LINEAR CONTROL SYSTEMS

Ali Karimpour
Professor
Ferdowsi University of Mashhad

Lecture 5

Time domain analysis of control systems

Topics to be covered include:

- Introduction
- Steady state error.
- Transient response of a some prototype systems.
- Different region of S plane.
- Transient response of a position control system.
- Dominant poles and approximation of high-order systems by low-order systems.
- Effect of zeros on the transfer function of a system.

Introduction

Steady state behavior.

Absolute stability.

Stability.

Steady state error.

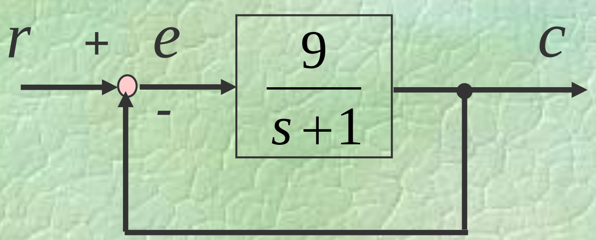
Dynamic behavior.

Relative stability.

Speed of response.

Deviation of response.

First-Order Loop Transfer Function System



$$T(s) = \frac{c(s)}{r(s)} = ?$$

$$T(s) = \frac{9}{s+10}$$

C(s)=T(s)R(s)

Step response?

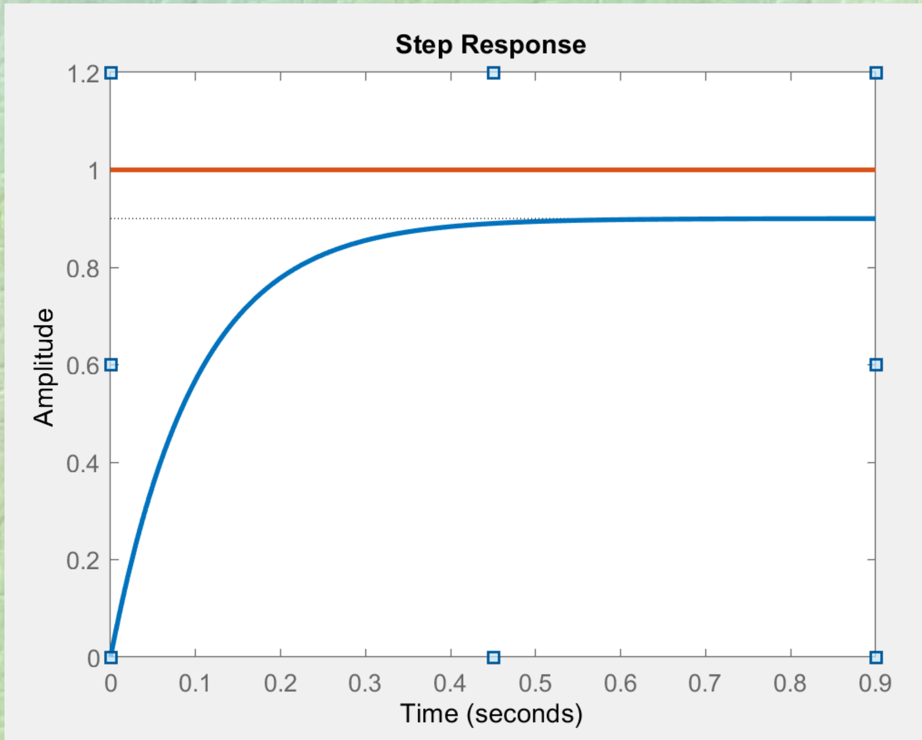
$$C(s) = \frac{9}{s+10}\frac{1}{s}$$

c(t)=?

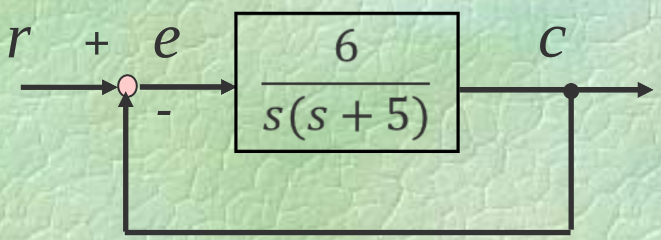
MATLAB

step(9,[1 10])

hold on;step(1,1)



Second-Order Loop Transfer Function System



$$T(s) = \frac{c(s)}{r(s)} = ?$$

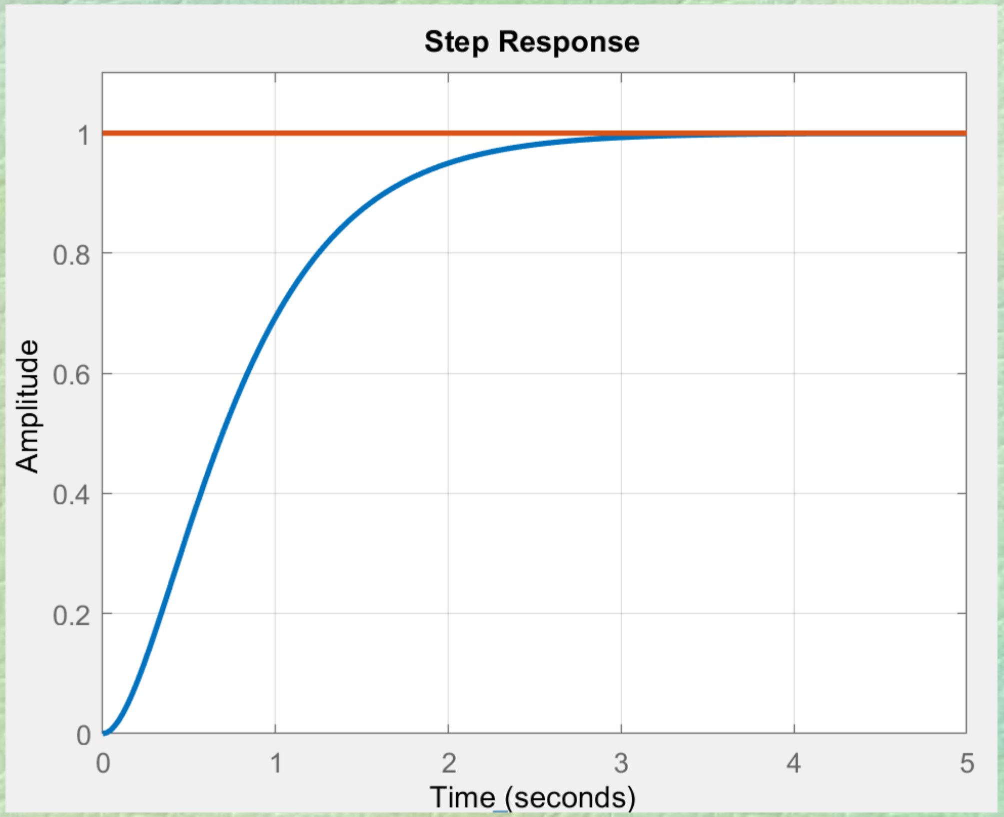
$$T(s) = \frac{6}{s^2 + 5s + 6}$$

C(s)=T(s)R(s)

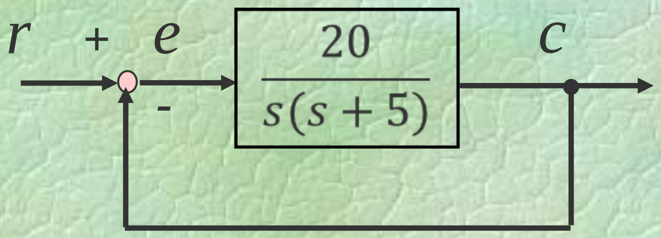
Step response?

$$C(s) = \frac{6}{s^2+5s+6}\frac{1}{s}$$

c(t)=?



Second-Order Loop Transfer Function System



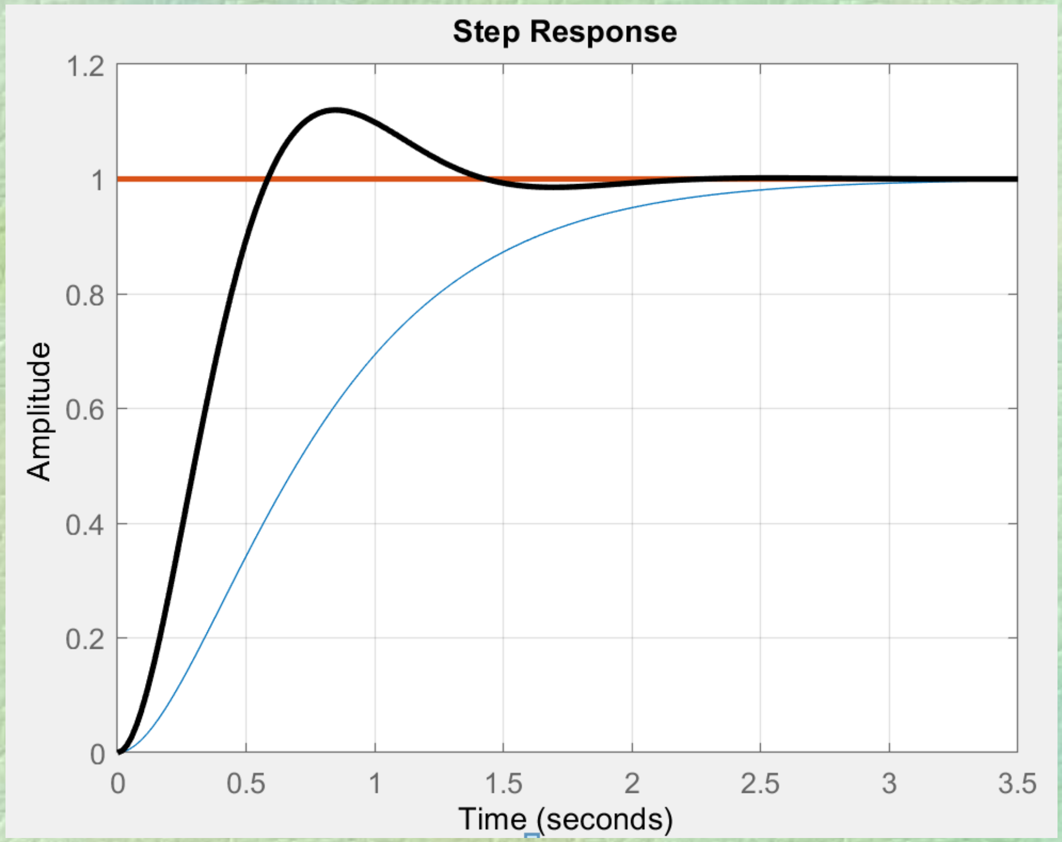
$$T(s)=\frac{c(s)}{r(s)}=?\quad T(s)=\frac{20}{s^2+5s+20}$$

C(s)=T(s)R(s)

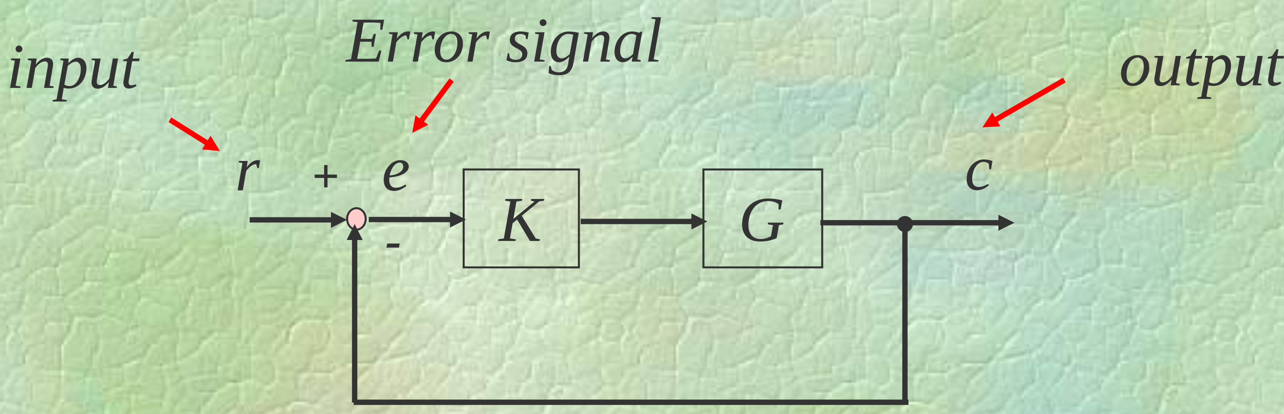
Step response?

$$C(s)=\frac{20}{s^2+5s+20}\frac{1}{s}$$

c(t)=?



