ADVANCED CONTROL

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Reference:

Chi-Tsong Chen, “Linear System Theory and Design”, 1999.

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Lecture 5

Stability

Topics to be covered include:

- Introduction.
- Input-Output Stability of LTI systems.
- Internal Stability.
- Lyapunov Theorem.
- Stability of Linear Time-Varying(LTV) Systems

What you will learn after studying this section

- **Input-output stability (BIBO)**
- **Input-output stable systems**
- **Internal stability(in the sense of Lyapunov and asymptotic)**
- **Marginal and asymptotic stability conditions**
- **Internal stability by Lyapunov equation**
- **Stability analysis for LTV state equation**

Introduction

Linear System property

The total response of a linear system, can be expressed as the sum of the zero-state response and the zero-input response.

$$y_{total}(t) = y_{zs}(t) + y_{zi}(t)$$

1. Input-output stability of linear systems is known as **BIBO (bounded input-bounded output)** stability, and it is concerned with the zero-state response.

2. Internal stability of linear systems is known as **asymptotic stability**, and it is concerned with the zero-input response.

Input output stability of LTI system

Input output relation for LTI systems

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau = \int_0^t g(\tau)u(t-\tau)d\tau \qquad (I)$$

Definition 1: A linear system is BIBO (bounded input-bounded output) stable if any bounded input leads to a bounded output. This stability concerns the zero-state response and assumes that the system is initially relaxed.

Theorem 1: A SISO system defined by Equation I is BIBO stable if and only if its impulse response is absolutely integrable, i.e.,

$$\int_0^\infty |g(t)|dt \le M < \infty$$

Where M is a positive constant number.

Input output stability of LTI system

Proof: It is evident that to prove the theorem, both sides must be shown.

g(t) is absolutely integrable $\Rightarrow$ system is BIBO stable

system is BIBO stable $\Rightarrow$ g(t) is absolutely integrable

First, we prove the first part.

Let $g(t)$ be absolutely integrable. We need to show that the system is BIBO stable, which requires demonstrating that any bounded input leads to a bounded output. Let u_m be the upper bound of the input. Then,

$$|y(t)| = \left|\int_0^t g(\tau)u(t-\tau)d\tau\right| \le \int_0^\infty |g(\tau)||u(t-\tau)|d\tau \le u_m\int_0^\infty |g(\tau)|d\tau \le u_m M$$

So, output is bounded.

Input output stability of LTI system

Proof: It is evident that to prove the theorem, both sides must be shown.

g(t) is absolutely integrable $\Rightarrow$ system is BIBO stable

system is BIBO stable $\Rightarrow$ g(t) is absolutely integrable

Now, we prove the second part.

Let the system be BIBO stable. We need to show that $g(t)$ is absolutely integrable. We will use the method of contradiction. Assume $g(t)$ is not absolutely integrable. Then, for any large M, there exists a time t_1 such that:

$$\int_0^{t_1} |g(\tau)| d\tau > M$$

Choose a bounded input as:

$$u(t-\tau) = \begin{cases} 1 & \text{if } g(\tau) \geq 0 \\ -1 & \text{if } g(\tau) < 0 \end{cases}$$

The output is:

$$y(t_1) = \int_0^{t_1} g(\tau) u(t-\tau) d\tau = \int_0^{t_1} |g(\tau)| d\tau > M$$

This is a contradiction because the output must be bounded.

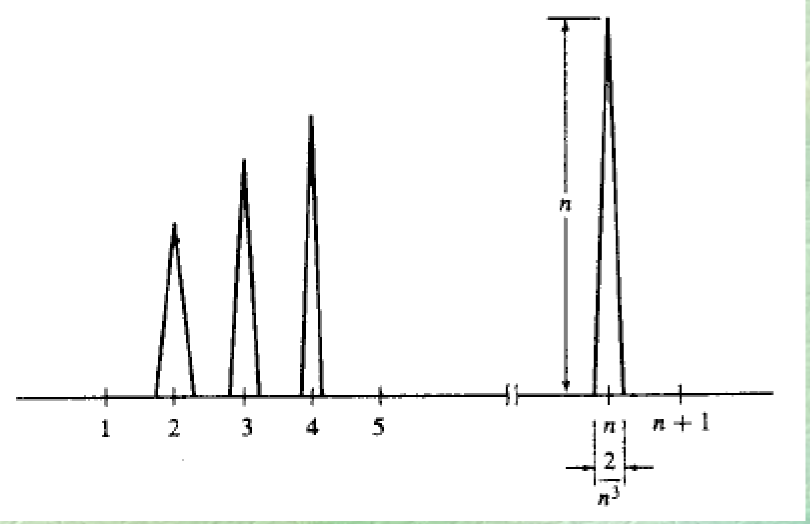
Input output stability of LTI system

Does an absolutely integrable impulse response imply a bounded impulse response?

