Engineering Mathematics

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Content of this course

1. Fourier Series and Fourier Integral.

2. Partial Differential Equation and Its Solutions.

3. Complex Analysis. (The theory of functions of a complex variable)

Complex Analysis (The theory of functions of a complex variable)

- Fundamentals
- Analytic Functions and Differentiability
- Integration in the Complex Plane
- Complex Series
- Residue Theory and Calculation of Real Integrals

Fundamentals

The term "**complex number**" refers to a number in the form $z=x+iy$, where x and y are real numbers, and i is known as the imaginary unit, defined as follows.

$$i^2 = -1$$

The real number x is called the **real part** or the real component of z and is represented as follows:

$$Re\,(z) = x$$

The real number y is called the **imaginary part** or the imaginary component of z and is represented as follows:

$$Im(z) = y$$

Fundamentals

Algebraic operations on complex numbers are defined as follows:

Negative of a complex number $-(a+ib)=-a-ib$

Conjugate of a complex number $z=a+ib \qquad \bar{z}=a-ib$

Addition of complex numbers $(a+ib)+(c+id)=(a+c)+(b+d)i$

Subtraction of complex numbers $(a+ib)-(c+id)=(a-c)+(b-d)i$

Fundamentals

Algebraic operations on complex numbers are defined as follows:

Multiplication of complex numbers

$$(a+ib)(c+id) = (ac-bd) + (bc+ad)i$$

Multiplication of a complex number by its conjugate

$$z\bar{z} = (a+ib)(a-ib) = a^2 + \ b^2$$

Division of complex numbers

$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id}\,\frac{c-id}{c-id} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

Example 1:

$i^2 = -1$ $\quad i^3 = -i$ $\quad i^4 = i^2 i^2 = 1$ $\quad i^5 = i$

Fundamentals

Exercise 1: Show that the real part of a complex number is obtained from the following relation:

$$Re(z) = \frac{z + \bar{z}}{2}$$

Exercise 2: Show that the imaginary part of a complex number is obtained from the following relation:

$$Re(z) = \frac{z - \bar{z}}{2i}$$

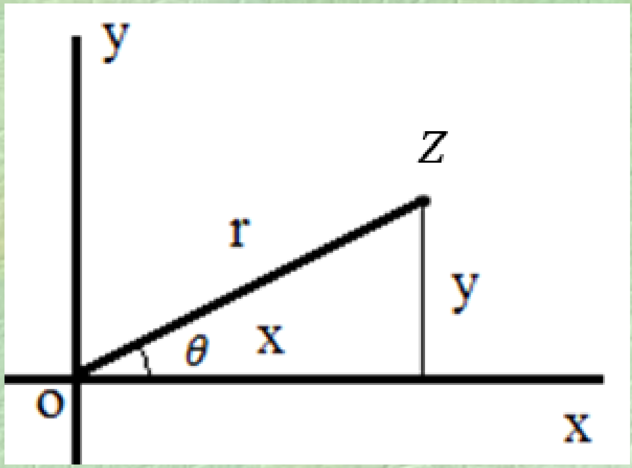
Exercise 3: Show that: $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

Exercise 4: Show that:

$$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2 \qquad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2 \qquad \overline{z_1 / z_2} = \frac{\bar{z}_1}{\bar{z}_2} \qquad z_2 \neq 0$$

Fundamentals

Every complex number can be represented as a point z where its real and imaginary components are the coordinates of the point. This representation is known as the complex plane, or the z-plane.



Cartesian Form of a Complex Number $z = x + iy$

Absolute value or magnitude or modulus of Z: $r = |z| = \sqrt{x^2 + y^2}$

Argument or angle of Z: $\theta = \arg(z)\,; tan\theta = \frac{y}{x}$

Polar or trigonometric form of a complex number: $z = r(cos\theta + isin\theta) = re^{i\theta}$

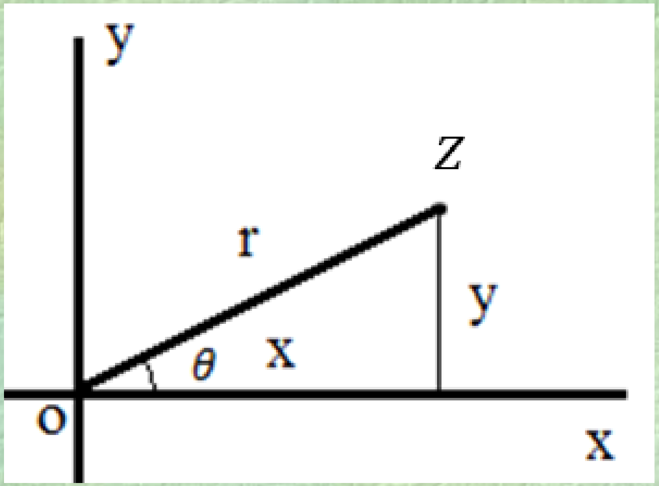
Fundamentals

Cartesian Form of a Complex Number

$$z = x + iy$$

Polar or trigonometric form of a complex number:

$$z = r(cos\theta + \ isin\theta) = re^{i\theta}$$



Representation of the Cartesian form in terms of the polar form:

$$x = rcos\theta \qquad y = rsin\theta$$

Representation of the polar form in terms of the Cartesian form:

$$r = |z| = \sqrt{x^2 + y^2} \qquad \theta = \arg(z) = \begin{cases} tan^{-1}\dfrac{y}{x} & x \geq 0 \\ tan^{-1}\dfrac{y}{x} + \pi & x < 0 \end{cases}$$

Fundamentals

Exercise 5: Find the polar form of the given complex number: $z = -1 - i1$

Exercise 6: Find the polar form of the given complex number: $z = -1 + i2$

Exercise 7: Find the two polar forms of the given complex number: $z = i1$

Exercise 8: What region of the z-plane does the following equation represent?

$$Im(z) \leq 1$$

Exercise 9: What region of the z-plane does the following equation represent?

$$Re(z) = 1$$

Exercise 10: What region of the z-plane does the following equation represent?

$$|z| \leq 1$$

Exercise 11: What region of the z-plane does the following equation represent?

$$|z - z_0| \leq 1$$

Fundamentals

Triangle inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$

Multiplication of a complex numbers in polar coordinates:

$$z_1 z_2 = [r_1(cos\theta_1 + isin\theta_1)][r_2(cos\theta_2 + isin\theta_2)]$$

$$= r_1 r_2[(cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2) + i(sin\theta_1 cos\theta_2 + cos\theta_1 sin\theta_2)]$$

$$= r_1 r_2[\cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2)] \qquad z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

Division of complex numbers in polar coordinates:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2)]$$

$$z_1/z_2 = r_{1/}r_2 e^{i(\theta_1-\theta_2)}$$

Fundamentals

De Moivre's Theorem:

$$z_1 z_2 \ldots z_n = r_1 r_2 \ldots r_n[\cos(\theta_1 + \theta_2 + \cdots + \theta_n) + isin(\theta_1 + \theta_2 + \cdots + \theta_n)]$$

$$z^n = r^n(cosn\theta + isinn\theta) \qquad z^n = r^n e^{in\theta}$$

Example 2: Find the eighth power of the following complex number:

$$z = 1 + i1$$

$$z = \sqrt{2}(cos\frac{\pi}{4} + isin\frac{\pi}{4}) = \sqrt{2}e^{i\frac{\pi}{4}}$$

$$z^8 = \sqrt{2}^8 \,(cos8\frac{\pi}{4} + isin8\frac{\pi}{4}) = 16$$

Fundamentals

De Moivre's Theorem:

$$z_1 z_2 \dots z_n = r_1 r_2 \dots r_n[\cos(\theta_1 + \theta_2 + \cdots + \theta_n) + isin(\theta_1 + \theta_2 + \cdots + \theta_n)]$$

$$z^n = r^n(cosn\theta + isinn\theta) \qquad z^n = r^n e^{in\theta}$$

Root Calculation

$$w^n = z \qquad w = ?$$

Suppose

$$w = Re^{i\varphi} \qquad z = re^{i(\theta+2k\pi)}$$

$$w^n = z \qquad w^n = R^n e^{in\varphi} = z = re^{i(\theta+2k\pi)}$$

$$R = r^{1/n} \qquad \varphi = \frac{\theta + 2k\pi}{n}$$

Fundamentals

Example 3: Calculate the fifth roots of z=1

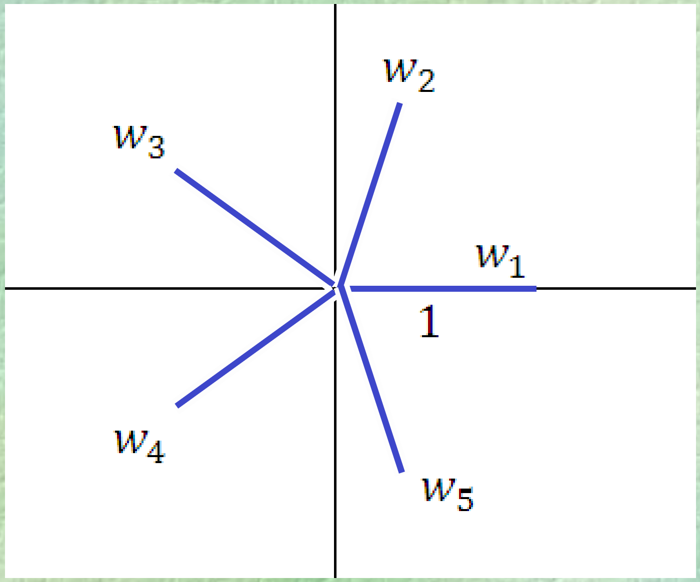
$$z = 1e^{i2k\pi} \qquad R = r^{1/n} \qquad \varphi = \frac{2k\pi}{n}$$

$$R = r^{1/5} \qquad \varphi = \frac{2k\pi}{5}$$

$$w_1 = 1e^{i0} = 1 \qquad w_4 = 1e^{i\frac{6\pi}{5}}$$

$$w_2 = 1e^{i\frac{2\pi}{5}} \qquad w_5 = 1e^{i\frac{8\pi}{5}}$$

$$w_3 = 1e^{i\frac{4\pi}{5}}$$



Complex Analysis (The theory of functions of a complex variable)

- Fundamentals
- Analytic Functions and Differentiability
- Integration in the Complex Plane
- Complex Series
- Residue Theory and Calculation of Real Integrals

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Analytic Functions and Differentiability

Consider the following two complex variables:

$$z = x + iy \qquad\qquad w = u + iv$$

If for each value of z in a part of the complex plane one or more values of www are defined, then w is called a function of z, and it is written as:

$$w = f(z)$$

In the context of complex functions, there are multi-valued and single-valued functions.

