ADVANCED CONTROL

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Reference:

Chi-Tsong Chen, “Linear System Theory and Design”, 1999.

I thank my students, Nima Vaezi and Alireza Bemani for their help in making slides of this lecture.

Lecture 6

Controllability and Observability

Topics to be covered include:

- Introduction.
- Controllability.
- Observability.
- Canonical Decomposition.
- Controllability and Observability in Jordan forms.
- Controllability and Observability in LTV systems.

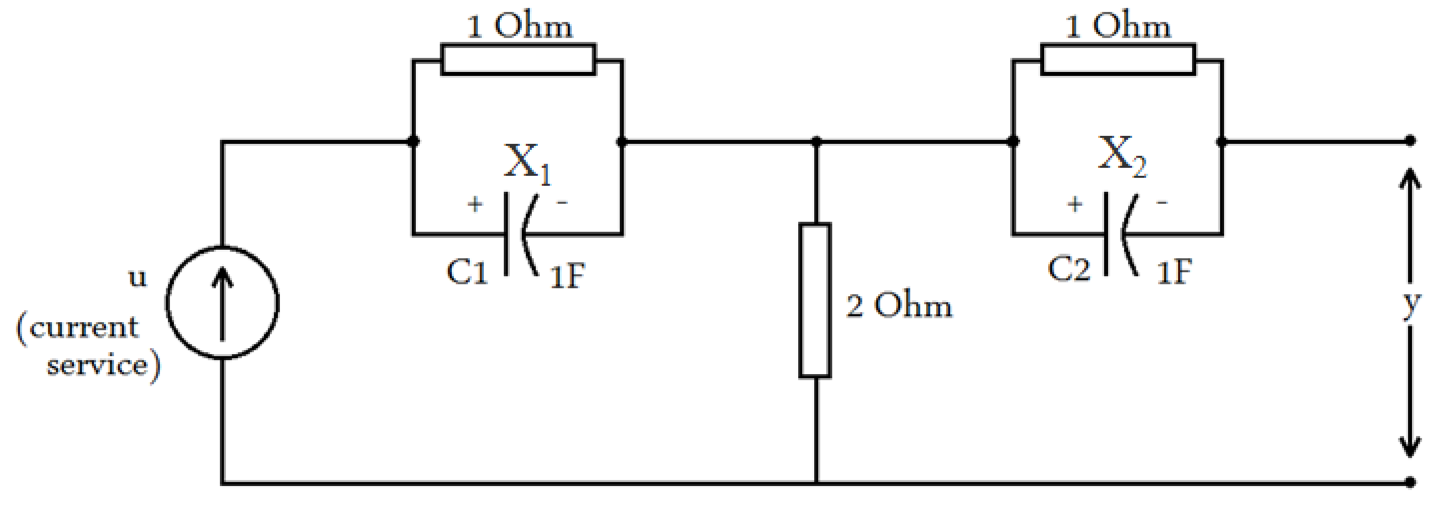
What you will learn after studying this section

- **Controllability and observability ideas**
- **Controllability and observability detection**
- **Application of controllability and observability**
- **Input determination in controllable systems**
- **Controllability and observability indices**
- **Duality of controllability and observability**
- **Effect of equivalent transformation on controllability and observability**
- **Controllability and observability in Jordan froms**
- **Controllability and observability in LTV systems**

Introduction

Controllability refers to the ability to control the states of a system through input.

Observability refers to the ability to estimate the states of a system by observing its inputs and outputs.



Controllability

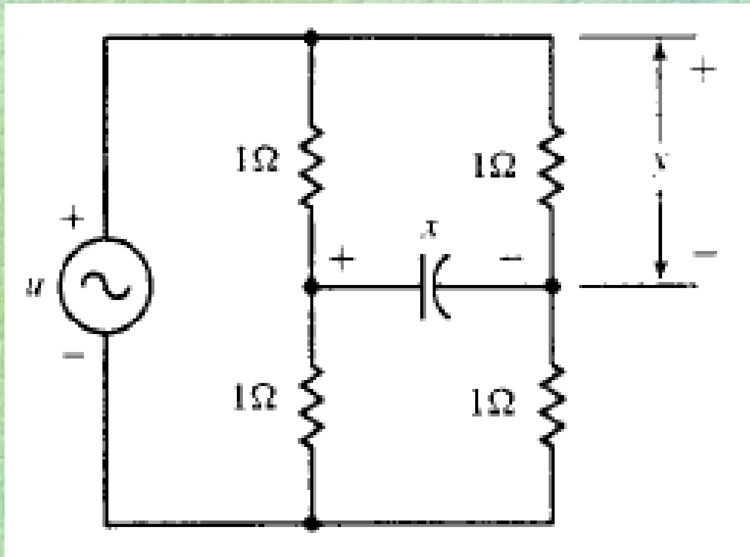
Consider following equation:

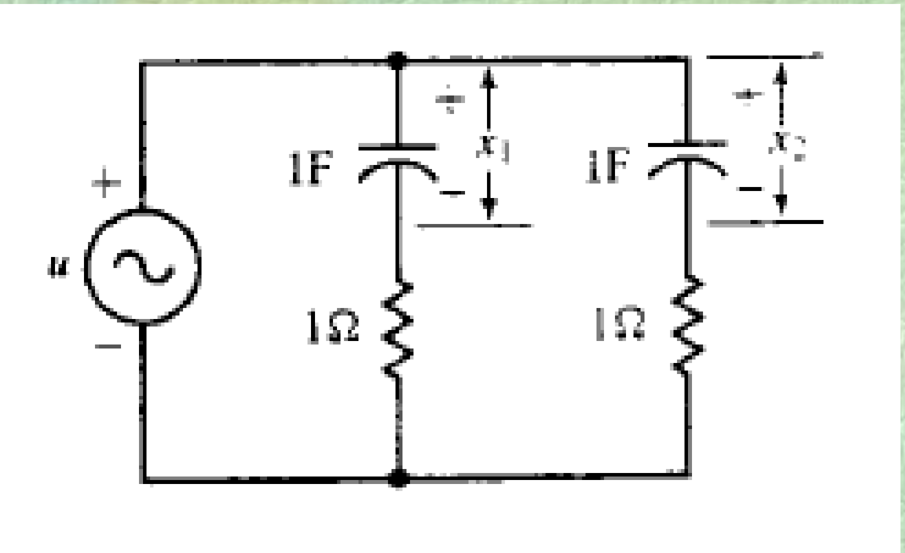
$$\dot{x} = Ax + Bu \qquad (I)$$
$$y = Cx + Du$$

Definition 1: The state equation (I) or the pair (A,B) is said to be controllable if for any initial state x_0 and any final state x_1, there exists an input that transfers x0 to x1 in a finite time. Otherwise (I) or (A,B) is said to be uncontrollable

Controllability

Example 1: Is it controllable?





It is clear that detecting controllability or uncontrollability is not an easy task just by observing the apparent view of the system.

Controllability test

Theorem 1: Following statements for the given system are equivalent:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

1- The pair (A,B) is controllable.

2- The following $n \times n$ matrix is non-singular for all t>0

$$W_c(t) = \int_0^t e^{A\tau} BB' e^{A'\tau} d\tau = \int_0^t e^{A(t-\tau)} BB' e^{A'(t-\tau)} d\tau$$

3- The $n \times np$ controllability matrix C has rank n of full row rank.

$$C = [B \quad AB A^2 B \quad \ldots \quad A^{n-1}B]$$

4- The matrix $[A - \lambda I \quad B]$ with dimension $n \times (n+p)$ has full row rank.

5- If, in addition to all the eigenvalues of A having negative real parts, the unique solution of the following equation is also positive definite. $AW_c + W_c A' = -BB'$

Controllability test

Proof: First, the equivalence of expressions 1 and 2 is examined.