Error in control systems



$e(t)=r(t)-c(t)$

$E(s)=R(s)-C(s)$

Loop transfer function

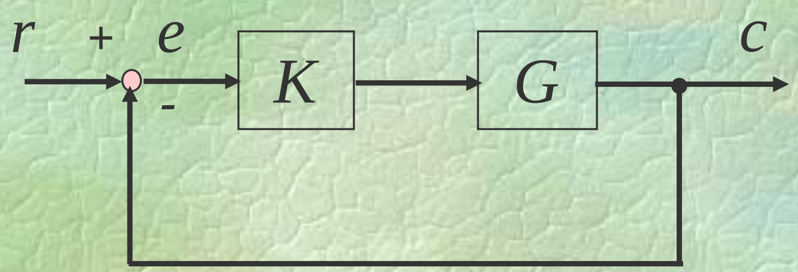
$$G(s)K(s) = \frac{k(1+\tau_1 s)(1+\tau_2 s)\ldots\ldots(1+\tau_m s)}{s^j(1+\tau_{d1} s)(1+\tau_{d2} s)\ldots\ldots(1+\tau_{dn} s)} e^{-T_d s}$$

type of system

Type of system

Relative degree

Error in control systems



$$T(s)=\frac{G(s)K(s)}{1+G(s)K(s)}$$

$$E(s)=R(s)-T(s)R(s)$$

If the system is stable:
(very important)

1

$$e_{ss}=\lim_{t\to\infty}e(t)=\lim_{s\to 0}sE(s)=\boxed{\lim_{s\to 0}s\big(R(s)-T(s)R(s)\big)}$$

$$e_{ss}=\lim_{t\to\infty}e(t)=\lim_{s\to 0}sE(s)=\boxed{\lim_{s\to 0}\frac{s}{1+G(s)K(s)}R(s)}$$

2

Error in control systems for step input

1

$$e_{ss} = \lim_{s\to 0} sE(s) = \boxed{\lim_{s\to 0} s\big(R(s) - T(s)R(s)\big)}$$

$$R(s) = \frac{R}{s}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \boxed{\lim_{s\to 0} \frac{s}{1+G(s)K(s)} R(s)}$$

2

$$\vdots$$

$$e_{ss} = R\big(1 - \lim_{s\to 0} T(s)\big)$$

$$e_{ss} = \frac{R}{1+K_p}$$

$$K_p = ?$$

Position constant

Error in control systems for velocity input

1

$$e_{ss} = \lim_{s\to 0} sE(s) = \boxed{\lim_{s\to 0} s\left(R(s) - T(s)R(s)\right)}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \boxed{\lim_{s\to 0} \frac{s}{1+G(s)K(s)} R(s)}$$

2

$$R(s) = \frac{R}{s^2}$$

$$\vdots$$

$$e_{ss} = R\lim_{s\to 0}\left(\frac{1-T(s)}{s}\right)$$

$$e_{ss} = \frac{R}{K_v}$$

$$K_v = ?$$

Velocity constant

Error in control systems for parabolic input

1

$$e_{ss} = \lim_{s\to 0} sE(s) = \boxed{\lim_{s\to 0} s\big(R(s) - T(s)R(s)\big)}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \boxed{\lim_{s\to 0} \frac{s}{1+G(s)K(s)} R(s)}$$

2

$$R(s) = \frac{R}{s^3}$$

$$\vdots$$

$$e_{ss} = R\lim_{s\to 0}\left(\frac{1-T(s)}{s^2}\right)$$

$$e_{ss} = \frac{R}{K_a}$$

$$K_a = ?$$

Acceleration constant

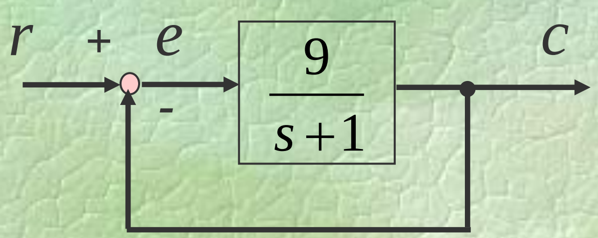
Error in control systems

$$G(s)K(s)=\frac{k(1+\tau_1 s)(1+\tau_2 s).......(1+\tau_m s)}{s^j(1+\tau_{d1}s)(1+\tau_{d2}s).......(1+\tau_{dn}s)}e^{-T_d s}$$

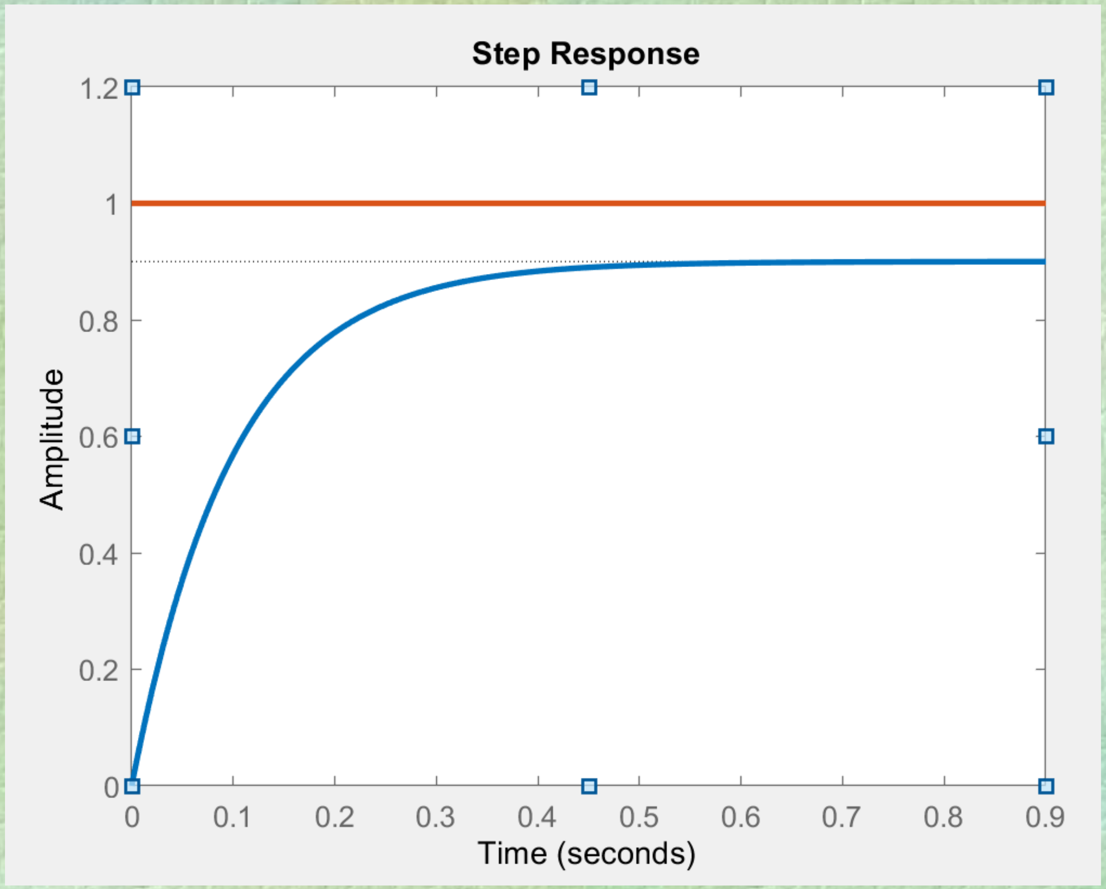
System Type	K_p	K_v	K_a	position e_{ss}	velocity e_{ss}	acceleration e_{ss}
0	k	0	0	$\frac{R}{1+k}$	∞	∞
1	∞	k	0	0	$\frac{R}{k}$	∞
2	∞	∞	k	0	0	$\frac{R}{k}$
3	∞	∞	∞	0	0	0

Error in control systems

Example 1: Step and velocity response

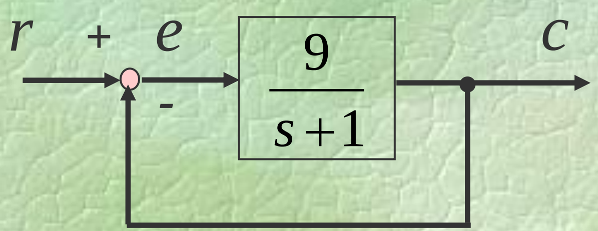


$$T(s) = \frac{9}{s+10}$$



Error in control systems

Example 1: Step and velocity response

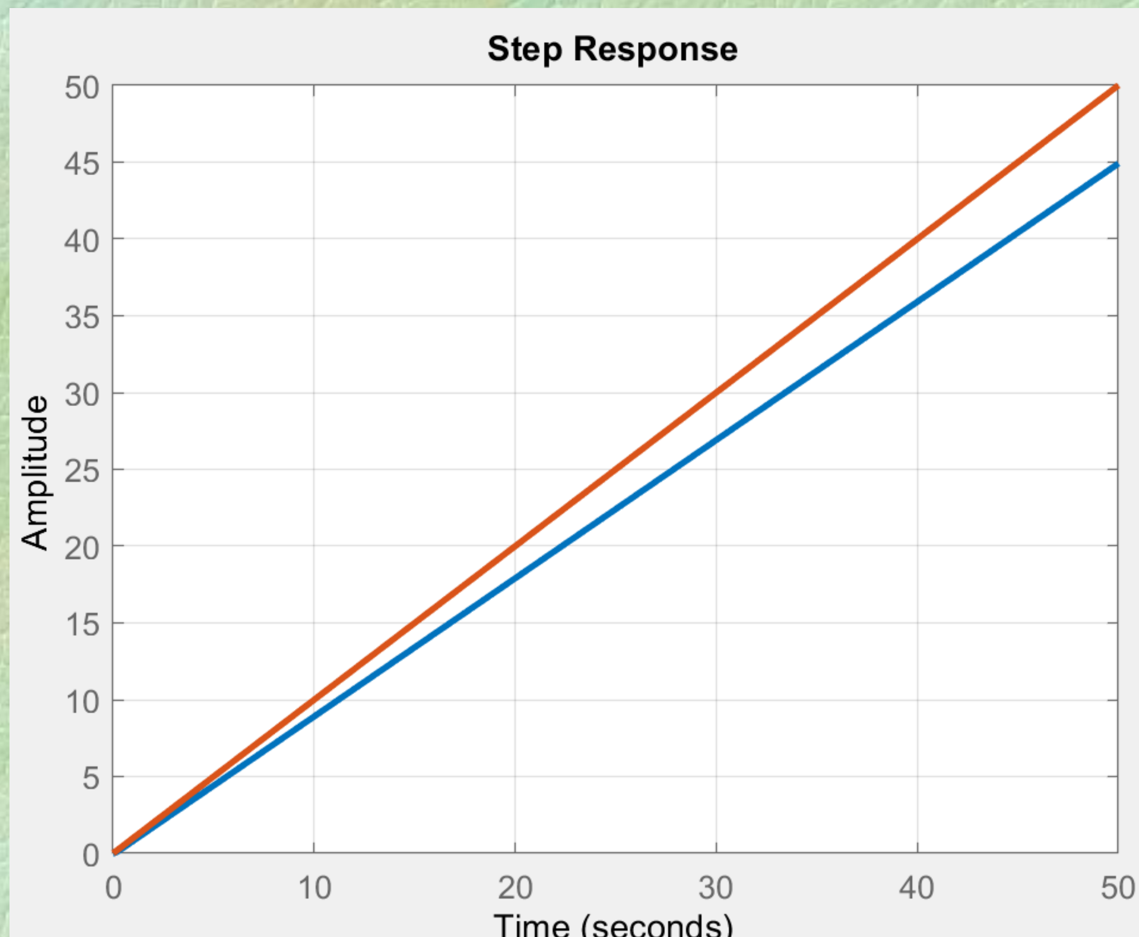


$$T(s) = \frac{9}{s+10}$$

MATLAB

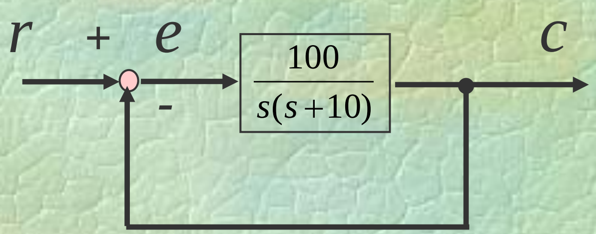
step(9,[1 10 0])

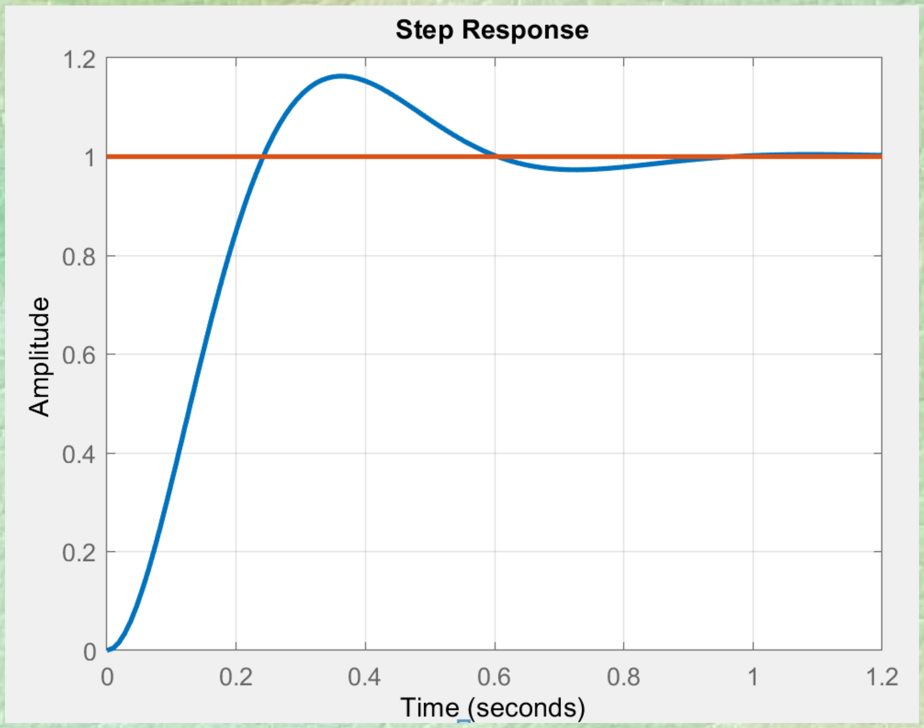
hold on;step(1,[1 0])