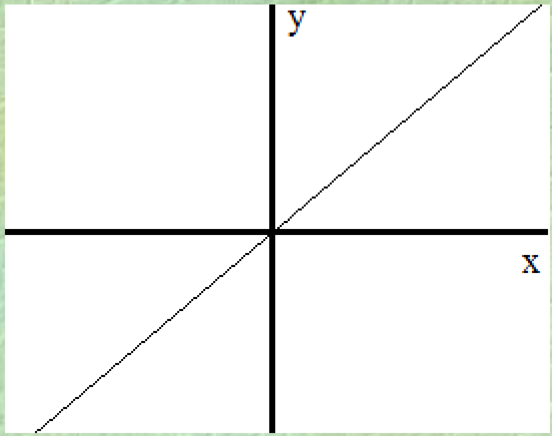
If w is separated into its real and imaginary parts, it can be expressed as:

$$w = u(x, y) + iv(x, y)$$

Analytic Functions and Differentiability

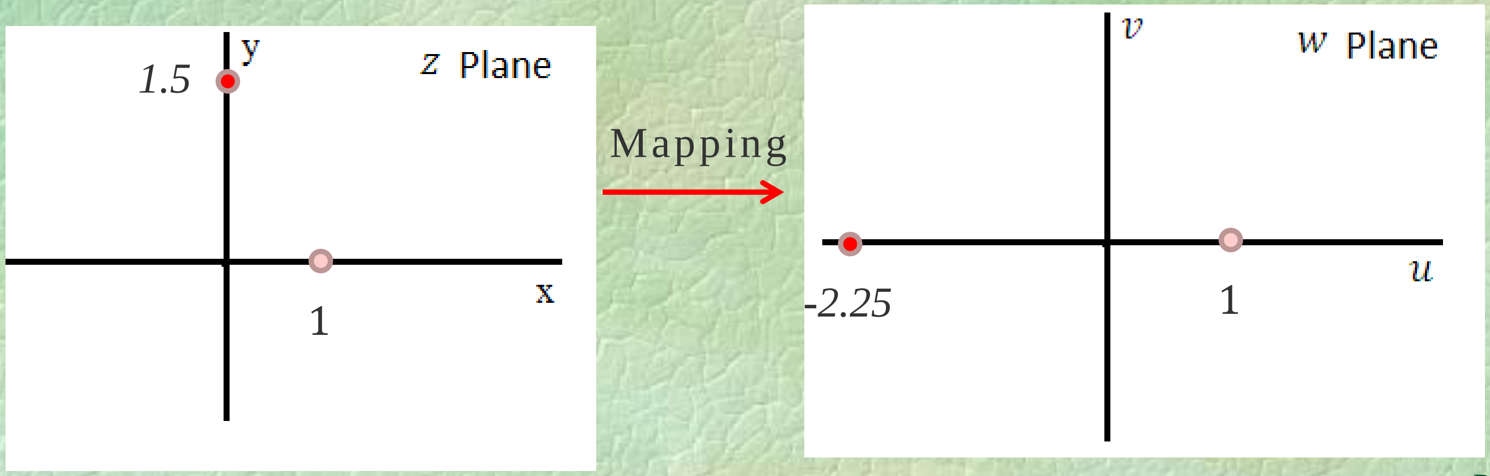
Real function

$$y = f(x) \quad \text{i.e.} \quad y = x$$

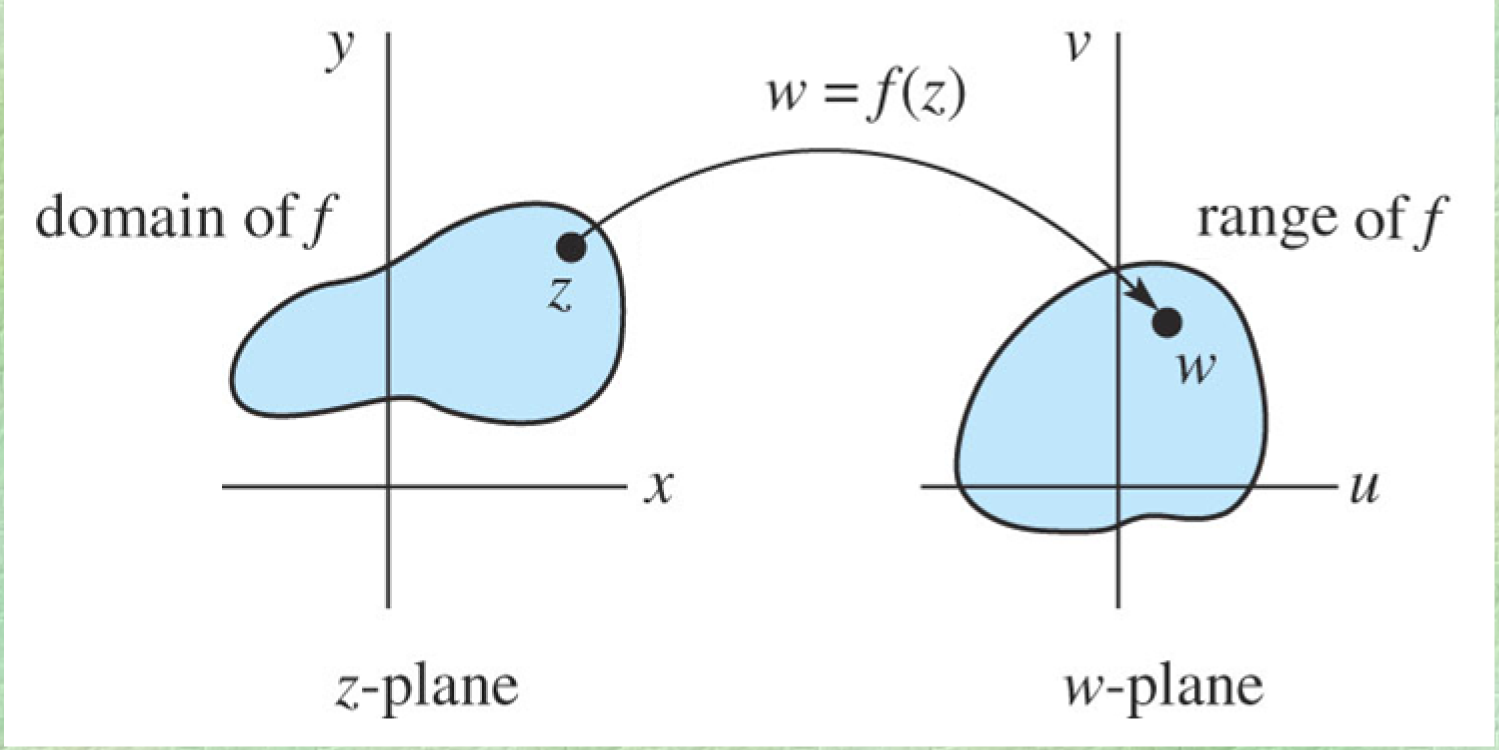


Complex function

$$w = f(z) \quad \text{i.e.} \quad w = z^2$$



Analytic Functions and Differentiability

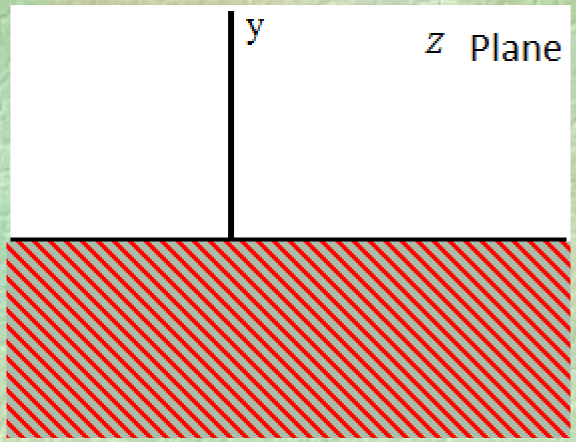


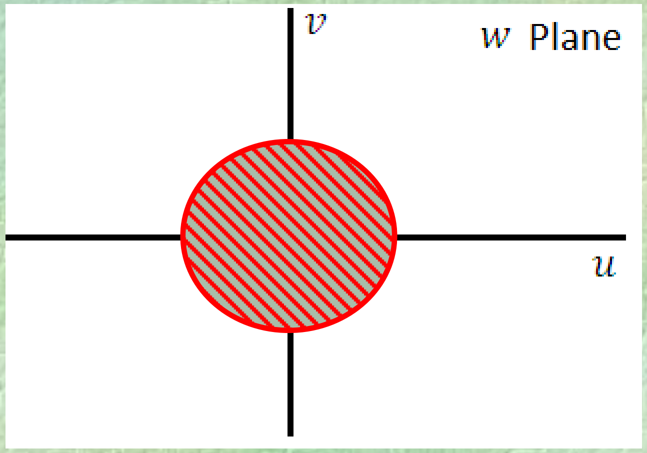
Analytic Functions and Differentiability

Example 4: Consider the following function.

$$w = \frac{z+i}{iz+1}$$

Show that the mapping of the lower half-plane in the *z*-plane is transformed into the unit disk centered at the origin in the *w*-plane.