$W_c(t)$ is invertible $\Rightarrow$ The pair (A,B) is controllable

The pair (A,B) is controllable $\Rightarrow$ $W_c(t)$ is invertible

First, the initial part of the proof is presented.

Since, $W_c(t)$ is invertible, for any t_1, $W_c(t_1)$ is invertible, we assert that following input transfers the system from an arbitrary initial point x_0 to an arbitrary final point x_1.

$$u(t) = -B' e^{A'(t_1-t)} W_c^{-1}(t_1)\left[e^{At_1} x_0 - x_1\right]$$

$$x(t_1) = e^{At_1} x_0 + \int_0^{t_1} e^{A(t_1-\tau)} B u(\tau) d\tau$$

$$x(t_1) = e^{At_1} x_0 - \int_0^{t_1} e^{A(t_1-\tau)} BB' e^{A'(t_1-\tau)} W_c^{-1}(t_1)[e^{At_1} x_0 - x_1] d\tau$$

$$x(t_1) = e^{At_1} x_0 - W_c(t_1) W_c^{-1}(t_1)[e^{At_1} x_0 - x_1] \quad \Rightarrow \quad x(t_1) = x_1$$

Controllability test

Proof: First, the equivalence of expressions 1 and 2 is examined.

$W_c(t)$ is invertible $\Rightarrow$ The pair (A,B) is controllable

The pair (A,B) is controllable $\Rightarrow$ $W_c(t)$ is invertible

Now, the proof of other side is presented.

We use contradiction. Assume $W_c(t)$ is not invertible at t_1. Then, there is a non-zero vector v such that:

$$v'W_c(t_1)v = \int_0^{t_1} v' e^{A(t_1-\tau)} BB' e^{A'(t_1-\tau)} v d\tau = 0 \Rightarrow \int_0^{t_1} \left\| B' e^{A'(t_1-\tau)} v \right\|^2 d\tau = 0$$

$$B' e^{A'(t_1-\tau)} v = 0 \ and \ v' e^{A(t_1-\tau)} B = 0 \quad \forall \ \tau \in [0, t_1]$$

Controllability allows easily transfer from $x_0 = e^{At_1} v$ to x_1=0.

$$0 = x(t_1) = e^{At_1} e^{-At_1} v + \int_0^{t_1} e^{A(t_1-\tau)} Bu(\tau) d\tau$$

$$0 = v'v + \int_0^{t_1} v' e^{A(t_1-\tau)} Bu(\tau) d\tau = v'v + 0 = 0 \qquad \|v\|^2 = 0$$

Contradiction!

Controllability test

Proof: First, the equivalence of expressions 1 and 2 is examined.

$W_c(t)$ is invertible $\Rightarrow$ The pair (A,B) is controllable

The pair (A,B) is controllable $\Rightarrow$ $W_c(t)$ is invertible

Now, the equivalence of expressions 2 and 3 is examined.

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Now, the equivalence of expressions 3 and 4 is examined.

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Finally, the equivalence of expression 5 with one of the others must be examined.

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Controllability test

Example 2: Check the controllability of following system.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 1 \quad 0] x$$

$$C = [b \quad Ab \quad A^2 b] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{bmatrix}$$

$$|C| = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{vmatrix} = -1$$

It is controllable canonical form.

Controllability test

Example 3: Check the controllability of each mode.

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r(t)$$

$$|sI - A| = \begin{vmatrix} s+2 & -1 \\ 0 & s+1 \end{vmatrix} = (s+1)(s+2) \rightarrow \lambda_1 = -1, \lambda_2 = -2$$

$$y = [1 \quad 1]x$$

$$C = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \qquad |C| = 0 \quad \rightarrow$$

It is not completely controllable

Controllability of $\lambda_1 = -1$:

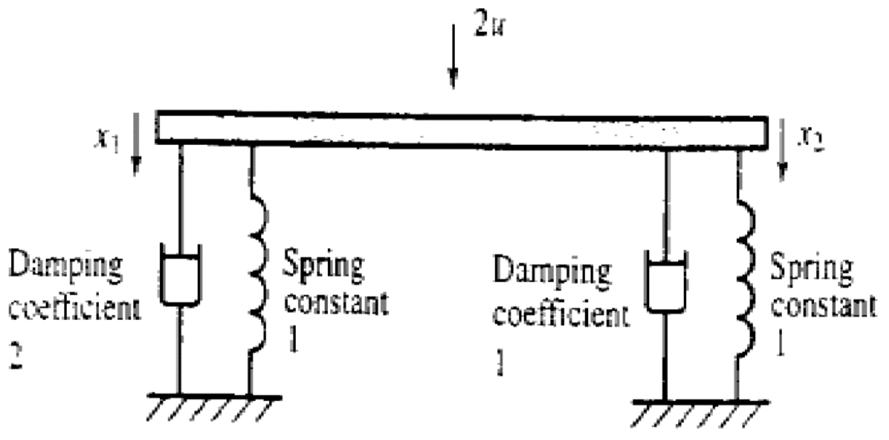
$$\left[\begin{array}{cc:c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{not full row rank}} \lambda_1 = -1 \text{ is not controllable}$$

Controllability of $\lambda_2 = -2$:

$$\left[\begin{array}{cc:c} 0 & -1 & 1 \\ 0 & -1 & 0 \end{array}\right] \xrightarrow{\text{full row rank}} \lambda_2 = -2 \text{ is controllable}$$

Controllability test

Example 4: Consider the following suspension platform. If the displacement of each spring from the equilibrium position is considered as the state of the system, the state-space equations are expressed as:



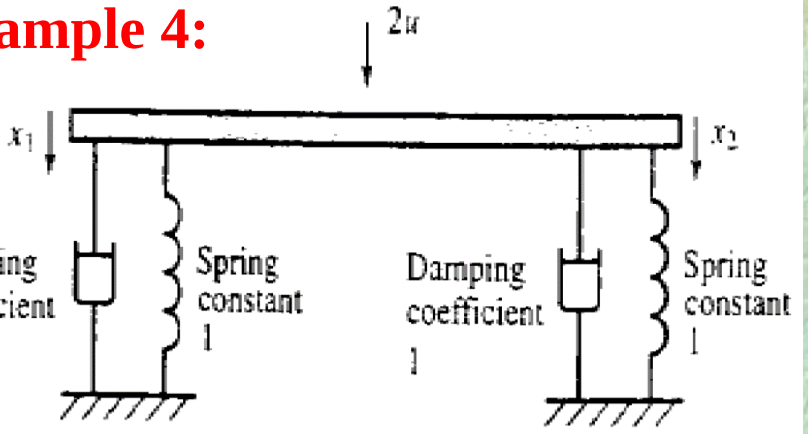
$$\dot{x} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u$$

If the initial displacement is non-zero and no force is applied, the suspension platform will exponentially approach equilibrium. Theoretically, the states will reach zero only after an infinite duration.

If $x_1(0)=10$ and $x_2(0)=-1$ is there a suitable force that can bring the suspension plate to equilibrium within 2 seconds?

Controllability test

Example 4:



$$\dot{x} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u$$

If $x_1(0)=10$ and $x_2(0)=-1$ is there a suitable force that can bring the suspension plate to equilibrium within 2 seconds?

