ADVANCED CONTROL

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Reference:

Chi-Tsong Chen, “Linear System Theory and Design”, 1999.

I thank my student, Alireza Bemani for his help in correction slides of this lecture.

Lecture 1

Mathematical Descriptions of Systems

Topics to be covered include:

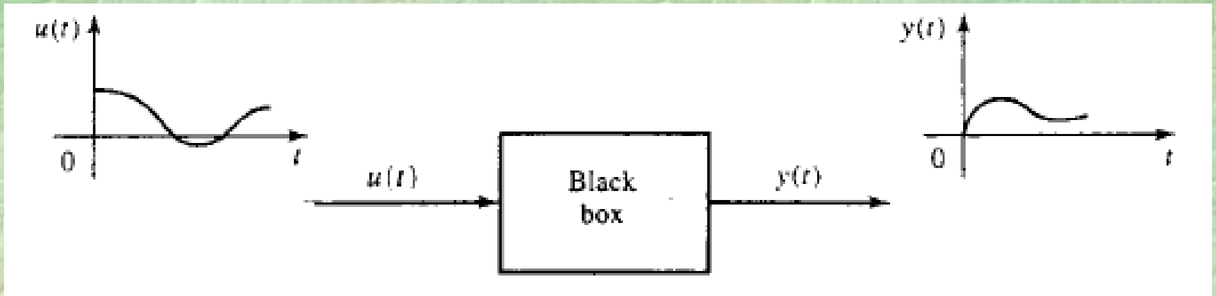
- Introduction.
- Linear Systems.
- Linear Time Invariant Systems.
- Op-Amp Circuit Implementation
- Linearization.
- Concluding Remarks

What you will learn after studying this section

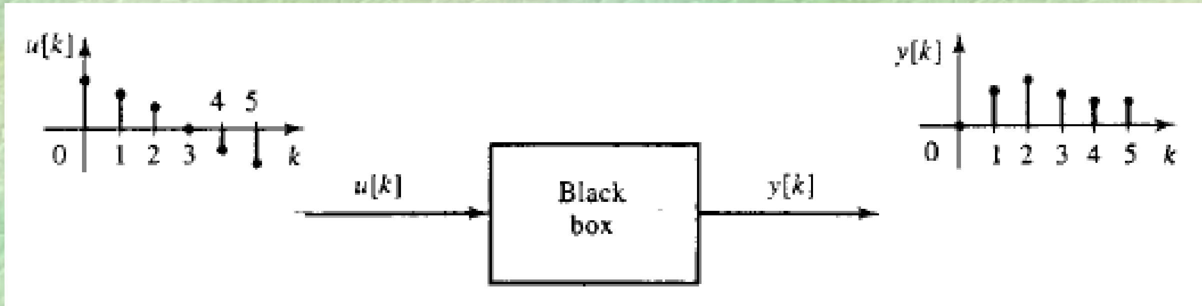
- Causal, lumped and lumpedness Systems
- Linear Systems and Its Property
- State Idea
- State-Input-Output Pair Idea
- Input-Output Relation
- State Space Representation for LTV Systems
- Time Invariant Systems and Transfer Function
- LTI State Space Representation
- Op-Amp Circuit Implementation
- Linearization of LTI Systems

Introduction

Continues System

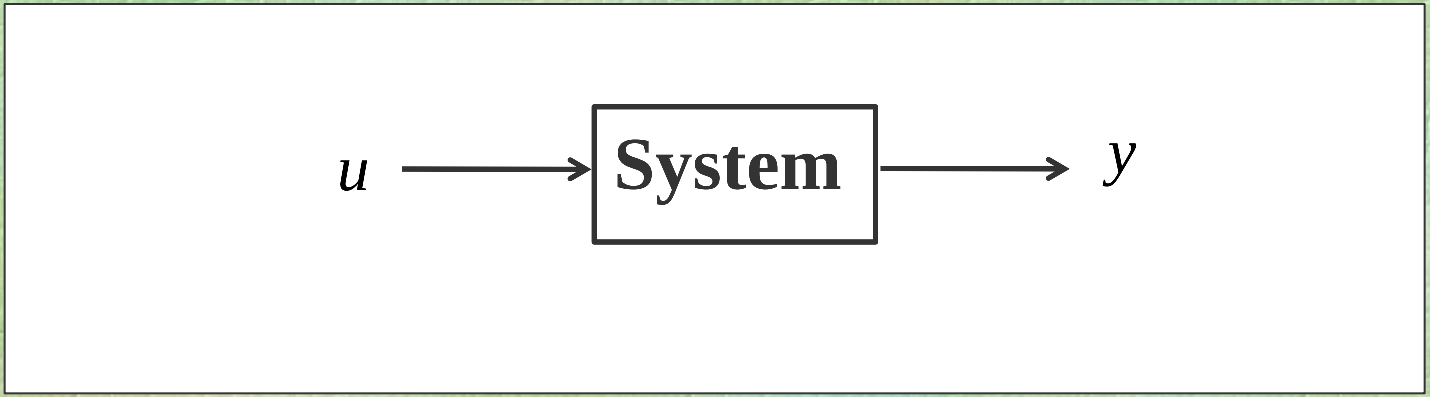


Discrete System

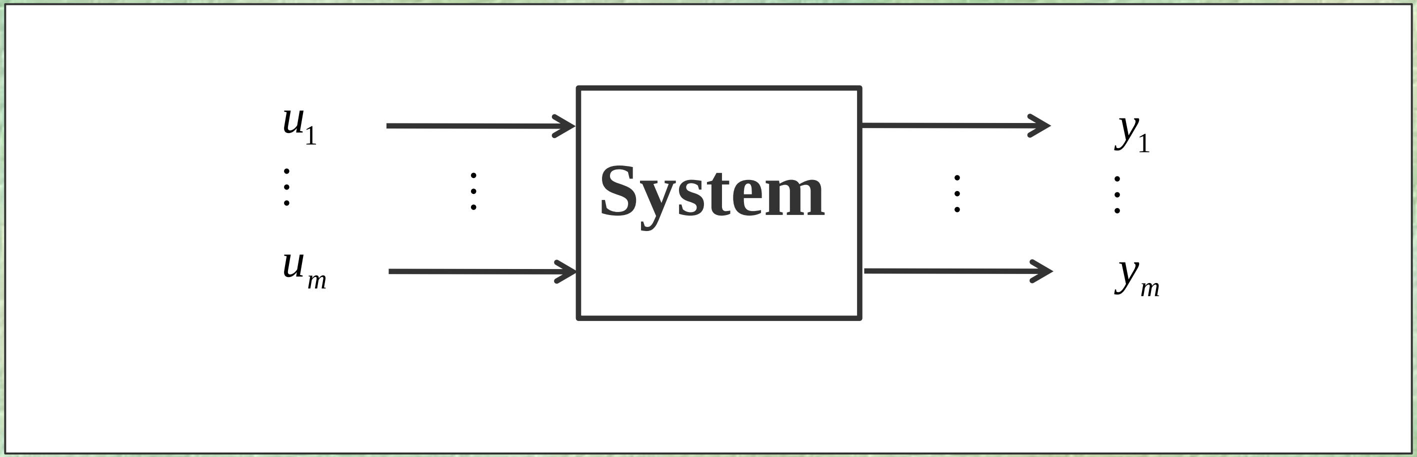


Introduction

SISO System



MIMO System



Introduction

Memory less System

A system is called **memoryless** if its output at any given moment depends only on the input at that same moment and is not related to any past or future inputs.

$$y(t) = 12.3u(t)$$

Causal system

A system is called **causal** if its output at time t_0 depends only on the input at time t_0 and earlier inputs, and is not related to any inputs after t_0 .

$$y(t) = u(t-1)$$

$$y(t) = u(t+1) \quad ?$$

Introduction

Input-Output Relation

$$u(t), \quad t \in (-\infty, +\infty) \quad \rightarrow \quad y(t)$$

Input-Output Relation in Casual System

$$u(t), \quad t \in (-\infty, t) \quad \rightarrow \quad y(t)$$

Definition 1 (State): The state $x(t_0)$ at time t_0 is the set of information that uniquely determines the output $y(t)$ for $t \geq t_0$ given the input $u(t)$ for $t \geq t_0$.