Analytic Functions and Differentiability

Example 4: Consider the following function.

$$w = \frac{z+i}{iz+1}$$

Show that the mapping of the lower half-plane in the *z*-plane is transformed into the unit disk centered at the origin in the *w*-plane.

$$|w|^2 = \frac{z+i}{iz+1}\,\overline{\frac{z+i}{iz+1}}$$

$$|w|^2 = \frac{z+i}{iz+1}\,\frac{\bar{z}+\bar{i}}{\bar{i}\bar{z}+1}$$

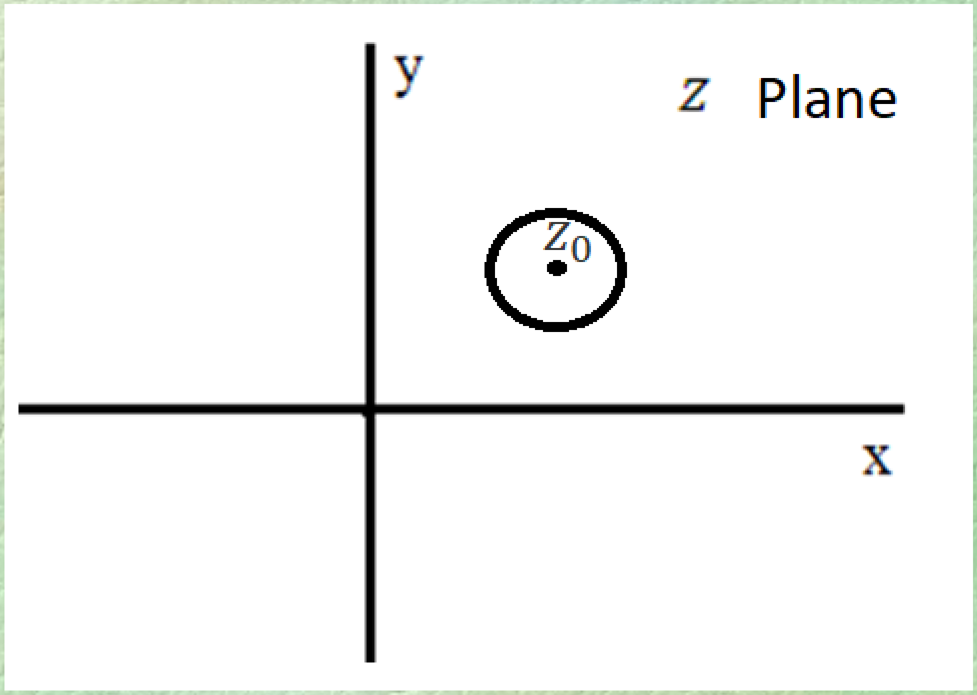
$$|w|^2 = \frac{z+i}{iz+1}\,\frac{\bar{z}-i}{-i\bar{z}+1} = \frac{(z\bar{z}+1)-i(z-\bar{z})}{(z\bar{z}+1)+i(z-\bar{z})} = \frac{|z|^2+1+2Im(z)}{|z|^2+1-2Im(z)}$$

$$if \quad Im(z) < 0 \;\rightarrow |w| \le 1$$

Analytic Functions and Differentiability

Neighborhood: A neighborhood of a point z_0 is a set of all points that satisfy an inequality of the following form:

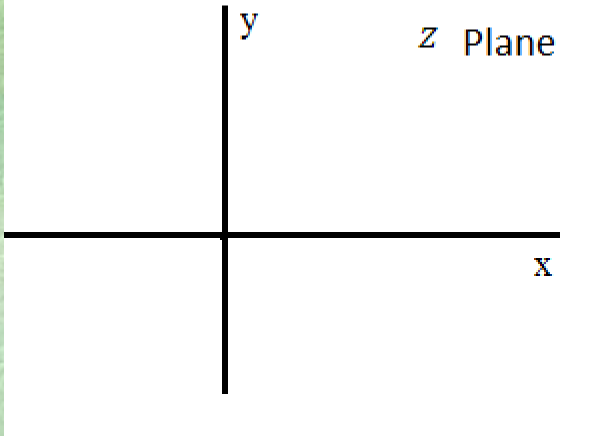
$$|z - z_0| < \varepsilon \qquad\qquad \varepsilon > 0$$



Analytic Functions and Differentiability

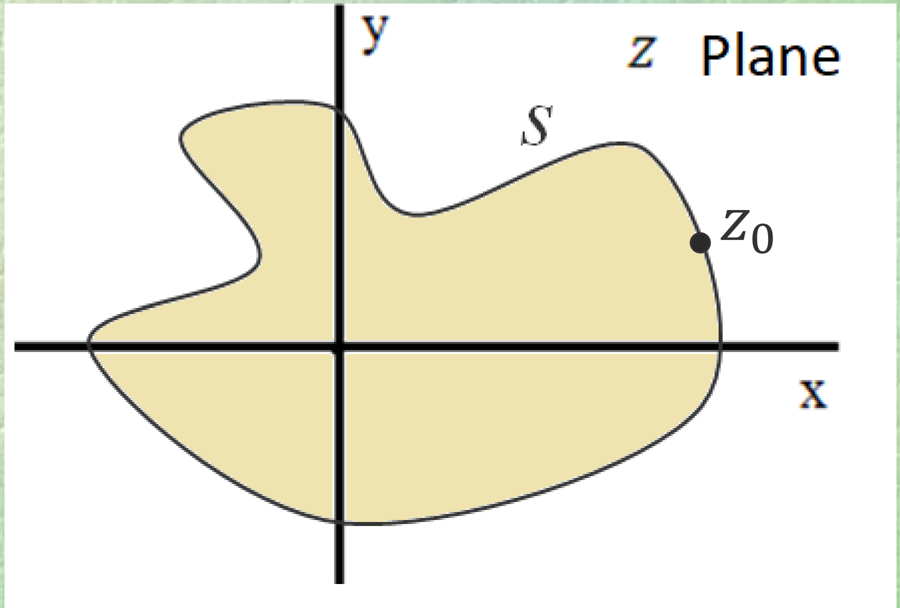
Interior Point: A point *z* in a set S is called an interior point if there exists at least one neighborhood around *z* such that all points within this neighborhood are also contained in S. In other words, there is a disk centered at *z* where every point in this disk belongs to the set S.

Exterior Point: A point *z* that does not belong to a set S is called an exterior point of S if there exists at least one neighborhood around *z* such that none of the points within this neighborhood belong to S. In other words, there is a disk centered at *z* where every point in this disk is outside the set S.



Analytic Functions and Differentiability

Boundary Point: A point z_0 is called a boundary point of a set *S* if every neighborhood of z_0 contains both points belonging to *S* and points not belonging to *S*. Depending on the definition of the set, boundary points may or may not belong to the set.



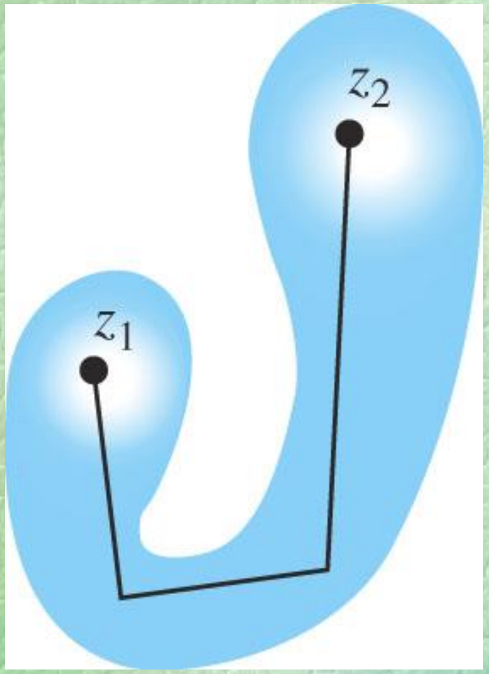
Limit Point: A point *z* is called a limit point of a set if every neighborhood of *z* contains at least one point of the set distinct from *z*.

Analytic Functions and Differentiability

Closed Set: A set that contains all its boundary points is called a closed set.

Open Set: A set that does not contain any of its boundary points is called an open set.

Connected Set: A set S is called connected if any pair of points in the set can be connected by a polygonal line (broken line) whose all points belong to the set.