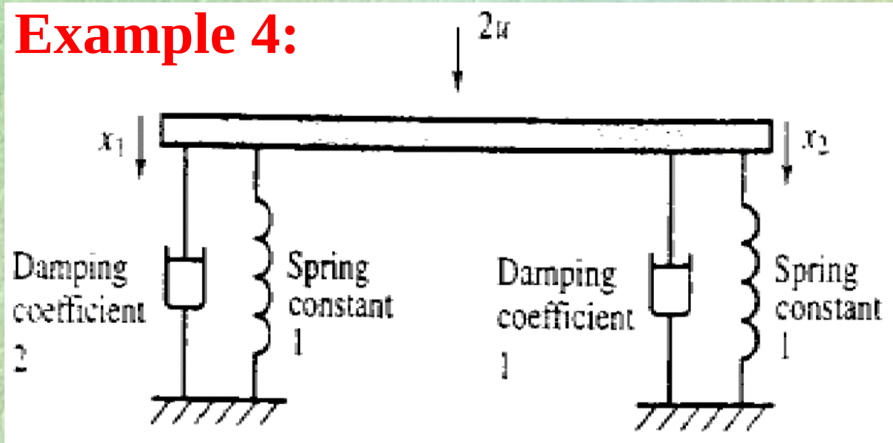
$$C = [b \quad Ab] = \begin{bmatrix} 0.5 & -0.25 \\ 1 & -1 \end{bmatrix}$$

$$|C| \neq 0$$

Thus, the suspension plate is controllable, and for any arbitrary initial condition, there exist a suitable input that can bring the plate to equilibrium.

Controllability test

Example 4:



$$\dot{x} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u$$

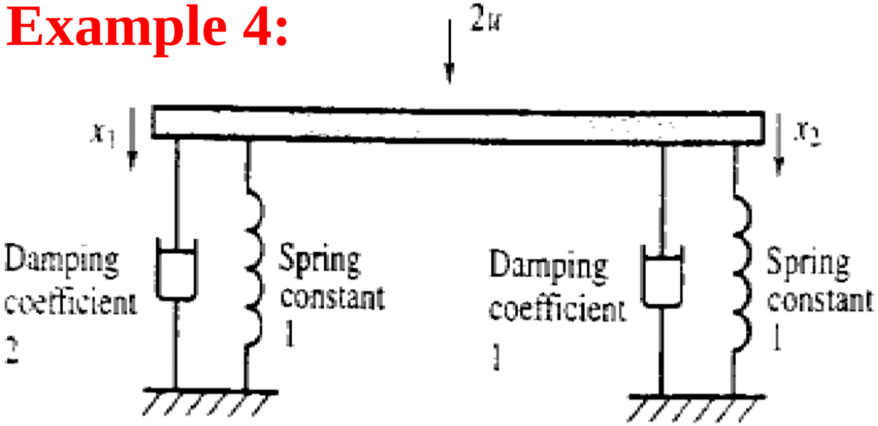
Now, we need to calculate $W_c(2)$ and $u(t)$.

$$W_c(2) = \int_0^2 \left(\begin{bmatrix} e^{-0.5\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} [0.5 \quad 1] \begin{bmatrix} e^{-0.5\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} \right) d\tau = \begin{bmatrix} 0.2162 & 0.3167 \\ 0.3167 & 0.4908 \end{bmatrix}$$

$$u(t) = -[0.5 \quad 1] \begin{bmatrix} e^{-0.5(2-t)} & 0 \\ 0 & e^{-(2-t)} \end{bmatrix} W_c^{-1}(2) \begin{bmatrix} e^{-1} & 0 \\ 0 & e^{-2} \end{bmatrix} \begin{bmatrix} 10 \\ -1 \end{bmatrix} = -58.8 e^{0.5t} + 27.96 e^{t}$$

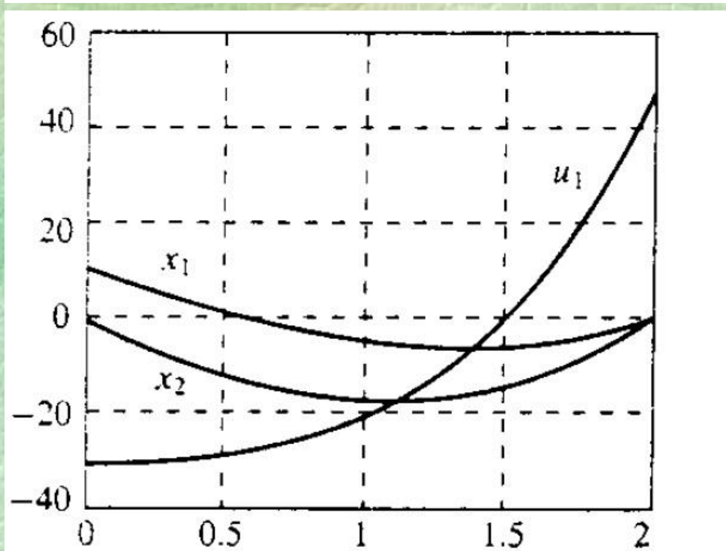
Controllability test

Example 4:



$$\dot{x} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u$$

$$u(t) = -58.8e^{0.5t} + 27.96e^{t}$$



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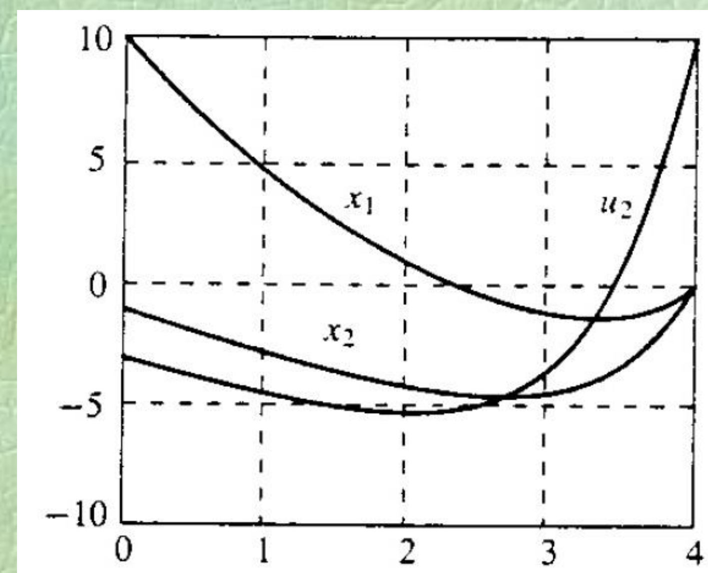
$$t_f = 2 \quad \Rightarrow \quad t_f = 4$$



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Controllability

Similarity transformation and controllability

Theorem 2: Controllability is invariant under similarity transformation.

Proof:

$$\dot{x} = Ax + bu$$
$$y = cx + du$$

$$\hat{A} = PAP^{-1} \qquad \hat{b} = Pb$$
$$\hat{c} = cP^{-1} \qquad \hat{d} = d$$

$$\dot{w} = \hat{A}w + \hat{b}u$$
$$y = \hat{c}w + \hat{d}u$$

Controllability Matrix

$$C = [b \;\; Ab \;\; A^2b \;\; \dots \;\; A^{n-1}b] \longleftarrow \qquad \longrightarrow \hat{C} = [\hat{b} \;\; \hat{A}\hat{b} \;\; \hat{A}^2\hat{b} \;\; \dots \;\; \hat{A}^{n-1}\hat{b}]$$

$$\hat{C} = [\hat{b} \;\; \hat{A}\hat{b} \;\; \hat{A}^2\hat{b} \;\; \dots \;\; \hat{A}^{n-1}\hat{b}] = [Pb \;\; PAP^{-1}Pb \;\; PA^2P^{-1}Pb \;\; \dots \;\; PA^{n-1}P^{-1}Pb] =$$

$$[Pb \;\; PAb \;\; PA^2b \dots PA^{n-1}b] = P[b \;\; Ab \;\; A^2b \dots A^{n-1}b] = PC$$

P is nonsingular $\Rightarrow \qquad \rho(\hat{C}) = \rho(C)$

Controllability indices

Suppose constant matrices *A* and *B* with suitable dimensions, and suppose *B* has full column rank (If *B* does not have full column rank, some inputs are excessive).