State-Input-Output Pair

$$\left.\begin{array}{l} x(t_0) \\ u(t), \quad t \geq t_0 \end{array}\right\} \rightarrow y(t), \; t \geq t_0$$

Introduction

Lumped System

A system is called **lumped** if it has a limited number of states.

Distributed System

A system is called **distributed** if it has an infinite number of states.

Example 1: ADistributed System

$$y(t) = u(t-1)$$

Linear Systems

Linear System

A system is called **linear** if, for every t_0 and for any two pairs of state-input-output, the following two conditions hold:

$$\left.\begin{array}{l} x_1(t_0) \\ u_1(t), \quad t \geq t_0 \end{array}\right\} \rightarrow y_1(t),\ t \geq t_0 \qquad \left.\begin{array}{l} x_2(t_0) \\ u_2(t), \quad t \geq t_0 \end{array}\right\} \rightarrow y_2(t),\ t \geq t_0$$

1- Additivity

$$\left.\begin{array}{l} x_1(t_0) + x_2(t_0) \\ u_1(t) + u_2(t), \quad t \geq t_0 \end{array}\right\} \rightarrow y_1(t) + y_2(t),\ t \geq t_0$$

2- Homogenity

$$\left.\begin{array}{l} \alpha x_1(t_0) \\ \alpha u_1(t), \quad t \geq t_0 \end{array}\right\} \rightarrow \alpha y_1(t),\ t \geq t_0$$

The two properties can be combined to result in the principle of superposition.

$$\left.\begin{array}{l} \alpha_1 x_1(t_0) + \alpha_2 x_2(t_0) \\ \alpha_1 u_1(t) + \alpha_2 u_2(t), \quad t \geq t_0 \end{array}\right\} \rightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t),\ t \geq t_0$$

Linear Systems

Linear System property

Consider the zero-state response of a system as follows:

$$\left.\begin{array}{l} x(t_0)=0 \\ u(t),\quad t\ge t_0 \end{array}\right\} \rightarrow y_{zs}(t),\ t\ge t_0$$

Consider the zero-input response of a system as follows:

$$\left.\begin{array}{l} x(t_0) \\ u(t)=0,\quad t\ge t_0 \end{array}\right\} \rightarrow y_{zi}(t),\ t\ge t_0$$

Now, for a linear system, we have:

$$\left.\begin{array}{l} x(t_0) \\ u(t),\quad t\ge t_0 \end{array}\right\} \rightarrow y_{zs}(t)+y_{zi}(t),\ t\ge t_0$$

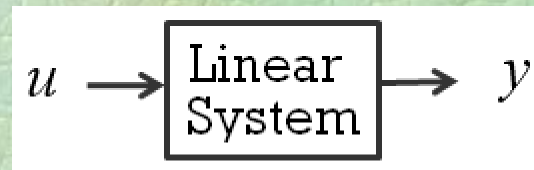
Therefore, in linear systems, we have:

$$y_{total}(t)=y_{zs}(t)+y_{zi}(t)$$

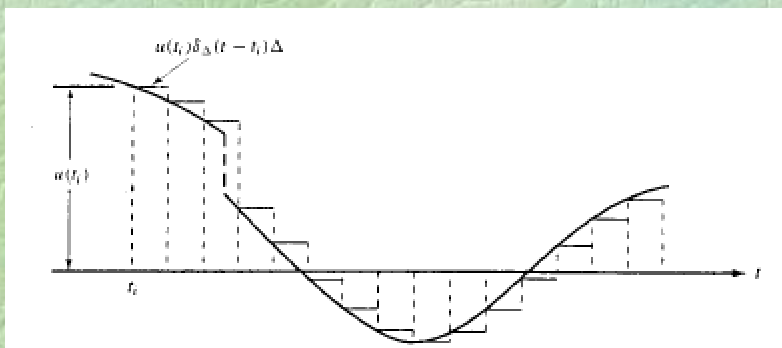
Zero-input response + zero-state response = complete response

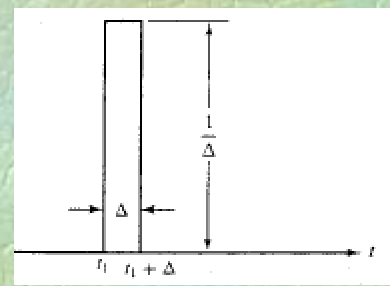
Linear Systems

Input-Output Description



Consider following input:





$$u(t) \approx \sum_i u(t_i)\delta_\Delta(t-t_i)\Delta$$

Suppose:

$$\delta_\Delta(t-t_i) \;\rightarrow\; g_\Delta(t,t_i)$$

According to homogeneity we have:

$$\delta_\Delta(t-t_i)u(t_i)\Delta \;\rightarrow\; g_\Delta(t,t_i)u(t_i)\Delta$$

According to additivity we have:

$$\sum_i \delta_\Delta(t-t_i)u(t_i)\Delta \;\rightarrow\; \sum_i g_\Delta(t,t_i)u(t_i)\Delta$$

So we have:

$$u(t) \;\rightarrow\; y(t) = \int_{-\infty}^{+\infty} g(t,\tau)u(\tau)d\tau$$

Linear Systems

Input-Output Description for Linear System

$$y(t) = \int_{-\infty}^{+\infty} g(t,\tau)u(\tau)d\tau$$

Input-Output Description for Causal Linear System

$$g(t,\tau) = 0 \quad if \quad t < \tau$$

$$y(t) = \int_{-\infty}^{t} g(t,\tau)u(\tau)d\tau$$

Input-Output Description for Causal and relaxed Linear System at t_0

$$y(t) = \int_{t_0}^{t} g(t,\tau)u(\tau)d\tau$$

In a MIMO case

$$\mathbf{y}(t) = \int_{t_0}^{t} \mathbf{G}(t,\tau)\mathbf{u}(\tau)d\tau$$

Linear Systems

State Space Description

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

Linear Time Invariant Systems

Linear Time Invariant Systems

A system is called time-invariant if, for every pair of state, input-output,

$$\left.\begin{array}{l} x(t_0) \\ u(t), \quad t \geq t_0 \end{array}\right\} \rightarrow y(t),\ t \geq t_0$$

and for any T, we have:

$$\left.\begin{array}{l} x(t_0+T) \\ u(t-T), \quad t \geq t_0+T \end{array}\right\} \rightarrow y(t-T),\ t \geq t_0+T$$