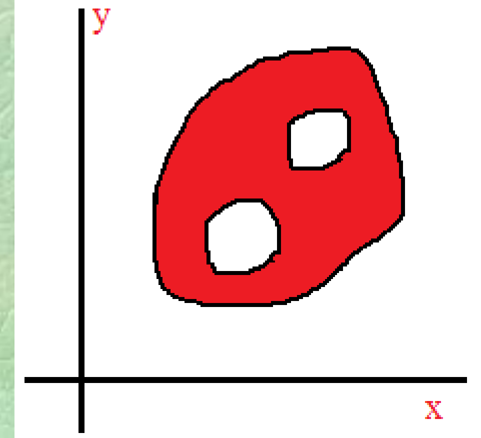
Domain: A connected open set is called a domain.

Region: A set composed of a domain along with some, all, or none of its boundary points is called a region.

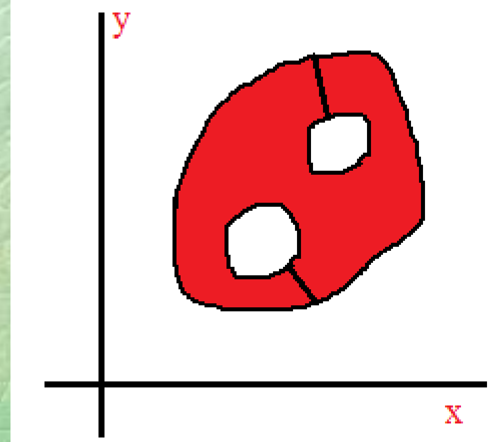
Analytic Functions and Differentiability

Simply Connected: A connected set S is called simply connected if every simple closed curve drawn within it encloses only points of S.

Multiply Connected: A connected set S is called multiply connected if there exists at least one simple closed curve within S that encloses one or more points not in S.



Multiply Connected



Simply Connected

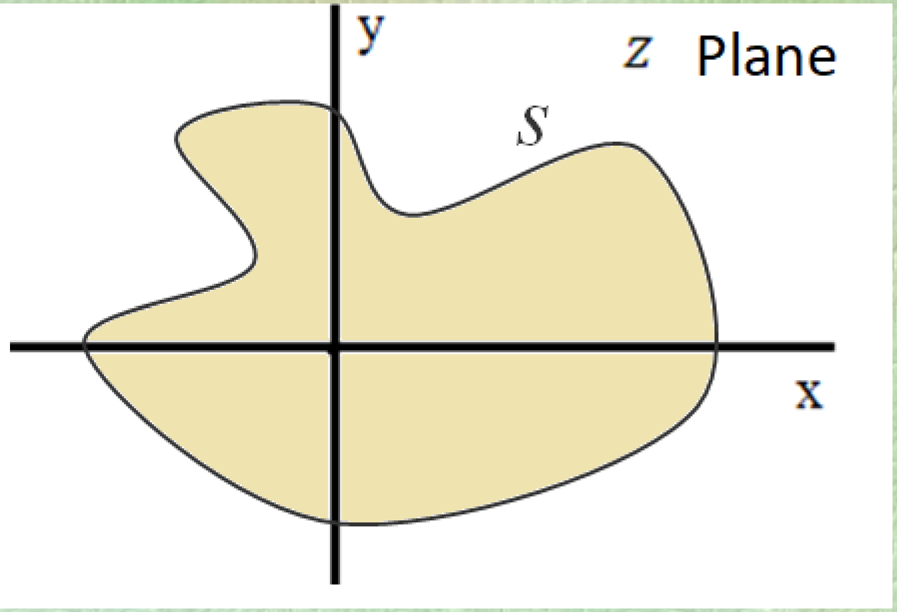
Transforming a Multiply Connected Region into a Simply Connected Region.

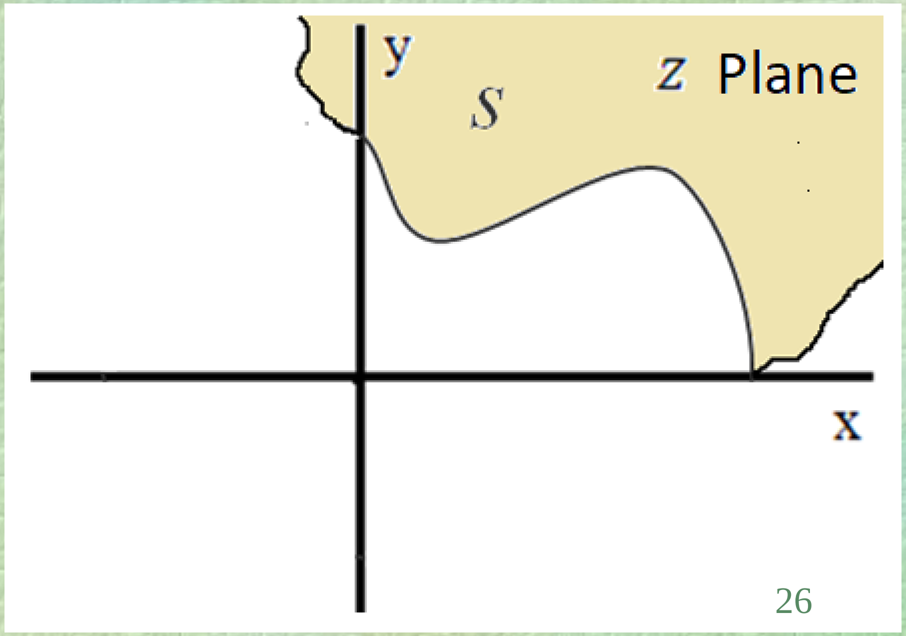
Analytic Functions and Differentiability

Bounded Set: A set S is called bounded if there exists a circle centered at the origin that encloses all points of the set S. In other words, there exists a number *d* such that

$$\forall \ z \in s \qquad\qquad |z| < d$$

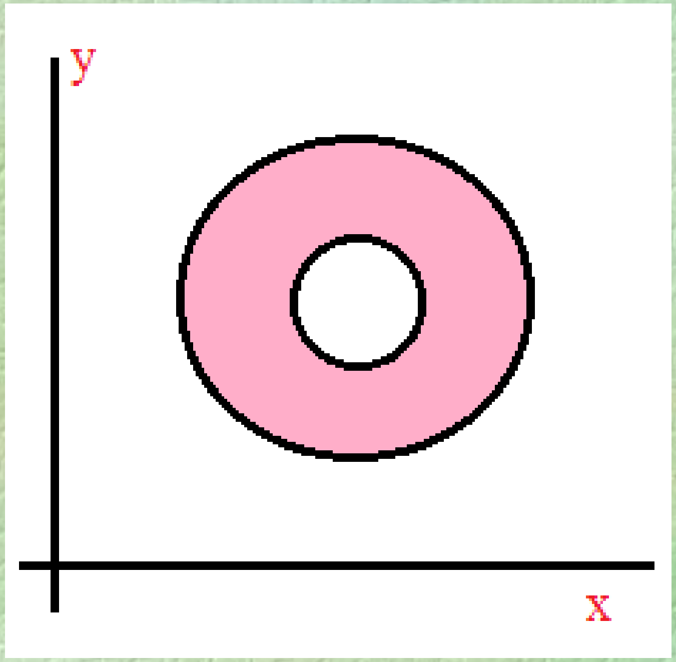
Then S is called bounded. A set that is not bounded is called **unbounded.**





Analytic Functions and Differentiability

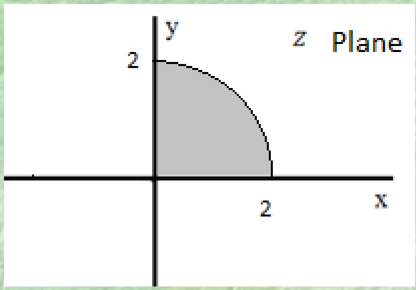
A **annular region** or **annulus** is a set composed of points between two concentric circles.



Exercises

Exercise 12: When we say "double-connected" or "triple-connected," what do the numbers 2 or 3 represent?

Exercise 13: What region in the *w*-plane does the region below in the *z*-plane map to under the function $w=z^2$?



Exercise 14: Find the seventh roots of the number 2 and plot them on a diagram.

Exercise 15: Consider the set of complex numbers whose imaginary part is zero. Determine the interior points, exterior points, boundary points, and limit points of this set. Is this set connected? If it is connected, is it simply connected or multiply connected?

Analytic Functions and Differentiability

Definition of a Limit: Let $f(z)$ be a single-valued function of z, and let w_0 be a complex constant. If, for every $\epsilon>0$, there exists a positive number δ such that for every z in the domain of f where $0<|z-z_0|<\delta$, we have $|f(z)-w_0|<\epsilon$, then w_0 is called the limit of $f(z)$ as z approaches z_0.

Analytic Functions and Differentiability

Example 5: If

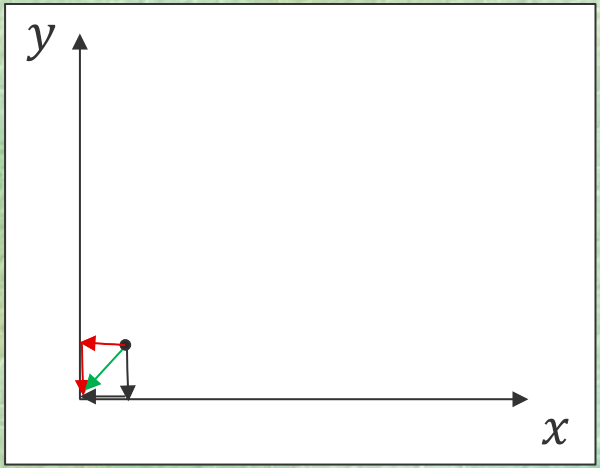
$$f(z) = \frac{(Re(z) + Im(z))^2}{|z|^2}$$

show that $\lim_{z\to 0} f(z)$ is not exist.

$$f(z) = \frac{(x+y)^2}{x^2+y^2}$$

$$\lim_{x\to 0}[\lim_{y\to 0} f(z)] = \lim_{x\to 0}[\lim_{y\to 0} \frac{(x+y)^2}{x^2+y^2}] =$$

$$\lim_{y\to 0}[\lim_{x\to 0} f(z)] = \lim_{y\to 0}[\lim_{x\to 0} \frac{(x+y)^2}{x^2+y^2}] =$$



$$\lim_{z\to 0} f(z) = \frac{(1+m)^2}{1+m^2}$$

Analytic Functions and Differentiability

Continuity: A function $f(z)$ is continuous at the point z_0 if and only if:

$$\lim_{z \to z_0} f(z) = f(z_0)$$

In other words, for a function to be continuous at a point z_0, it must:

- Be defined and have a value at that point.
- Have a limit as z approaches that point.
- The value of the function at that point must equal the limit as z approaches the point.