Example 1

$$f(t-n)=\begin{cases} n+(t-n)n^4 & for \quad n-1/n^3 \le t \le n \\ n-(t-n)n^4 & for \quad n < t \le n+1/n^3 \end{cases} \qquad n\in N,\, n\ge 2$$

Area under any triangle is: $1/n^2$



Integral of the absolute value of the function:

$$\sum_{n=2}^{\infty}\left(\frac{1}{n^2}\right)<\infty$$

$f(t)$ is absolutely integrable but not bounded.

Input output stability of LTI system

Theorem 2: If a system with impulse response $g(t)$ is BIBO (Bounded Input, Bounded Output) stable, then for t→∞

1- The output stimulated by $u(t) = a$ for t≥0 approaches $\hat{g}(0)a$.

2- The output stimulated to $u(t) = \sin \omega_0 t$ for t≥0 approaches $\hat{g}(j\omega_0)\sin(\omega_0 t + \angle \hat{g}(j\omega_0))$,where $\hat{g}(s)$ is the Laplace transform of $g(t)$, i.e.

$$\hat{g}(s) = \int_0^\infty g(\tau)e^{-s\tau}d\tau \qquad \text{(II)}$$

Proof of 1)

$$y(t) = \int_{-\infty}^{t} g(\tau)u(t-\tau)d\tau = a\int_0^t g(\tau)d\tau$$

According to Laplace transform definition at s=0:

$$y(t) \to a\int_0^\infty g(\tau)d\tau = a\hat{g}(0)$$

Input output stability of LTI system

Proof of 2)

For the output stimulated to $u(t) = \sin \omega_0 t$ for $t \geq 0$, the output is:

$$y(t) = \int_{-\infty}^{t} g(\tau)\sin\omega_0(t-\tau)d\tau = \int_0^t g(\tau)[\sin\omega_0 t\cos\omega_0\tau - \cos\omega_0 t\sin\omega_0\tau]d\tau$$

$$= \sin\omega_0 t\int_0^t g(\tau)\cos\omega_0\tau d\tau - \cos\omega_0 t\int_0^t g(\tau)\sin\omega_0\tau d\tau$$

For $t \to \infty$ we have:

$$y(t) \to \sin\omega_0 t\int_0^\infty g(\tau)\cos\omega_0\tau d\tau - \cos\omega_0 t\int_0^\infty g(\tau)\sin\omega_0\tau d\tau$$

Since the system is BIBO stable two integrals are bounded and:

$$\hat{g}(j\omega) = \int_0^\infty g(\tau)[\cos\omega\tau - j\sin\omega\tau]d\tau$$

$$\mathrm{Re}[\hat{g}(j\omega)] = \int_0^\infty g(\tau)\cos\omega\tau d\tau$$

$$\mathrm{Im}[\hat{g}(j\omega)] = -\int_0^\infty g(\tau)\sin\omega\tau d\tau$$

By substituting the real and imaginary parts into the equation, we have:

$$y(t) \to \sin\omega_0 t\,\mathrm{Re}\left(\hat{g}(j\omega_0)\right) + \cos\omega_0 t\,\mathrm{Im}\left(\hat{g}(j\omega_0)\right) = \left|\hat{g}(j\omega_0)\right|\sin\left(\omega_0 t + \angle\hat{g}(j\omega_0)\right)$$

Input output stability of LTI system

Theorem 3: An SISO system with proper real rational $\hat{g}(s)$ is BIBO stable if and only if all poles of $\hat{g}(s)$ have negative real parts; in other words, all poles must lie in the left half of the sss-plane.