Input-Output Description for LTI

$$g(t,\tau) = g(t+T, \tau+T)$$

$$g(t,\tau) = g(t-\tau, \tau-\tau) = g(t-\tau, 0) = g(t-\tau)$$

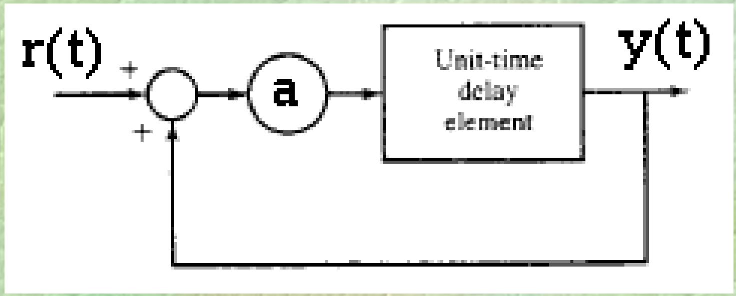
Very important $\quad g(t,\tau) = g(t-\tau)$ impulse response

Linear Time Invariant Systems

Example 2: The desired response of a system with a unit delay is the impulse response.

$$g(t) = \delta(t-1)$$

Example 3: The desired response of the following system:



$$g(t) = a\delta(t-1) + a^2\delta(t-2) + a^3\delta(t-3) + \ldots\ldots = \sum_{i=1}^{\infty} a^i \delta(t-i)$$

Example 4: The desired response of the system from the previous example to an arbitrary input *r*(*t*), which is zero for *t*<0 is:

$$y(t) = \int_0^t g(t-\tau) r(\tau) d\tau = \sum_{i=1}^{\infty} a^i \int_0^t \delta(t-i-\tau) r(\tau) d\tau$$

$$y(t) = \sum_{i=1}^{\infty} a^i r(\tau)\Big|_{\tau=t-i} = \sum_{i=1}^{\infty} a^i r(t-i)$$

Linear Time Invariant Systems

Input-Output Description for Linear Time Invariant System

$$y(t)=\int_{-\infty}^{+\infty} g(t,\tau)u(\tau)d\tau \quad \rightarrow$$

$$y(t)=\int_{-\infty}^{+\infty} g(t-\tau)u(\tau)d\tau=\int_{-\infty}^{+\infty} u(t-\tau)g(\tau)d\tau$$

Input-Output Description for Causal Linear Time Invariant System

$$y(t)=\int_{-\infty}^{t} g(t-\tau)u(\tau)d\tau$$

Input-Output Description for Causal and Relaxed Linear Time Invariant System

$$y(t)=\int_{0}^{t} g(t-\tau)u(\tau)d\tau$$

Transfer Function Matrix

The output of a system in the Laplace domain is represented as:

$$y(s) = \int_0^{\infty} y(t)e^{-st}dt$$

Using the input-output relationship,

$$y(s) = \int_{t=0}^{\infty} \left(\int_{\tau=0}^{\infty} g(t-\tau)u(\tau)d\tau \right) e^{-st}dt$$

$$y(s) = \int_{t=0}^{\infty} \left(\int_{\tau=0}^{\infty} g(t-\tau)u(\tau)d\tau \right) e^{-s(t-\tau)}e^{-s\tau}dt$$

$$y(s) = \int_{\tau=0}^{\infty} \left(\int_{t=0}^{\infty} g(t-\tau)e^{-s(t-\tau)}dt \right) u(\tau)e^{-s\tau}d\tau$$

$$y(s) = \int_{\tau=0}^{\infty} \left(\int_{v=-\tau}^{\infty} g(v)e^{-s(v)}dv \right) u(\tau)e^{-s\tau}d\tau$$

$$y(s) = \int_{\tau=0}^{\infty} g(s)u(\tau)e^{-s\tau}d\tau$$

$$y(s) = g(s)u(s)$$

Transfer Function Matrix

The transfer function = the input-output representation of a system in the Laplace domain.

$$y(s) = g(s)u(s)$$

Proper transfer function(tf): $g(s) \Leftrightarrow \deg D(s) \geq \deg N(s) \Leftrightarrow g(\infty) = cte$

Strictly proper tf: $g(s) \Leftrightarrow \deg D(s) > \deg N(s) \Leftrightarrow g(\infty) = 0$

Improper tf: $g(s) \Leftrightarrow \deg D(s) < \deg N(s) \Leftrightarrow g(\infty) = \infty$

Biproper tf: $g(s) \Leftrightarrow \deg D(s) = \deg N(s) \Leftrightarrow g(\infty) = cte \neq 0$

For a system with *p* inputs and *q* outputs, the transfer function is converted into a transfer matrix.

$$\begin{bmatrix} y_1(s) \\ y_2(s) \\ \vdots \\ y_q(s) \end{bmatrix} = \begin{bmatrix} g_{11}(s) & g_{12}(s) & \dots & g_{1p}(s) \\ g_{21}(s) & g_{22}(s) & \dots & g_{2p}(s) \\ \vdots & \vdots & \ddots & \vdots \\ g_{q1}(s) & g_{q2}(s) & \dots & g_{qp}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \\ \vdots \\ u_p(s) \end{bmatrix}$$

$$\mathbf{y}(s) = G(s)\mathbf{u}(s)$$

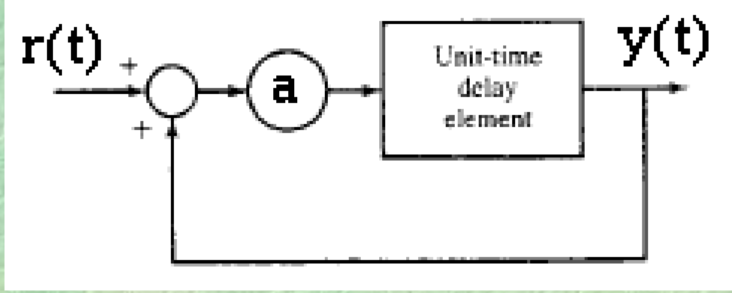
Transfer Function Matrix

Example 5: The transfer function of a system with a unit delay is:

$$g(t) = \delta(t-1)$$

$$g(s) = e^{-s}$$

Example 6: Find the transfer function of the following system:



$$g(s) = \frac{ae^{-s}}{1-ae^{-s}}$$

State Space Description for Linear Systems

State Space Description for LTI systems

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

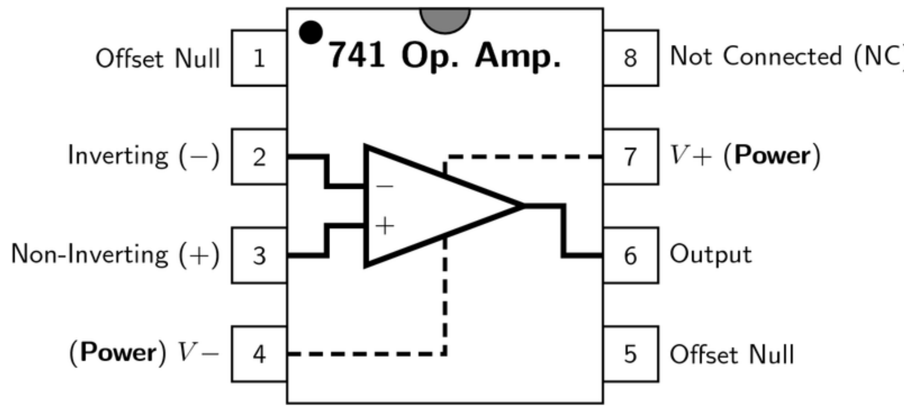
To calculate the transfer function, it is sufficient to take the Laplace transform of the state-space equations:

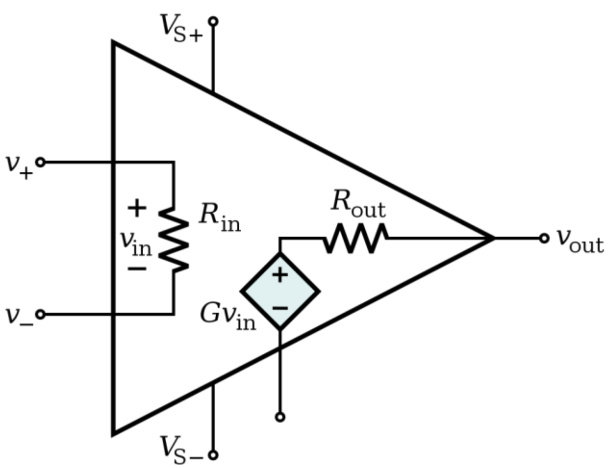
$$x(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}Bu(s)$$
$$y(s) = C(sI - A)^{-1}x(0) + C(sI - A)^{-1}Bu(s) + Du(s)$$

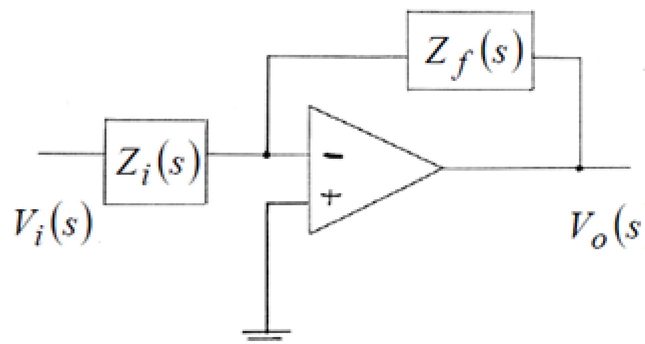
Therefore, the transfer function (assuming zero initial conditions) is given by:

$$g(s) = C(sI - A)^{-1}B + D$$

Op-Amp Circuit Implementation

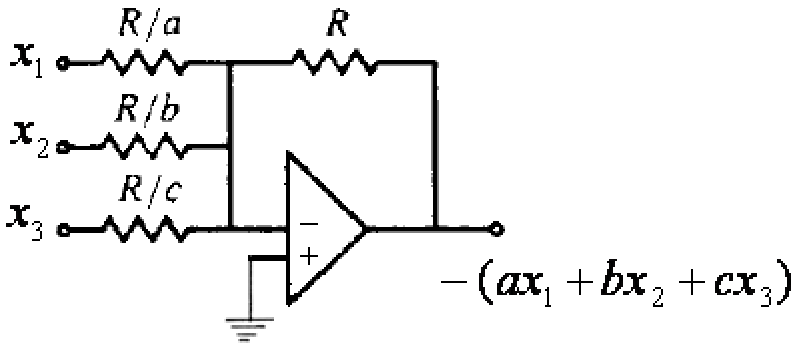


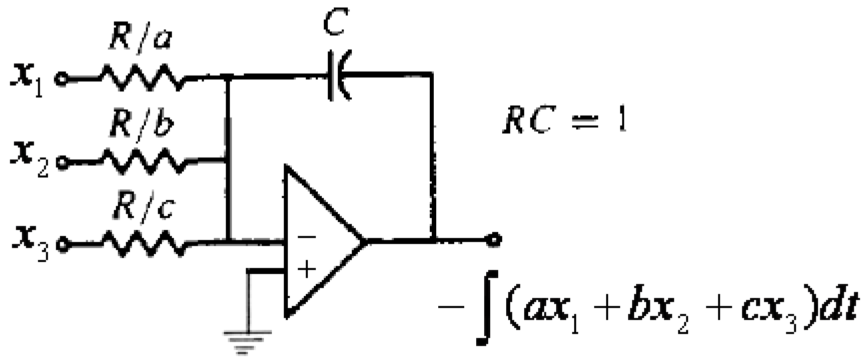




$$G(s) = \frac{V_o(s)}{V_i(s)} = -\frac{Z_f(s)}{Z_i(s)}$$

Op-Amp Circuit Implementation



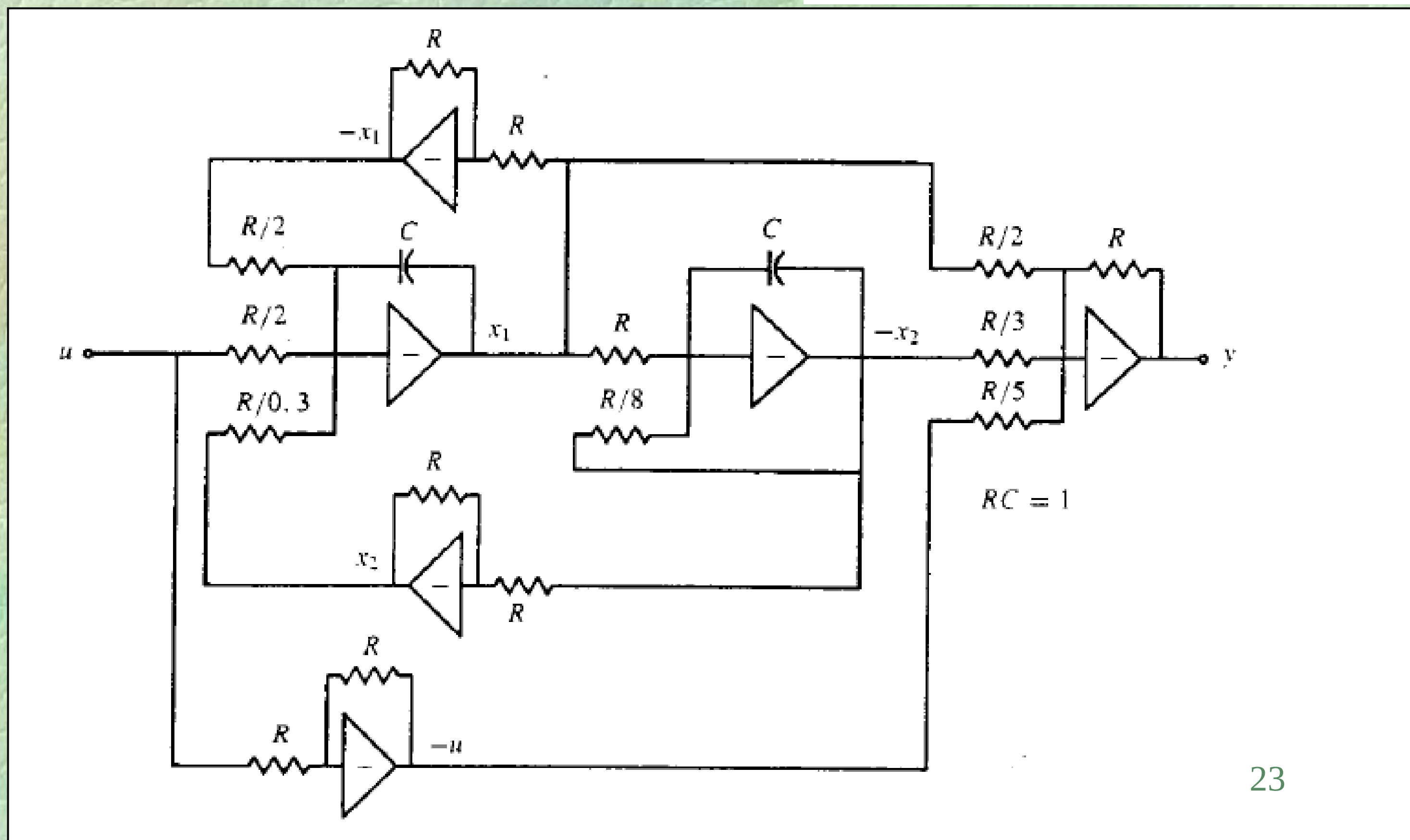


Op-Amp Circuit Implementation

Example 7: The implementation of an operational amplifier (op-amp) for the given system is desired.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}2 & -0.3\\ 1 & -8\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}-2\\ 0\end{bmatrix}u$$

$$y=\begin{bmatrix}-2 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+5u$$



Linearization

Although almost every real system includes nonlinear features, many systems can be reasonably described, at least within certain operating ranges, by linear models.

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = g(x(t), u(t))$$

Say that $\{x_Q(t), u_Q(t), y_Q(t)\}$ is a given set of trajectories that satisfy the above equations, so we have

$$\dot{x}_Q(t) = f(x_Q(t), u_Q(t)); \qquad x_Q(t_o) \text{ given}$$
$$y_Q(t) = g(x_Q(t), u_Q(t))$$

$$\dot{x}(t) \approx f(x_Q, u_Q) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}} (u(t) - u_Q)$$

$$y(t) \approx g(x_Q, u_Q) + \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}} (u(t) - u_Q)$$

Linearization

$$\dot{x}(t) \approx f(x_Q, u_Q) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}} (u(t) - u_Q)$$

$$y(t) \approx g(x_Q, u_Q) + \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}} (u(t) - u_Q)$$

$$\dot{x}(t) - f(x_Q, u_Q) \approx \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}} (u(t) - u_Q)$$

$$y(t) - g(x_Q, u_Q) \approx \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}} (u(t) - u_Q)$$

Linearization procedure

$$\dot{\delta x} = A\,\delta x + B\,\delta u$$

$$\delta y = C\,\delta x + D\,\delta u$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} ; \qquad B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}}$$

$$C = \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} ; \qquad D = \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}}$$

Linearization

Example 8: Consider the given system: $\frac{dx(t)}{dt} = f(x(t), u(t)) = -\sqrt{x(t)} + \frac{(u(t))^2}{3}$

Suppose the input has a small variation around 2; linearize the system around the given point.

$$u_Q = 2 \quad \Rightarrow \quad 0 = -\sqrt{x_Q} + \frac{2^2}{3} \quad \Rightarrow \quad x_Q = \frac{16}{9}$$

$$\text{Operating point}: \quad u_Q = 2, \; x_Q = \frac{16}{9}$$

Linearization procedure

$$\overset{\bullet}{\delta x} = A\,\delta x + B\,\delta u \qquad A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} ; \qquad B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}}$$

$$\delta y = C\,\delta x + D\,\delta u \qquad C = \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} ; \qquad D = \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}}$$

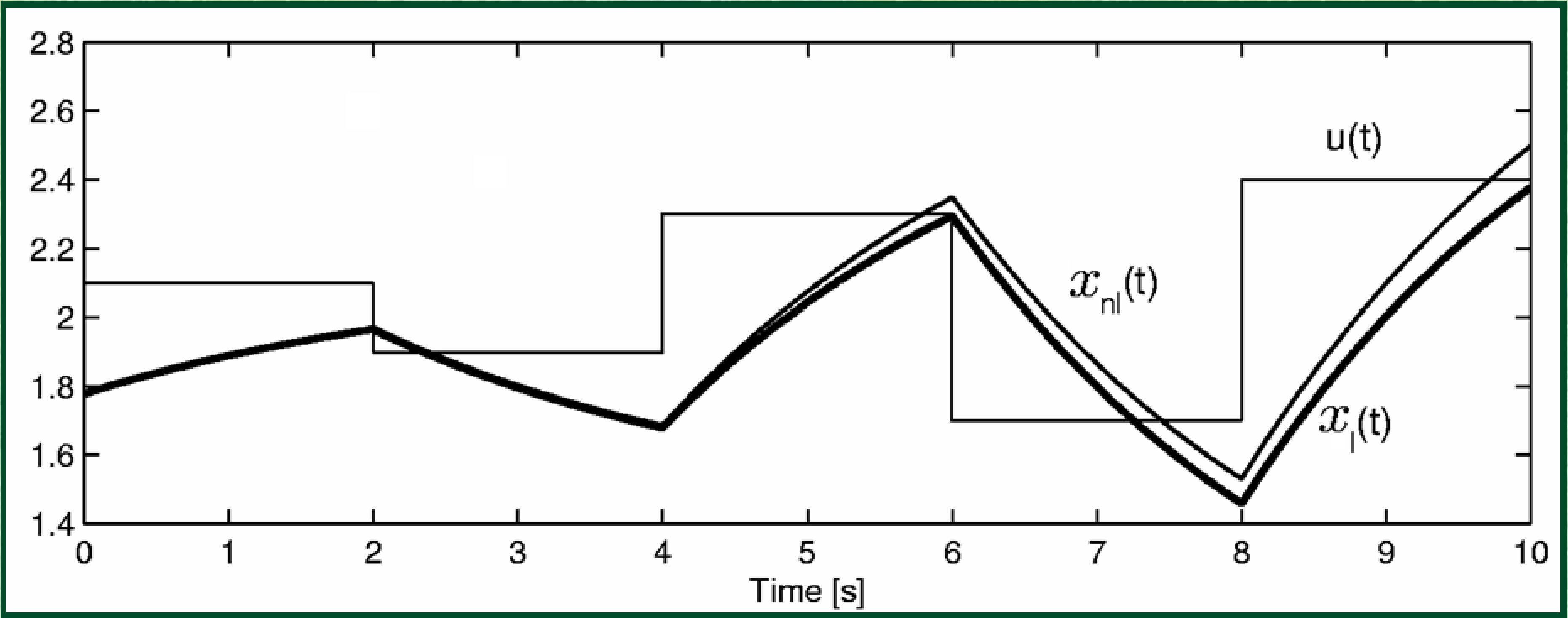
$$A = \frac{\partial f}{\partial x}\Big|_{x=x_Q, u=u_Q} = -\frac{1}{2\sqrt{x_Q}} = -\frac{3}{8}$$

$$B = \frac{\partial f}{\partial u}\Big|_{x=x_Q, u=u_Q} = \frac{2}{3}u_Q = \frac{4}{3}$$

$$\overset{\bullet}{\delta x} = -\frac{3}{8}\delta x + \frac{4}{3}\delta u$$

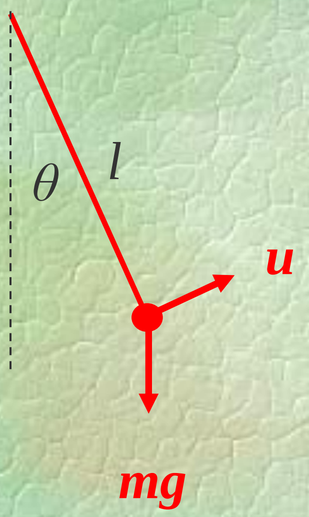
Linearization

Simulation



Linearization

Example 9: Linearize the following system around its equilibrium point.