If *A* and *B* are controllable, controllability matrix *C* has rank n, so, there is n linearly independent column in *C*.

$$C = [\ b_1 \ \dots \ b_p \ | \ Ab_1 \ \dots \ Ab_p \ | \dots | \ A^{n-1}b_1 \ \dots \ A^{n-1}b_p \ ]$$

Now we search for linearly independent columns of *C* from the left. Suppose μ_m is the number of independent columns of *C* corresponding to b_m.

$$b_m\ ,\ Ab_m\ ,\ \dots\ ,\ A^{\mu_m - 1}b_m$$

It is clear that if *C* has full column rank, then:

$$\mu_1 + \mu_2 + \dots + \mu_p = n$$

Controllability indices

The set $\{\mu_1,\mu_2,\dots,\mu_p\}$ represents the controllability indices.

The maximum element in the set of controllability indices is called the controllability index, and it is denoted by μ.

$$\mu = \max\,(\,\mu_1\,,\,\mu_2\,,\,\dots\,,\,\mu_p\,)$$

Equivalently, if the pair (A,B) is controllable, the controllability index is the smallest integer that:

$$\rho(C_\mu) = \rho([\,B \quad AB \quad \dots \quad A^{\mu-1}B\,]) = n$$

Controllability indices

Now we define a bound for μ. If $\mu_1 = \mu_2 = \ldots = \mu_p$, then we have:

$$n/p \le \mu$$

If all μ_i are equal to 1 except for one, which is different, then:

$$\mu = n - p + 1$$

Let $\bar{n}$ be the degree of the minimal polynomial. Then, there exists a set of α_i such that:

$$A^{\bar{n}} = \alpha_1 A^{\bar{n}-1} + \alpha_2 A^{\bar{n}-2} + \ldots + \alpha_{\bar{n}} I$$

So $A^{\bar{n}} B$ can be described by a linear combination of:

$$\{B, AB, \ldots, A^{\bar{n}-1} B\}$$

So, we have:

$$n/p \le \mu \le \min(\bar{n}, n - p + 1)$$

Controllability indices

Theorem 3: The pair (A,B), where B has a rank of p, is controllable if and only if following matrix has a rank n.

$$C_{n-p+1} = [B \;\; AB \;\; \dots \;\; A^{n-p}B]$$

Example 5: Consider following state space model. Derive controllability indices.

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} U \quad , \quad y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} X$$

$$[\; B \;\; AB \;\; A^2B \;] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 & 0 & -4 \end{bmatrix}$$

The rank of this matrix is 4, which implies that the above state-space model is controllable. It can be easily shown that the controllability indices are 2 and 2, and the controllability index is 2.

Observability

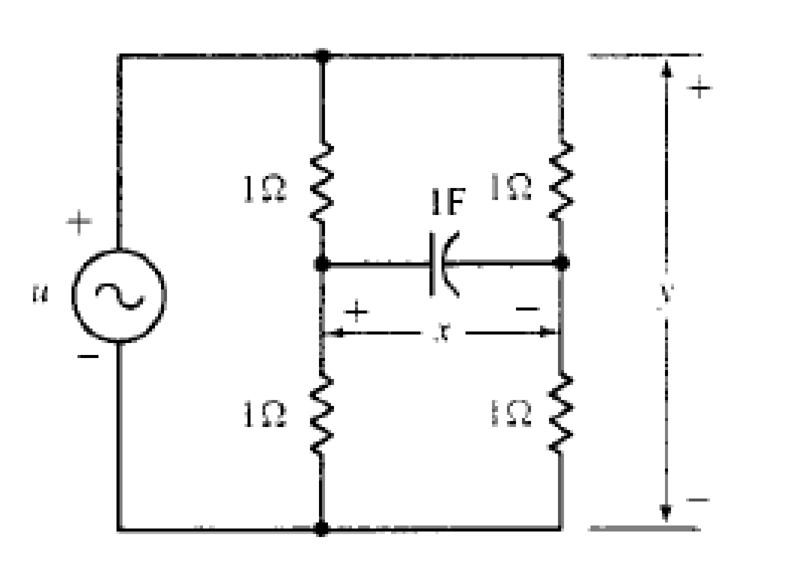
Consider following equation:

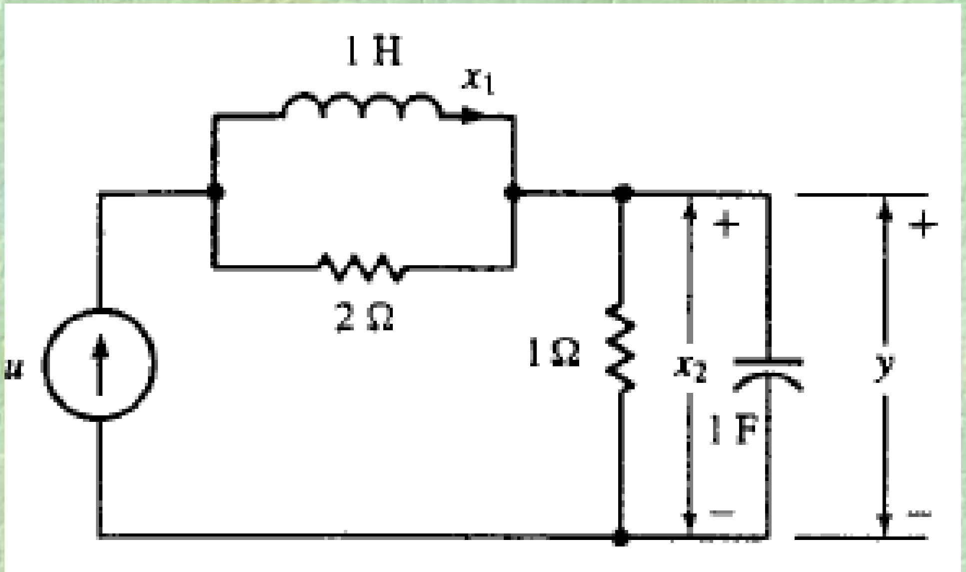
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du \qquad (I)$$

Definition 2:The state equation (I) or the pair (A,C) is said to be observable if for any unknown initial state x_0 , there exists a finite time $t_1 > 0$ such that the knowledge of the input u and the output y over $[0,t_1]$ suffices to determine Uniquely the initial state x_0. Other wise, the equation is unobservable.

Observability

Example 6: Unobservable systems.





It is clear that detecting controllability or uncontrollability is not an easy task just by observing the apparent view of the system.

Observability test

Theorem 4: A state-space system is observable if and only if the following n-dimensional matrix is nonsingular for all t>0.

$$w_o(t) = \int_0^t e^{A'\tau} C' C e^{A\tau} \, d\tau$$

Proof: Two side of the theorem must be examined.

$W_o(t)$ is invertible $\Rightarrow$ The pair (A,C) is observable

The pair (A,C) is observable $\Rightarrow$ $W_o(t)$ is invertible

First, the initial part of the proof is presented.

$$y(t) = Ce^{At} x_0 + C\int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau + Du(t)$$

$$Ce^{At} x_0 = \bar{y}(t) = y(t) - C\int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau - Du(t)$$

$$e^{A't} C' C e^{At} x_0 = e^{A't} C' \bar{y}(t) \qquad \int_0^{t_1} e^{A'\tau} C' C e^{A\tau} x_0 d\tau = \int_0^{t_1} e^{A'\tau} C' \bar{y}(\tau) d\tau$$

Observability test

$$e^{A't}C'Ce^{At}x_0 = e^{A't}C'\bar{y}(t) \quad \int_0^{t_1} e^{A'\tau}C'Ce^{A\tau}x_0 d\tau = \int_0^{t_1} e^{A'\tau}C'\bar{y}(\tau)d\tau$$

$$x_0 = W_0^{-1}(t_1)\int_0^{t_1} e^{A'\tau}C'\bar{y}(\tau)d\tau$$

Now, the proof of other side is presented.