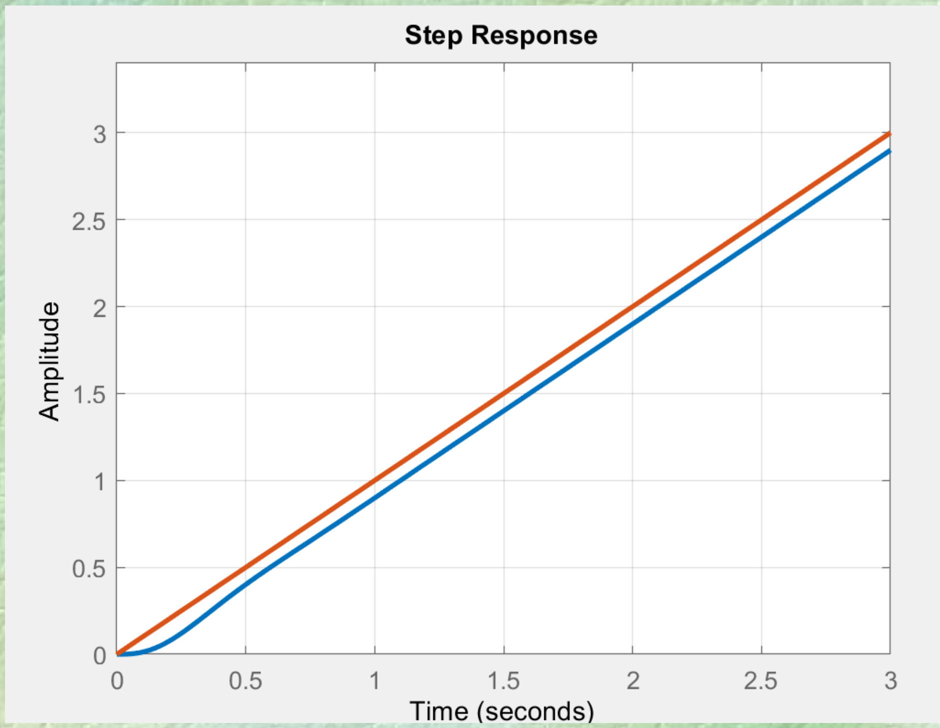


Error in control systems

Example 2: Derive errors to unit step, unit velocity and unit acceleration.

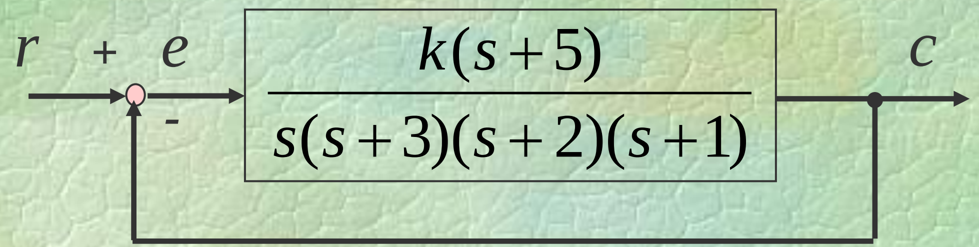






Error in control systems

Exercise1: Derive k such that the system error to unit velocity be 0.1.



…………. It is not possible.

Exercise 2: The closed loop transfer function of a system is given. Determine *a* such that the system error to step input is zero.

$$T(s)=\frac{a}{s^3+12s^2+6s+23}$$

…………..

Time domain analysis of control systems

Topics to be covered include:

- Introduction
- Steady state error.
- Introducing some performance criteria (ISE, ITSE, IAE and ITAE).
- Transient response of a some prototype systems.
- Different region of S plane.
- Transient response of a position control system.
- Dominant poles and approximation of high-order systems by low-order systems.
- Effect of zeros on the transfer function of a system.

Introducing some prototype systems

Introduction to a First-Order Sample System

$$r \longrightarrow \boxed{T(s)=\frac{k}{Ts+1}} \longrightarrow c$$

Step response …………..

Steady state error …………..

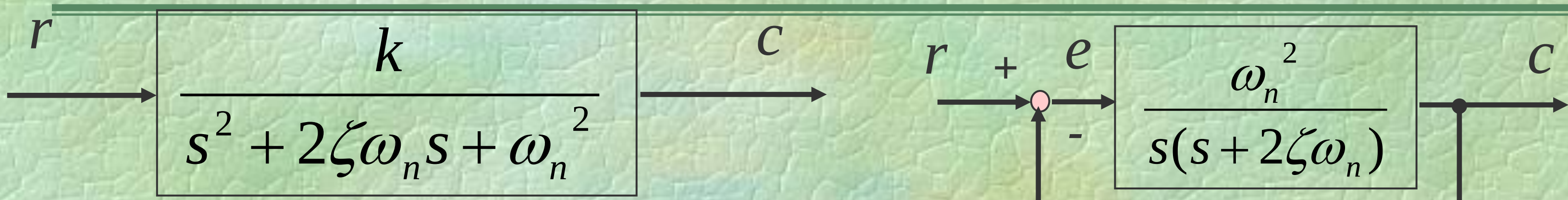
Time constant…………..

Settling time…………..

Effect of T on speed…………..

Introducing some prototype systems

Introduction to a Second-Order Sample System

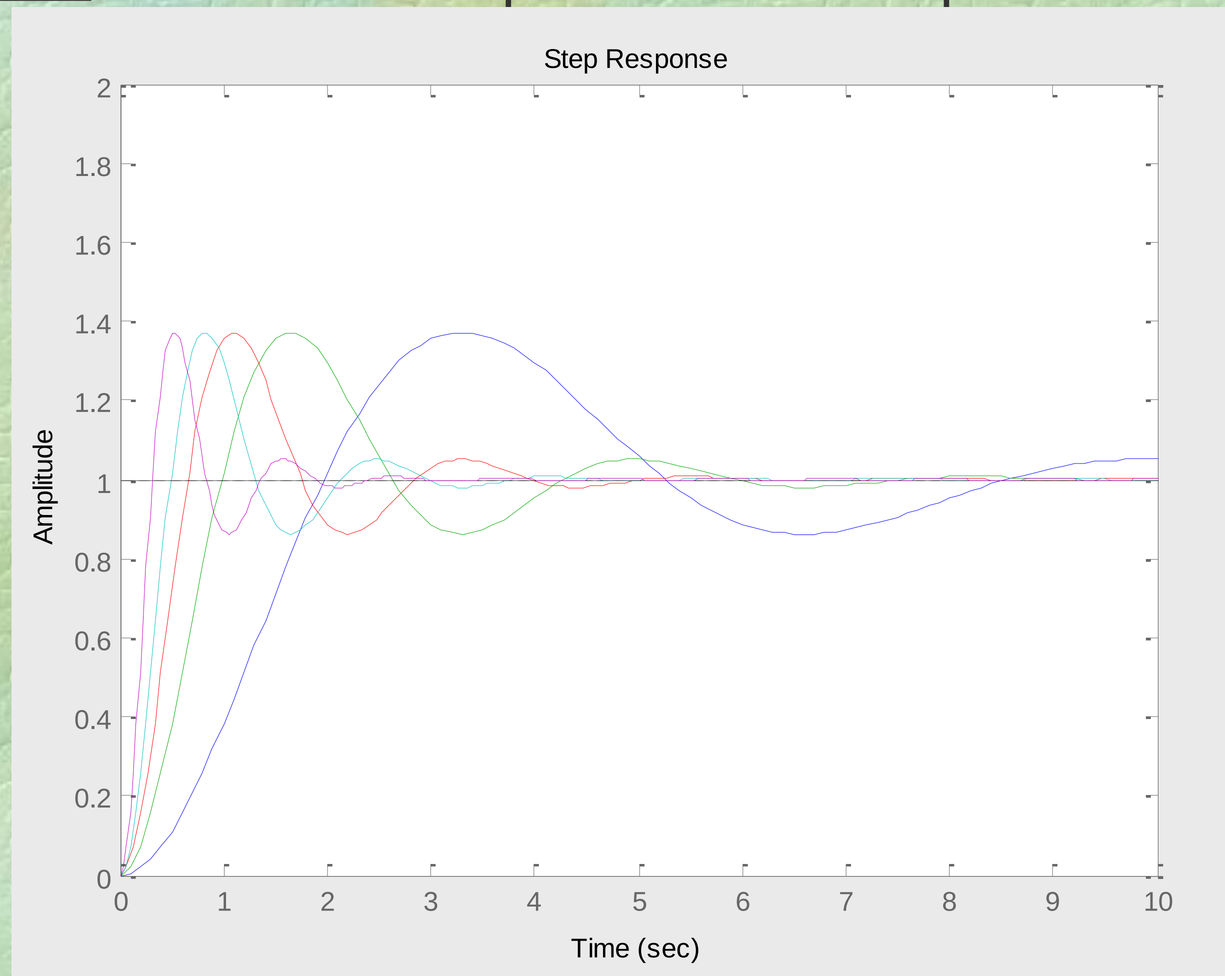


Step Response

$$\zeta = 0.3 \quad \omega_n = 1, 2, 3, 4, 6$$

MATLAB

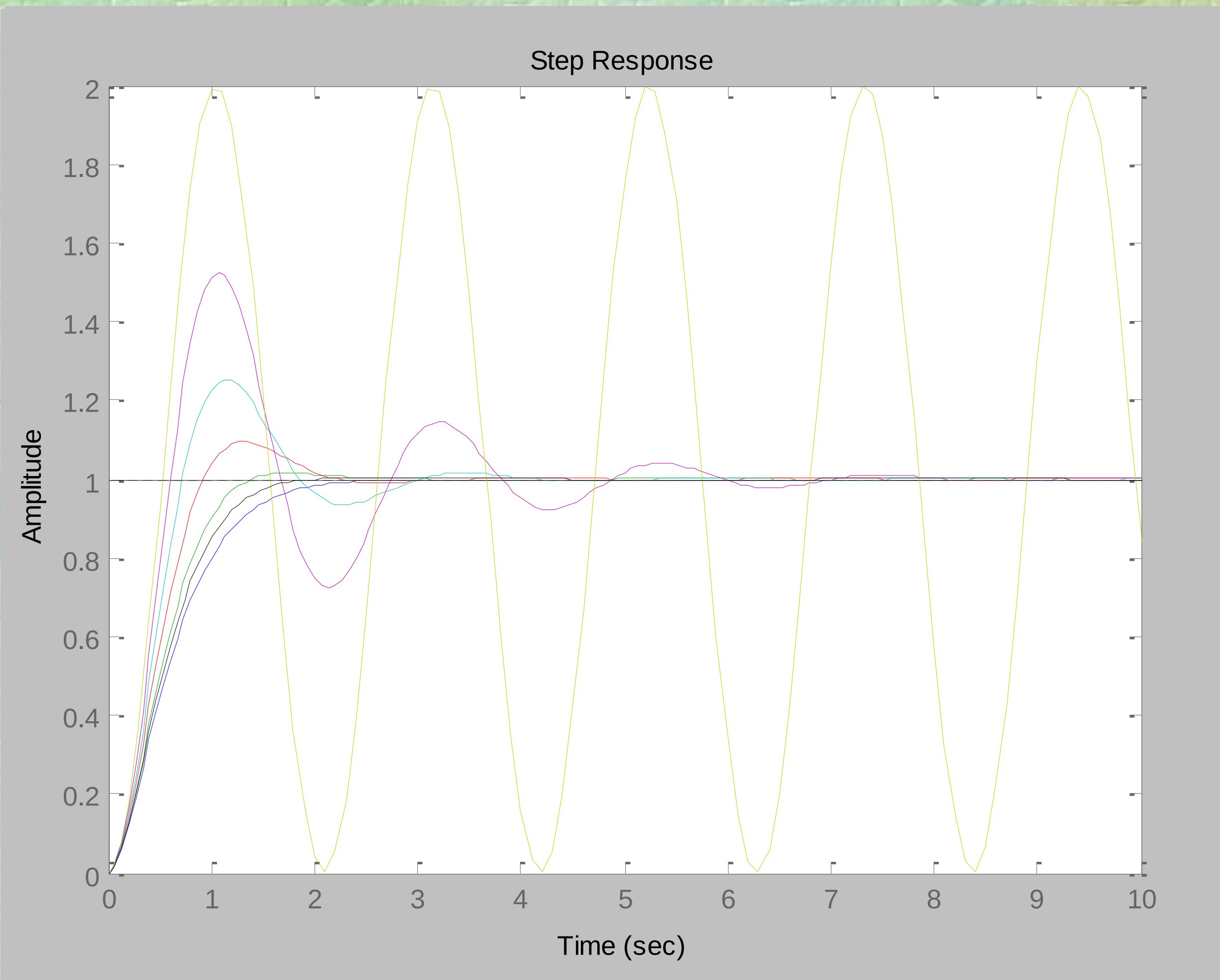
....