$$J\frac{d^2\theta}{dt^2} = ul - mgl\sin\theta, \quad J = ml^2$$

$$\frac{d^2\theta}{dt^2} = \frac{u}{ml} - \frac{g}{l}\sin\theta \qquad x_1 = \theta, x_2 = \dot{\theta}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{u}{ml} - \frac{g}{l}\sin x_1$$

$$x_{1Q} = x_{2Q} = u_Q = 0$$

is operating point

Linearization procedure

$$\dot{\delta} x = A\,\delta x + B\,\delta u \qquad A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}}; \qquad B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}}$$

$$\delta y = C\,\delta x + D\,\delta u \qquad C = \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}}; \qquad D = \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}}$$

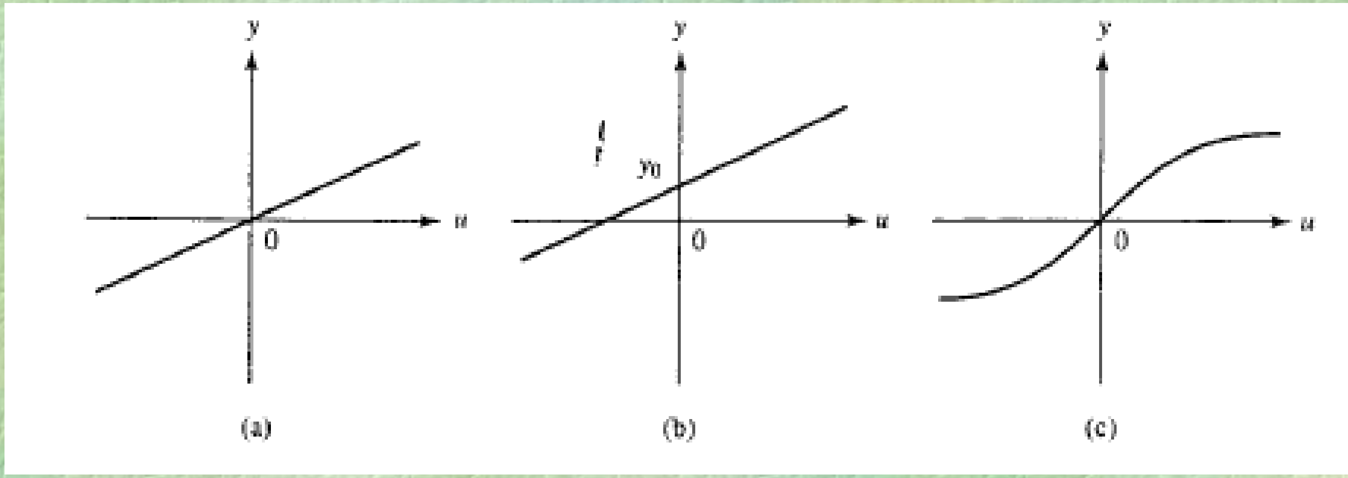
$$\begin{bmatrix}\delta\ddot{x}_1\\ \delta\ddot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{g}{l} & 0\end{bmatrix}\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{ml}\end{bmatrix}\delta u$$

Concluding Remarks

System Type	Internal Description	External Description
Distributed, linear	- - - - - - - - - - - -	$\mathbf{y}(t)=\int_{t_0}^{t}\mathbf{G}(t,\tau)\mathbf{u}(\tau)d\tau$
Lumped, linear	$\dot{x}(t)=A(t)x(t)+B(t)u(t)$ $y(t)=C(t)x(t)+D(t)u(t)$	$\mathbf{y}(t)=\int_{t_0}^{t}\mathbf{G}(t,\tau)\mathbf{u}(\tau)d\tau$
Distributed, linear time-invariant	- - - - - - - - - - - -	$\mathbf{y}(t)=\int_{0}^{t}\mathbf{G}(t-\tau)\mathbf{u}(\tau)d\tau$ $\mathbf{y}(s)=G(s)\mathbf{u}(s), G(s)\ irrational$
Lumped, linear time-invariant	$\dot{x}(t)=Ax(t)+Bu(t)$ $y(t)=Cx(t)+Du(t)$	$\mathbf{y}(t)=\int_{0}^{t}\mathbf{G}(t-\tau)\mathbf{u}(\tau)d\tau$ $\mathbf{y}(s)=G(s)\mathbf{u}(s), G(s)\ rational$

Exercises

Exercise 1: The following systems have zero initial conditions, and their input-output relationship is shown in the figure. Which system is linear? Why?



Exercise 2: The clipping operator is given by the following relationship. Is the system linear? Is the system time-invariant? Is the system causal? Provide a justification for each case.

$$y(t) = (P_\alpha u)(t) := \begin{cases} u(t) & \text{for } t \le \alpha \\ 0 & \text{for } t > \alpha \end{cases}$$

Exercises

Exercise 3: A linear system is subjected to the inputs $u_1(t)$, $u_2(t)$, and $u_3(t)$, In each case, the initial condition is $x(0)$. If we assume $x(0)=0$, which of the following statements is correct? Why?

If we assume $x(0)$, which of the following statements is correct? Why?

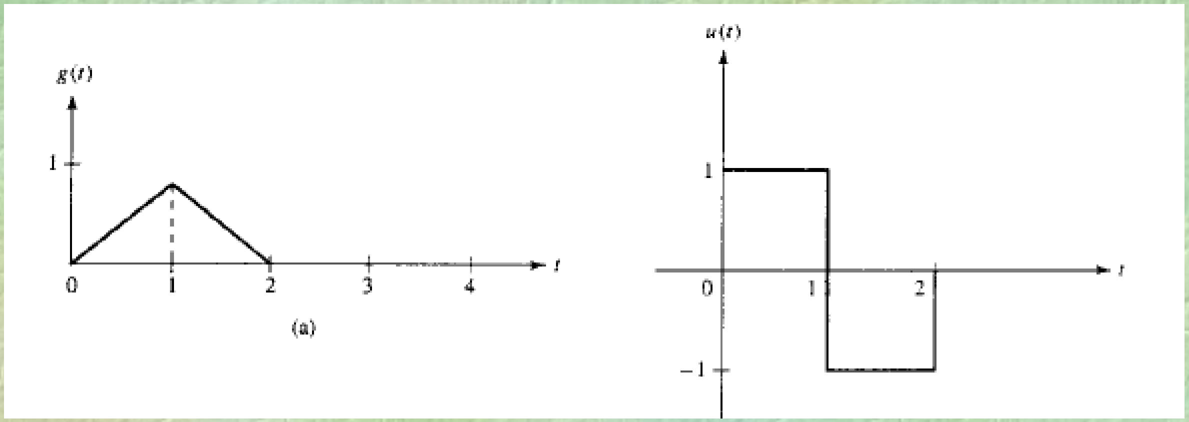
1. If $u_3 = u_1 + u_2$, then $y_3 = y_1 + y_2$.
2. If $u_3 = 0.5(u_1 + u_2)$, then $y_3 = 0.5(y_1 + y_2)$.
3. If $u_3 = u_1 - u_2$, then $y_3 = y_1 - y_2$.

