ADVANCED CONTROL

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Reference:

Chi-Tsong Chen, “Linear System Theory and Design”, 1999.

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Lecture 3

Basic Idea of Linear Algebra-Part II

Topics to be covered include:

- Functions of Square Matrix.
- Lyapunov Equation.
- Some Useful Formula.
- Quadratic Form and Positive Definiteness.
- Singular Value Decomposition.
- Norm of Matrices

What you will learn after studying this section

- **Calculation of Function of Square Matrix**
- **Minimal Polynomials and Characteristic Polynomials**
- **Cayley-Hamilton Theorem**
- **Equal Polynomials on the Spectrum of A**
- **Lyapunov Equation and its Solution**
- **Symmetric Matrix and Quadratic Form and Orthogonal Matrix**
- **Matrix and PD/ND Matrix**
- **Singular Value Decomposition**
- **Null Space and Range Space From SVD**
- **Norm of Matrices**

Function of Square Matrix

Polynomial of square matrices $f(\lambda)=\lambda^3+2\lambda^2-6 \;\rightarrow\; f(A)=A^3+2A^2-6I$

Block matrices

$$A=\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \rightarrow A^2=\begin{bmatrix} A_1^{\,2} & 0 \\ 0 & A_2^{\,2} \end{bmatrix} \cdots \rightarrow A^k=\begin{bmatrix} A_1^{\,k} & 0 \\ 0 & A_2^{\,k} \end{bmatrix} \Rightarrow f(A)=\begin{bmatrix} f(A_1) & 0 \\ 0 & f(A_2) \end{bmatrix}$$

Jordan form

$$A=Q\hat{A}Q^{-1}, \quad \hat{A}=Q^{-1}AQ$$

$$A^k=\left(Q\hat{A}Q^{-1}\right)\left(Q\hat{A}Q^{-1}\right)\ldots\ldots\ldots\ldots\ldots\left(Q\hat{A}Q^{-1}\right)=Q\hat{A}^kQ^{-1}$$

And in general

$$f(A)=Qf(\hat{A})Q^{-1}, \quad f(\hat{A})=Q^{-1}f(A)Q$$

Function of Square Matrix

Example 1: The matrix A, its diagonal form, and the corresponding transformation are given. Find $A^6+12A^4+3A^2$.

$$A=\begin{bmatrix}1 & 0 & 12\\ 0 & 1 & 1\\ 0 & 0 & 4\end{bmatrix} \qquad Q=\begin{bmatrix}12 & 1 & 0\\ 1 & 0 & 1\\ 3 & 0 & 0\end{bmatrix} \qquad \hat{A}=Q^{-1}AQ=\begin{bmatrix}4 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

We know

$$A^6+12A^4+3A^2=Q\left(\hat{A}^6+12\hat{A}^4+3\hat{A}^2\right)Q^{-1}$$

$$\hat{A}^6+12\hat{A}^4+3\hat{A}^2=\begin{bmatrix}4^6 & 0 & 0\\ 0 & 1^6 & 0\\ 0 & 0 & 1^6\end{bmatrix}+12\begin{bmatrix}4^4 & 0 & 0\\ 0 & 1^4 & 0\\ 0 & 0 & 1^4\end{bmatrix}+3\begin{bmatrix}4^2 & 0 & 0\\ 0 & 1^2 & 0\\ 0 & 0 & 1^2\end{bmatrix}=\begin{bmatrix}7216 & 0 & 0\\ 0 & 16 & 0\\ 0 & 0 & 16\end{bmatrix}$$

$$A^6+12A^4+3A^2=\begin{bmatrix}12 & 1 & 0\\ 1 & 0 & 1\\ 3 & 0 & 0\end{bmatrix}\begin{bmatrix}7216 & 0 & 0\\ 0 & 16 & 0\\ 0 & 0 & 16\end{bmatrix}\begin{bmatrix}12 & 1 & 0\\ 1 & 0 & 1\\ 3 & 0 & 0\end{bmatrix}^{-1}=\begin{bmatrix}16 & 0 & 28800\\ 0 & 16 & 2400\\ 0 & 0 & 7216\end{bmatrix}$$

Function of Square Matrix

Monic polynomial

A polynomial whose leading coefficient is equal to one is called a **monic polynomial**. For example:

$$\lambda^6 + 12\lambda^4 + 3\lambda^2 + 5$$

Minimal polynomial

A monic polynomial of the smallest degree that nullifies a matrix *A* is called the **minimal polynomial** of the matrix *A* and it is denoted by:

$$\psi(\lambda)$$

Characteristic polynomial

The **characteristic polynomial** of an n×n matrix *A* is given by:

$$\Delta(\lambda) = |\lambda I - A| = \prod_i (\lambda - \lambda_i)^{n_i} \qquad \sum_i n_i = n$$