Step response

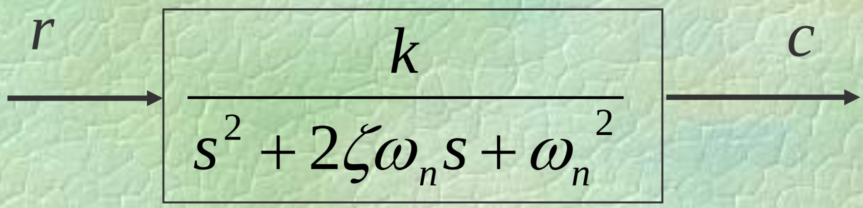
$$r \longrightarrow \boxed{\dfrac{\omega_n^{\,2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}}} \longrightarrow c$$

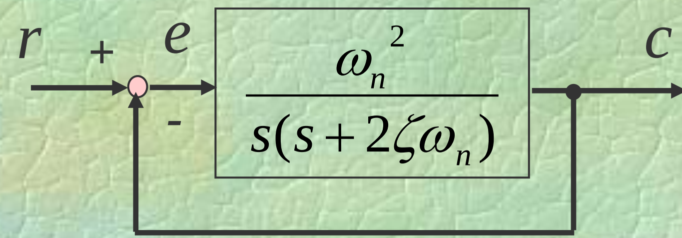
Step response $\omega_n = 3 \quad \zeta = 1, 0.8, 0.6, 0.4, 0.2,\ 0$



Introducing some prototype systems

Introduction to a Second-Order Sample System





Step response …………..

Steady state error …………..

Rise time, Peak time, settling time, Percent overshoot …………..

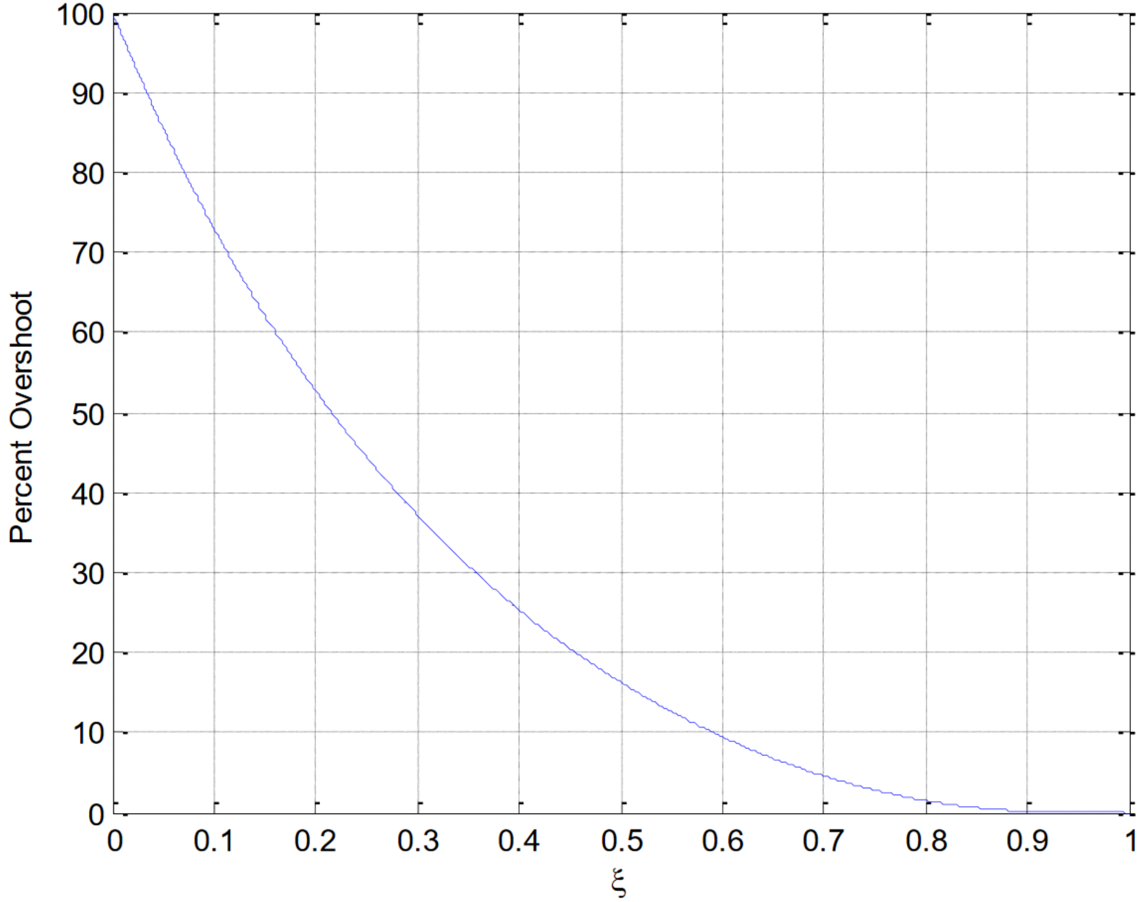
Speed of sytem …………….

$$t_r = \frac{0.8 + 2.5\zeta}{\omega_n}$$

$$t_r = \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n} \qquad 0 < \zeta < 1$$

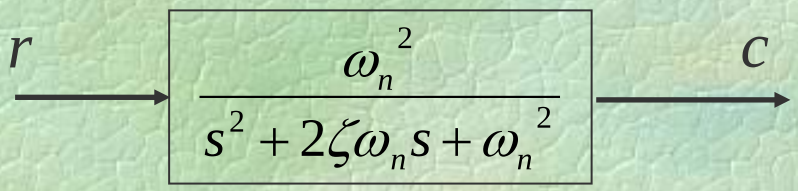
Introducing some prototype systems

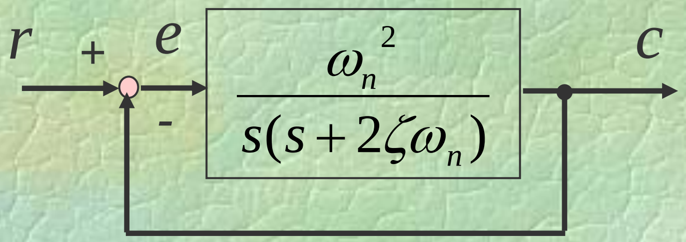
$$P.O. = \%100 e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$



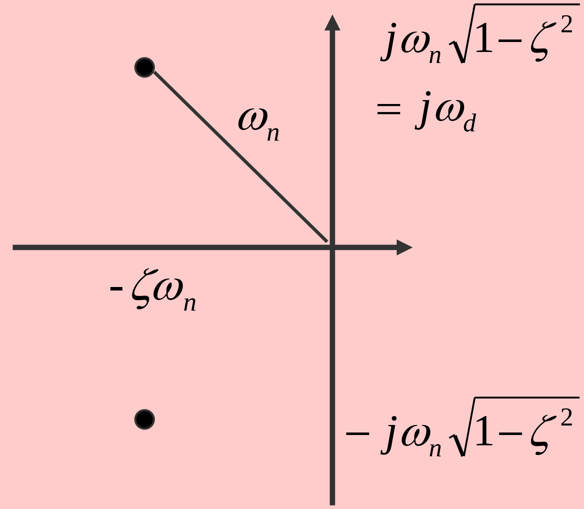
ζ	$P.O.$
0	100%
0.100	73%
0.200	53%
0.300	37%
0.400	25%
0.500	16%
0.707	4.3%
1	0%

Introducing a prototype second order system.





$$\text{Poles are: } -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_d \quad \text{if } 0 \le \zeta \le 1$$



ω_n Natural frequency

$\zeta\omega_n$ Damping factor

ζ Damping ratio

$\omega_d = \omega_n\sqrt{1-\zeta^2}$ Natural damped frequency

Constant loci.

Constant natural frequency loci.

…………

Constant damped frequency loci.

…………

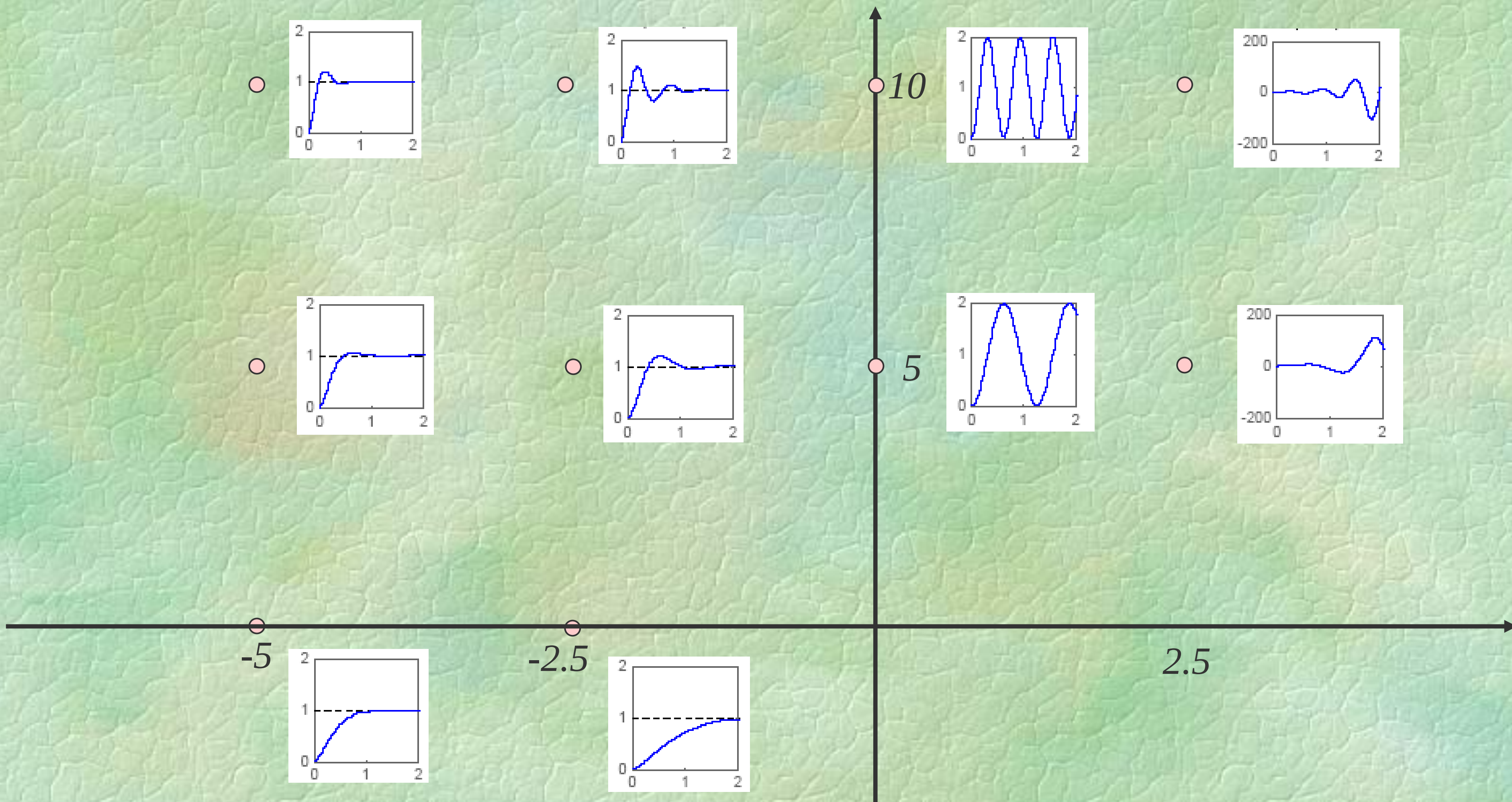
Constant damping factor loci.

…………

Constant damping ratio loci.

…………

Effect of roots loci on step response



Conjugate root is not shown

Time domain analysis

Exercise 3: Find the loci that the percent overshoot is less than 16% and settling time is less than 1 sec (according to 5% bound).