If $f(z)$ is continuous at every point in a region, it is said to be continuous throughout that region.

Analytic Functions and Differentiability

Theorem 1: The sum, difference, and product of continuous functions are continuous. Additionally, the quotient of continuous functions is continuous, provided the denominator is non-zero.

Theorem 2: A continuous function of a continuous function is continuous.

Theorem 3: A necessary and sufficient condition for a function

$$f(z) = u(x, y) + iv(x, y)$$

to be continuous is that the real functions $u(x,y)$ and $v(x,y)$ are continuous.

Theorem 4: If $f(z)$ is continuous at a point z_0 and $f(z_0)\neq 0$, then there exists a neighborhood around z_0 where $f(z)$ is non-zero throughout.

Theorem 5: If $f(z)$ is continuous in a closed, bounded region R, then there exists a positive constant M such that for every value z in R, we have:

$$|f(z)| < M$$

Analytic Functions and Differentiability

A complex function $f(z)$ is said to be differentiable at a point z_0 if the following limit exists:

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

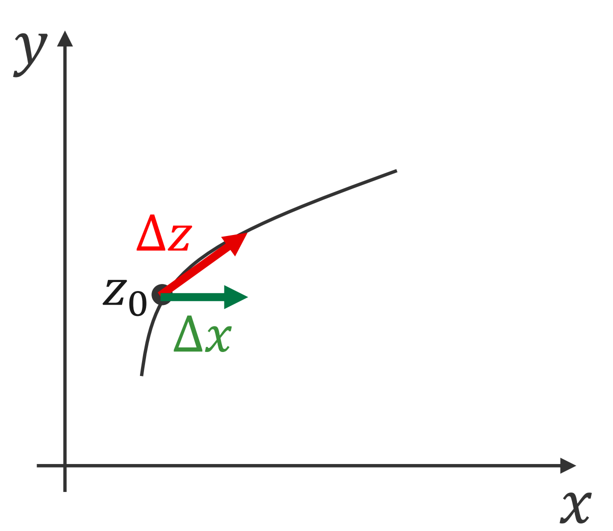
The value of the above limit is called the derivative of the complex function $f(z)$ at the point z_0 .

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Analytic Functions and Differentiability

Necessary Condition for the Existence of the Limit:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$



$$\Delta z = \Delta x + i\Delta y$$

The first path $\quad \Delta z = \Delta x$

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x}$$

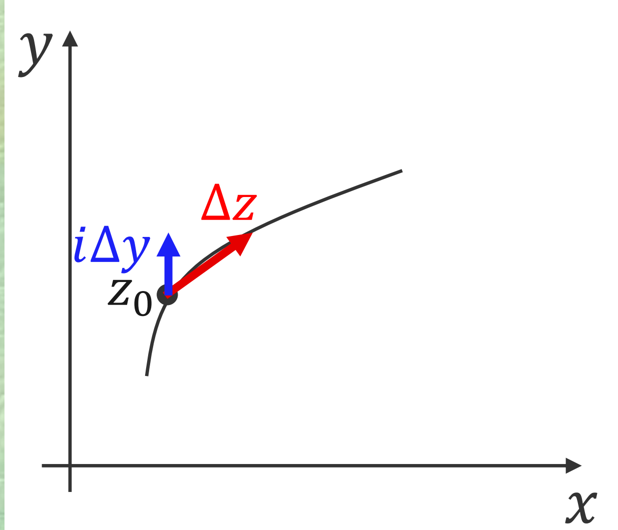
$$= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + \lim_{\Delta x \to 0} \frac{iv(x_0 + \Delta x, y_0) - iv(x_0, y_0)}{\Delta x}$$

$$\boxed{= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}$$

Analytic Functions and Differentiability

Necessary Condition for the Existence of the Limit:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$



$$\Delta z = \Delta x + i\Delta y$$

The second path $\Delta z = i\Delta y$

$$f'(z_0) = \lim_{i\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\Delta y}$$

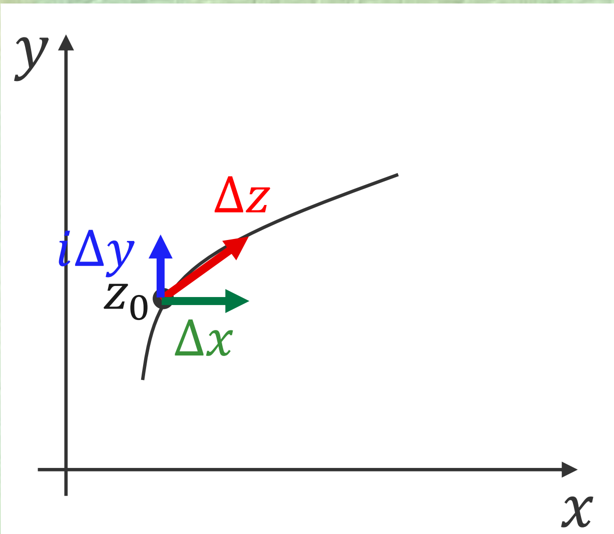
$$f'(z_0) = \lim_{i\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + \lim_{i\Delta y \to 0} \frac{iv(x_0, y_0 + \Delta y) - iv(x_0, y_0)}{i\Delta y}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Analytic Functions and Differentiability

Necessary Condition for the Existence of the Limit:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$



The first path

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

The second path

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

The Cauchy-Riemann equations (necessary conditions for the differentiability of $f(z)$)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Analytic Functions and Differentiability

Analytic Function: A complex function $f(z)$ is analytic at the point z_0 if

- A complex function $f(z)$ is differentiable at the point z_0 and
- A complex function $f(z)$ is differentiable at every point in a neighborhood of z_0 .

Analytic Functions and Differentiability

Theorem: If *u* and *v* are real-valued, single-valued functions of *x* and *y* and their first four partial derivatives are continuous throughout a region R, then the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are both necessary and sufficient conditions for the function to be analytic.

$$f(z) = u(x, y) + iv(x, y)$$

in R. The derivative is given by:

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}} \qquad or \qquad \boxed{f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}}$$

Analytic Functions and Differentiability

Example 6: Check the analyticity of the following function.

$$f(z) = \bar{z} = x - iy$$

Answer:

$$u(x, y) = x \qquad\qquad v(x, y) = -y$$

The Cauchy-Riemann equations are:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$1 \neq -1 \qquad\qquad 0 = 0$$

The function $f(z)$ is not analytic at any point.

Analytic Functions and Differentiability

Example 7: Check the analyticity of the following function.

$$f(z) = z\bar{z} = x^2 + y^2$$

Answer:

$$u(x,y) = x^2 + y^2 \qquad v(x,y) = 0$$

The Cauchy-Riemann equations are:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$2x = 0 \qquad 2y = 0$$

The function $f(z)$ is differentiable in z=0 but not analytic at any point.

Analytic Functions and Differentiability

Example 8: Check the analyticity of the following function.

$$f(z) = z^2 = x^2 - y^2 + i2xy$$

Answer:

$$u(x,y) = x^2 - y^2 \qquad v(x,y) = 2xy$$

The Cauchy-Riemann equations are:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$2x = 2x \qquad -2y = -2y$$

The function is differentiable and analytic throughout the entire complex plane. The derivative is:.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i2y = 2z$$

Analytic Functions and Differentiability

Analytic Function: A complex function $f(z)$ is analytic at the point z_0 if

- A complex function $f(z)$ is differentiable at the point z_0 and
- A complex function $f(z)$ is differentiable at every point in a neighborhood of z_0 .

In this case, the point z_0 is an ordinary point of this function.

Singular Point: If a complex function $f(z)$ is not analytic at the point z_0 but every neighborhood of z_0 contains points where f $f(z)$ is analytic, then z_0 is called a singular point of the function.

For example, any point for $f(z)=z^2$ is ordinary point and z=1 in $f(z) = \frac{1}{z-1}$ is singular point.

Analytic Functions and Differentiability

Property 1 of Analytic Functions: If both the real part and the imaginary part of an analytic function have continuous second partial derivatives, then they satisfy the Laplace equation:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

Proof: Suppose $w=u(x,y)+iv(x,y)$ is an analytic function of z. In this case, u and v must satisfy the Cauchy-Riemann equations, which are:

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

If we take the partial derivative of one of the equations with respect to x and the partial derivative of the other with respect to y, and then add the results, we obtain:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Analytic Functions and Differentiability

Harmonic Function: A function that has continuous second partial derivatives and satisfies Laplace's equation is called a harmonic function.