Function of Square Matrix

The **characteristic polynomial** of an n×n matrix *A* is given by:

$$\Delta(\lambda) = \left|\lambda I - A\right| = \prod_i (\lambda - \lambda_i)^{n_i} \qquad \sum_i n_i = n$$

Calculation of the **minimal polynomial**(According to Nilpotent property):

$$\psi(\lambda) = \prod_i (\lambda - \lambda_i)^{\bar{n}_i} \qquad \sum_i \bar{n}_i \le \sum_i n_i = n$$

$\overline{n_i}$ is is the size of the largest block corresponding to λ_i in the Jordan form.

Theorem 1: (Cayley-Hamilton Theorem): The matrix *A* satisfies its own characteristic equation.

Proof: We know

$$\Delta(\lambda) = \prod_i (\lambda - \lambda_i)^{n_i} = \psi(\lambda) h(\lambda)$$

By Nilpotent property):

$$\Delta(A) = \psi(A) h(A) = 0.h(A) = 0$$

Function of Square Matrix

The **characteristic polynomial** of an n×n matrix *A* is given by:

$$\Delta(\lambda) = |\lambda I - A| = \prod_i (\lambda - \lambda_i)^{n_i} \qquad \sum_i n_i = n$$

The **minimal polynomial** of an n×n matrix *A* is given by:

$$\psi(\lambda) = \prod_i (\lambda - \lambda_i)^{\bar{n}_i} \qquad \sum_i \bar{n}_i \le \sum_i n_i = n$$

Example 2: Find the characteristic polynomial and the minimal polynomial of the following matrices:

$$I) \quad A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

$$\Delta(\lambda) = |\lambda I - A| = \prod_i (\lambda - \lambda_i)^{n_i} = (\lambda - \lambda_1)^4 (\lambda - \lambda_2)$$

$$\psi(\lambda) = \prod_i (\lambda - \lambda_i)^{\bar{n}_i} = (\lambda - \lambda_1)^4 (\lambda - \lambda_2)$$

$$II) \quad A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

$$\Delta(\lambda) = |\lambda I - A| = \prod_i (\lambda - \lambda_i)^{n_i} = (\lambda - \lambda_1)^4 (\lambda - \lambda_2)$$

$$\psi(\lambda) = \prod_i (\lambda - \lambda_i)^{\bar{n}_i} = (\lambda - \lambda_1)^3 (\lambda - \lambda_2)$$

Function of Square Matrix

Example 2: Find the characteristic polynomial and the minimal polynomial of the following matrices:

$$III)\quad A=\begin{bmatrix}\lambda_1 & 1 & 0 & 0 & 0\\ 0 & \lambda_1 & 0 & 0 & 0\\ 0 & 0 & \lambda_1 & 1 & 0\\ 0 & 0 & 0 & \lambda_1 & 0\\ 0 & 0 & 0 & 0 & \lambda_2\end{bmatrix}$$

$$\Delta(\lambda)=\left|\lambda I-A\right|=\prod_i(\lambda-\lambda_i)^{n_i}=(\lambda-\lambda_1)^4(\lambda-\lambda_2)$$

$$\psi(\lambda)=\prod_i(\lambda-\lambda_i)^{\bar{n}_i}=(\lambda-\lambda_1)^2(\lambda-\lambda_2)$$

$$IV)\quad A=\begin{bmatrix}\lambda_1 & 1 & 0 & 0 & 0\\ 0 & \lambda_1 & 0 & 0 & 0\\ 0 & 0 & \lambda_1 & 0 & 0\\ 0 & 0 & 0 & \lambda_1 & 0\\ 0 & 0 & 0 & 0 & \lambda_2\end{bmatrix}$$

$$\Delta(\lambda)=\left|\lambda I-A\right|=\prod_i(\lambda-\lambda_i)^{n_i}=(\lambda-\lambda_1)^4(\lambda-\lambda_2)$$

$$\psi(\lambda)=\prod_i(\lambda-\lambda_i)^{\bar{n}_i}=(\lambda-\lambda_1)^2(\lambda-\lambda_2)$$