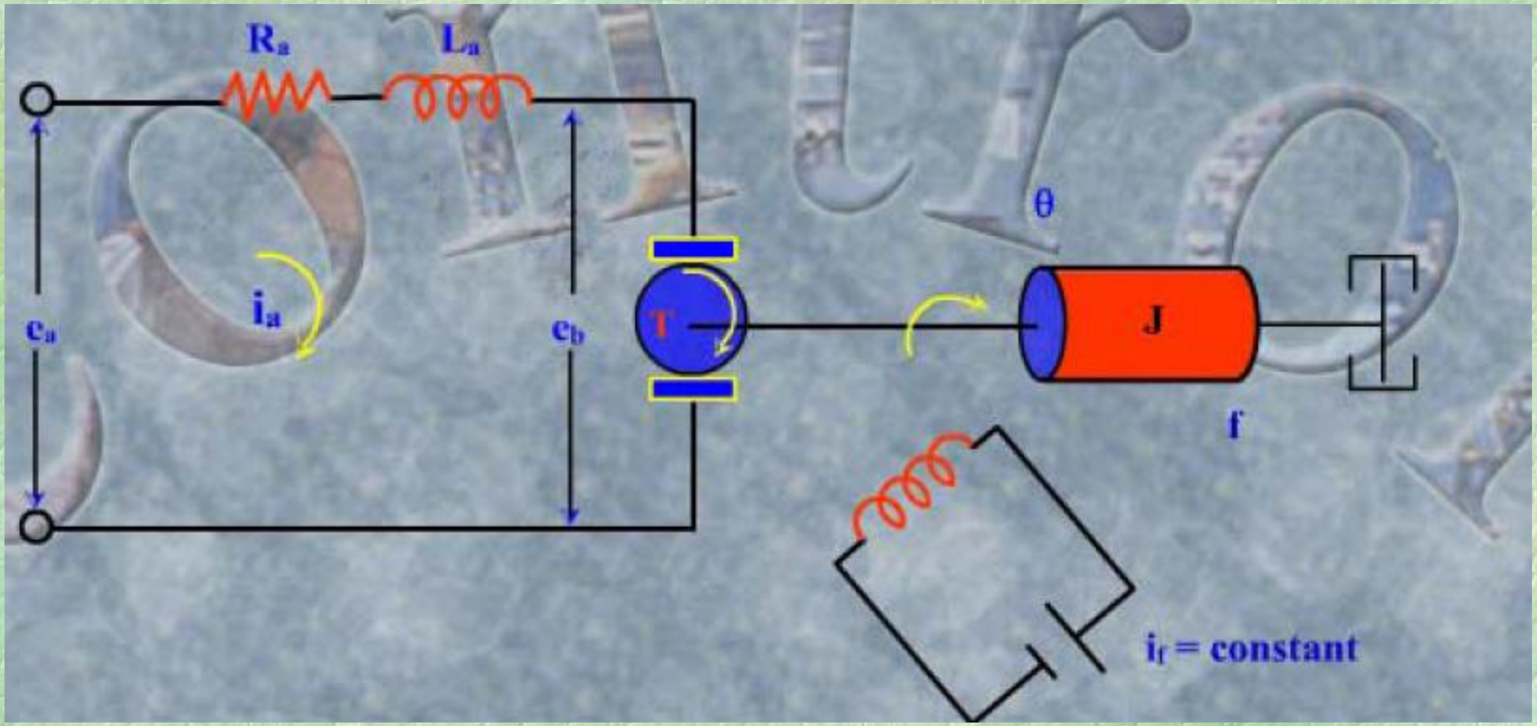
…………

Time domain analysis of control systems

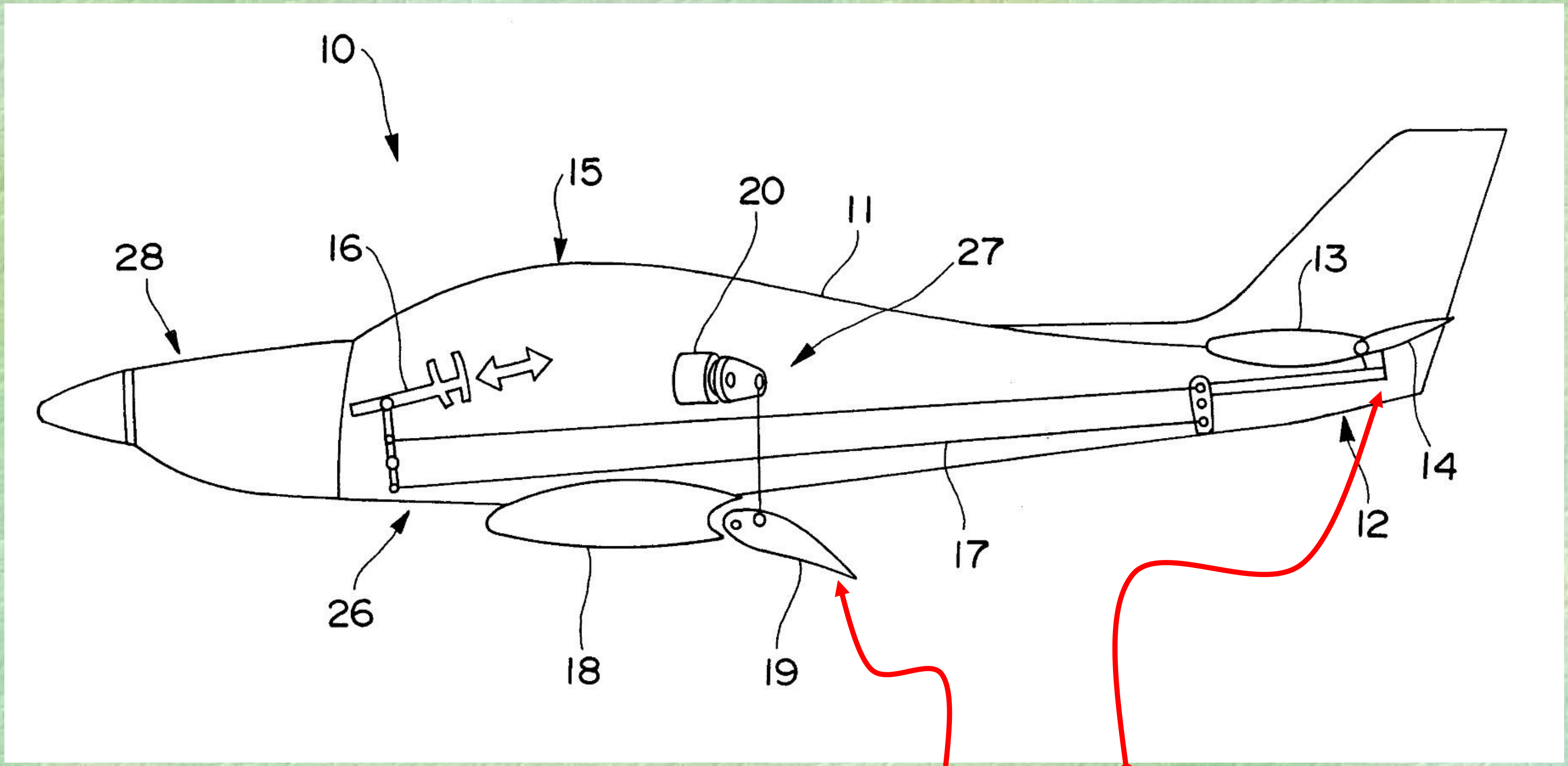
Topics to be covered include:

- Steady state error.
- Introducing some performance criteria (ISE, ITSE, IAE and ITAE).
- Transient response of a prototype second order system.
- Different region of S plane.
- Transient response of a position control system.
- Dominant poles and approximation of high-order systems by low-order systems.
- Effect of zeros on the transfer function of a system.

Dynamics of electromechanical system

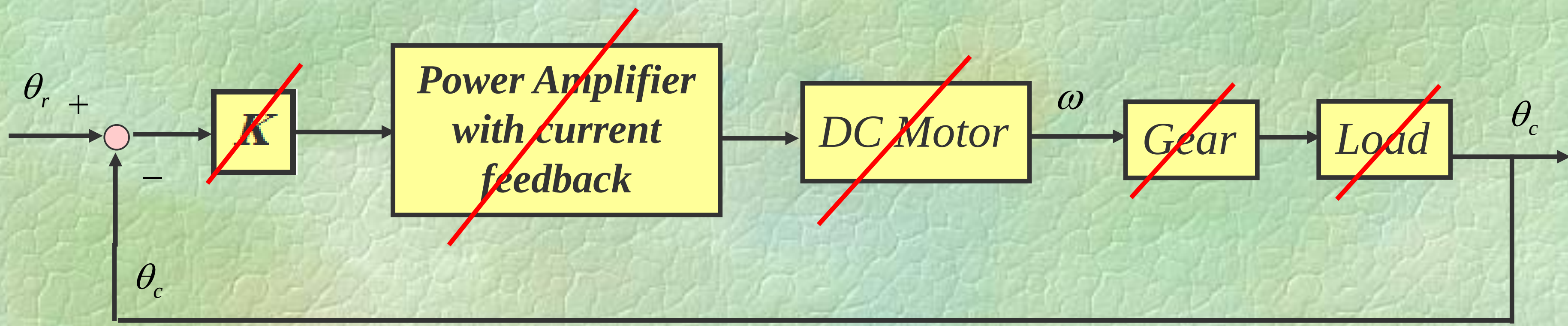


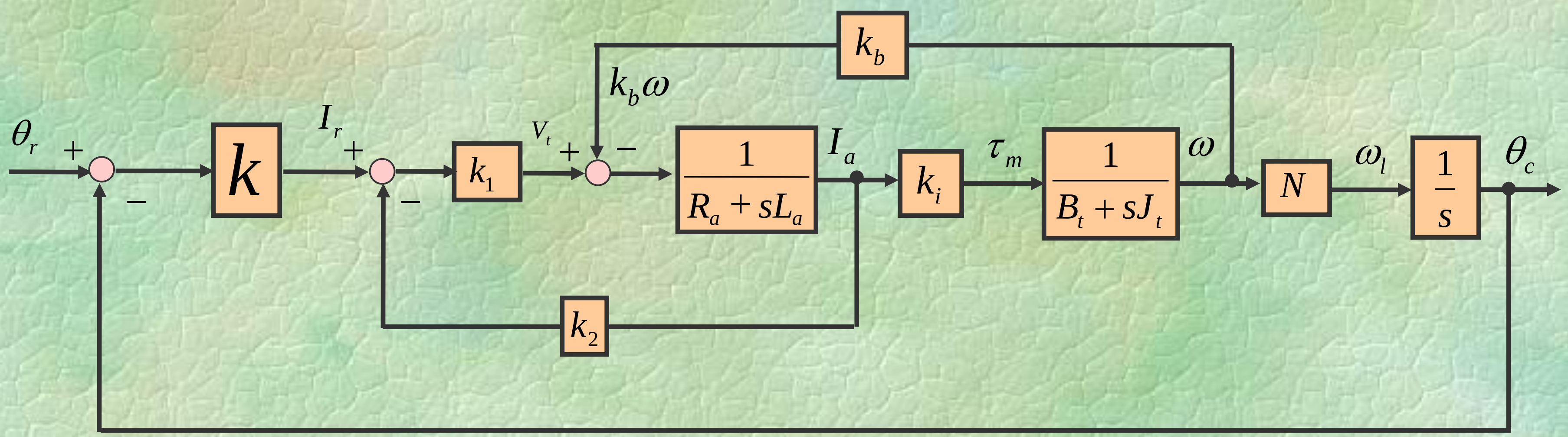
A simplified aeroplane (position control system)



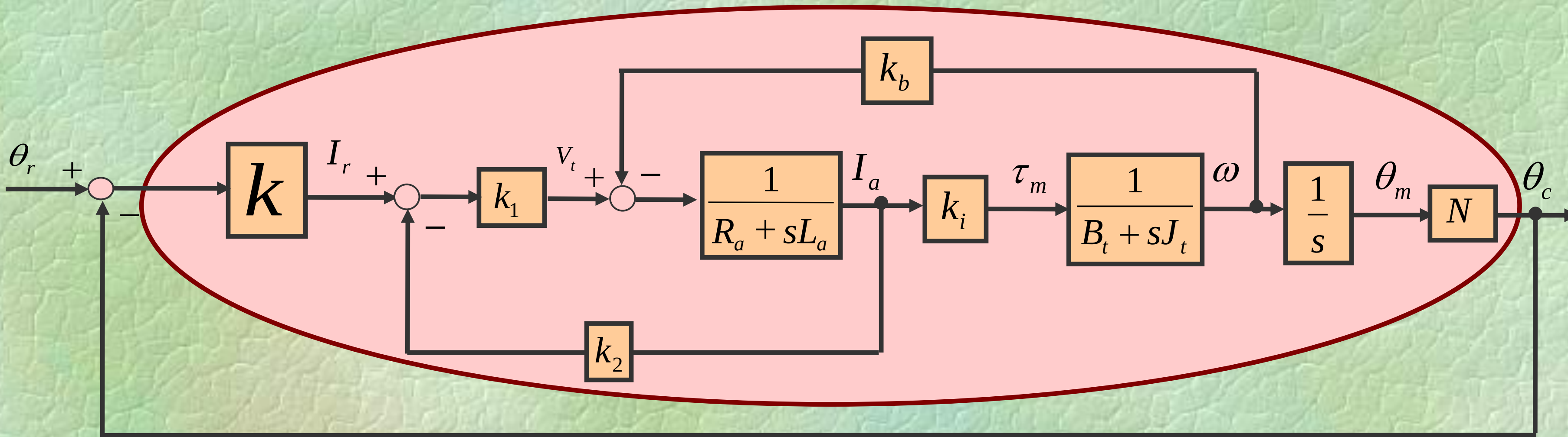
Position control

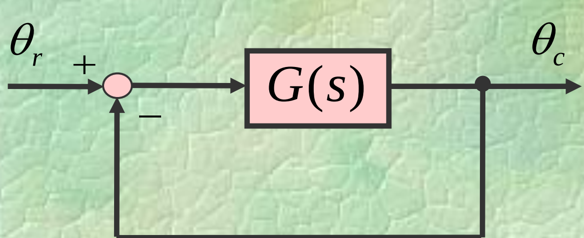
Block diagram of a position control system