If $\hat{g}(s)$ has poles p_i with a multiplicity of m_i , its partial fraction expansion will include the following terms.

$$\frac{1}{s-p_i}, \frac{1}{(s-p_i)^2}, \ldots, \frac{1}{(s-p_i)^{m_i}}$$

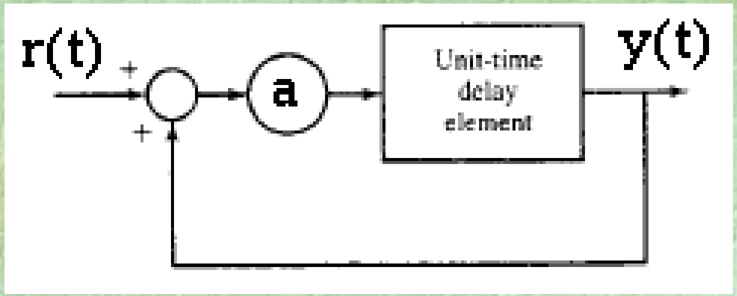
So, the inverse Laplace of $\hat{g}(s)$ is:

$$e^{p_i t}, te^{p_i t}, \ldots, t^{m_{i-1}} e^{p_i t}$$

It can be shown that all of this terms are absolutely integrable if and only if p_i has negative real part.

Input output stability of LTI system

Example 2: Check BIBO stability of system.



$$g(t) = a\delta(t-1) + a^2\delta(t-2) + a^3\delta(t-3) + ... = \sum_{i=1}^{\infty} a^i \delta(t-i)$$

$$|g(t)| = \sum_{i=1}^{\infty} |a^i| \delta(t-i)$$

$$\int_0^{\infty} |g(t)|\,dt = \sum_{i=1}^{\infty} |a|^i = \begin{cases} \infty & \text{if } |a| \geq 1 \\ |a|/(1-|a|) < \infty & \text{if } |a| < 1 \end{cases}$$

So it is BIBO stable for $|a|<1$.

Its transfer function is given as follows, but BIBO stability cannot be determined from it because …..

$$\hat{g}(s) = \frac{ae^{-s}}{1-ae^{-s}}$$

Input output stability of LTI system

Theorem 4: A MIMO system described with impulse response $G(\mathrm{t}) = [g_{ij}(t)]$ is BIBO stable if and only if the absolute value of $g_{ij}(t)$ is integrable over $[0,\infty)$.

Theorem 5: A MIMO system described with proper real rational transfer function $G(s) = [g_{ij}(s)]$ is BIBO stable if and only if every pole of each $g_{ij}(t)$ has a negative real part.

Determination of input-output stability of LTI systems using state-space equations.

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$G(s) = C(sI - A)^{-1}B + D$$

$$G(s) = C\frac{adj(sI - A)}{|sI - A|}B + D$$

So if all eigenvalues of A have negative real parts, then ….. But if ….

Input output stability of LTI system

Example 3: Check BIBO stability of the given system.

$$\dot{x}(t) = x(t) + 0 \times u(t)$$
$$y(t) = 0.5x(t) + 0.5u(t)$$

The matrix A has an eigenvalue of 1. So, its real part is positive.

Transfer function of system is:

$$\hat{g}(s) = 0.5(s-1)^{-1} \times 0 + 0.5 = 0.5$$

The transfer function has no pole so, it is BIBO stable.

Input output stability of LTI system

Definition 2: The zero input response of $\dot{x} = Ax$ is stable in the sense of Lyapunov (marginal stability) if every bounded initial condition x_0 leads to bounded response. Additionally, if the response approaches zero it is asymptotic stability.

Theorem 6:

1) The equation $\dot{x} = Ax$ is stable in the sense of Lyapunov (marginal stability) if and only if all eigenvalues of A have zero or negative real parts, and those with zero real parts must be simple roots of the minimal polynomial of A.

2) The equation $\dot{x} = Ax$ is asymptotiacally stable if and only if all eigenvalues of A has negative real parts.

Input output stability of LTI system

Proof of part 1)

$$\bar{x} = Px$$

$$\dot{\bar{x}} = \bar{A}\bar{x} = PAP^{-1}\bar{x}$$

So, similarity transformation do not change the stability of a system.

One can check the stability of matrix A by studying matrix $\bar{A}$, as the eigenvalues of A and $\bar{A}$ are similar.