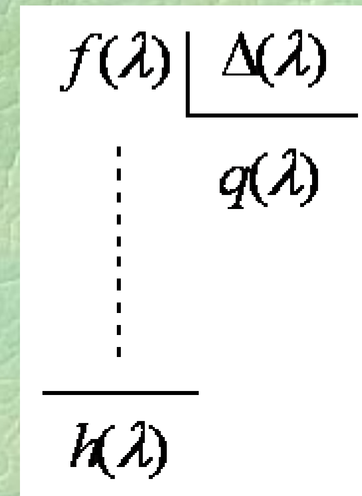
$$V)\quad A=\begin{bmatrix}\lambda_1 & 0 & 0 & 0 & 0\\ 0 & \lambda_1 & 0 & 0 & 0\\ 0 & 0 & \lambda_1 & 0 & 0\\ 0 & 0 & 0 & \lambda_1 & 0\\ 0 & 0 & 0 & 0 & \lambda_2\end{bmatrix}$$

$$\Delta(\lambda)=\left|\lambda I-A\right|=\prod_i(\lambda-\lambda_i)^{n_i}=(\lambda-\lambda_1)^4(\lambda-\lambda_2)$$

$$\psi(\lambda)=\prod_i(\lambda-\lambda_i)^{\bar{n}_i}=(\lambda-\lambda_1)(\lambda-\lambda_2)$$

Function of Square Matrix

Consider an arbitrary polynomial $f(\lambda)$ and a matrix A of size n×n.



Function f(λ) can be expressed as:

$$f(\lambda) = q(\lambda)\Delta(\lambda) + h(\lambda)$$

Now, to compute f(A)f(A)f(A), we have:

$$f(A) = q(A)\Delta(A) + h(A)$$

And according to the Cayley-Hamilton theorem:

$$f(A) = q(A).0 + h(A) = h(A) \qquad \Rightarrow \qquad f(A) = h(A)$$

The polynomial $h(\lambda)$ that is equivalent to $f(\lambda)$ on the spectrum of A is called the polynomial equivalent to $f(\lambda)$ on the spectrum of A.

Note: The degree of h(λ)?

Note: Calculation of h(λ)?

Function of Square Matrix

Calculation of *h*(*λ*) for the case where the matrix *A* has non-repeated eigenvalues.

$$f(\lambda) = q(\lambda)\Delta(\lambda) + h(\lambda) \qquad f(\lambda) = q(\lambda)\Delta(\lambda) + \beta_{n-1}\lambda^{n-1} + \ldots + \beta_1\lambda + \beta_0$$

By substituting the eigenvalues of *A* into the above equation, we get:

$$f(\lambda_1) = q(\lambda_1)\Delta(\lambda_1) + \beta_{n-1}\lambda_1^{n-1} + \ldots + \beta_1\lambda_1 + \beta_0 \quad \rightarrow \quad f(\lambda_1) = \beta_{n-1}\lambda_1^{n-1} + \ldots + \beta_1\lambda_1 + \beta_0$$

$$f(\lambda_2) = q(\lambda_2)\Delta(\lambda_2) + \beta_{n-1}\lambda_2^{n-1} + \ldots + \beta_1\lambda_2 + \beta_0 \quad \rightarrow \quad f(\lambda_2) = \beta_{n-1}\lambda_2^{n-1} + \ldots + \beta_1\lambda_2 + \beta_0$$

....

$$f(\lambda_n) = q(\lambda_n)\Delta(\lambda_n) + \beta_{n-1}\lambda_n^{n-1} + \ldots + \beta_1\lambda_n + \beta_0 \quad \rightarrow \quad f(\lambda_n) = \beta_{n-1}\lambda_n^{n-1} + \ldots + \beta_1\lambda_n + \beta_0$$

After solving the nnn equations with nnn unknowns, the values of the unknowns are obtained.

$$\beta_{n-1},\ \beta_{n-2},\ \ldots, \beta_1,\ \beta_0$$

Function of Square Matrix

Calculation of $h(\lambda)$ for the case where the matrix A has non-repeated eigenvalues.

Theorem 2: Consider the equation $f(\lambda)$ and the matrix A with dimensions $n\times n$ with the following characteristic equation.

$$\Delta(\lambda)=\prod_{i=1}^{m}(\lambda-\lambda_i)^{n_i} \quad where \;\; n=\sum_{i=1}^{m} n_i$$

The polynomial $h(\lambda)$ of degree $n-1$, equivalent to $f(\lambda)$ over the spectrum of A, is defined as follows.

$$h(\lambda)=\beta_0+\beta_1\lambda+....+\beta_{n-1}\lambda^{n-1}$$

After solving the following n equations with n unknowns, the unknown coefficients of $h(\lambda)$ are calculated.

$$f^{\,l}(\lambda_i)=h^{\,l}(\lambda_i) \;\; \text{for } l=0,1,\ldots,n_i-1 \;\text{ and }\; i=1,2,\ldots,m$$