Exercise 4: The system below has an initial condition of zero, and its input-output relationship is shown in the figure. Examine the properties of additivity and homogeneity.

$$y(t) = \begin{cases} u^2(t)/u(t-1) & \text{if } u(t-1) \neq 0 \\ 0 & \text{if } u(t-1) = 0 \end{cases}$$

Exercises

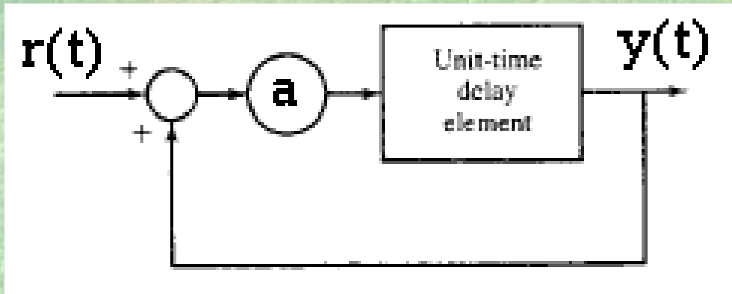
Exercise 5: The impulse response of a linear system and its input are shown in the figure below. Find the zero-state response of the system.



Exercise 6: Find the transfer function and impulse response of the system below.

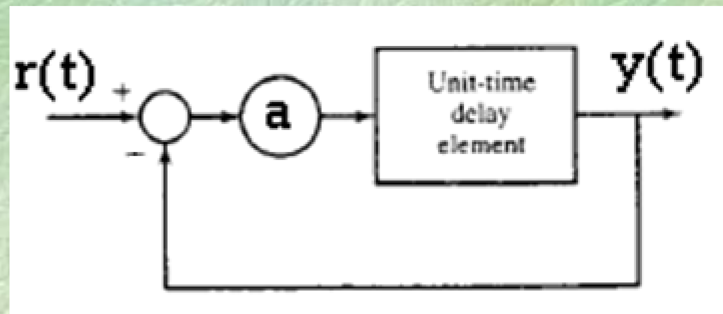
$$\ddot{y} + 2\dot{y} - 3y = \dot{u} - u$$

Exercise 7: Find the step response of the system for a=1and a=0.5.



Exercises

Exercise 8: Find the step response of the system for a=1and a=0.5.

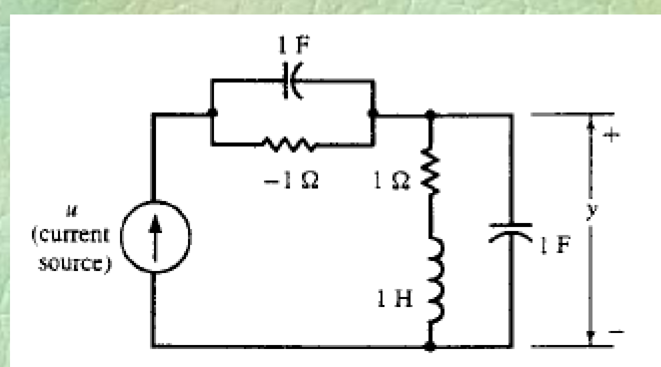


Exercise 9: Obtain the Bode diagram of the system below.

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 4 \\ 0 & 5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ -4 \end{bmatrix} u$$

$$y = [3 \quad 10]\mathbf{x} - 2u$$

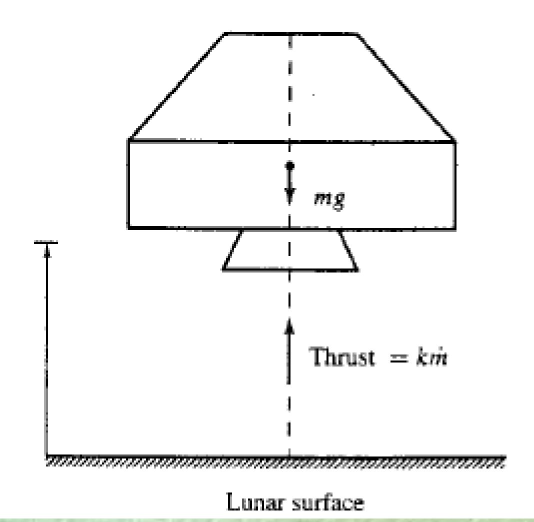
Exercise 10: Find the state-space equations and transfer function of the system below.



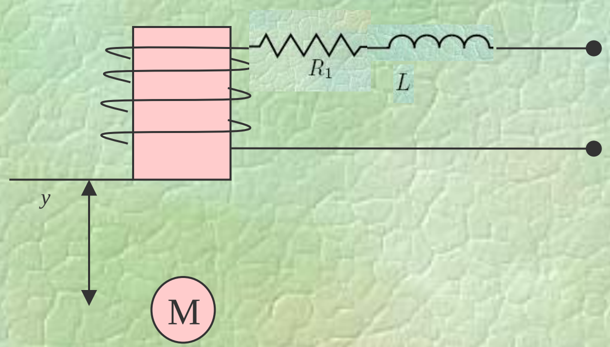
Exercises

Exercise 11: Find state-space equation for following system:

The soft landing phase of a lunar module descending on the moon can be modeled as shown in Fig. 2.24. The thrust generated is assumed to be proportional to $\dot{m}$, where m is the mass of the module. Then the system can be described by $m\ddot{y} = -k\dot{m} - mg$, where g is the gravity constant on the lunar surface. Define state variables of the system as $x_1 = y$, $x_2 = \dot{y}$, $x_3 = m$, and $u = \dot{m}$. Find a state-space equation to describe the system.



Example 10: Suppose electromagnetic force is i^2/y and find linearzed model around $y=y_0$



$$M\frac{d^2y}{dt^2} = Mg - \frac{i^2(t)}{y}$$

$$e(t) = R_1 i + L\frac{di}{dt}$$

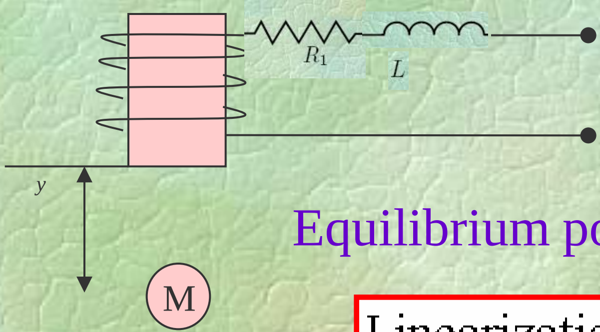
$$x_1 = y,\ x_2 = \dot{y},\ x_3 = i$$

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = g - \frac{1}{M}\frac{x_3{}^2}{x_1} \qquad \dot{x}_3 = -\frac{R_1}{L}x_3 + \frac{e(t)}{L}$$

Equilibrium point:

$$x_{1Q} = y_0 \qquad x_{2Q} = 0 \qquad x_{3Q} = \mathrm{i}_Q = \sqrt{Mgy_0} \qquad e_Q = R_1\sqrt{Mgy_0}$$

Example 10: Suppose electromagnetic force is i^2/y and find linearzed model around $y=y_0$