For example, $e^x \cos y$, $x^2 - y^2$, and xy are harmonic functions.

Conjugate Harmonic Functions: Two harmonic functions *u* and *v* are called conjugate harmonic functions if they are related in such a way that *u*+*iv* is an analytic function.

Analytic Functions and Differentiability

Example 9: Show that the following function is harmonic. Find an analytic complex function whose real part is the given function.

$$e^x \cos y$$

Answer: Let

$$u(x,y) = e^x \cos y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x cosy \qquad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Therefore, the function is harmonic. By the Cauchy-Riemann equations

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \qquad \frac{\partial v}{\partial x} = e^x \sin y \qquad \rightarrow \qquad v = e^x \sin y + g(y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad e^x \cos y = e^x \cos y + g'(y) \qquad \rightarrow \qquad g(y) = c$$

$$\rightarrow \qquad f(z) = e^x \cos y + i(e^x \sin y + c)$$

Analytic Functions and Differentiability

Property 2 of Analytic Functions: If $w=u(x,y)+iv(x,y)$ is an analytic function of z, then the level curves of $u(x,y)=c$ are orthogonal to the level curves of $v(x,y)=k$, and vice versa.

Proof: The slope of the level curves of the family $u(x,y)=c$ is given by implicit differentiation:

$$\partial u/_{\partial x}\,dx + \partial u/_{\partial y}\,dy = 0 \qquad \frac{dy}{dx} = -\frac{\partial u/_{\partial x}}{\partial u/_{\partial y}}$$

Similarly, the slope of the level curves of the family $v(x,y)=k$ is given by:

$$\frac{dy}{dx} = -\frac{\partial v/_{\partial x}}{\partial v/_{\partial y}} \xrightarrow{\text{By Cauchy-Riemann}} \frac{dy}{dx} = \frac{\partial u/_{\partial y}}{\partial u/_{\partial x}}$$

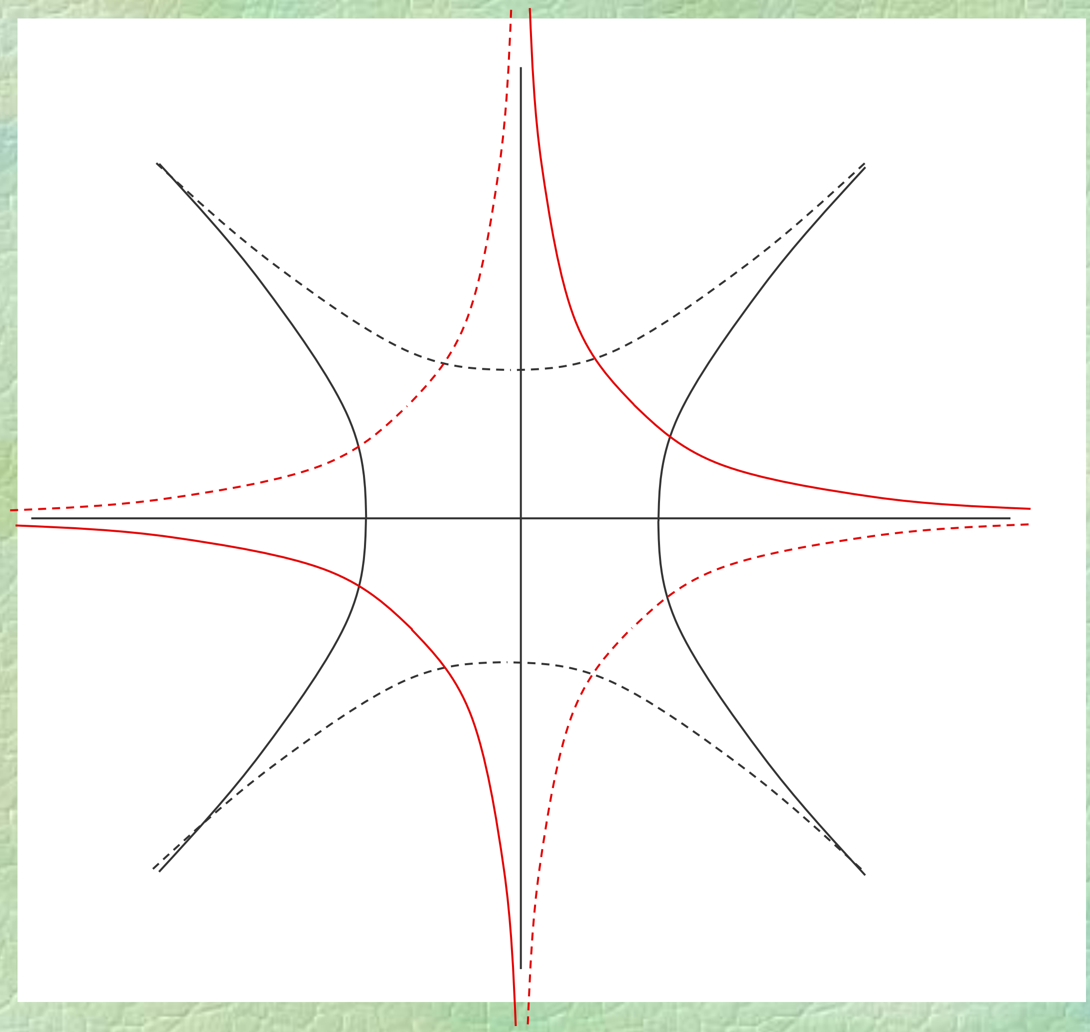
Comparing the two expressions shows that the level curves of the families $v(x,y)=k$ and $u(x,y)=c$ are orthogonal.

Analytic Functions and Differentiability

Property 2 of Analytic Functions: If $w=z^2$, then the level curves are:

$$u(x,y) = x^2 - y^2 = c$$

$$v(x,y) = 2xy = k$$



Analytic Functions and Differentiability

Property 3 of analytical functions: If in any analytical function $w=u(x,y)+iv(x,y)$, we substitute the variables x and y with their equivalents in terms of $x=\frac{z+\bar{z}}{2}$ and $y=\frac{z-\bar{z}}{2i}$, then w becomes a function of z."

Proof:

$$w=u(x,y)+iv(x,y)=u\left(\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2i}\right)+iv\left(\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2i}\right)$$

Now we need:

$$\partial w/_{\partial\bar{z}}=0$$

$$\frac{\partial w}{\partial\bar{z}}=\frac{\partial(u+iv)}{\partial\bar{z}}=\frac{\partial u}{\partial\bar{z}}+i\frac{\partial v}{\partial\bar{z}}=\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial\bar{z}}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial\bar{z}}\right)+i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial\bar{z}}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial\bar{z}}\right)$$

$$\frac{\partial w}{\partial\bar{z}}=\left(\frac{1}{2}\frac{\partial u}{\partial x}+\frac{i}{2}\frac{\partial u}{\partial y}\right)+i\left(\frac{1}{2}\frac{\partial v}{\partial x}+\frac{i}{2}\frac{\partial v}{\partial y}\right)=\frac{1}{2}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

Analytic Functions and Differentiability

Definition of Function e^z

A: e^z should be single-valued and analytical.

B: $\frac{de^z}{dz} = e^z$

C: When Im(z)=0, e^z becomes e^x

Analytic Functions and Differentiability

Objective: Find a function that satisfies the desirable properties of e^z.

$$e^z = u + iv$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u + iv$$

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \qquad \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = u + iv$$

$$\frac{\partial u}{\partial x} = u \qquad \frac{\partial v}{\partial x} = v \qquad \frac{\partial v}{\partial y} = u \qquad -\frac{\partial u}{\partial y} = v$$

Analytic Functions and Differentiability

$$\frac{\partial u}{\partial x} = u \longrightarrow u = e^{x}\varphi(y)$$

$$-\frac{\partial u}{\partial y} = v \longrightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\partial v}{\partial y}$$

Cauchy-Riemann equations

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial u}{\partial x}$$

$$\varphi''(y) = -\varphi(y) \longrightarrow \varphi(y) = Acosy + Bsiny$$

$$u = e^{x}(Acosy + Bsiny)$$

Analytic Functions and Differentiability

$$u = e^{x}(A\cos y + B\sin y)$$

$$v = -\frac{\partial u}{\partial y} = -e^{x}(-A\sin y + B\cos y)$$

$$\mathrm{e}^{\mathrm{z}} = \mathrm{u} + \mathrm{iv} = e^{x}[(\mathrm{A}\cos y + \mathrm{B}\sin y) + \mathrm{i}(\mathrm{A}\sin y - \mathrm{B}\cos y)]$$

$$e^{x} = e^{x}(A - iB) \longrightarrow A = 1 \quad B = 0$$

$$\mathrm{e}^{\mathrm{z}} = e^{x+iy} = e^{x}(\cos y + i\sin y)$$

$$\mathrm{mod}\ \mathrm{e}^{\mathrm{z}} = |e^{z}| = e^{x} \qquad \arg \mathrm{e}^{\mathrm{z}} = y$$

$$u(x,y) = e^{x}\cos y \qquad v(x,y) = e^{x}\sin y$$

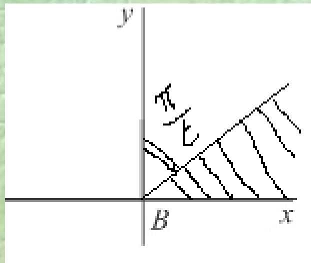
Exercises

Exercise 16: Examine the analyticity of the following functions.

$$f(z) = z\bar{z} \qquad g(z) = z|z|^2 \qquad k(z) = z^2 + 3z + 2$$

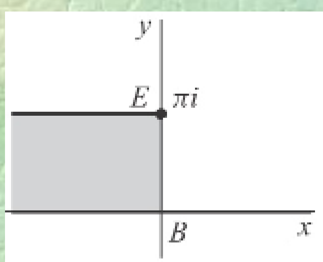
Exercise 17: Examine the harmonicity of the function *cosxcoshy* and find a complex function whose real part is the given function.