In this relation:

$$f^{\,l}(\lambda)=\frac{d^{\,l} f(\lambda)}{d\lambda^{l}}, \quad h^{\,l}(\lambda)=\frac{d^{\,l} h(\lambda)}{d\lambda^{l}}$$

And finally:

$$f(A)=h(A)$$

Function of Square Matrix

Example 3: Determine A^{100}.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

Let $f(\lambda)=\lambda^{100}$

The eigenvalues of *A* should now be calculated.

$$\Delta(A) = |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda+2 \end{vmatrix} = \lambda^2 + 2\lambda + 1 \qquad \lambda_1 = \lambda_2 = -1$$

Now, $h(\lambda)$ should be considered as follows:

$$h(\lambda) = \beta_0 + \beta_1 \lambda$$

$$f(-1) = h(-1) \quad \Rightarrow \quad (-1)^{100} = \beta_0 - \beta_1$$

$$f'(-1) = h'(-1) \quad \Rightarrow \quad 100(-1)^{99} = \beta_1$$

Now, $h(\lambda)$ is given by:

$$h(\lambda) = -99 - 100\lambda \qquad A^{100} = -99\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 100\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -99 & -100 \\ 100 & 101 \end{bmatrix}$$

Function of Square Matrix

Example 4: Determine e^{At}.

$$A=\begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

Let $f(\lambda t)=e^{\lambda t}$

The eigenvalues of A should now be calculated.

$$\Delta(A)=|\lambda I-A|=(\lambda-1)^2(\lambda-2) \qquad \lambda_1=\lambda_2=1,\quad \lambda_3=2$$

Now, $h(\lambda)$ should be considered as follows:

$$h(\lambda)=\beta_0+\beta_1\lambda+\beta_2\lambda^2$$

$$f(1)=h(1) \quad \Rightarrow \quad e^t=\beta_0+\beta_1+\beta_2 \qquad \beta_0=-2te^t+e^{2t}$$

$$f'(1)=h'(1) \quad \Rightarrow \quad te^t=\beta_1+2\beta_2 \qquad \beta_1=3te^t+2e^t-2e^{2t}$$

$$f(2)=h(2) \quad \Rightarrow \quad e^{2t}=\beta_0+2\beta_1+4\beta_2 \qquad \beta_2=-te^t-e^t+e^{2t}$$

Now $f(A)$ is:

$$e^{At}=\beta_0 I+\beta_1 A+\beta_2 A^2=....=\begin{bmatrix} 2e^t-e^{2t} & 0 & 2e^t-2e^{2t} \\ 0 & e^t & 0 \\ e^{2t}-e^t & 0 & 2e^{2t}-e^t \end{bmatrix}$$

Function of Square Matrix

Example 5: Determine e^{At}.

$$A=\begin{bmatrix}0 & 2 & -2\\ 0 & 1 & 0\\ 1 & -1 & 3\end{bmatrix}$$

Let $f(\lambda t)=e^{\lambda t}$

The eigenvalues of A should now be calculated.

$$\Delta(A)=|\lambda I-A|=(\lambda-1)^2(\lambda-2) \qquad \lambda_1=\lambda_2=1,\quad \lambda_3=2$$

Now, $h(\lambda)$ should be considered as follows:

$$h(\lambda)=\beta_0+\beta_1\lambda+\beta_2\lambda^2$$

$$f(1)=h(1) \quad\Rightarrow\quad e^t=\beta_0+\beta_1+\beta_2 \qquad \beta_0=-2te^t+e^{2t}$$

$$f'(1)=h'(1) \quad\Rightarrow\quad te^t=\beta_1+2\beta_2 \qquad \beta_1=3te^t+2e^t-2e^{2t}$$

$$f(2)=h(2) \quad\Rightarrow\quad e^{2t}=\beta_0+2\beta_1+4\beta_2 \qquad \beta_2=-te^t-e^t+e^{2t}$$

Now $f(A)$ is:

$$e^{At}=\beta_0 I+\beta_1 A+\beta_2 A^2=....=\begin{bmatrix}2e^t-e^{2t} & 2te^t & 2e^t-2e^{2t}\\ 0 & e^t & 0\\ e^{2t}-e^t & -te^t & 2e^{2t}-e^t\end{bmatrix}$$

Comparison with the previous example!

Function of Square Matrix

Example 6: Determine $e^{\hat{A}t}$.

$$\hat{A} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix}$$

Let $f(\lambda t)=e^{\lambda t}$

The eigenvalues of A should now be calculated.