$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = g - \frac{1}{M}\frac{x_3^{\,2}}{x_1} \qquad \dot{x}_3 = -\frac{R_1}{L}x_3 + \frac{e(t)}{L}$$

Equilibrium point: $\left(x_{1Q}, x_{2Q}, x_{3Q}, e_Q\right) = \left(y_0, 0, \sqrt{Mgy_0}, R_1\sqrt{Mgy_0}\right)$

Linearization procedure

$$\delta\dot{x} = A\,\delta x + B\,\delta u \qquad A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} ; \qquad B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}}$$

$$\delta y = C\,\delta x + D\,\delta u \qquad C = \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} ; \qquad D = \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}}$$

$$\delta\dot{x}_1 = \delta x_2$$

$$\delta\dot{x}_2 = \frac{g}{y_0}\delta x_1 - 2\sqrt{\frac{g}{My_0}}\,\delta x_3$$

$$\delta\dot{x}_3 = -\frac{R_1}{L}\delta x_3 + \frac{\delta e(t)}{L}$$

$$\begin{bmatrix}\delta\dot{x}_1\\ \delta\dot{x}_2\\ \delta\dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ \frac{g}{y_0} & 0 & -2\sqrt{\frac{g}{My_0}}\\ 0 & 0 & -\frac{R_1}{L}\end{bmatrix}\begin{bmatrix}\delta x_1\\ \delta x_2\\ \delta x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \frac{1}{L}\end{bmatrix}\delta e(t)$$

Example 11: Consider the following nonlinear system. Suppose $u(t)=0$ and initial condition is $x_{10}=x_{20}=1$. Find the linearized system around response of system.

$$\dot{x}_1(t) = \frac{-1}{x_2(t)^2}$$

$$\dot{x}_2(t) = u(t)x_1(t)$$

$$\dot{x}_2(t) = 0.x_1(t) = 0 \quad \Rightarrow \quad x_2(t) = a = 1$$

$$\dot{x}_1(t) = -1 \quad \Rightarrow \quad x_1(t) = -t + b = -t + 1$$

Linearization procedure

$$\dot{\delta} x = A\,\delta x + B\,\delta u \qquad A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} ; \qquad B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}}$$

$$\delta y = C\,\delta x + D\,\delta u \qquad C = \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} ; \qquad D = \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}}$$

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1-t \end{bmatrix} \delta u(t)$$

Example 12: (Inverted pendulum)

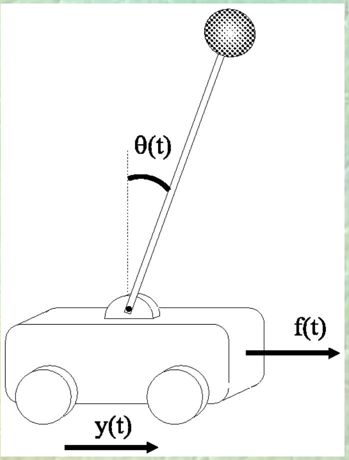
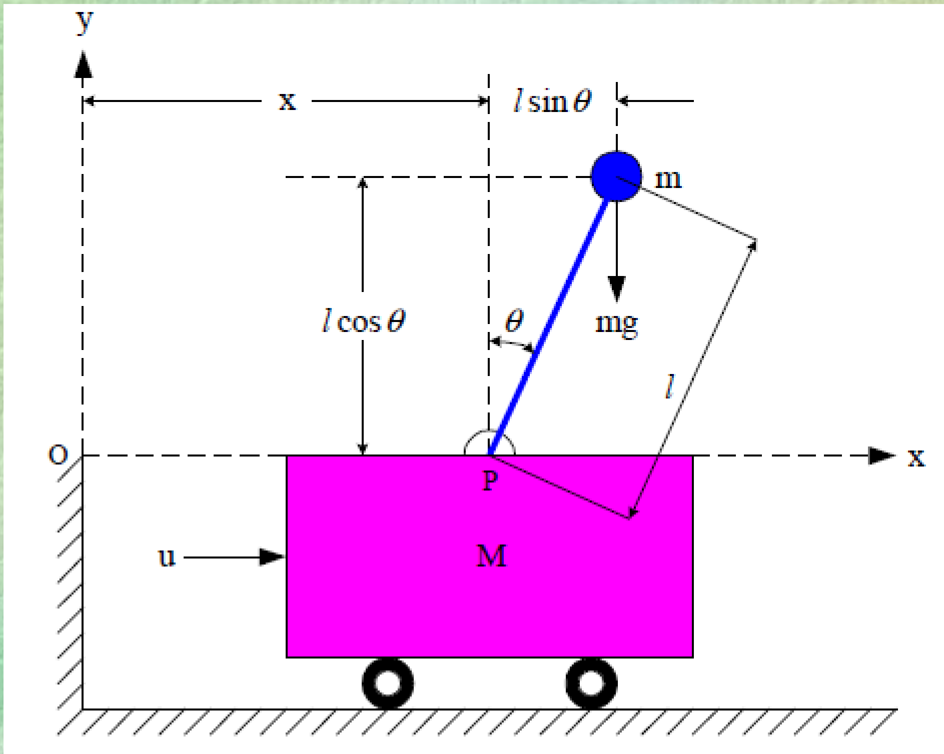


Figure : *Inverted pendulum*

$y(t)$	-	distance from some reference point
$\theta(t)$	-	angle of pendulum
M	-	mass of cart
m	-	mass of pendulum (assumed concentrated at tip)
l	-	length of pendulum
$f(t)$	-	forces applied to pendulum

Inverted Pendulum

Inverted Pendulum



$$M\frac{d^2x}{dt^2}+m\frac{d^2x_g}{dt^2}=u-c\frac{dx}{dt} \qquad x_g=x+l\sin\theta$$

$$(M+m)\frac{d^2x}{dt^2}+c\frac{dx}{dt}+ml\cos\theta\ddot{\theta}-ml\sin\theta\dot{\theta}^2=u$$

Inverted Pendulum

Application of Newtonian physics to this system leads to the following model:

$$\ddot{y} = \frac{1}{\lambda_m + \sin^2\theta(t)}\left[\frac{f(t)}{m} + \dot{\theta}^2(t)\ell\sin\theta(t) - g\cos\theta(t)\sin\theta(t)\right]$$

$$\ddot{\theta} = \frac{1}{\ell\lambda_m + \sin^2\theta(t)}\left[-\frac{f(t)}{m}\cos\theta(t) + \dot{\theta}^2(t)\ell\sin\theta(t)\cos\theta(t) + (1-\lambda_m)g\sin\theta(t)\right]$$

where $\lambda_m = (M/m)$

This is a linear state space model in which A, B and C are:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{M\ell} & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{M\ell} \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

Answers to selected problems

Answer 1: Linear, nonlinear, and nonlinear
Answer 2: Linnear, time invariant and casual.
Answer 3: For nonzero initial condition, no, yes, and no
For zero initial condition yes, yes, and yes
Answer 5: fot t<0 and t>4 the output is zero and

$$y(t)=\begin{cases} 0.5t^2 & for \quad 0\le t<1 \\ -1.5t^2+4t-2 & for \quad 1\le t\le 2 \\ -t(4-t) & for \quad 2<t\le 4 \end{cases}$$

Answer 6:

$$g(s)=\frac{1}{s+3}\ ,\ \ g(t)=e^{-3t} \quad for\ t\ge 0$$