The pair (A,C) is observable $\Rightarrow$ $W_o(t)$ is invertible

We use contradiction. Assume $W_O(t)$ is not invertible at t_1. Then, there is a non-zero vector v such that:

$$v'W_o(t_1)v = \int_0^{t_1} v'e^{A'\tau}C'Ce^{A\tau}v d\tau = 0 \qquad \int_0^{t_1}\left\|Ce^{A\tau}v\right\|^2 d\tau = 0$$

Now consider: $Ce^{At}v = 0 \;\forall\; t \in [0,t_1]$

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau + Du(t)$$

Two different initial conditions, x_0 and v, with zero input both result in y=0, so the initial condition cannot be uniquely determined.

Observability test

Theorem 5(Duality): The pair (A,B) is controllable if and only if the pair (A',B') is observable.

Theorem 6(Duality): The pair (A',C') is controllable if and only if the pair (A,C) is observable.

Example 7: Check observability of the given system.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 1 \quad 0]x$$

$$c' = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ A' = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}$$

$$C = \begin{bmatrix} c' & A'c' & A'^2 c' \end{bmatrix} = \begin{bmatrix} 1 & 0 & -6 \\ 1 & 1 & -11 \\ 0 & 1 & -5 \end{bmatrix}$$

The rank of matrix is not 3, thus ….

Observability test

Theorem 7: Following statements for the given system are equivalent:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

1- The pair (A,C) is observable.

2- The following $n \times n$ matrix is non-singular for all t>0

$$W_O(t) = \int_0^t e^{A'\tau} C' C \, e^{A\tau} \, d\tau$$

3- The $nq \times n$ observability matrix O has rank n of full row rank.

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

4- The matrix $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ with dimension ($n+q) \times n$) has full column rank.

5- If, in addition to all the eigenvalues of A having negative real parts, the unique solution of the following equation is also positive definite. $A' W_o + W_o A = -C' C$

Observability test

Example 8: Check observability of the following system.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 1 \quad 0] x$$

$$V = \begin{bmatrix} c \\ cA \\ cA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -6 & -11 & -5 \end{bmatrix}$$

$$|V| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -6 & -11 & -5 \end{vmatrix} = 1(-5+11) - 1(0+6) = 0$$

So .....

Observability

Effect of similarity transformation on observability

Theorem 8: Observability is invariant under similarity transformation.

Proof:

$$\dot{x} = Ax + bu$$
$$y = cx + du$$

$$\hat{A} = PAP^{-1} \qquad \hat{b} = Pb$$
$$\hat{c} = cP^{-1} \qquad \hat{d} = d$$

$$\dot{w} = \hat{A}w + \hat{b}u$$
$$y = \hat{c}w + \hat{d}u$$

$$O = \begin{bmatrix} c \\ cA \\ . \\ . \\ cA^{n-1} \end{bmatrix} \quad \longleftarrow \quad \text{Observability Matrix} \quad \longrightarrow \quad \hat{O} = \begin{bmatrix} \hat{c} \\ \hat{c}\hat{A} \\ . \\ . \\ \hat{c}\hat{A}^{n-1} \end{bmatrix}$$

$$\hat{O} = OP^{-1}$$

P is nonsingular $\Rightarrow$ $\rho(\hat{O}) = \rho(O)$

Observability indices

Suppose we have constant matrices A and C with suitable dimensions, and suppose C has full row rank. If C does not have full row rank, some outputs are linear combinations of others, meaning no new information is provided.

If A and C are observable, observability matrix O has rank n, so, there is n linearly independent row in O.

Now we search for linearly independent rows of O from the top. Suppose v_m is the number of independent rows of O corresponding to c_m.

It is clear that if O has full row rank, then:

$$v_1 + v_2 + \ldots + v_q = n$$

The set $\{v_1, v_2, \ldots, v_q\}$ represents the observability indices.

The maximum element in the set of observability indices is called the observability index, and it is denoted by v.

$$v = \max\left(v_1, v_2, \ldots, v_q\right)$$

Observability indices

Equivalently, if the pair (A,C) is observable,
the observability index is the smallest integer that:

$$\rho(O_v) = \rho\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{v-1} \end{bmatrix}\right) = n$$

Similar to controllability we have:

$$n/q \le v \le \min\,(\bar{n}\ , n-q+1)$$

Theorem 9: The pair (A,C), where C has a rank of q, is observable if and only if following matrix has a rank n.

$$O_{n-q+1} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-q} \end{bmatrix}$$

Canonical Decomposition

$$\dot{x} = Ax + bu$$
$$y = cx + du$$

Suppose $w = Px$ then

$$\dot{w} = \bar{A}w + \bar{b}u$$
$$y = \bar{c}w + \bar{d}u$$

Where

$$\bar{A} = PAP^{-1} \qquad \bar{b} = Pb$$
$$\bar{c} = cP^{-1} \qquad \bar{d} = d$$

We know that stability, controllability, and observability are preserved under similarity transformations.

$$\bar{C} = PC \quad , \quad \bar{O} = OP^{-1}$$

Canonical Decomposition

Theorem 10: Consider following system is not controllable

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

so

$$\rho(C) = \rho(\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}) = n_1 \prec n$$

Then we form the following matrix:

$$P^{-1} = \begin{bmatrix} q_1 & q_2 & \dots q_{n1} \dots\dots & q_n \end{bmatrix}$$

The first n_1 columns of P^{-1} are n_1 independent columns of the controllability matrix C, and the remaining columns are chosen such that P is non-singular. Then, similarity transformation leads to:

$$\begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + Du$$

Canonical Decomposition

$$\begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + Du$$

Where

$$\bar{A}_{c\ \ n_1 \times n_1} \quad , \quad \bar{A}_{\bar{c}\ \ (n-n_1)\times(n-n_1)}$$

And n_1 dimensional state-space sub equation is:

$$\dot{\bar{x}}_c = \bar{A}_c \bar{x}_c + \bar{B}_c u$$

$$y = \bar{C}_c \bar{x}_c + Du$$

The new system is controllable and has the same transfer function as the first system (zero-state equivalent).

Canonical Decomposition

Proof: We know that

$$P^{-1} = \left[q_1 \quad q_2 \quad \ldots q_{n1} \; \ldots. \quad q_n \right]$$

The i[th] column of $\bar{A}$ is the representation of Aq_i in terms of the columns of P^{-1}.

$$\bar{A} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix}$$

The column of $\bar{B}$ is the representation of the columns of B in terms of column of P^{-1}.

$$\bar{B} = \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}$$

So, the converted system is:

$$\begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u$$

$$y = \left[\bar{C}_c \quad \bar{C}_{\bar{c}} \right] \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + Du$$

Canonical Decomposition

Proof: Continue

$$\begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + Du$$

We know:

$$\rho(C) = \rho(\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}) = n_1 = \rho(\bar{C})$$

So, we have:

$$n_1 = \rho(\bar{C}) = \rho(\begin{bmatrix} \bar{B}_c & \bar{A}_c\bar{B}_c & \dots & \bar{A}_c^{\,n-1}\bar{B}_c \\ 0 & 0 & \dots & 0 \end{bmatrix}) = \rho(\begin{bmatrix} \bar{B}_c & \bar{A}_c\bar{B}_c & \dots & \bar{A}_c^{\,n_1-1}\bar{B}_c \\ 0 & 0 & \dots & 0 \end{bmatrix})$$

$$= \rho(\begin{bmatrix} \bar{B}_c & \bar{A}_c\bar{B}_c & \dots & \bar{A}_c^{\,n_1-1}\bar{B}_c \end{bmatrix}) = n_1$$

And this is the controllability matrix of reduced system.

$$\dot{\bar{x}}_c = \bar{A}_c \bar{x}_c + \bar{B}_c u$$

$$y = \bar{C}_c \bar{x}_c + Du$$

Canonical Decomposition

Proof: Continue

$$\begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + Du$$

$$\dot{\bar{x}}_c = \bar{A}_c \bar{x}_c + \bar{B}_c u$$

$$y = \bar{C}_c \bar{x}_c + Du$$

On the other hand, the transfer function of systems is given by:

$$G(s) = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} sI - \bar{A}_c & \bar{A}_{12} \\ 0 & sI - \bar{A}_{\bar{c}} \end{bmatrix}^{-1} \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} + D$$

$$G(s) = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} (sI - \bar{A}_c)^{-1} & M \\ 0 & (sI - \bar{A}_{\bar{c}})^{-1} \end{bmatrix} \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} + D$$

$$G(s) = \bar{C}_c (sI - \bar{A}_c)^{-1} \bar{B}_c + D$$

Canonical Decomposition

Example 9: Consider following state space model.