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_1$$

Now, $h(\lambda)$ should be considered as follows:

$$h(\lambda) = \beta_0 + \beta_1(\lambda-\lambda_1) + \beta_2(\lambda-\lambda_1)^2 + \beta_3(\lambda-\lambda_1)^3$$

$$f(\lambda_1) = h(\lambda_1) \quad \Rightarrow \quad f(\lambda_1) = \beta_0 \qquad f^1(\lambda_1) = h^1(\lambda_1) \quad \Rightarrow \quad f^1(\lambda_1) = \beta_1$$

$$f^2(\lambda_1) = h^2(\lambda_1) \quad \Rightarrow \quad f^2(\lambda_1) = 2\beta_2 \qquad f^3(\lambda_1) = h^3(\lambda_1) \quad \Rightarrow \quad f^3(\lambda_1) = 6\beta_3$$

$$f(\hat{A}) = \begin{bmatrix} f(\lambda_1) & f^1(\lambda_1)/1! & f^2(\lambda_1)/2! & f^3(\lambda_1)/3! \\ 0 & f(\lambda_1) & f^1(\lambda_1)/1! & f^2(\lambda_1)/2! \\ 0 & 0 & f(\lambda_1) & f^1(\lambda_1)/1! \\ 0 & 0 & 0 & f(\lambda_1) \end{bmatrix} \qquad e^{\hat{A}t} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & t^2 e^{\lambda_1 t}/2! & t^3 e^{\lambda_1 t}/3! \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & t^2 e^{\lambda_1 t}/2! \\ 0 & 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & 0 & e^{\lambda_1 t} \end{bmatrix}$$

Function of Square Matrix

Example 7: Determine $(sI - A)^{-1}$, e^{At}.

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

Based on the previous example, we have:

$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & t^2 e^{\lambda_1 t}/2! & 0 & 0 \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & 0 & 0 \\ 0 & 0 & e^{\lambda_1 t} & 0 & 0 \\ 0 & 0 & 0 & e^{\lambda_2 t} & te^{\lambda_2 t} \\ 0 & 0 & 0 & 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s-\lambda_1} & \frac{1}{(s-\lambda_1)^2} & \frac{1}{(s-\lambda_1)^3} & 0 & 0 \\ 0 & \frac{1}{s-\lambda_1} & \frac{1}{(s-\lambda_1)^2} & 0 & 0 \\ 0 & 0 & \frac{1}{s-\lambda_1} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{s-\lambda_2} & \frac{1}{(s-\lambda_2)^2} \\ 0 & 0 & 0 & 0 & \frac{1}{s-\lambda_2} \end{bmatrix}$$

Function of Square Matrix

Exponential series: $$e^{\lambda t} = 1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \ldots + \frac{\lambda^n t^n}{n!} + \ldots \qquad (I)$$

By substituting A into the above equation, we have:

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \ldots + \frac{t^n}{n!}A^n + \ldots$$

Important property of e^{At}

$$e^{0} = I \qquad e^{A(t_1+t_2)} = e^{At_1}e^{At_2}$$

$$\left[e^{At}\right]^{-1} = e^{-At} \qquad \frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$$

And a very important property: $$e^{(A+B)t} \neq e^{At}e^{Bt}$$

But in the special case: ….

Function of Square Matrix

Exponential series: $$e^{\lambda t} = 1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \ldots + \frac{\lambda^n t^n}{n!} + \ldots \qquad (I)$$

By substituting *A* into the above equation, we have:

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \ldots + \frac{t^n}{n!}A^n + \ldots$$

We know

$$L\left(\frac{t^k}{k!}\right) = s^{-(k+1)}$$

So:

$$L\left(e^{At}\right) = s^{-1}I + s^{-2}A + s^{-3}A^2 + \ldots + s^{-n-1}A^n + \ldots$$

With some simplification, we have:

$$L\left(e^{At}\right) = (sI - A)^{-1} \qquad e^{At} = L^{-1}\left((sI - A)^{-1}\right)$$

Lyapunov Equation

Consider the following equation:

$$\underset{\uparrow\; n\times n}{A}\overset{n\times m\;\downarrow}{M} + M\underset{\uparrow\; m\times m}{B} = \overset{n\times m\;\downarrow}{C}$$

This equation is called the **Lyapunov equation** and actually has *nm* equations and *nm* unknowns (the elements of the matrix *M*)