Exercise 18: Determine the mapping of the region below using the function $f(z)=z^2$.



Examine the conformality of the mapping using the resulting figure and the related condition.

Exercise 19: Determine the mapping of the region below using the function $f(z)=e^z$.



Examine the conformality of the mapping using the resulting figure and the related condition.

Exercise 20: What region of the *z*-plane is mapped to the annular region between two circles with radii 2 and 3 by the function $f(z)=e^z$?

Analytic Functions and Differentiability

Trigonometric function *cosz* :

$$e^{i\theta} = cos\theta + \mathrm{isin}\theta$$

$$e^{-i\theta} = cos\theta - \mathrm{isin}\theta$$

$$cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$cosz = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \frac{e^{-y}e^{ix} + e^{y}e^{-ix}}{2}$$

$$= \frac{e^{-y}(cosx + isinx) + e^{y}(cosx - isinx)}{2}$$

$$= cosx\frac{e^{y} + e^{-y}}{2} - isinx\frac{e^{y} - e^{-y}}{2}$$

$$cosz = \cos(x + iy) = cosxcoshy - isinxsinhy$$

Q1: u(x,y) ?

Q2: v(x,y) ?

Q3: Analytic ?

Q4: y=0 ?

Q5: $f'(z)$?

Analytic Functions and Differentiability

Trigonometric function *sinz* :

$$e^{i\theta} = cos\theta + \mathrm{isin}\theta$$

$$sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$e^{-i\theta} = cos\theta - \mathrm{isin}\theta$$

$$sinz = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= \frac{e^{-y}e^{ix} - e^{y}e^{-ix}}{2i} = \frac{e^{-y}(cosx + isinx) - e^{y}(cosx - isinx)}{2i}$$

$$= sinx\frac{e^{y} + e^{-y}}{2} + icosx\frac{e^{y} - e^{-y}}{2}$$

$$sinz = \sin(x + iy) = sinxcoshy + icosxsinhy$$

Q1: u(x,y) ?

Q2: v(x,y) ?

Q3: Analytic ?

Q4: y=0 ?

Q5: $f'(z)$?

Analytic Functions and Differentiability

Hyperbolic functions

$$sinhz = \frac{e^z - e^{-z}}{2}$$

$$coshz = \frac{e^z + e^{-z}}{2}$$

Q1: u(x,y) ?

Q2: v(x,y) ?

Q3: Analytic ?

Q4: y=0 ?

Q5: $f'(z)$?

Analytic Functions and Differentiability

Other functions

$$tanz = \frac{sinz}{cosz} \qquad tanhz = \frac{sinhz}{coshz} \qquad secz = \frac{1}{cosz}$$

$$sechz = \frac{1}{coshz} \qquad \ldots \ldots \ldots \ldots$$

Q1: u(x,y) ?

Q2: v(x,y) ?

Q3: Analytic ?

Q4: y=0 ?

Q5: $f'(z)$?

Analytic Functions and Differentiability

Other functions

$$w = \frac{1}{z}$$

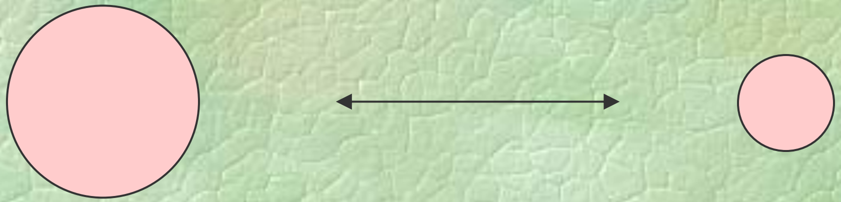
Q1: u(x,y) ?

Q2: v(x,y) ?

Q3: Analytic ?

Q4: y=0 ?

Q5: $f'(z)$?



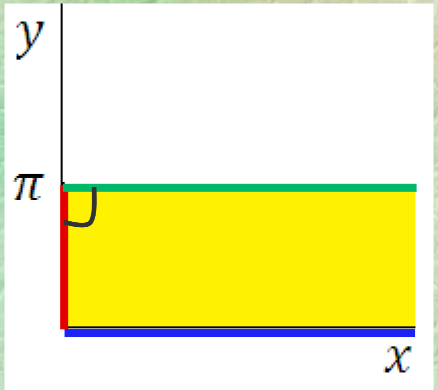
Note: A line is also a circle with an infinite radius.

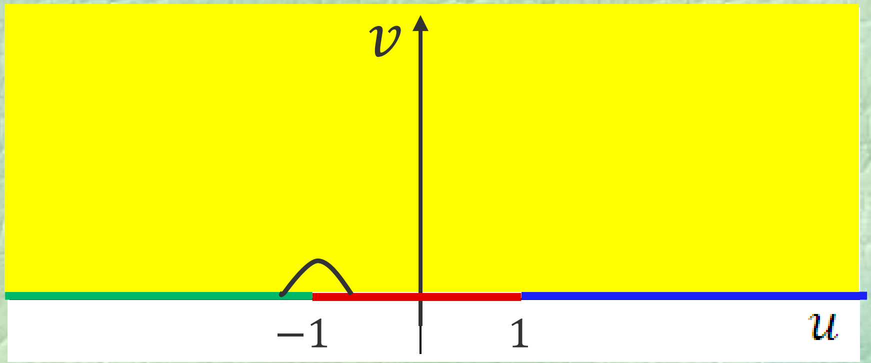
Analytic Functions and Differentiability

The mapping of the function *cosh*(*z*)

$$coshz = \cos h(x + iy) = coshxcosy + isinhxsiny$$

Exercise 21: Determine the mapping of the shaded region using the transformation *cosh*(*z*).



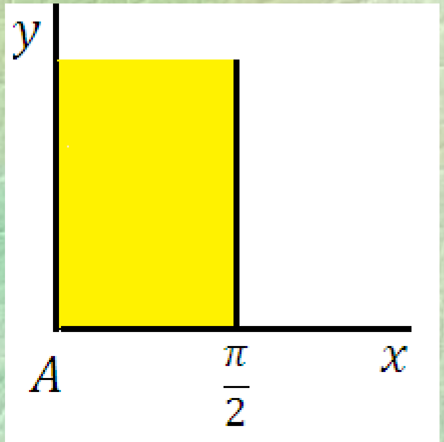


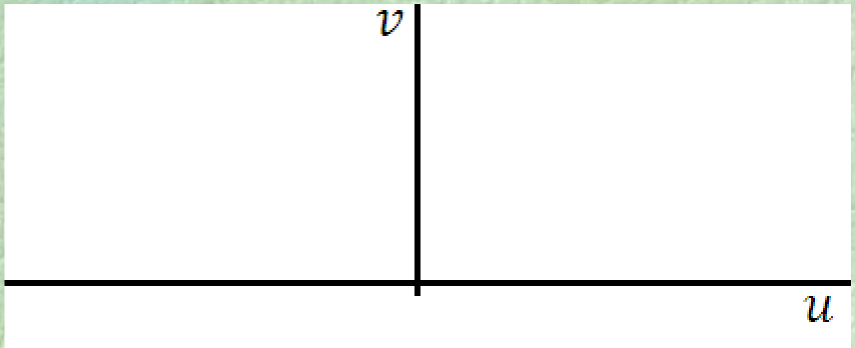
Analytic Functions and Differentiability

The mapping of the function *sin*(*z*)

$$sinz = \sin(x + iy) = sinxcoshy + icosxsinhy$$

Exercise 22: Determine the mapping of the shaded region using the transformation *sin*(z).





Analytic Functions and Differentiability

Linear fractional or bilinear function

$$w = \frac{az+b}{cz+d} \qquad ad - bc \neq 0 \ ?$$

$$w = \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{cz+d}$$

$$z \xrightarrow{\text{Expansion, contraction, and rotation}} cz \xrightarrow{\text{Shift}} cz+d \xrightarrow{\text{Circle to circle}} \frac{1}{cz+d}$$

$$\xrightarrow{\text{Expansion, contraction, and rotation}} \frac{bc-ad}{c}\frac{1}{cz+d} \xrightarrow{\text{Shift}} w = \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{cz+d}$$

Note: A bilinear transformation maps a circle to a circle.