Since P is nonsingular, if x is bounded, $\bar{x}$ is also bounded. Additionally, if x approaches zero as $t\to\infty$, $\bar{x}$ also approaches zero as $t\to\infty$.

$$\dot{\bar{x}} = \bar{A}\bar{x} \Rightarrow \bar{x}(t) = e^{\bar{A}t}\bar{x}(0)$$

This response is bounded if and only if every entry of matrix $e^{\bar{A}t}$ is bounded for all $t\geq 0$.

$$A: Jordan\ Form \xrightarrow{chapter 3} e^{\bar{A}t} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & t^2 e^{\lambda_1 t}/2! & t^3 e^{\lambda_1 t}/3! \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & t^2 e^{\lambda_1 t}/2! \\ 0 & 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & 0 & e^{\lambda_1 t} \end{bmatrix}$$

Internal stability of LTI Systems

If all eigenvalues of $\bar{A}$ have negative real part, then every entry of $e^{\bar{A}t}$ is bounded and approaches zero as $t\rightarrow\infty$.

If any of eigenvalues of $\bar{A}$ has zero real part, and there is no Jordan block larger than one corresponding to that eigenvalue, then $e^{\bar{A}t}$ will contain a constant or a sinusoid term.

If there is an eigenvalues of $\bar{A}$ that has a positive real part, then there is a term in $e^{\bar{A}t}$ that approach to infinity as $t\rightarrow\infty$.

If there is an eigenvalues of $\bar{A}$ that has zero real part, and there is Jordan block larger than one corresponding to that eigenvalue, then there is a term in $e^{\bar{A}t}$ that approach to infinity as $t\rightarrow\infty$.

Proof of part 2)

For asymptotic stability every entry of $e^{\bar{A}t}$ must approaches to zero as $t\rightarrow\infty$.

$\Rightarrow$ Eigenvalues with zero or positive real parts are not permissible.

Internal stability of LTI Systems

Example 4: Check asymptotic and Lyapunov stability of following system.

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} x$$

$$\Delta(\lambda) = \lambda^2(\lambda+1)$$

$$\psi(\lambda) = \lambda(\lambda+1)$$

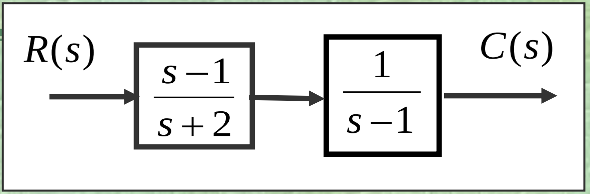
Repeat example 4 for following system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} x$$

$$\Delta(\lambda) = \lambda^2(\lambda+1)$$

$$\psi(\lambda) = \lambda^2(\lambda+1)$$

Internal stability and input-output stability



Example 5: Check asymptotic and BIBO stability of following system.

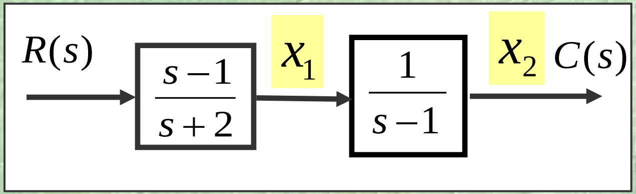
BIBO stability:

$$G(s)=\frac{C(s)}{R(s)}=\frac{1}{s+2}$$

So it is BIBO stable

Internal stability:

For internal stability we need state-space model so we have:



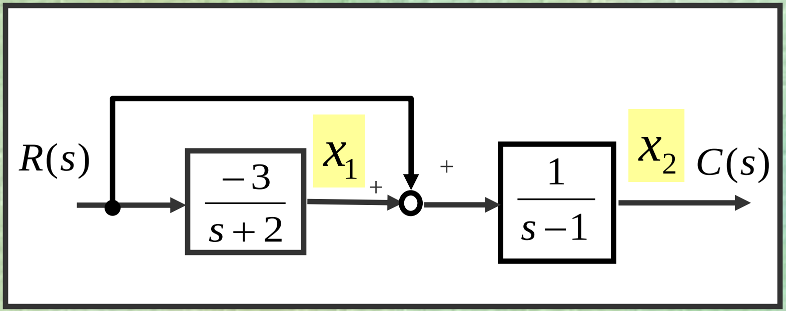
$$x_2(s)=\frac{1}{s-1}x_1(s) \longrightarrow \dot{x}_2=x_1+x_2$$

$$x_1(s)=\frac{s-1}{s+2}R(s) \longrightarrow \dot{x}_1=-2x_1+\dot{r}-r$$

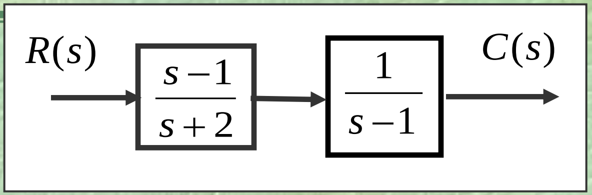
?