Reminder:

$$\underset{\uparrow\; m\times n}{A}\overset{n\times 1\;\downarrow}{\mathbf{x}} = \underset{\uparrow\; m\times 1}{\mathbf{y}}$$

$$A:\ R^n \mapsto R^m$$

The Lyapunov equation can also be represented as follows:

$$\underset{\uparrow\; n\times n}{A}\overset{n\times m\;\downarrow}{M} + M\underset{\uparrow\; m\times m}{B} = \overset{n\times m\;\downarrow}{C}$$

$$\mathcal{A}(\mathbf{M}) = C$$

$$\mathcal{A}:\ \mathrm{R}^{n\times m} \mapsto \mathrm{R}^{n\times m}$$

Solution of the Lyapunov equation: $M = lyap\,(A, B, -C)$

Lyapunov Equation

Linear algebraic equation: $A\mathbf{x} = \mathbf{y}$

A scalar λ is an eigenvalue of A if there exists a non-zero vector v such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

Lyapunov equation

$$\underset{\substack{\uparrow \\ n\times n}}{A}\overset{\substack{n\times m \\ \downarrow}}{M} + \overset{\substack{n\times m \\ \downarrow}}{M}\underset{\substack{\uparrow \\ m\times m}}{B} = C$$

$$\mathcal{A}(\mathbf{M}) = C \qquad \mathcal{A}:\ \mathbf{R}^{n\times m} \mapsto \mathbf{R}^{n\times m}$$

A scalar μ is an eigenvalue of $\mathcal{A}$ if there exists a non-zero matrix V such that

$$\mathcal{A}(\mathrm{V}) = \mu\,\mathrm{V}$$

Some Useful Formula

Suppose A and B are square matrices, then:

$$\rho(AB) \le \min\ \left(\rho(A),\ \rho(B)\right)$$

Suppose C and D are arbitrary non-singular matrices, then:

$$\rho(AC) = \rho(A) = \rho(DA)$$

Let A be an m×n matrix and B be an n×m matrix, then:

$$\det(\ I_m + AB\ ) = \det(\ I_n + BA\ )$$

For proof define:

$$N = \begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix} \qquad Q = \begin{bmatrix} I_m & 0 \\ -B & I_n \end{bmatrix} \qquad P = \begin{bmatrix} I_m & -A \\ B & I_n \end{bmatrix}$$

$$\det(I_m + AB) = \det(\ NP\ ) = \det(\ QP\ ) = \det(I_n + BA)$$

Quadratic Form and Orthogonal Matrix

Symmetric Matrices and Quadratic Form (Square) and Orthogonal Matrix (Unitary)

Definition 1: A matrix $M \in R^{n \times m}$ is symmetric if

$$M = M^T$$

Definition 2: For a symmetric matrix M and any vector x, the expression $x^T M x$ is called a quadratic form.

Definition 3: A matrix $M \in R^{n \times n}$ is called orthogonal (or unitary, in the complex case) if all of its columns are orthonormal, meaning each column is of unit length and orthogonal to the other columns. We have

$$M^T M = I, \quad M^T = M^{-1}$$

Quadratic Form and Positive Definiteness

Theorem 3: For any real symmetric matrix *M*, there exists an orthogonal matrix *Q* such that:

$$M = QDQ^T \quad or \quad D = Q^T MQ$$

Matrix *D* is diagonal, with its diagonal elements being the eigenvalues of *M*, and the columns of *Q* are the eigenvectors of *M*.

Proof: It is clear that D is a similarity transformation of M. Therefore, to prove the theorem, we need to show:

- The eigenvalues of *M* are real.
- There are no generalized eigenvectors.
- *Q* is orthogonal.

Suppose λ is an eigenvalue of *M*. Then:

$Mv = \lambda v \qquad v^* Mv = v^* \lambda v \qquad v^* Mv = \lambda v^* v$ **Real** → **λ is real**

…….

Quadratic Form and Positive Definiteness

Definite Matrices

Definition 4: A symmetric matrix M is called positive definite if, for any nonzero vector $x \in R^n$

$$x^T M x \in R^+$$

Definition 5: A symmetric matrix M is called negative definite if, for any nonzero vector $x \in R^n$

$$x^T M x \in R^-$$

Definition 6: A symmetric matrix M is called positive semi definite if, for any nonzero vector $x \in R^n$

$$x^T M x \in R^+ \cup \{0\}$$

Definition 7: A symmetric matrix M is called negative semi definite if, for any nonzero vector $x \in R^n$

$$x^T M x \in R^- \cup \{0\}$$

Quadratic Form and Positive Definiteness

Theorem 4: A real symmetric matrix *M* is **positive definite** (positive semi-definite) if and only if any of the following conditions hold.

1- Positive Eigenvalues: All eigenvalues of *M* are positive(positive or zero).

2- Positive Quadratic Form: For any non-zero vector *x*, the quadratic form x^TMx is positive (positive or zero), i.e., $x^TMx > 0$.

3- Positive Quadratic Form: For any non-zero vector *x*, the quadratic form x^TMx is positive (positive or zero), i.e., $x^TMx > 0$.