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

Rank of matrix B is 2 so:

$$\rho(C_2) = \rho([B \quad AB]) = \rho\left(\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}\right) = 2 \prec 3$$

So, the system is not controllable, if we choose:

$$P^{-1} = Q = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The new system is:

$$\dot{\bar{x}} = \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right] \bar{x} + \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hdashline 0 & 0 \end{array}\right] u$$

$$y = \left[\begin{array}{cc:c} 1 & 2 & 1 \end{array}\right] \bar{x}$$

$$\dot{\bar{x}}_c = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \bar{x}_c + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} \bar{x}_c$$

Canonical Decomposition

Theorem 11: Consider following system is not observable

$$\dot{x} = Ax + bu$$
$$y = cx + du$$

so

$$\rho(O) = \rho\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n_2 < n$$

Then we form the following matrix:

$$P = \begin{bmatrix} p_1 \\ \vdots \\ p_{n_2} \\ \vdots \\ p_n \end{bmatrix}$$

The first n_2 rows of P are n_2 independent rows of the observability matrix O, and the remaining rows are chosen such that P is non-singular. Then, similarity transformation leads to:

$$\begin{bmatrix} \dot{\bar{x}}_o \\ \dot{\bar{x}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{bmatrix} u \quad , \quad y = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix} + du$$

Canonical Decomposition

$$\begin{bmatrix} \dot{\bar{x}}_o \\ \dot{\bar{x}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{bmatrix} u \quad , \quad y = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix}$$

Where

$$\bar{A}_o \quad {}_{n_2 \times n_2} \qquad \bar{A}_{\bar{o}} \quad {}_{(n-n_2)\times(n-n_2)}$$

And n_2 dimensional state-space sub equation is:

$$\dot{\bar{x}}_o = \bar{A}_o \bar{x}_o + \bar{B}_o u$$
$$\bar{y} = \bar{C}_o \bar{x}_o + Du$$

The new system is observable and has the same transfer function as the first system (zero-state equivalent).

Proof: Similar to previous theorem.

Canonical Decomposition

Theorem 12: Any state-space equation can be transformed into the following canonical form using suitable similarity transformation.

$$\begin{bmatrix} \dot{\bar{x}}_{co} \\ \dot{\bar{x}}_{c\bar{o}} \\ \dot{\bar{x}}_{\bar{c}o} \\ \dot{\bar{x}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_{co} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{\bar{c}o} & 0 \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_{co} \\ \bar{B}_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_{co} & 0 & \bar{C}_{\bar{c}o} & 0 \end{bmatrix} \bar{x} + Du$$

Where

$$\bar{x}_{co} = \text{Controllable and observable}$$

$$\bar{x}_{c\bar{o}} = \text{Controllable but not observable}$$

$$\bar{x}_{\bar{c}o} = \text{Observable but not controllable}$$

$$\bar{x}_{\bar{c}\bar{o}} = \text{Neither observable nor controllable}$$

Canonical Decomposition

Theorem 12: Any state-space equation can be transformed into the following canonical form using suitable similarity transformation.

$$\begin{bmatrix} \dot{\bar{x}}_{co} \\ \dot{\bar{x}}_{c\bar{o}} \\ \dot{\bar{x}}_{\bar{c}o} \\ \dot{\bar{x}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_{co} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{\bar{c}o} & 0 \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_{co} \\ \bar{B}_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_{co} & 0 & \bar{C}_{\bar{c}o} & 0 \end{bmatrix} \bar{x} + Du$$

A minimal-order zero-state reduced-order controllable and observable system is:

$$\dot{\bar{x}}_{co} = \bar{A}_{co} \bar{x}_{co} + \bar{B}_{co} u$$

$$\bar{y} = \bar{C}_{co} \bar{x}_{co} + Du$$

Controllabiltiy and Observability in Jordan Form

We show that controllability and observability preserved under similarity transformation. If we change the system to Jordan canonical form controllability and observability easily derived. Now consider:

$$\dot{x} = Jx + Bu \qquad III$$
$$y = Cx + Du$$

Where J is in Jordan form. Let J have two distinct eigenvalues, λ_1 and λ_2, and it is in the following form:

$$J = diag\ (\ J_1\ ,\ J_2\ )$$

Now suppose J_1 is corresponding to λ_1 and J_2 corresponding to λ_2, and let J_1 have three Jordan blocks and J_2 have two Jordan blocks as:

$$J_1 = diag\ (\ J_{11}\ ,\ J_{12}\ ,\ J_{13}\ ) \qquad J_2 = diag\ (\ J_{21}\ ,\ J_{22}\ )$$

Suppose b_{lij} is the last row of J_{ij} and c_{fij} is the first column of J_{ij}

Controllabiltiy and Observability in Jordan Form

Theorem 13: 1- State-space equation III is controllable if and only if three row vectors,

$$\{ b_{l11} , b_{l12} , b_{l13} \}$$

and two row vectors:

$$\{ b_{l21} , b_{l22} \}$$

are linearly independent.

2- State-space equation III is observable if and only if three column vectors,

$$\{ c_{f11} , c_{f12} , c_{f13} \}$$

and two column vectors:

$$\{ c_{f21} , c_{f22} \}$$

are linearly independent.

Controllabiltiy and Observability in Jordan Form

Remark 1: If a state-space model is in Jordan form, the controllability of states corresponding to one eigenvalue can be examined independently from controllability of other eigenvalues.

Remark 2: The controllability of states corresponding to a particular eigenvalue depends only on the rows of B associated with the last rows of the Jordan blocks corresponding to that eigenvalue, and it is independent of the other rows.

Remark 3: A similar statement can be made about observability. However, in this case, the columns of C determine the observability of states corresponding to a particular eigenvalue, rather than the rows of B.

Controllabiltiy and Observability in Jordan Form

Example 10: Consider following Jordan form state space model.

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 3 & 0 & 2 & 0 \end{bmatrix} x$$

Controllable?

Observable?

Controllabiltiy and Observability in Jordan Form

Theorem 14: The Jordan canonical form of a single-input state-space model is controllable if and only if, for each eigenvalue, there is exactly one Jordan block, and the last row of the vector *b* corresponding to that Jordan block is non-zero.

Theorem 15: The Jordan canonical form of a single-output state-space model is observable if and only if, for each eigenvalue, there is exactly one Jordan block, and the first column of the vector *c* corresponding to that Jordan block is non-zero.

Controllabiltiy and Observability in Jordan Form

Example 11: Consider following state space model.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 10 \\ 9 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 2 \end{bmatrix} x$$

Controllable?

Observable?

