrix}2 & 2 & 4 & -3\\ -3 & -2 & -6 & -2\end{bmatrix}x+\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}u$$

Answers to selected problems

Answer 8:

$$\mathbf{X}(t) = \begin{bmatrix} 1 & \int_0^t e^{0.5\tau^2} d\tau \\ 0 & e^{0.5t^2} \end{bmatrix}$$

$$\Phi(t, t_0) = \begin{bmatrix} 1 & -e^{0.5t^2} \int_{t_0}^t e^{0.5\tau^2} d\tau \\ 0 & e^{0.5(t^2 - t_0^2)} \end{bmatrix}$$

Answer 9:

$$\mathbf{X}(t) = \begin{bmatrix} e^{-t} & e^t \\ 0 & 2e^{-t} \end{bmatrix}$$

$$\Phi(t, t_0) = \begin{bmatrix} e^{-(t-t_0)} & 0.5(e^t e^{t_0} - e^{-t} e^{t_0}) \\ 0 & e^{-(t-t_0)} \end{bmatrix}$$

Answer 10:

$$\Phi(t, t_0) = \begin{bmatrix} e^{\cos t - \cos t_0} & 0 \\ 0 & e^{-\sin t + \sin t_0} \end{bmatrix}$$

Answer 12:

$$\dot{\mathbf{x}} = \mathbf{0} \cdot \mathbf{x} + \begin{bmatrix} t^2 e^{-\lambda t} \\ -2t e^{-\lambda t} \\ e^{-\lambda t} \end{bmatrix} u \quad y = [e^{\lambda t} \;\; t e^{\lambda t} \;\; t^2 e^{\lambda t}]\mathbf{x}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 3\lambda & -3\lambda^2 & \lambda^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \quad y = [0 \;\; 0 \;\; 2]\mathbf{x}$$