4- Existence of a Non-Singular Matrix *N*: There exists a non-singular matrix *N* such that $M = N^TN$(There exists a matrix *N* such that $M = N^TN$, where *N* can be non-singular or rectangular with dimensions $m \times n$ where $m \leq n$).

Quadratic Form and Positive Definiteness

Theorem 5:

1- A matrix H of size $m\times n$ with m≥n has rank n if and only if the matrix H^TH of size n×n has rank n or det(H^TH)≠0.

2- A matrix H of size $m\times n$ with m≤n has rank m if and only if the matrix HH^T of size $m\times m$ has rank m or det(HH^T)≠0.

Proof: We prove both sides of part 1, and part 2 is similar to part 1.

$$(I) \qquad \rho(H^TH)=n \;\Rightarrow\; \rho(H)=n$$

If rank(H)<n, non-zero vector v exists such that:

$$Hv=0 \qquad \Rightarrow \qquad H^THv=0 \qquad \Rightarrow \qquad \text{contradiction}$$

$$(II) \qquad \rho(H)=n \;\Rightarrow\; \rho(H^TH)=n$$

If rank(HTH)<n, a non-zero vector v such that:

$$H^THv=0 \Rightarrow v^TH^THv=0 \Rightarrow (Hv)^THv=\|Hv\|_2^2=0 \Rightarrow Hv=0 \Rightarrow \text{contradiction}$$

Singular Value Decomposition (SVD)

Theorem 6: Suppose $M \in C^{l\times m}$, then there exist unitary matrices $\Sigma \in R^{l\times m}$, $Y \in C^{l\times l}$, and $U \in C^{m\times m}$ such that:

$$M = Y\Sigma U^H$$

$$\Sigma = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \qquad S = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & \sigma_r \end{bmatrix} \qquad \sigma_1 \geq \sigma_2 \geq \dots\dots\dots \geq \sigma_r > 0$$

$$Y = [y_1, y_2, \dots\dots, y_l], \; U = [u_1, u_2, \dots\dots, u_m]$$

Where σ_i are singular values

Columns of matrix Y are ….

Columns of matrix U are ….

Singular Value Decomposition (SVD)

Example 8: Decompose the singular values of the given matrix.

$$M=\begin{bmatrix}1 & 2 & -1\\ 3 & 4 & 1\\ 4 & 2 & 8\end{bmatrix}$$

$$M=\begin{bmatrix}0.04 & -0.53 & -0.85\\ 0.38 & -0.77 & 0.51\\ 0.92 & 0.34 & -0.17\end{bmatrix}.\begin{bmatrix}9.77 & 0 & 0\\ 0 & 4.53 & 0\\ 0 & 0 & 0\end{bmatrix}.\begin{bmatrix}0.50 & -0.33 & -0.80\\ 0.35 & -0.77 & 0.53\\ 0.79 & 0.55 & 0.27\end{bmatrix}^H$$

$$u_1=\begin{bmatrix}0.50\\ 0.35\\ 0.79\end{bmatrix}\qquad Mu_1=9.77\begin{bmatrix}0.04\\ 0.38\\ 0.92\end{bmatrix}=9.77y_1\qquad u_2=\begin{bmatrix}-0.33\\ -0.77\\ 0.55\end{bmatrix}\qquad Mu_2=4.53\begin{bmatrix}-0.53\\ -0.77\\ 0.34\end{bmatrix}=4.53y_2$$

$$u_3=\begin{bmatrix}-0.80\\ 0.53\\ 0.27\end{bmatrix}$$ **Has no affect on the output or** $Mu_3=0$

The range space of matrix M is: …

The null space of matrix M is: …

Norm of vectors

p-norm is: $$\|x\|_p = \left(\sum_i |a_i|^p \right)^{\frac{1}{p}} \qquad p \geq 1$$

For p=1 we have 1-norm or sum norm $$\|x\|_1 = \left(\sum_i |a_i| \right)$$

For p=2 we have 2-norm or euclidian norm $$\|x\|_2 = \left(\sum_i |a_i|^2 \right)^{1/2}$$

For p=∞ we have ∞-norm or max norm $$\|x\|_\infty = \max_i \left\{ |a_i| \right\}$$

Norm of matrices

The notion of norms can also be extended to matrices.

Sum matrix norm (extension of 1-norm of vectors) is: $\|A\|_{sum} = \sum_{i,j} |a_{ij}|$

Frobenius norm (extension of 2-norm of vectors) is: $\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2}$

Max element norm (extension of max norm of vectors) is: $\|A\|_{\max} = \max_{i,j} |a_{ij}|$

Induced matrix norm

A norm for matrices is called a **matrix norm** if it has the following property:

$$\|AB\| \leq \|A\|.\|B\|$$

The induced norm is defined as follows:

$$\|A\|_{ip} = \max_{\|x\|_p = 1} \|Ax\|_p$$

Every induced norm is a matrix norm.