$$x_2(s)=\frac{1}{s-1}\left(x_1(s)+R(s)\right) \longrightarrow \dot{x}_2=x_1+x_2+r$$

$$x_1(s)=\frac{-3}{s+2}R(s) \longrightarrow \dot{x}_1=-2x_1-3r$$



Internal stability and input-output stability

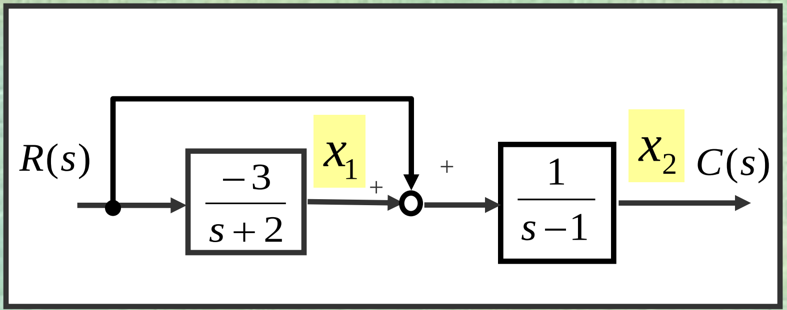


Example 5: Check asymptotic and BIBO stability of the following system.

It is BIBO stable.

Internal stability:

For internal stability we need state-space model so we have:



$\dot{x}_1 = -2x_1 - 3r$ $\qquad$ $\dot{x}_2 = x_1 + x_2 + r$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-2 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}-3\\ 1\end{bmatrix} r$$

$$c = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$\Rightarrow$

$$|sI - A| = \begin{vmatrix}s+2 & 0\\ -1 & s-1\end{vmatrix} = (s-1)(s+2)$$

$$\rightarrow \lambda_1 = +1, \lambda_2 = -2$$

The system is not internally stable (neither asymptotic nor Lyapunov stable).

Very important note: If RHP poles and zeros between different part of system omitted then the system is internally unstable although it may be BIBO stable.

BIBO stability of LTV systems

Input-output relationship for a single input-single output linear time-varying system.

$$y(t) = \int_{t_0}^{t} g(t,\tau)u(\tau)d\tau$$

A linear system is BIBO (bounded input-bounded output) stable if any bounded input leads to a bounded output.

The BIBO stability condition is:

$$\int_{t_0}^{t} \left|g(t,\tau)\right| d\tau \le M < \infty \quad \forall t,\ t_0 \ and\ t \ge t_0$$

Input-output relationship for a mutivariable linear time-varying system.

$$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$$

The BIBO stability condition is that every entry of G is absolutely integrable, or:

$$\int_{t_0}^{t} \left\|G(t,\tau)\right\| d\tau \le M < \infty \quad \forall t,\ t_0 \ and\ t \ge t_0$$

BIBO stability of LTV systems

Consider following state space model

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t) + D(t)u(t)$$

Impulse response is:

$$G(t,\tau) = C(t)\Phi(t,\tau)B(\tau) + D(t)\delta(t-\tau)$$

Zero state response is:

$$y(t) = \int_{t_0}^{t} \left(C(t)\Phi(t,\tau)B(\tau) + D(t)\delta(t-\tau)\right)u(\tau)d\tau$$

Zero-state response is BIBO stable if and only if there exist M_1 and M_2 such that:

$$\int_{t_0}^{t} \left\| G(t,\tau) \right\| d\tau \le M_2 < \infty$$

$$\left\| D(t) \right\| \le M_1 < \infty$$

$$\forall t, t_0 \;\; and \;\; t \ge t_0$$

Internal stability of LTV systems

For internal stability we must use →

$$\dot{x} = A(t)x$$

Its response is:

$$x(t) = \Phi(t,t_0)x(t_0)$$

The zero-input response of the system $\dot{x} = Ax$ is **marginally stable** if any bounded initial condition leads to bounded states.