Controllabiltiy and Observability in LTV Systems

Consider an n×n dimensional LTV state-space model with p inputs and q outputs:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t) + D(t)u(t)$$

Definition 3: The state equation or the pair $(A(t),B(t))$ is said to be controllable at t_0 if for any initial state $x(t_0)= x_0$ and any final state x_1, there exists an input that transfers x_0 to x_1 in a time t_1. Otherwise, the state equation or $(A(t),B(t))$ is said to be uncontrollable at t_0.

Remark: In LTI systems, if the state-space model is controllable, then it is controllable for any t_0 and for any $t_1 > t_0$. Therefore, there is no need to specify t_0 and t_1. However, in LTV systems, it is necessary to specify t_0 and t_1.

Controllabiltiy and Observability in LTV Systems

Theorem 16: The n-dimensional pair (A(t),B(t)) is said to be controllable at t_0 if and only if there exists $t_1 > t_0$ such that the following n-dimensional matrix is non-singular.

$$W_c(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)B'(\tau)\Phi'(t_1,\tau)d\tau$$

$\Phi(t,\tau)$ is the state transition matrix for $\dot{x} = A(t)x$.

Proof:

$W_c(t_0,t_1)$ is invertible $\Rightarrow$ The pair (A(t),B(t)) is controllable

The pair (A(t),B(t)) is controllable $\Rightarrow$ $W_c(t_0,t_1)$ is invertible

Since $W_c(t_0,t_1)$ is invertible, $W_c^{-1}(t_0,t_1)$ exists. Thus we assert that the input $u(t)$ can transfer an arbitrary x_0 to an arbitrary x_1 at time t_1 :

$$u(t) = -B'(t)\Phi'(t_1,t)W_c^{-1}(t_0,t_1)\left[\Phi(t_1,t_0)x_0 - x_1\right]$$

Controllabiltiy and Observability in LTV Systems

$W_c(t_0,t_1)$ is invertible $\Rightarrow$ The pair (A(t),B(t)) is controllable

The pair (A(t),B(t)) is controllable $\Rightarrow$ $W_c(t_0,t_1)$ is invertible

Since $W_c(t_0,t_1)$ is invertible, $W_c^{-1}(t_0,t_1)$ exists. Thus we assert that the input $u(t)$ can transfer an arbitrary x_0 to an arbitrary x_1 at time t_1 :

$$u(t) = -B'(t)\Phi'(t_1,t)W_c^{-1}(t_0,t_1)\left[\Phi(t_1,t_0)x_0 - x_1\right]$$

We know that the states follow from the following relation:

$$x(t_1) = \Phi(t_1,t_0)x_0 + \int_{t_0}^{t_1}\Phi(t_1,\tau)B(\tau)u(\tau)\,d\tau$$

$$= \Phi(t_1,t_0)x_0 - \int_{t_0}^{t_1}\Phi(t_1,\tau)B(\tau)\,B'(\tau)\,\Phi'(t_1,\tau)W_c^{-1}(t_0,t_1)[\Phi(t_1,t_0)x_0 - x_1]\,d\tau$$

$$= \Phi(t_1,t_0)x_0 - W_c(t_0,t_1)W_c^{-1}(t_0,t_1)[\Phi(t_1,t_0)x_0 - x_1] \Rightarrow x(t_1) = x_1$$

Controllabiltiy and Observability in LTV Systems

$W_c(t_0,t_1)$ is invertible $\Rightarrow$ The pair (A(t),B(t)) is controllable

The pair (A(t),B(t)) is controllable $\Rightarrow$ $W_c(t_0,t_1)$ is invertible

Now, we must prove the other side of theorem:

Suppose that $W_c(t_0, t_1)$ is not invertible for all $t_1>t_0$, so there exists a vector $v\neq 0$ such that:

$$v'W_c(t_0,t_1)v = \int_{t_0}^{t_1} v'\Phi(t_1,\tau)B(\tau)B'(\tau)\Phi'(t_1,\tau)v\,d\tau = 0$$

$$\Rightarrow \int_{t_0}^{t_1} \left\| B'\Phi'(t_1,\tau)v \right\|^2 d\tau = 0$$

$$B'(\tau)\Phi'(t_1,\tau)v = 0 \; and \;\; v'\Phi(t_1,\tau)B(\tau) = 0 \quad \forall\, \tau \in [t_0,t_1]$$

Since the system is controllable, we can easily transfer from $x_0 = \Phi(t,\tau)$ to x_1=0 so:

$$0 = x(t_1) = \Phi(t_1,t_0)\Phi(t_0,t_1)v + \int_{t_0}^{t_1}\Phi(t_1,\tau)B(\tau)u(\tau)\,d\tau$$

$$0 = v'v + \int_{t_0}^{t_1} v'\Phi(t_1,\tau)B(\tau)u(\tau)\,d\tau = v'v + 0 = 0 \quad \|v\|^2 = 0$$

Contradiction

Controllabiltiy and Observability in LTV Systems

To use the previous theorem, we need the state transition matrix $\Phi(t,\tau)$. Therefore, we need a controllability test that is independent of the state transition matrix.

Suppose A(t) and B(t) are (n-1) times continuously differentiable. Then define:

$$M_0(t) = B(t)$$

Now, define $M_m(t)$ as:

$$M_{m+1}(t) = -A(t)M_m(t) + \frac{d}{dt}M_m(t)$$

Clearly, for all t_2 we have:

$$\Phi(t_2,t)B(t) = \Phi(t_2,t)M_0(t)$$

$$\frac{\partial}{\partial t}\left[\Phi(t_2,t)B(t)\right] = \frac{\partial}{\partial t}\left[\Phi(t_2,t)\right]B(t) + \Phi(t_2,t)\frac{d}{dt}B(t)$$

Controllabiltiy and Observability in LTV Systems

Suppose A(t) and B(t) are (n-1) times continuously differentiable. Then define:

$$M_0(t) = B(t)$$

$$M_{m+1}(t) = -A(t)M_m(t) + \frac{d}{dt}M_m(t)$$

$$\Phi(t_2,t)B(t) = \Phi(t_2,t)M_0(t)$$

$$\frac{\partial}{\partial t}\left[\Phi(t_2,t)B(t)\right] = \frac{\partial}{\partial t}\left[\Phi(t_2,t)\right]B(t) + \Phi(t_2,t)\frac{d}{dt}B(t)$$

$$= -\Phi(t_2,t)A(t)M_0(t) + \Phi(t_2,t)\frac{d}{dt}M_0(t)$$

$$= \Phi(t_2,t)M_1(t)$$

Controllabiltiy and Observability in LTV Systems

By more differentiation we have:

$$\frac{\partial^m}{\partial t^m}\left[\Phi(t_2,t)B(t)\right]=\Phi(t_2,t)\,M_m(t)$$

Following theorem is sufficient but not necessary condition for controllability.