Simplification





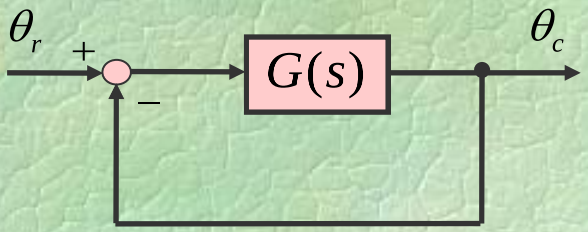
$$G(s)=\frac{\dfrac{kk_1k_iN}{s(R_a+sL_a)(B_t+sJ_t)}}{1+\dfrac{k_1k_2}{R_a+sL_a}+\dfrac{k_ik_b}{(R_a+sL_a)(B_t+sJ_t)}}$$

$$=\frac{kk_1k_iN}{s\left((R_a+sL_a)(B_t+sJ_t)+k_1k_2(B_t+sJ_t)+k_ik_b\right)}$$

Simplification

$$G(s)=\frac{k_s k_1 k_i N}{s\left((R_a+sL_a)(B_t+sJ_t)+k_1k_2(B_t+sJ_t)+k_ik_b\right)}$$

$$\tilde{G}(s)=\frac{\dfrac{k_s k_1 k_i N}{(R_a+k_1k_2)J_t}}{s\left(s+\dfrac{R_aB_t+k_1k_2B_t+k_ik_b}{(R_a+k_1k_2)J_t}\right)} \quad \text{by ignoring } L_a$$



$k_1=10 \quad k_2=0.5 \quad k_i=9$

$k_b=0.0636 \quad R_a=5 \quad L_a=0.003 \quad N=0.1$

$B_m=0.0005 \quad B_l=1 \quad J_l=0.01 \quad J_m=0.0001$

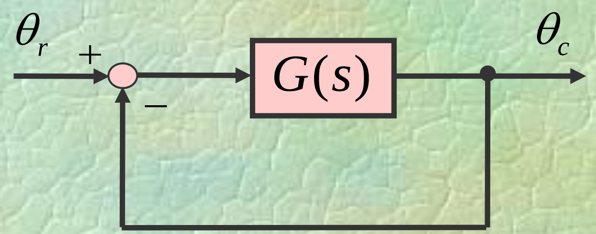
$$B_t=B_m+N^2B_l=0.005+\frac{1}{100}=0.015$$

$$J_t=J_m+N^2J_l=0.0001+\frac{0.01}{100}=0.0002$$

$$G(s)=\frac{1.5\times10^7 k}{s\left(s^2+3408.3s+1204000\right)}=\frac{1.5\times10^7 k}{s(s+400.3)(s+3008)}$$

$$\tilde{G}(s)=\frac{4500k}{s\left(s+361.2\right)}$$

Block diagram of a position control system



$$G(s)=\frac{1.5\times 10^{7}k}{s(s+400.3)(s+3008)}$$

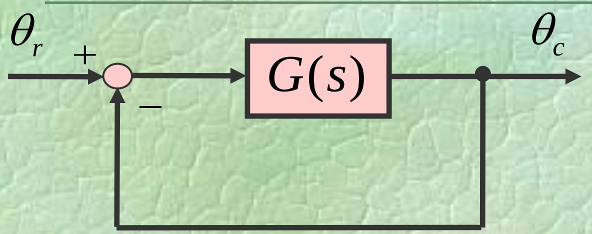
$$\tilde{G}(s)=\frac{4500k}{s\left(s+361.2\right)}$$

Stability analysis …………..

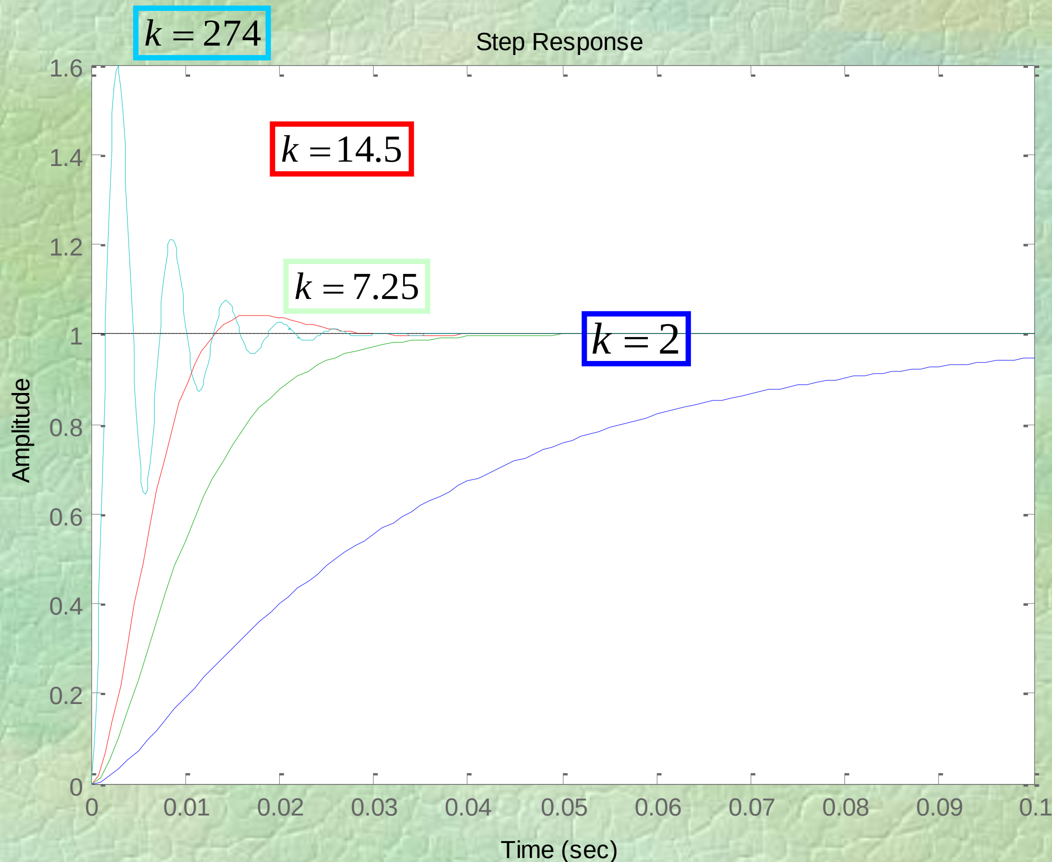
Step response of second order system …………..

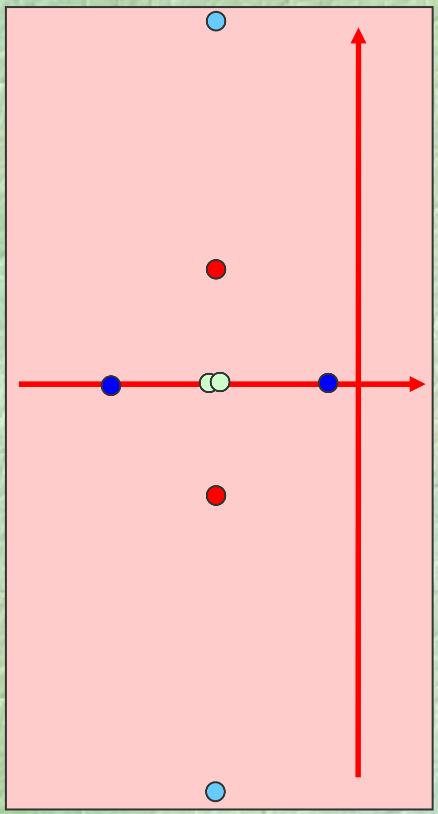
Step response of third order system …………..

Step response



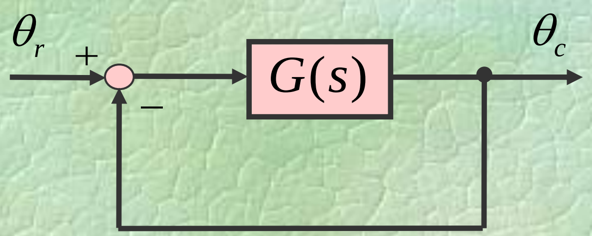
$$\tilde{G}(s) = \frac{4500k}{s(s+361.2)}$$



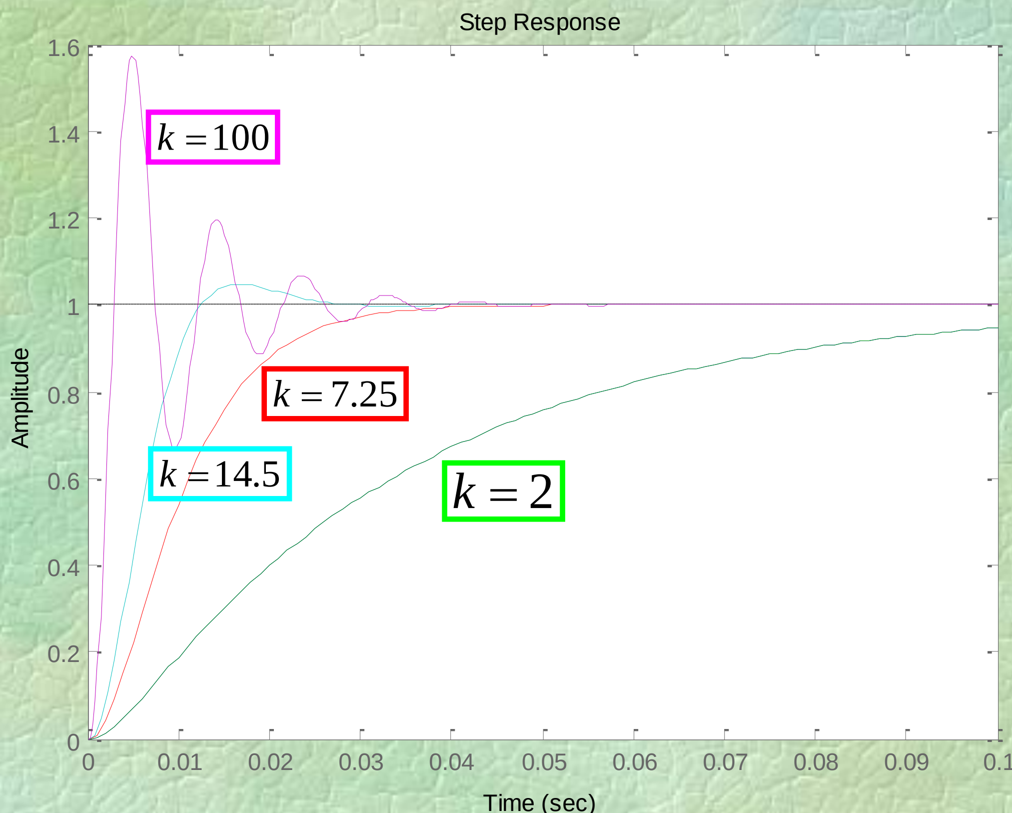


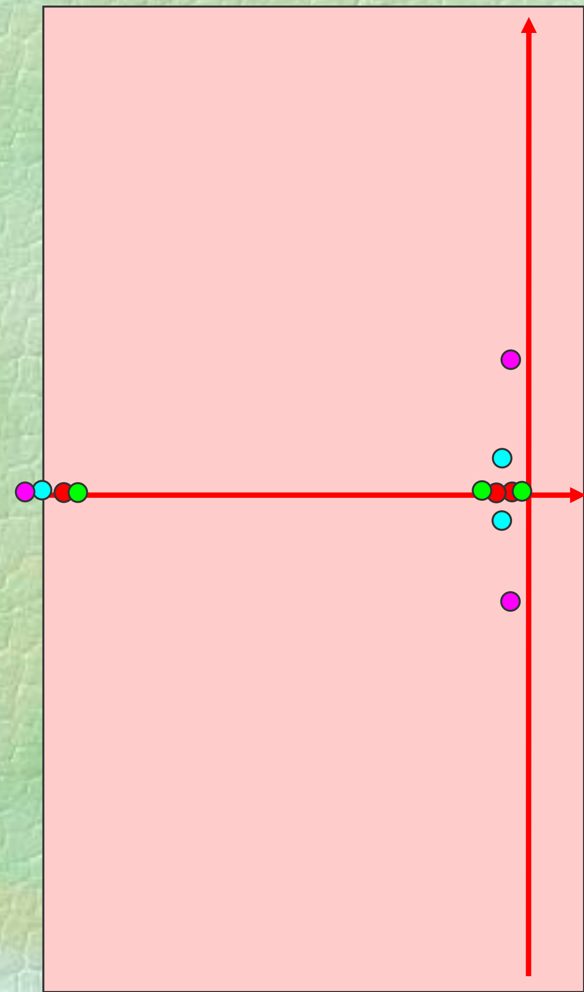
Which k is ok?