Matrix norm for matrices

$$\|A\|_{ip} = \max_{\|x\|_p = 1} \|Ax\|_p$$

Assuming p=1 in the induced norm formula, we have:

$$\|A\|_{i1} = \max_{\|x\|_1 = 1} \|Ax\|_1 = \max_j \sum_i |a_{ij}|$$

Maximum column sum

Assuming p= ∞ in the induced norm formula, we have:

$$\|A\|_{i\infty} = \max_{\|x\|_\infty = 1} \|Ax\|_\infty = \max_i \sum_j |a_{ij}|$$

Maximum row sum

Assuming p=2 in the induced norm formula, we have:

$$\|A\|_{i2} = \max_{\|x\|_2 = 1} \|Ax\|_2 = \max_{\|x\|_2 \neq 1} \frac{\|Ax\|_2}{\|x\|_2} = \sigma_1(A) = \sigma_{\max}(A) = \overline{\sigma}(A)$$

Exercises

Exercise 1: With use of $e^{\lambda t} = 1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \cdots + \frac{\lambda^n t^n}{n!} + \ldots$ prove followings:

$$e^0 = I \qquad e^{A(t_1+t_2)} = e^{At_1} e^{At_2} \qquad \left[e^{At}\right]^{-1} = e^{-At} \qquad \frac{d}{dt} e^{At} = A e^{At} = e^{At} A$$

Exercise 2: Show that the eigenvalues of matrix $\mathcal{A}$ are all possible sums of the eigenvalues of matrices *A* and *B*. Additionally, demonstrate that the matrix V can be formed from the product of the right eigenvectors of *A* and the left eigenvectors of *B*.

Exercise 3: Show that for a square symmetric matrix, there are no generalized eigenvectors, and the matrix can be diagonalized using an orthogonal matrix. (Hint: Proof by contradiction)

Exercise 4: Show that if λ is an eigenvalue of matrix *A* with *x* as the corresponding eigenvector, then $f(\lambda)$ is an eigenvalue of the matrix $f(A)$, and x is the corresponding eigenvector.

Exercise 5: Show that functions of a matrix have the commutative property, i.e:

f(A)g(A)=g(A)f(A)

Exercises

Exercise 6: Determine B such that $e^B=C$. Show that if $\lambda_i=0$, then B does not exist.

$$C = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Now let $C = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

Determine B such that $e^B=C$. Is it true that for every nonsingular matrix C, there exists a matrix B such that $e^B=C$?

Exercise 7: If matrix A is symmetric, what is the relationship between its eigenvalues and singular values? (Hint: For symmetric matrices, $A= A^T$)

Exercise 8: Show that if all eigenvalues of A are distinct, then

$$(sI - A)^{-1} = \sum \frac{1}{s-\lambda_i} q_i p_i$$

where qi and pi are right and left eigenvectors od A associate with λ_i

Exercises

Exercise 9: Find M to meet the Lyapunov equation with

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \qquad \mathrm{B} = 3 \qquad \mathrm{C} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

What are the eigenvalues of the Lyapunov equation? Is the Lyapunov equation singular? Is the solution unique?

Exercise 10: Repeat exercise 9 for

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \qquad \mathrm{B} = 1 \qquad \mathrm{C1} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \qquad C2 = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

For two different C.

Exercise 11: Check to see the following matrices are positive definite or semidefinite:

$$\begin{bmatrix} 2 & 3 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} a_1 a_1 & a_1 a_2 & a_1 a_3 \\ a_2 a_1 & a_2 a_2 & a_2 a_3 \\ a_3 a_1 & a_3 a_2 & a_3 a_3 \end{bmatrix}$$

Exercise 12: Compute the singular values of following matrices:

$$\begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Exercises

Exercise 13: Show that:

$$det\left(I_n + \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix} [b_1 \quad b_2 \quad \dots \quad b_n]\right) = 1 + \sum_{m=1}^{n} a_m b_m$$

Answer of exercise 6:

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{ln\lambda_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{ln\lambda_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{ln\lambda_3} \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{ln\lambda} & \boldsymbol{1/\lambda} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{ln\lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{ln\lambda} \end{bmatrix}$$

Answer of exercise 10: Eigenvalues: 0, 0. No solution for C_1. For any m1, $[m_1 \quad 3- m_1]^T$ is a solution for C_2.