Theorem 17: Let A(t) and B(t)) are (n-1) times continuously differentiable. Then pair (A(t),B(t)) is said to be controllable at t_0 if there exists a finite time $t_1 > t_0$ such that:

$$rank\left[M_0(t_1)\quad M_1(t_1)\;\;\ldots\;\;M_{n-1}(t_1)\right]=n$$

Controllabiltiy and Observability in LTV Systems

Theorem 17: Let A(t) and B(t)) are (n-1) times continuously differentiable. Then pair (A(t),B(t)) is said to be controllable at t_0 if there exists a finite time $t_1 > t_0$ such that:

$$rank\left[M_0(t_1) \quad M_1(t_1) \;\; \dots \;\; M_{n-1}(t_1)\right] = n$$

Proof: We will show that if the rank of the above matrix is n, then $W_C(t_0,t)$ is non-singular for all $t>t_1$. We will prove this by contradiction. Suppose $W_C(t_0, t_2)$ is singular for some $t_2 > t_1$, then

$$v'W_c(t_0,t_2)v = \int_{t_0}^{t_2} v'\Phi(t_2,\tau)B(\tau)B'(\tau)\Phi'(t_2,\tau)v\,d\tau = 0$$

$$\Rightarrow \quad \int_{t_0}^{t_2} \left\| B'\Phi'(t_2,\tau)v \right\|^2 d\tau = 0$$

So:

$$B'(\tau)\Phi'(t_2,\tau)v = 0 \; and \;\; v'\Phi(t_2,\tau)B(\tau) = 0 \quad \forall\, \tau \in [t_0,t_2]$$

By differentiating with respect to τ and substituting t_1 instead, we have

$$v'\Phi(t_2,t_1)\left[M_0(t_1) \quad M_1(t_1) \quad \dots \quad M_{n-1}(t_1)\right] = 0$$

Contradiction

Controllabiltiy and Observability in LTV Systems

Example 12: Check the controllability of following state space model.

$$\dot{x} = \begin{bmatrix} t & -1 & 0 \\ 0 & -t & t \\ 0 & 0 & t \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

Let

$$M_0 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad M_1 = -AM_0 + \frac{d}{dt} M_0 = -\begin{bmatrix} -1 \\ 0 \\ t \end{bmatrix} \quad M_2 = -AM_1 + \frac{d}{dt} M_1 = \begin{bmatrix} -t \\ t^2 \\ t^2 - 1 \end{bmatrix}$$

$$| M_0 \quad M_1 \quad M_2 | = \begin{vmatrix} 0 & 1 & -t \\ 1 & 0 & t^2 \\ 1 & -t & t^2 - 1 \end{vmatrix} = t^2 + 1$$

The determinant is non-zero for all t; therefore, it is invertible for all *t*, thus the system is controllable.

Controllabiltiy and Observability in LTV Systems

Example 13: Check the controllability of following state space models.

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \qquad (I)$$

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} e^{t} \\ e^{2t} \end{bmatrix} u \qquad (II)$$

State-space equation (I) is LTI and in Jordan form. Clearly it is controllable.

State-space equation (II) is an LTV system in Jordan form. Since all elements of *B* are non-zero for all *t*, it might be incorrectly considered controllable, but...

$$M_0 = \begin{bmatrix} e^{t} \\ e^{2t} \end{bmatrix} \qquad M_1 = -AM_0 + \frac{d}{dt} M_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \rho([M_0 \quad M_1]) = 1 < 2$$

So …………………

Controllabiltiy and Observability in LTV Systems

Consider an $n\times n$ dimensional LTV state-space model with p inputs and q outputs:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t) + D(t)u(t)$$

Definition 4: The state equation or the pair $(A(t),C(t))$ is said to be observable at t_0 if there exists a finite time $t_1>t_0$ such that, for any $x(t_0)= x_0$, the input and output information in $[t_0, t_1]$ is sufficient to uniquely determine x_0 .

Controllabiltiy and Observability in LTV Systems

Theorem 18: The n-dimensional pair (A(t),C(t)) is said to be observable at t_0 if and only if there exists $t_1 > t_0$ such that the following n-dimensional matrix is non-singular.

$$W_o(t_0,t_1) = \int_{t_0}^{t_1} \Phi'(\tau,t_0)C'(\tau)C(\tau)\Phi(\tau,t_0)\,d\tau$$

Theorem 19: Let A(t) and C(t)) are (n-1) times continuously differentiable. Then pair (A(t),C(t)) is said to be observable at t_0 if there exists a finite time $t_1 > t_0$ such that:

$$rank\begin{bmatrix} N_0(t_1) \\ N_1(t_1) \\ \vdots \\ N_{n-1}(t_1) \end{bmatrix} = n$$

Where:

$$N_0(t) = C(t), \qquad N_{m+1}(t) = N_m(t)A(t) + \frac{d}{dt}N_m(t)$$

Exercises

Exercise 1: Examine the controllability and observability of the following system.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

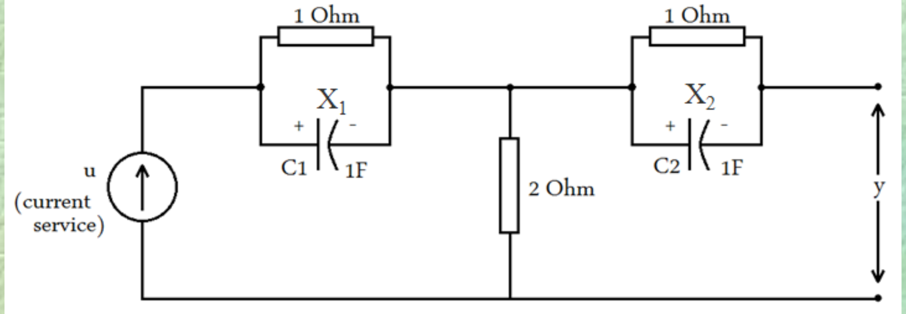
$$y = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} x$$

Exercise 2: Examine the controllability and observability of the following system.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x$$

Exercise 3: Describe the state-space model and then examine the controllability and observability of the following system.



Exercises

Exercise 4: Examine the controllability and observability indices of the following system.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} x$$

Exercise 5: Examine the controllability and observability indices of the following system.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x$$

Exercise 6: Examine the controllability and observability indices of the following system. *I* is identity matrix.

$$\dot{x} = Ax + Iu$$

Exercises

Exercise 7: Reduce the following system to a controllable form. Is the reduced system observable?

$$\dot{x} = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

Exercise 8: Reduce the following system to a controllable and observable form.

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \end{bmatrix} x$$

Exercises

Exercise 9: Examine the controllability and observability of the following system.

$$\dot{x} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 2 & 1 & 3 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} x$$

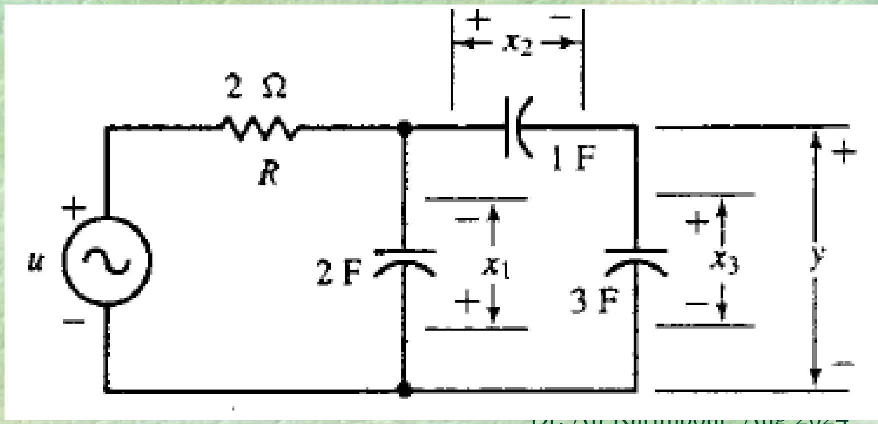
Exercises

Exercise 10: Is it possible to set the B and C matrices such that the system is controllable?
What about observable?

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \end{bmatrix} u$$

$$y = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \end{bmatrix} x$$

Exercise 11: Derive a two-dimensional and three-dimensional state-space model for the given system. Check the controllability and observability of each.



Exercises

Exercise 12: Examine the controllability and observability of the following system.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & t \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Exercise 13: Examine the controllability and observability of the following system.

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & e^{-t} \end{bmatrix} x$$

Exercise 14: Derive a zero-state equivalent with the least degree for the given system(Final 2014).

$$x = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 3] x$$

Exercises

Exercise 15: Examine the controllability and observability of the following system(Final 2013).

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 3 & 0 & 2 & 0 \end{bmatrix} x$$

Answers to selected problems

Answer 2: Controllable and observable.

Answer 3: Neither controllable nor observable.

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & -1 \end{bmatrix} x + 2u$$