Step response



$$G(s)=\frac{1.5\times10^{7}k}{s(s+400.3)(s+3008)}$$

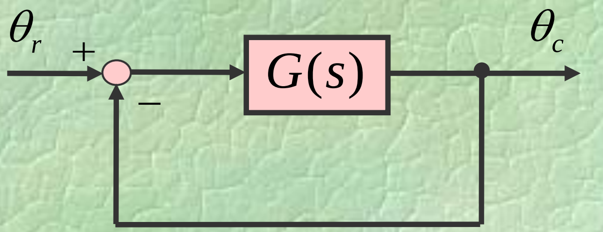


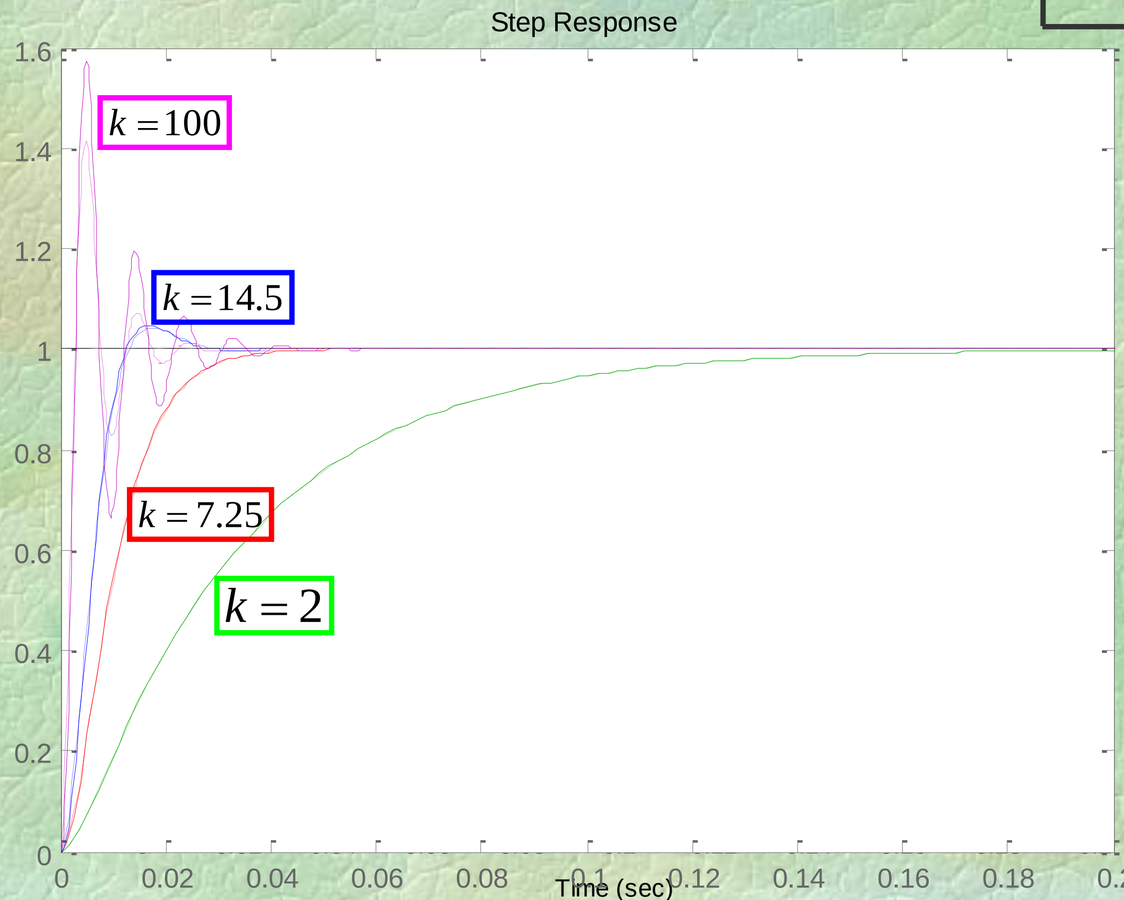


Step response comparison

$$G(s)=\frac{1.5\times10^{7}k}{s(s+400.3)(s+3008)}$$

$$\tilde{G}(s)=\frac{4500k}{s(s+361.2)}$$





Time domain analysis of control systems

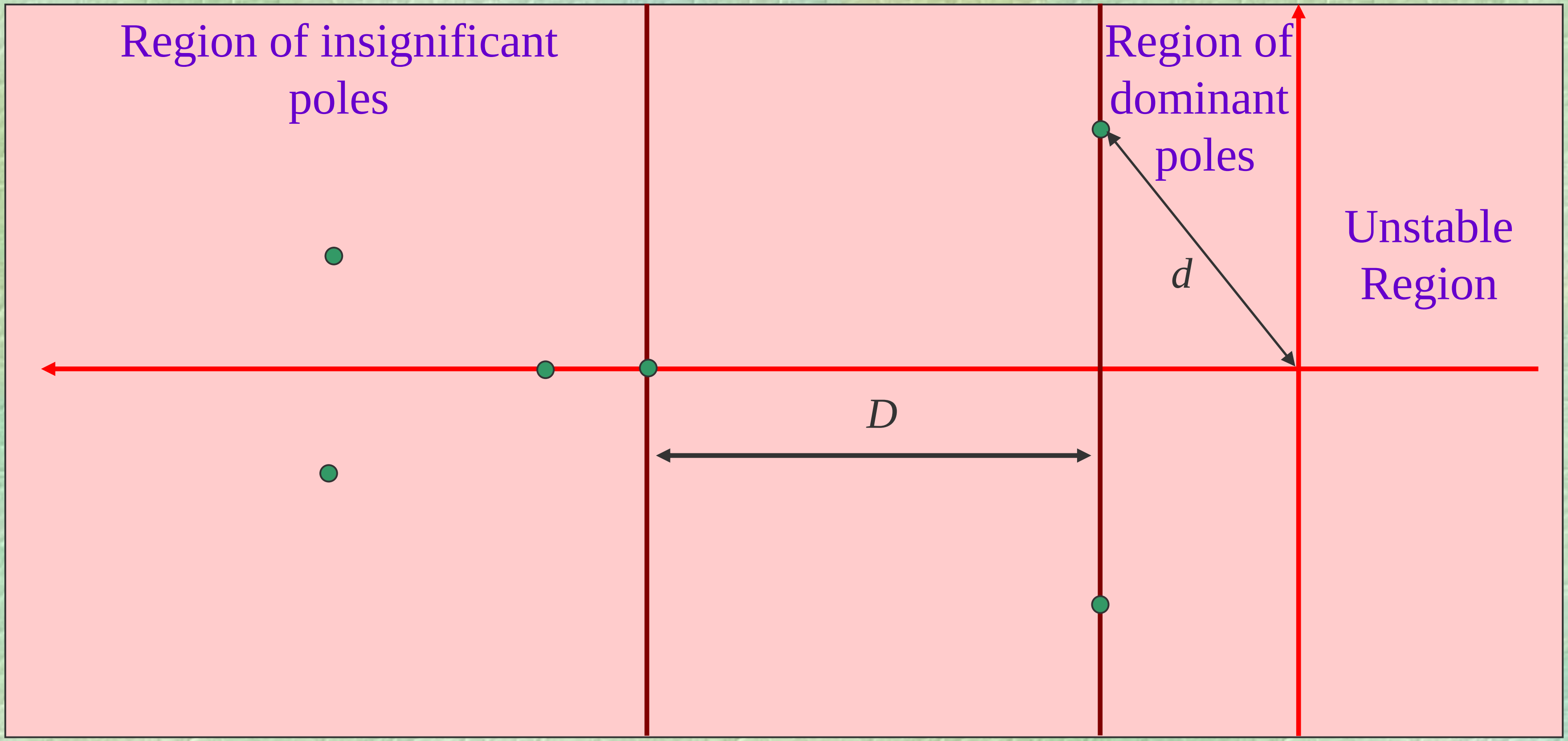
Topics to be covered include:

- Steady state error.
- Introducing some performance criteria (ISE, ITSE, IAE and ITAE).
- Transient response of a prototype second order system.
- Different region of S plane.
- Transient response of a position control system.
- Dominant poles and approximation of high-order systems by low-order systems.
- Effect of zeros on the transfer function of a system.

Model order reduction

Dominant Complex Poles in the Transfer Function

……………

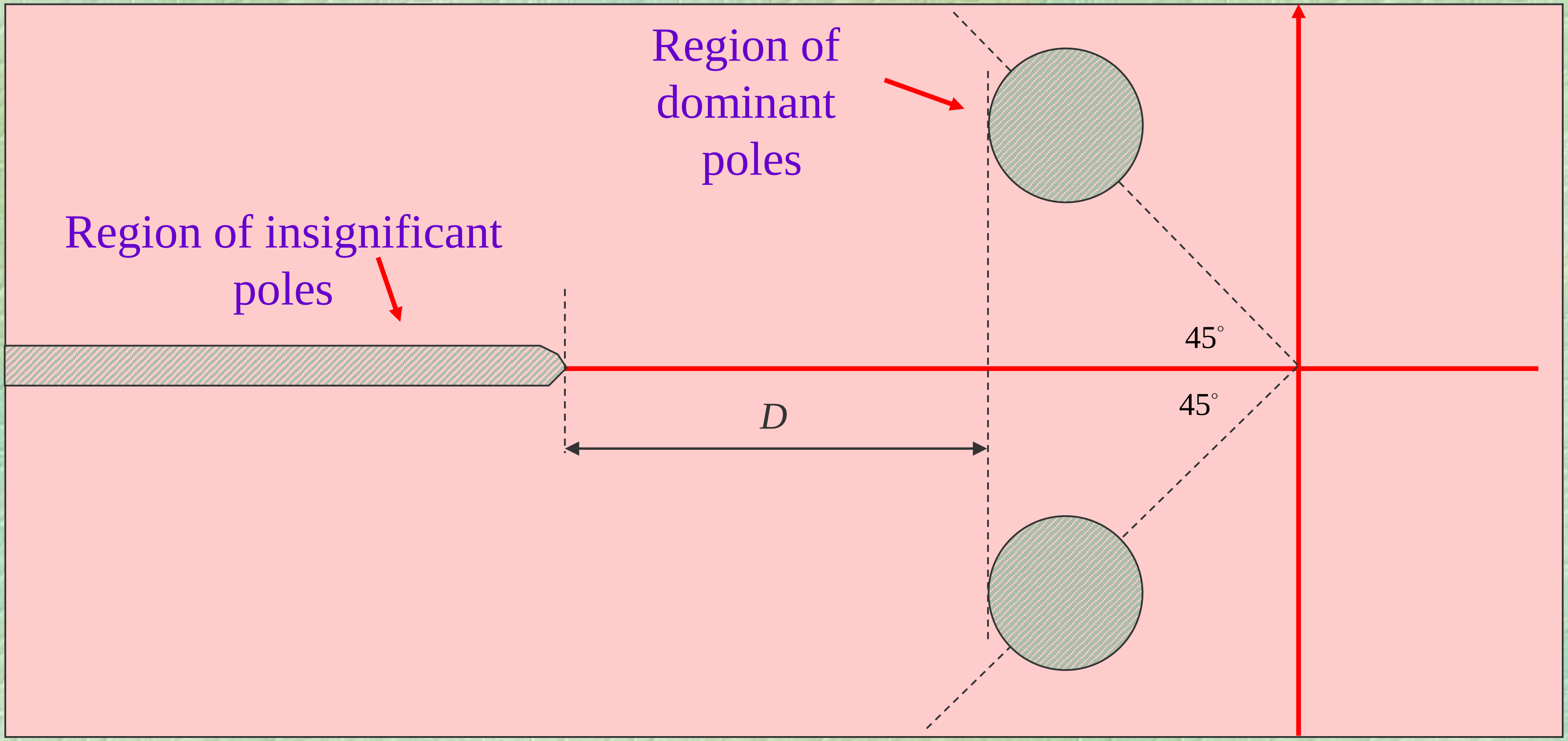


What about D?

D > 5 times of d.

Design procedure

For design purposes, such as in the pole placement design we try to put poles on:



Model order reduction

Exercise 4: What are the dominant poles of the following transfer function.

$$M(s) = \frac{32}{(s+2)(s+16)}$$

Exercise 5: What are the dominant poles of the following transfer function.

$$M(s) = \frac{15.24(s+2.1)}{(s+16)(s+2)}$$

Exercise 6: Compare the step response of M and its approximation for different values of k.