The zero-input response of the system $\dot{x} = Ax$ is **marginally stable** if and only if there exists a bounded constant M such that:

$$\left\|\Phi(t,t_0)\right\| \leq M < \infty, \ \forall t,t_0 \ \ and \ \ t \geq t_0$$

Internal stability of LTV systems

For internal stability we must use →

$$\dot{x} = A(t)x$$

Its response is:

$$x(t) = \Phi(t,t_0)x(t_0)$$

The zero-input response of the system $\dot{x} = Ax$ is **asymptotically stable** if any bounded initial condition leads to bounded states that approaches zero as $t \to \infty$.

The zero-input response of the system $\dot{x} = Ax$ is **asymptotically stable** if there exist bounded constant M such that:

$$\left\|\Phi(t,t_0)\right\| \le M < \infty, \ \forall t, t_0 \ \ and \ \ t \ge t_0$$

$$\left\|\Phi(t,t_0)\right\| \to 0 \quad as \quad t \to \infty$$

Internal stability of LTV systems

Example 6: Check asymptotic and marginal stability of the following system.

$$\dot{x} = A(t)x = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} x$$

Characteristic equation: $\Delta(\lambda) = (\lambda + 1)^2$

For all t there are two eigenvalues at -1.

The state transition matrix is:

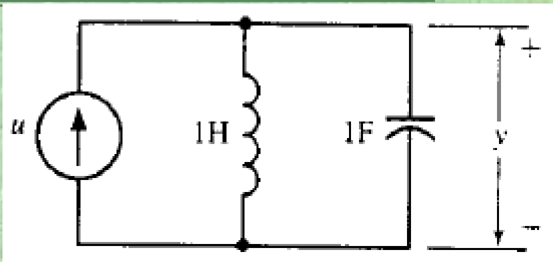
$$\Phi(t,0) = \begin{bmatrix} e^{-t} & \frac{1}{2}(e^{t} - e^{-t}) \\ 0 & e^{-t} \end{bmatrix}$$

It is clear that there is neither marginal stability nor asymptotic stability.

Theorem 7: Marginal stability and asymptotic stability of $\dot{x} = Ax$ doesn’t change with Lyapunov similarity transformation.

Exercises

Exercise 1: Is the following system BIBO stable? If not, find a bounded input that leads to an unbounded output.



Exercise 2: Is the system with impulse response $g(t) = \frac{1}{1+t}$ BIBO stable? What can you say about $g(t) = te^{-t}$

Exercise 3: Consider $g(s) = \frac{s-2}{s+1}$. Derive the steady-states response to $3u(t)$ and sint.u(t).

Exercise 4: Is the following system BIBO stable? What about marginal stability? What about asymptotic stability?

$$\dot{x} = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} -2 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} -2 & 3 \end{bmatrix} x - 2u$$

Exercises

Exercise 5: Is following the homogenous systems marginally stable? What about asymptotic stability?

a) $\dot{x} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x$ b) $\dot{x} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x$

Exercise 6: Does the system with impulse response $g(t,\tau) = e^{-2(|t|-|\tau|)}$) BIBO stable? What can you say about $g(t,\tau) = sinte^{-(t-\tau)}cos\tau$.

Exercise 7: Is the following LTV system BIBO stable? What about marginal stability? What about asymptotic stability?

$$\dot{x} = 2tx + u \qquad y = e^{-t^2} x$$

Exercises

Exercise 8: Consider the following system.

$$\dot{x} = 2tx + u \qquad y = e^{-t^2} x$$

Show that the transformation $P(t) = e^{-t^2}$ changes the system to the following:

$$\dot{\hat{x}} = 0.\hat{x} + e^{-t^2} u \qquad y = \hat{x}$$

Is the new system BIBO stable? What about marginal stability? What about asymptotic stability?

Exercise 9: Is following the homogenous system marginally stable for for $t_0 \geq 0$?What about asymptotic stability?

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ -e^{-3t} & 0 \end{bmatrix} x$$

Exercises

Exercise 10: Consider the following system.(Final 2014)

$$x = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [0 \quad 0 \quad 3] x$$

a) Check the asymptotic stability of the system.
b) Check the marginal stability of the system.
c) Check The BIBO stability of system.

Answers to selected problems

Answer 1: It is not BIBO stable since $u(t) = \sin t \;\rightarrow\; y(t) = 0.5t\sin t$

Answer 2: No, Yes

Answer 3: y(t)→ 1.26sin(2t+1.25) and y(t)→-6

Answer 5: It is not asymptotically sable. It is marginally stable.

Answer 6: It is not asymptotically sable and it is not marginally stable.