Analytic Functions and Differentiability

Linear fractional or bilinear function

$$w = \frac{az+b}{cz+d} \qquad ad - bc \neq 0$$

Transformation of three arbitrary points z_1, z_2, and z_3 in the z-plane to three arbitrary points w_1, w_2, and w_3 in the w-plane.

$$z_1 \to w_1 \qquad z_2 \to w_2 \qquad z_3 \to w_3$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Example 10: Find a transformation that:

$$0 \to 1 \qquad 1 \to -1 \qquad \infty \to i$$

$$\frac{(w-1)(-1-i)}{(w-i)(-1-1)} = \frac{(z-0)(1-\infty)}{(z-\infty)(1-0)} \qquad \frac{(w-1)(1+i)}{2(w-i)} = z \qquad w = \frac{2iz-1-i}{2z-1-i}$$

Analytic Functions and Differentiability

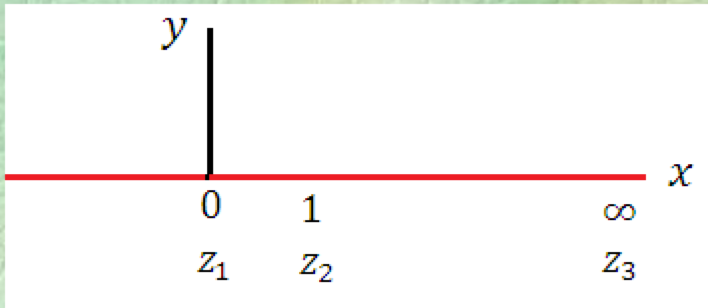
Linear fractional or bilinear function

$$w = \frac{az+b}{cz+d} \qquad ad-bc \neq 0$$

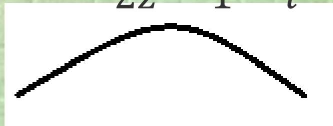
Example 10: Find a transformation that:

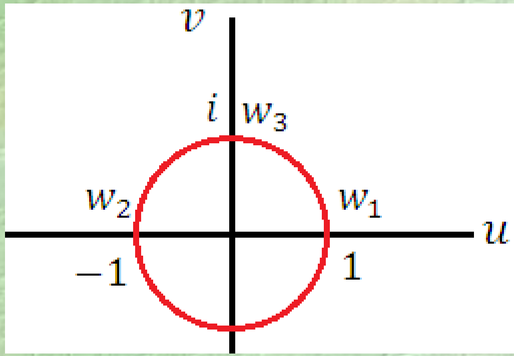
$$0 \to 1 \qquad 1 \to -1 \qquad \infty \to i$$

$$w = \frac{2iz-1-i}{2z-1-i}$$



$$w = \frac{2iz-1-i}{2z-1-i}$$





Analytic Functions and Differentiability

Logarithmic function

$w = lnz$ $\qquad$ $e^{w} = z$

$w = u + iv$ $\qquad$ $z = re^{i\theta}$

$e^{w} = z$ $\qquad$ $e^{u+iv} = e^{u}e^{iv} = re^{i\theta}$

$e^{u} = r \qquad v = \theta$

$w = u + iv = lnr + \mathrm{i}\theta = ln|z| + iargz$

$\qquad\qquad u \qquad\qquad v$

Analytic Functions and Differentiability

Logarithmic function

$$w = lnz = u + iv = lnr + \mathrm{i}\theta = ln|z| + iargz$$

Example 11: Find the logarithm of z=1+i1.

$$w = lnz = ln|z| + iargz = ln\sqrt{2} + i(\frac{\pi}{4} + 2n\pi)$$

Thus, the *lnz* is a multi-valued function.

The principal value of the *lnz* for the given example is:

$$w = lnz = ln|z| + iargz = ln\sqrt{2} + i\frac{\pi}{4}$$

Analytic Functions and Differentiability

Logarithmic function

$$w = lnz = u + iv = lnr + \mathrm{i}\theta = ln|z| + iargz$$

lnz is a multi-valued function.

$$w = lnz = lnr + \mathrm{i}\theta = ln|z| + i(\theta + 2n\pi) \qquad -\pi < \theta \leq \pi$$

The principal value of the lnz is:

$$w = lnz = lnr + \mathrm{i}\theta = ln|z| + i\theta \qquad -\pi < \theta \leq \pi$$

Analytic Functions and Differentiability

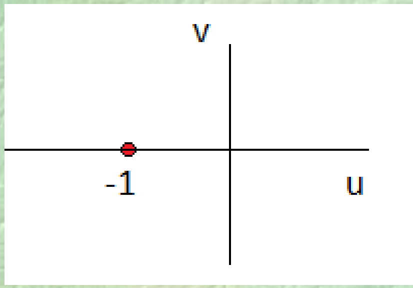
Logarithmic function

$$w = lnz = lnr + \mathrm{i}\theta = ln|z| + i\theta \qquad -\pi < \theta \leq \pi$$

It is evident that $ln(z)$ is discontinuous at z=0.

Consider an arbitrary point on the negative real axis, for example, -1.

$$w = \ln(-1) = lnr + \mathrm{i}\theta = 0 + i\pi \quad or \quad 0 + i(-\pi)^{+}$$



Therefore, $ln(z)$ is also discontinuous at every point on the negative real axis.

Exercise 23: Show that $ln(z)$ is analytic in the entire z-plane except at the origin and the negative real axis.

Analytic Functions and Differentiability

Theorem: The principal value of *ln*(*z*) satisfies the following relationships:

$$lnz_1z_2 = \begin{cases} lnz_1 + lnz_2 - 2i\pi & \pi < argz_1 + argz_2 \leq 2\pi \\ lnz_1 + lnz_2 & -\pi < argz_1 + argz_2 \leq \pi \\ lnz_1 + lnz_2 + 2i\pi & -2\pi < argz_1 + argz_2 \leq -\pi \end{cases}$$

$$ln\frac{z_1}{z_2} = \begin{cases} lnz_1 - lnz_2 - 2i\pi & \pi < argz_1 - argz_2 \leq 2\pi \\ lnz_1 - lnz_2 & -\pi < argz_1 - argz_2 \leq \pi \\ lnz_1 - lnz_2 + 2i\pi & -2\pi < argz_1 - argz_2 \leq -\pi \end{cases}$$

$$lnz^m = mlnz - 2ki\pi$$ Where m is an integer number

where k is a unique integer such that:

$$\left(\frac{m}{2\pi}\right)argz - \frac{1}{2} \leq k < \left(\frac{m}{2\pi}\right)argz + \frac{1}{2}$$

Analytic Functions and Differentiability

Complex exponential form

$$z^a = \exp(alnz)$$

Since $ln(z)$ is multi-valued, the expression above is generally such that:

$$z^a = \exp(alnz) = \exp\{a[ln|z| + i(\theta_1 + 2n\pi)]\}$$

$$= e^{aln|z|}e^{ia(\theta_1+2n\pi)}$$

Analytic Functions and Differentiability

Example 12: Find the principal value of $(1+i)^{2-i}$.

Answer: By definition:

$$(1+i)^{2-i} = \exp[(2-i)\ln(1+i)] = \exp\{(2-i)\left[ln\sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right)\right]\}$$

The principal value of this expression, obtained by setting n=0, is:

$$\exp\left[(2-i)\left(ln\sqrt{2} + i\frac{\pi}{4}\right)\right] = \exp\left[\left(2ln\sqrt{2} + \frac{\pi}{4}\right) + \mathrm{i}\left(-ln\sqrt{2} + \frac{\pi}{2}\right)\right]$$

$$= exp\left(ln2 + \frac{\pi}{4}\right)\left[cos\left(\frac{\pi}{2} - ln\sqrt{2}\right) + isin\left(\frac{\pi}{2} - ln\sqrt{2}\right)\right]$$

$$= e^{1.479}(cos1.224 + isin1.224) = 1.491 + 4.127i$$

Analytic Functions and Differentiability

Inverse trigonometric and hyperbolic functions

$$w = \cos^{-1} z$$

$$z = cosw = \frac{e^{iw} + e^{-iw}}{2}$$

$$e^{2iw} - 2ze^{iw} + 1 = 0$$

$$e^{iw} = z \pm \sqrt{z^2 - 1}$$

$$iw = \ln[z \pm \sqrt{(z^2 - 1)}]$$

$$w = \cos^{-1} z = -i \ln[z \pm \sqrt{(z^2 - 1)}]$$

Analytic Functions and Differentiability

Exercise 24: Derive the following formulas.

$$\sin^{-1} z = -i \ln[iz \pm \sqrt{(1 - z^2)}]$$

$$\tan^{-1} z = \frac{i}{2} \ln \frac{i + z}{i - z}$$

$$\sinh^{-1} z = \ln[z \pm \sqrt{(z^2 + 1)}]$$

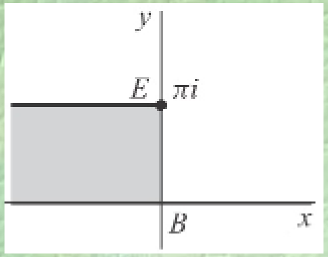
$$\cosh^{-1} z = \ln[z \pm \sqrt{(z^2 - 1)}]$$

$$\tanh^{-1} z = \frac{1}{2} \ln \frac{1 + z}{1 - z}$$

Exercises

Exercise 25: What curve does the transformation $1/z$ map the line $y=2x+2$ to in the w-plane? The exact relationship is needed.

Exercise 26: Determine the mapping of the opposite region using the function $f(z)=cosh(z)$.



Examine the conformality of the mapping using the resulting figure and the related condition.

Exercise 27: In linear control theory, it is sometimes necessary to map the left half of the y-axis in the z-plane into a circle centered at the origin with a unit radius in the w-plane. Introduce at least two functions (transformations) that achieve this.

Exercise 28: Find the principal values of the following functions at the point $z=2+i3$.

$w_1 = coshz$ $\quad w_2 = sinz$ $\quad w_3 = e^z$ $\quad w_4 = lnz$

$w_5 = z^{2-i}$ $\quad w_6 = e^{sinz}$ $\quad w_7 = e^{sinz}$ $\quad w_8 = sinh^{-1}z$

References

- Advanced Engineering Mathematics , E. Kreyszig
- Advanced Engineering Mathematics, C. R. Wylie
- Complex Variables and Applications, J. Brown and R. Churchill