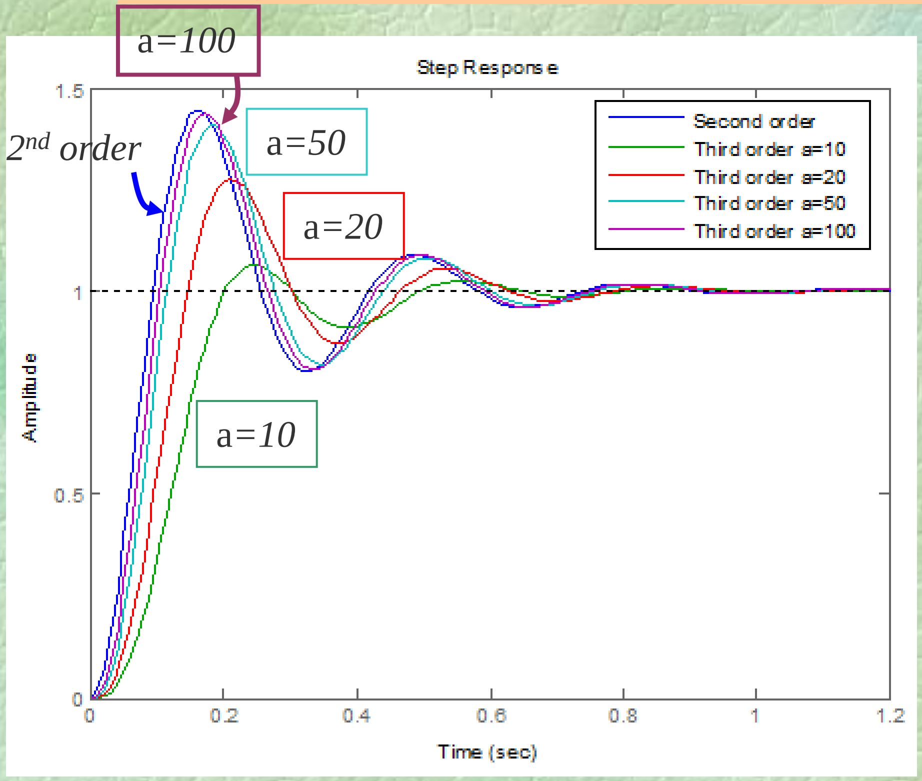
$$M(s) = \frac{400a}{(s+a)(s^2+10s+400)}$$

Model order reduction

Example 3: Compare the step response of M and its approximation for different values of k.

$$M(s) = \frac{400a}{(s+a)(s^2+10s+400)}$$

$$\tilde{M}(s) = \frac{400}{(s^2+10s+400)}$$



MATLAB

```
step(400,[1 10 400]);hold on
a=10;step(400*a,conv([1 a],[1 10 400])
a=20;step(400*a,conv([1 a],[1 10 400]))
a=50;step(400*a,conv([1 a],[1 10 400]))
a=100;step(400*a,conv([1 a],[1 10 400]))
```

Time domain analysis of control systems

Topics to be covered include:

- Steady state error.
- Introducing some performance criteria (ISE, ITSE, IAE and ITAE).
- Transient response of a prototype second order system.
- Different region of S plane.
- Transient response of a position control system.
- Dominant poles and approximation of high-order systems by low-order systems.
- Effect of zeros on the transfer function of a system.

Effect of zero on the closed loop system

Importance of zeros in transfer functions

We see that the performance of system is concerned to :

Poles and zeros not just poles

Exercises

Exercise 8: Find the error of following system to step input.

$$\dot{x} = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -3 & 2 \\ 1 & 4 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = [1 \quad 2 \quad 0] x$$

Exercise 9: Find the error of the following systems to step input.

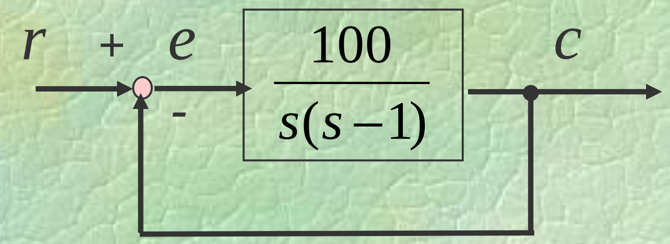
$$a)\ M(s) = \frac{2000}{s^3 + 15s^2 + 50s + 2000} \qquad b)\ M(s) = \frac{200}{s^3 + 15s^2 + 50s + 200} \qquad c)\ M(s) = \frac{500}{s^3 + 15s^2 + 50s + 600}$$

Exercise 10: Find the error of the following systems to velocity input.

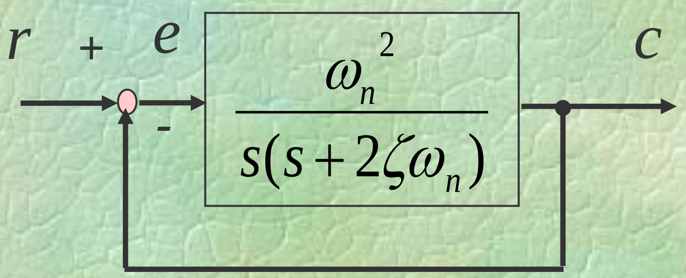
$$a)\ M(s) = \frac{50s + 2000}{s^3 + 15s^2 + 50s + 2000} \qquad b)\ M(s) = \frac{50s + 200}{s^3 + 15s^2 + 50s + 200}$$

Exercises

Exercise 11: Find the error of following system to unit step input(Final 1391).



Exercise 12: Consider following system.

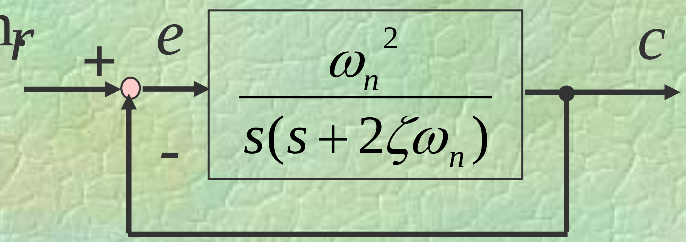


a) Find the step response of the system for $\omega_n = 12.56$, $\zeta = 0.3$

b) Find the rise time, settling time, overshoot, and percent overshoot.

Exercises

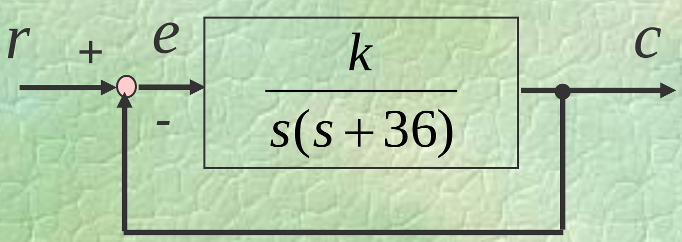
Exercise 13: Consider following system



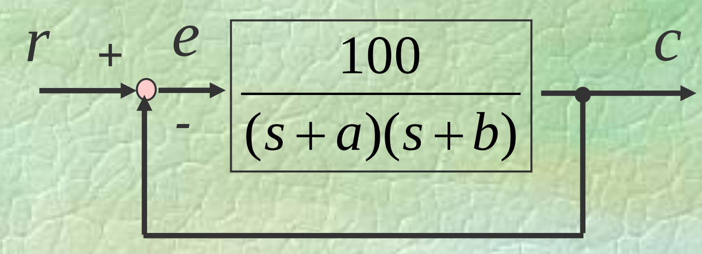
a) Find the step response of the system for $\omega_n = 12.56\ ,\ \zeta = 0.9$

b) Find the rise time, settling time, overshoot, and percent overshoot.

Exercise 14: In the following system set k such that the percent overshoot of system be 4.3%



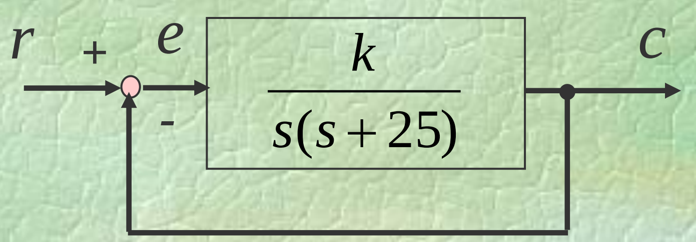
Exercise 15: In the following system set a and b such that the percent Overshoot of system be 4.3% and the steady state error to step input be 0.



Exercises

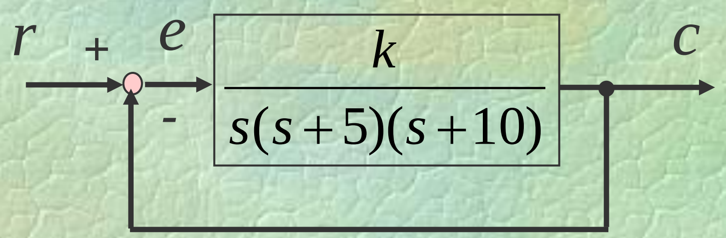
Exercise 16: In the system of problem 1 set *k* such that
a) The error to ramp input be 0.01
b) The percent overshoot of system be 4.3%
c) The error to ramp input be 0.01 and the percent overshoot of system be 4.3%

Exercise 17: In the following system
a) For k=200 derive settling time, rise time and percent overshoot. Confirm your result with step response.
b) For k=1000 derive settling time and percent overshoot. Confirm your result with step response.



Exercises

Exercise 18: In the following system set the *k* such that the imaginary poles have 0.707 damping ratio.



Exercise 19: Find the roots of following system for -301<k<301 and show them on the s plane.

$$\boxed{G(s)=\frac{4500k}{s(s+361.2)}}$$

Let k=-300,-280,-260,……….260,280,300

Exercise 20: Find the roots of following system for -301<k<301 and show them on the s plane.

$$\boxed{G(s)=\frac{1.5\times10^{7}k}{s(s+400.3)(s+3008)}}$$

Let k=-300,-280,-260,……….260,280,300

Exercises

Exercise 21: Consider following system. Find the dominant poles and insignificant poles of system.

$$M(s) = \frac{96}{(s+16)(s+2)(s+3)}$$

Exercise 22: Derive a suitable second order system for following system.

$$M(s) = \frac{96}{(s+16)(s+2)(s+3)}$$

Exercise 23: Compare the step response of the system in problem 15 and step response of its second order approximation.

Exercise 24: Consider following system. Find the dominant poles and insignificant poles of system.

$$M(s) = \frac{3150(s+3.05)}{(s+160)(s+20)(s+3)(s+1)}$$

Exercises

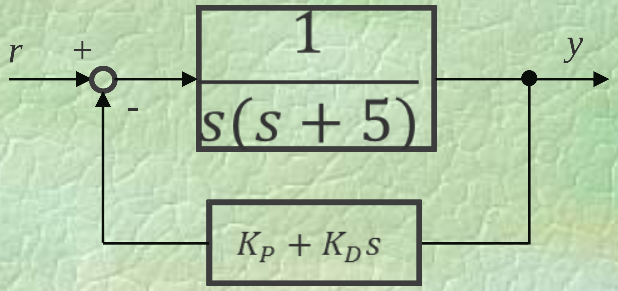
Exercise 25: Derive a suitable second order system for following system.

$$M(s) = \frac{3150(s+3.05)}{(s+160)(s+20)(s+3)(s+1)}$$

Exercise 26: Compare the step response of the system in problem 18 and step response of its second order approximation.

Exercise 27: a)Find K_P and K_D such that the steady state error to unit ramp be 1.

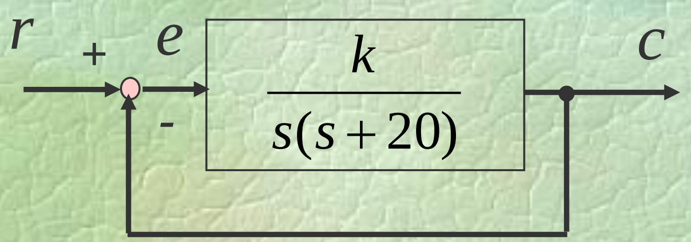
b) Plot exact step response of system. Define error as r(t)-y(t) (Final 1396/10/28)



Answer: K_P=1, K_D=-4.

Examples

Example 4: Find the poles of the following systems and its corresponding step response for k=75, 100, 200 and 1000



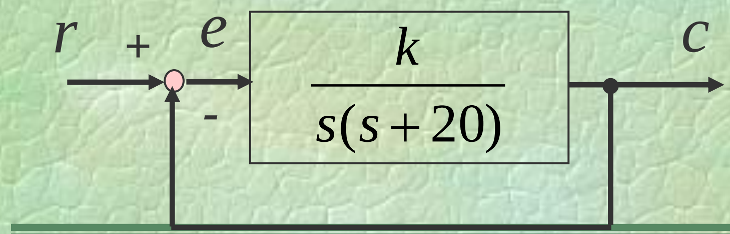
$$M(s)=\frac{c(s)}{r(s)}=\frac{k}{s^2+20s+k}$$

$$k=75 \quad \Rightarrow \quad p_1=-5,\ p_2=-15$$

$$k=100 \quad \Rightarrow \quad p_1=-10,\ p_2=-10$$

$$k=200 \quad \Rightarrow \quad p_1=-10+10j,\ p_2=-10-10j$$

$$k=1000 \quad \Rightarrow \quad p_1=-10+30j,\ p_2=-10-30j$$



$$M(s) = \frac{c(s)}{r(s)} = \frac{k}{s^2 + 20s + k}$$